Möbius function for modules and thin representations

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March 12, 2024

Abstract

This paper studies the Möbius function and related questions about the finiteness of the poset of submodules of semisimple and general modules. We show how to calculate the Möbius function for semisimple modules based on endomorphism rings of simple submodules. We discuss the Möbius function for representations of bounded path algebras in detail.

Keywords: Möbius function, semisimple modules, thin representations,

Mathematics Subject Classification: 16G20, 16D60; Secondary: 16P10, 06A07

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1 Introduction

The Möbius function and the Möbius inversion formula are classical tools in number theory. It was later realised that the Möbius function can be defined for arbitrary locally finite posets, seeing the *number-theoretic* Möbius function as a particular case for the poset of natural numbers. This combinatorial view of Möbius's function is usually associated with G.C. Rota and his article [6], ca. 1964. However, it first appeared in the mid-1930s in independent works of P. Hall and L. Weisner, motivated by the study of p-groups. Starting from the late 1970s, several generalisations of the Möbius function for categories appeared, each assuming some *finitness* condition on the category; see [5] for a bibliography on the subject and detailed discussion of several versions of the Möbius function for categories.

The category of modules, even when restricted to a full subcategory of modules with finitely many submodules, is too *large* for categorical techniques of Möbius inversion to work. Instead, for a module, we consider a poset of its submodules, partially ordered by inclusion, using Rota's definition of the Möbius function; see Section 4. This is inspired by the work of Honold and Nechaev, who used Möbius inversion in algebraic coding theory. In their article, [4], the Möbius function of a finite module over a finite ring is studied. This result was later used in [3] to give a combinatorial characterisation of finite Frobenius rings.

For a module with finitely many submodules, we show how to calculate its Möbius function in Section 4.2. We use the same combinatorial techniques as in [4], formulated in Subsection 4.1, and their results can be seen as a special case of the results. However, their proofs strongly depend on the fact that they work over finite and hence semilocal rings.

For this reason, we present a series of module-theoretic observations that might be of independent interest in Section 3. In particular, Subsection 3.1 studies when the poset of submodules of a direct product of two modules is a product of their respective lattices. Subsection 3.2 then studies finite direct powers of simple modules based on the endomorphism ring of the said simple module. The Subsection 3.3 then discusses when a modular lattice with chain conditions is finite.

*This work is a part of project SVV-2023-260721

Section 5 discusses a particular case of bounded path algebras. We show how to calculate the Möbius function directly from the dimension vector. Using results from Section 3.3, it is shown that a representation of a bound quiver algebra over an infinite field has finitely many subrepresentations if and only if it is a thin representation.

2 Prelimanaries and notation

This section recalls some properties of modules, posets and representations that will be used throughout the text.

By a ring, we always mean an *associative ring with unity*, and R always denotes a ring. All modules are assumed to be left modules; all ideals are assumed to be left ideals - the right version is analogous.

For an element $a \in M$ we write $Ann(a) = \{r \in R \mid ra = 0\}$ viewed as a left ideal. For a module M and $a \in M$ and an R-homorphsism $\phi\colon M \to N$ there si an inclusion $Ann(a) \subseteq Ann(\phi(a))$. If the morphism ϕ is an isomorphism, then $Ann(a) = Ann(\phi(a))$.

In several places, we will implicitly use Shur's lemma in several forms. For a simple R-module S, a nonzero homomorphism whose domain (codomain) is S is a monomorphism (epimorphism). Consequently, the ring of endomorphisms, $End_R(S)$, is a division ring, as all nonzero endomorphisms are isomorphisms.

2.1 Poset of submodules

Let P be a poset and x,$y \in P$. The *interval* $[x,y]$ is a subposet of elements z such as $x \leq z \leq y$. A poset P is *locally finite* if all intervals in P are finite. A locally finite bounded poset is finite. It is exactly locally finite posets for which the Möbius function, in the sense of [6], is definable; see Section 4.

For an R-module M, its poset of submodules is denoted by $\mathcal{L}(M)$. It is a complete modular lattice. In particular, $\mathcal{L}(M)$ is a bounded lattice; hence, it is locally finite if and only if it is finite. In that case, module M has a finite composition length, i.e., the lattice $\mathcal{L}(M)$ satisfies both ascending and descending chain conditions. In particular, if such a module is nonzero, it has nonzero socle, $Soc\ M$, and maximal submodules. Minimal (maximal) submodules correspond to *atoms* (coatoms) in $\mathcal{L}(M)$.

By the Correspondance theorem, for a module M and its submodule $N \leq M$, there is a canonical lattice isomorphism between the interval $[N, M]$ in $\mathcal{L}(M)$ and the lattice $\mathcal{L}(M/N)$. Its restriction to maximal modules containing N is a bijection.

Recall that an R-module is *semisimple* iff M is generated by simple submodules, iff it is a direct sum of simple modules, iff all its submodules are direct summands.

2.2 Representations of bound quivers

This subsection formulates terminology and some properties of representations of bound quivers. We refer to [ASS] for missing terminology. Throughout this text, K always denotes a field.

A *quiver* Q is a quadruple (Q_0, Q_1,s,t) where Q_0 is a nonempty *finite* set of *vertices*, Q_1 a finite set of *arrows* and s and t are two maps $Q_1 \to Q_0$ mapping an arrow to its *source* and *target*, respectively. Vertex a is called a *sink* if there is no arrow with source in a. Quiver is called *acyclic* if it contains no oriented cycles. A *full subquiver* of Q is a quadruple (Q_0', Q_1', s', t') such that $Q_0' \subseteq Q_0$ and Q_1' consists of all arrows in Q_1 such that their target and source lie in Q_0'. Maps s' and t' are restrictions of s and t respectively.

For a field K, a quiver Q, and I an admissible ideal in the path algebra KQ, we consider category $rep_K(Q, I)$ of finite-dimensional K-linear representations of Q bounded by relations in I. A representation $M \in rep_K(Q,I)$ consists of a collection of finite-dimensional vector spaces M_a for each $a \in Q_0$ and K-linear *structural* maps M_α for each $\alpha \in Q_1$. A morphism Φ between two representations N and M is given by a collection of K-linear maps $\phi_a\colon N_a \to M_a$ for each $a \in Q_0$ commuting with structural maps. For a bounded path algebra KQ/I, the category *mod*-KQ/I is equivalent with $rep_K(Q, I)$.

We will use the convention that vertices in Q_0 are labelled by natural numbers $1, \ldots, |Q_0|$. *Dimension vector* of a K-representation M is then $dim(M) = [dim_K(M_1), \ldots, dim_K(M_{|Q_0|})]$. A representation M is called *thin* if $dim_K(M_a) \leq 1$ for all $a \in Q_0$.

There is a bijection between the isomorphism classes of simple modules and Q_0. We denote $S(a)$ the simple representation such that $S(a)_a = K$ and $S(a)_b = 0$ for $a \neq b \in Q_0$.

Let $a \in Q_0$ and consider ϵ_a the primitive idempotent in KQ/I corresponding to a stationary path a. For any $k \in K$, the element ke_a induces an endomorphism of $S(a)$. Thus we get

$$|End_{KQ/I}(S)| = |K|$$

for any simple representation $S \in rep_K(Q,I)$.

Throughout the text, we will use the following lemma

Lemma 1 (1, Lemma III.2.2)**.** *Let Q be a quiver, I an admissible ideal in KQ and let $M \in rep_K(Q,I)$ then*

(a) M is semisimple iff $M_\alpha = 0$ for all arrows $\alpha \in Q_1$.

(b) Soc M is a subrepresentation of M such that Soc $M_a = M_a$ if a is a sink and

$$Soc\ M_a = \bigcap_{\substack{\alpha \in Q_0 \\ s(\alpha)=a}} Ker(\alpha),$$

otherwise. Structural maps in Soc M are restriction of structural maps in M.

3 Poset of submodules

This section gathers the module-theoretic properties used in the sequel. Results from Subsection 3.1 will allow us to reduce the calculation of the Möbius function to the case of finite direct powers of simple modules. These modules are then discussed in Subsection 3.2. The final subsection 3.3 then provides a criterion for deciding whether a module has finitely many submodules that will be used in Section 5.

Bounded representations of quivers are used as examples to illustrate the results of this section. For the used notation, we refer the reader to section 2.2 or the monography [1] for more detailed exposition.

3.1 Orthocyclic modules

This section investigates when the poset of submodules of a direct product of modules can be viewed as a product of their respective posets. We show that there is no nonzero homomorphism between such modules or their submodules. Proposition 7 then shows that for semisimple modules, the opposite implication is also true.

Definition 2. *Let $(P, \leq_1)$ and $(Q, \leq_2)$ be posets. Then the* product poset *is $(P \times Q, \leq)$ where*

$$(a,b) \leq (c,d) \ \Leftrightarrow \ a \leq_1 c \text{ and } b \leq_2 d$$

Let M,N be R-modules. Then there is a canonical poset monomorphism

$$\nu \colon \mathcal{L}(M) \times \mathcal{L}(N) \hookrightarrow \mathcal{L}(M \oplus N)$$

given by

$$(N',\ M') \mapsto N' \oplus M' \quad \text{for } N' \leq N,\ M' \leq M$$

We say that M and N are poset-orthogonal *if ν is a poset isomorphism.*

Note that two modules M, N are poset-orthogonal if and only if for any $L \subseteq M \oplus N$ there are modules $M' \leq M$ and $N' \leq N$ such that $L = M' \oplus N'$.

Example 3. *Let K be any field and let A be a path K-algebra given by the quiver*

$$1 \xleftarrow{\alpha} 2 \xrightarrow{\beta} 3$$

and consider representations

$$L\colon \quad (K \xleftarrow{id} K \xrightarrow{0} 0)$$
$$M\colon \quad (0 \xleftarrow{0} K \xrightarrow{id} K)$$
$$N\colon \quad (K \xleftarrow{id} K \xrightarrow{id} K)$$

then L and M are poset-orthogonal, whereas $M \oplus N$ contains $(0 \xleftarrow{0} 0 \xrightarrow{0} K^2)$ as a submodule.

Definition 4. *Let M,N be two R-modules.*

We say that M and N are orthocyclic *if for all submodules $M' \leq M$ and $N' \leq N$ we have $Hom_R(M',N) = 0 = Hom_R(N',M)$.*

The name *orthocyclic* is motivated by the observation that the above condition can be restricted only to cyclic modules. For $M' \leq M$, any nonzero homomorphism $\phi\colon M' \to N$ can be restricted to a nonzero map $Ra \to R\phi(a)$ for some $a \in M'$ not contained in the kernel of ϕ.

Example 5. *Let K be any field and let A be a path K-algebra given by the quiver*

$$1 \xleftarrow{\alpha} 2 \xrightarrow{\beta} 3$$

and consider representations

$$M' \hookrightarrow M: \quad (0 \xleftarrow{0} 0 \xrightarrow{0} K) \hookrightarrow (0 \xleftarrow{0} K \xrightarrow{id} K)$$
$$N\colon \quad (K \xleftarrow{id} K \xrightarrow{id} K)$$

Then $Hom_A(N,M) \cong K$ whereas $Hom_A(M,N) = 0$. Although $Hom_A(M,N)$ is trivial, there is a nonzero homomorphism from a subrepresentation of M' to N and $Hom(M',N) \cong K$.

Proposition 6. *Let M,N be R-modules.*

If they are poset-orthogonal, then they are also orthocyclic.

Proof. We use proof by contraposition. WLOG assume that there is a cyclic submodule $Ra \leq M$ and a nonzero homomorphism $\phi\colon Ra \to N$ and consider a submodule of $M \oplus N$ generated by $(a,\phi(a))$. Then $Ann(a) \subseteq Ann(\phi(a))$ hence for any $r \in R$ if $r\phi(a) = 0$ then $r(a,\phi(a)) = 0$. Therefore, no nonzero submodule of $R(a,\phi(a))$ is strictly contained in N. In particular, N does not contain a nonzero *direct summand* of $R(a,\phi(a))$. $\square$

For semisimple modules, the opposite implication is also true.

Proposition 7. *Let M be a semisimple module, S be a simple module and $t \geq 1$ a natural number. TFAE*

(1) M and S^t are orthocyclic.

(2) M and S^t are poset-orthogonal.

(3) None of the simple direct summands of M is isomorphic to S.

Proof. By the previous proposition, (2) implies (1). To show that (1) implies (3), assume that there is a simple direct summand $T \leq_{\oplus} M$ isomorphic to S. Composing the isomorphism $T \cong S$ with an inclusion $S \hookrightarrow S^t$ gives a nonzero homomorphism in $Hom_R(T,S^t) \neq 0$, so M is not orthocyclic.

It remains to show that (3) implies (2). Because all submodules of $M \oplus S^t$ are semisimple, it is enough to prove that for any simple submodule $T \leq M \oplus S^t$, it is either contained in M or S^t. Let (a,b) be a generator of T and assume $a \neq 0 \neq b$. If $Ann(a) = Ann(b)$ then we have a series of isomorphisms:

$$Ra \cong R/Ann(a) = R/Ann(b) \cong Rb$$

but $Ra \subseteq M$ and $Rb \subseteq S^t$, hence the isomorphism $Ra \cong Rb$ contradictis (3).

For the case, $Ann(a) \neq Ann(b)$, first assume that there exists $r \in Ann(a) \setminus Ann(b)$. Then $r(a,b) = (0,rb)$. Because $T = R(a,b)$ is simple, and $rb \neq 0$ we get $Rrb = T$ hence $T \subseteq S^t$. If there is $r \in Ann(b) \setminus Ann(a)$, we get $T \subseteq M$. $\square$

3.2 Finite direct powers of simple modules

This section characterises when a semisimple module has only finitely many submodules. Following Proposition 7, it is enough to discuss the case where the module is of type S^t for some simple module S and $t \in \mathbb{N}$. If S is finite, then the module S^t has only finitely many modules. For bound quiver algebras, modules of form S^t have finitely many submodules iff S is finite, i.e., the base field of the algebra is finite.

In general, the number of submodules of a module of type S^t follows from the cardinality of the division ring $End_R(S)$ as shown in Lemma 9 and Corollary 11. The following example shows that an infinite simple module may have a finite ring of endomorphisms.

Example 8. *Let F be a finite field and κ an infinite cardinal, $n \in \mathbb{N}$.*

(1) Module $S := F^{(\kappa)}$ is an infinite simple $End_F(S)$-module with a finite ring of endomorphisms.

(2) Module $T := F^n$ is a simple $End_F(F^n)$-module and its endomorphisms are in bijection with elements of F.

Proof. We only prove (1). Let $R := End_F(S)$ and consider $\phi \in End_R(S)$, i.e., an R-linear map $F^{(\kappa)} \to F^{(\kappa)}$ commuting with any endomorphism of $F^{(\kappa)}$. Then, in particular, it commutes with any restrictions of an endomorphism of $F^{(\kappa)}$ to a finite-dimensional subspace. Thus, ϕ must be a map that multiplies an element by some scalar $f \in F$.

On the other hand, any such map is an endomorphism, as elements from R commute with multiplication by scalars. Thus $End_R(S) \cong F$.

□

Lemma 9. *Let S be a simple R-module such that $End_R(S)$ is infinite.*

Then $\mathcal{L}(S^2)$ is infinite.

Proof. We fix some nonzero element $a \in S$, and for each nonzero $\phi \in End_R(S)$, we consider a cyclic module $R(a,\phi(a))$. This module is simple: consider a map

$$R \to R(a,\phi(a)) \qquad r \mapsto (ra,r\phi(a))$$

its kernel is $Ann(a) \cap Ann(\phi(a))$. Because ϕ is an isomorphism we get $Ann(a) = Ann(\phi(a))$ hence $R/Ann(a) \cong R(a,\phi(a))$ is a simple module.

Let $\phi, \psi \in End_R(S)$ be two isomoprhisms. If $R(a,\phi(a)) = R(a,\psi(a))$ then in particular $R(a,\phi(a))$ also contains $(a,\psi(a))$, hence $(0,\phi(a) - \psi(a)) \in R(a,\phi(a))$. Then we get the following inclusions of modules

$$0 \subseteq R(0,\phi(a) - \psi(a)) \subsetneq R(a,\phi(a)).$$

Because $R(a,\phi(a))$ is a simple module we get $R(a,\phi(a) - \psi(a)) = 0$, so in particular $\phi(a) = \psi(a)$. But a generates simple module S, hence $\phi = \psi$.

□

We can easily calculate the number of submodules of any fixed length for a simple module with a finite endomorphism ring. We start by counting simple submodules.

Lemma 10. *Let S be a simple R-module $t,q \in N$ such that $|End_R(S)| = q$.*

Then S^t contains $1 + q + q^2 + \cdots + q^{t-1}$ simple submodules.

Proof. Consider the semisimple ring $Q := End_R(S^t)$ isomorphic to a ring of $t \times t$-matrices over division ring $End_R(S)$. Then, we can view $Hom_R(S, S^t)$ as a *simple* Q-module where the action of an element of $End_R(S^t)$ is given by post-composition. Hence $Hom_R(S,S^t)$ is isomorphic (as a Q-module) to a $1 \times t$-matrix module over the division ring $End_R(S^t)$. Thus, if $End_R(S)$ is a finite field with q elements, there is q^t R-homomorphisms from S to S^t. In particular, $q^t - 1$ monomorphism $S \hookrightarrow S^t$.

Any simple submodule T of S^t is isomorphic with S, so there are $q - 1$ morphisms in $Hom_R(S,S^t)$ with image T corresponding to non-zero elements in $End_R(S)$. Thus there is

$$\frac{q^t - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{t-1}$$

distinct, simple submodules of S^t.

□

There is a bijection between the simple and maximal submodules.

Corollary 11. *Let S be a simple R-module l,t,$q \in N$ such that $|End_R(S)| = q$.*

Then S^t contains

$$\frac{s_{t-l+1} + \cdots + s_t}{s_1 + \cdots + s_l}$$

submodules of length l, where $s_t = q^{t-1} + q^{t-2} + \cdots + q + 1$

An analogous statement is well known for abelian p-groups (here, we use q instead of p); see [2, 48-49]. Once we know the number of simple submodules, the proof is similar. The set of simple submodules $T_1, \ldots, T_k$ of S^t is *independent* iff $l(\sum_{i=1}^{k} T_i) = k$, or equivalently, iff no T_i is in the submodule generated by the remaining modules. Then, we can calculate the number of submodules of fixed length l by calculating the number of *independent* sets of size l contained in S^t and S^l.

Corollary 12. *Let M be a semisimple module. Then $\mathcal{L}(M)$ is finite if and only if for any simple submodule $S \leq M$ such that $End_R(S)$ is infinite, no submodule of M is isomorphic to S^2.*

Any *square-free* semisimple module, e.g. a socle of a commutative Frobenius ring, has finitely many submodules.

Example 13. *Let K be an infinite field, Q a quiver, and M be a K-linear representation of Q. Then $\mathcal{L}(Soc\ M)$ is finite if and only if $Soc\ M$ is a thin representation.*

3.3 Modules with finitely many submodules

For an R-module M, if $\mathcal{L}(M)$ is finite, then M is a finite length module. If R is finite, the opposite implication is also true. As seen in the previous subsection, over a general ring, for example, an infinite field, even a finite-length semisimple module can have infinitely many submodules.

This section shows that a module M has an infinite poset of submodules if and only if there is a factor module M/N such that its socle has infinitely many submodules. This characterisation will be used in Section 5. As a corollary, we obtain a characterisation of rings whose all finite-length modules have finitely many submodules.

Proposition 14. *Let M be a finite-length R-module such that $\mathcal{L}(M)$ is infinite.*

Then, there exists a simple R-module S and a submodule $K \leq M$ such that M/K contains a submodule isomorphic to S^2 and $End_R(S)$ is infinite.

Proof. Consider set

$$\mathcal{M} := \{M' \leq M \mid \mathcal{L}(M') \ is\ infinite\}$$

partially ordered by inclusion. Module M is artinian, so $\mathcal{M}$ has a minimal module N. By the minimality of N, no maximal submodule of N contains infinitely many submodules. Because N has a finite length, this implies that N has infinitely many maximal submodules.

Let N_0 be a maximal submodule of N. By the minimality of N in $\mathcal{M}$ the poset $\mathcal{L}(N_0)$ is finite, and so is the set

$$\{N_0 \cap N' \mid N'\ maximal\ in\ N\} \subseteq \mathcal{L}(N).$$

But N has infinitely many maximal submodules, so there exists a submodule $K \leq N'$ such that the set

$$\mathcal{N} := \{N'\ maximal\ in\ N \mid N' \cap N_0 = K\}$$

is infinite. Note that K is maximal in N_0 and in any module from $\mathcal{N}$.

The canonical projection $\pi\colon M \to M/K$ then induces a lattice isomorphism between interval $[K,M]$ in $\mathcal{L}(M)$ and $\mathcal{L}(M/K)$. Modules from $\mathcal{N}$ all contain K as a maximal submodule, so their images are distinct, simple modules in M/K.

□

It seems natural to ask whether it is enough only to investigate socles of *some* factors - such as socles in the socle series. We will show a module with infinitely many submodules whose all socles in the socle series have only finitely many submodules in Example 26.

Corollary 15. *Let R be a ring.*

All finite-length R-modules have finitely many submodules if and only if all simple R-modules have a finite ring of R-endomorphisms.

To ensure that all *finitely generated* R-modules have finitely many submodules, we need that the regular module R is finite length. This implies that R is left artinian and the factor $R/rad\ R$ is finite. Hence the ring R is finite.

4 Möbius function

This section defines the Möbius function and completely determines the Möbius function for a module with finitely many submodules. We will use [6] as a reference.

For a locally finite poset P and a field $\mathbb{R}$, we consider an *incidence algebra* $\mathbb{R}P$ consisting of all real-valued functions with domain P^2. For two elements $\alpha,\beta \in \mathbb{R}P$ the multiplication $*$ is then defined

$$(\alpha * \beta)(x,y) = \sum_{x \leq z \leq y} \alpha(x,z)\beta(z,y),$$

if $x \leq y$ and zero otherwise. Kronecker delta is then the multiplicative unit in $\mathbb{R}P$.

The *zeta function* $\zeta_P(x,y)$ is defined as 1 if $x \leq y$ and 0 otherwise. By [Rota, Prop. 1.], the zeta function has a two-sided inverse, called the *Möbius function*, denoted by μ_P.

For a ring R and an R-module M with finitely many submodules, we define the *Möbius function of* M, denoted by $\mu_R(M)$, as the value of the Möbius function $\mu_{\mathcal{L}(M)}$ on $(0, M)$ in the incidence algebra $\mathbb{R}\mathcal{L}(M)$. This allows us to define the Möbius function recursively using the fact that the Möbius function is a left inverse of the zeta function in $\mathbb{R}\mathcal{L}(M)$.

Definition 16. *Let M be an R-module such that $\mathcal{L}(M)$ is finite.*

We define the Möbius function *of M, denoted by $\mu_R(M)$, recursively by setting $\mu_R(0) := 1$ and if $M \neq 0$ then $\mu_R(M)$ is the unique integer such that*

$$\sum_{N \in \mathcal{L}(M)} \mu_R(N) = 0$$

Example 17. *If S is a simple module, then $\mu_R(S) = -1$. Let M be a module of length 2 with $\mathcal{L}(M)$ finite. If M has a simple socle then $\mu_R(M) = 0$. If M is semisimple with n simple submodules then $\mu_R(M) = n - 1$.*

Remark 1. *Let $N \leq M$ be two R-modules such that $\mathcal{L}(M)$ (and thus also $\mathcal{L}(N)$) is finite , then*

$$\mu_R(N) = \mu_{\mathcal{L}(N)}(0,N) = \mu_{\mathcal{L}(M)}(0,N).$$

Using the correspondence theorem, for two modules $N \leq M$, we have

$$\mu_{\mathcal{L}(M)}(N,M) = \mu_R(M/N).$$

4.1 Combinatorial properties of Möbius function

The following three lemmas are a module-theoretic reformulation of well-known properties of the Möbius function. The reference for this subsection is [6] but Lemma 18 is usually attributed to P. Hall and Lemma 21 to L. Weisner.

Lemma 18 (6, Prop. 3.2)**.** *Let M be a nonzero R-module.*

If M is not semisimple then $\mu_R(M) = 0$.

In the following subsection, we will see that the opposite implication is also true. In a general finite lattice $\mathcal{L}$, there might be an element x that is a join of atoms, yet $\mu_{\mathcal{L}}(0,x) = 0$.

Example 19. *Consider a 6-element lattice $\mathcal{L}$ with the upper and lower bounds 0 and 1 three atoms a,b,c and two more elements $a \vee b$ and $b \vee c$ with $\mu_{\mathcal{L}}(0,a \vee b) = \mu_{\mathcal{L}}(0,b \vee c) = 1$.*

In this lattice, $1 = a \vee b \vee c = a \vee c$ and $\mu(0,1) = 0$.

Lemma 20 (6, Prop. 3.5)**.** *Let M, N be two poset-orthogonal modules such that $\mathcal{L}(M)$ and $\mathcal{L}(N)$ are finite posets.*

Then for any $L \leq M \oplus N$ we have

$$\mu_R(L) = \mu_R(M') \cdot \mu_R(N')$$

where $M' \leq M$ and $N' \leq N$ such that $L = M' \oplus N'$.

Lemma 21 (6, Prop. 5.4)**.** *Let M be an R-module such that $\mathcal{L}(M)$ is finite, and let T be a simple submodule. Then*

$$\mu_R(M) = \sum_{\substack{T \nleq N \leq M \\ N \ maximal}} \mu_R(N)$$

4.2 Calculation of the Möbius function

We assume that all modules in this subsection have only finitely many submodules. By Lemma 18, the Möbius function of a nonzero module that is not semisimple is zero.

Let M be a semisimple module with decomposition

$$M = S_1^{t_1} \oplus \dots S_n^{t_n} \tag{1}$$

where $S_1, \dots, S_n$ are non-isomorphic simple modules and t_i natural numbers.

Becuase we asssume that $\mathcal{L}(M)$ is finite, for any $i \leq m$ the inequality $t_i > 1$ implies that $End_R(S_i)$ is finite by Corollary 12

By Proposition 7 we have

$$\mathcal{L}(M) \cong \mathcal{L}(S_1^{t_1}) \times \dots \times \mathcal{L}(S_t^{t_n}),$$

so by repeatedly apllying Lemma 20 we get

$$\mu_R(M) = \prod_{1 \leq i \leq n} \mu_R(S_i^{t_i}).$$

The following lemma then completes the calculation of the Möbius function.

Lemma 22. *Let S be a simple R-module, t and q natural numbers such that $|End_R(S)| = q$. Then*

$$\mu_R(S^t) = (-1)^t q^{\frac{t(t-1)}{2}}.$$

Proof. We apply Lemma 21. By Corollary 11 we get that S^t contains $1+q+\dots+q^{t-1}$ maximal submodules.

Let T be some simple submodule of S^t. There is a bijection between maximal submodules of S^t containing T and maximal submodules of S^t/T. From an R-module isomorphism $S^t/T \cong S^{T-1}$ then follows, using again Corollary 11, that $1 + q + \dots + q^{t-2}$ maximal submodules of S^t contain fixed simple submodule T. Thus, q^{t-1} maximal submodules do not contain T. By Lemma 21 we see that

$$\mu_R(S^t) = -q^{t-1}\mu_R(S^{t-1}).$$

The conclusion then follows by induction.

□

Using the structure of the above calculation, we prove that Morita equivalence preserves the Möbius function.

Lemma 23. *Let R and R' be two Morita equivalent rings, and let $G : Mod\text{-}R \to Mod\text{-}S$ be an equivalence of categories. Let $M \in Mod\text{-}R$ be a semisimple module with finitely many submodules.*

Then $G(M)$ has finitely many submodules and $\mu_{R'}(G(M)) = \mu_R(M)$.

Proof. Let S be a simple R-module and t a natural number. We first prove the statement for modules of form S^t. Because equivalence preserves direct limits, we get that $G(S^t) \cong G(S)^t$. By Shur's lemma, $G(S)$ is a simple R'-module if and only if S is a simple R-module. Because equivalence is a full and faithful functor, there is a bijection $Hom_R(S, S) \leftrightarrow Hom_{R'}(G(S), G(S))$, i.e., G preserves sizes of endomorphism rings. The statement then follows from Lemma 22.

Now let $M \cong S_1^{t_1} \oplus \dots S_n^{t_n}$ where S_i are simple pairwise non-isomorphic modules, then we get

$$G(M) \cong G(S_1)^{t_1} \oplus \cdots \oplus G(S_n)^{t_n},$$

where $G(S_i)$ are pairwise non-isomorphic simple modules. Thus, $G(M)$ is a semisimple module with finitely many submodules, and the statement then follows from Lemma 20 and Proposition 7.

□

4.3 Möbius inversion formula

This brief section discusses the Möbius inversion formula. The following is a reformulation of a *poset version* of the Möbius inversion formula [6, Prop. 2] for modules.

Proposition 24. *Let M be a module with finitely many submodules. Let f,g be real-valued functions on $\mathcal{L}(M)$ such the value $g(M)$ equals the sum of values of f on all submodules of M. Then*

$$f(M) = \sum_{N \leq M} g(N)\mu_R(M/N).$$

Recall that the radical of a finite-length module is zero if and only if such a module is semisimple. Thus, for a submodule $N \leq M$, we get M/N is semisimple if and only if $rad\ M \leq N$. Using Lemma 18, we get the following reformulation of the Möbius inversion formula:

$$f(M) = \sum_{rad\ M \leq N \leq M} g(N)\mu_R(M/N).$$

5 Möbius function for finite-dimensional algebras

This section studies the Möbius function for K-linear representations of *bound quiver algebras* for acyclic quivers, in the sense defined in [1]. If the field K is infinite, a representation has only finitely many subrepresentations if it is thin. Example 27 shows this is untrue for quivers with oriented cycles, even if the resulting bound quiver algebra is finite-dimensional. The section ends with calculating the Möbius function of semisimple representation directly from its dimension vectors.

We need some preliminary observations. For a quiver Q, we say that a full subquiver Q' is a *sinking* subquiver if any arrow from Q_1 with source in Q'_0 has also target in Q'_0.

For a representation $M \in rep_K(Q,I)$ and a sinking subquiver Q', we define a subrepresentation $R_M(Q') \in rep_K(Q,I)$ as follows: for $a \in Q_0$ we set $R_M(Q')_a = M_a$ if $a \in Q'_0$ and $R_M(Q')_a = 0$ otherwise. Similarly for $\alpha \in Q_1$ we define $R_M(Q')_\alpha = M_\alpha$ if $\alpha \in Q'_1$ and $R_M(Q')_\alpha = 0$ otherwise. The representation $R_M(Q')$ is a subrepresentation of M as witnessed by a homomorphism $\phi\colon R_M(Q') \hookrightarrow M$ such that $\phi_a = id_{M_a}$ if $a \in Q'_0$ and $\phi_a = 0$ otherwise. Because Q' is a sinking quiver, ϕ is monomorphism: let $\alpha \in Q_1$, then either $s(\alpha) \in Q'_0$ and because Q' is a sinking subquiver of Q then $t(\alpha) \in Q'_0$ and both maps $\phi_{s(\alpha)}$ and $\phi_{t(\alpha)}$ are isomorphisms. Otherwise, $M_{s(\alpha)} = 0$, so the corresponding maps commute trivially.

Proposition 25. *Let Q be an acyclic quiver, K an infinite field, I an admissible ideal in KQ and let $M \in rep_K(Q,I)$.*

Then $\mathcal{L}(M)$ is finite if and only if M is thin.

Proof. Using Lemma 1 and Corollary 12, a semisimple representation has finitely many subrepresentations if and only if it is a thin representation. Any factor of a thin representation is thin and thus has a thin socle.

Now assume M is not thin, i.e., there is a vertex $a \in Q_0$ such that $dim_K(M_a) = t > 1$. If a is a sink, then by Lemma 1, $Soc\ M$ contains an isomorphic copy of $S(a)^t$; thus, it has infinitely many subrepresentations.

Assume a is not a sink and let $b_1, \dots, b_k$ be the set of all vertices that are targets of arrows with the source a. To each b_i, we assign the minimal sinking subquiver containing the vertex b_i. We denote it by Q^i. Because Q is acylic, the minimality of Q^i implies that it does not contain vertex a.

For a sinking quiver $Q' = \cup_{i=1}^{k} Q^i$ consider a subrepresentation $R_M(Q')$. We claim that factor representation $M/R_M(Q')$ has a socle with infinitely many subrepresentations. Consider an arrow $\alpha \in Q_1$, if its source is a, then the map $(M/R_M(Q'))_\alpha$ is a zero map because its codomain,$(M/R_M(Q'))_{t(\alpha)}$, is the zero K-vector space. So, using again Lemma 1, the socle of $M/R_M(Q')$ contains an isomorphic copy of $S(a)^t$.

$\square$

Example 26. *Let K be an infinite field and*

$$Q\colon \begin{tikzcd} & 2 \arrow[dr, "\gamma"] & \\ 1 \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & 4 \\ & 3 \arrow[ur, "\delta"'] & \end{tikzcd}$$

a quiver with representation

$$M\colon \begin{tikzcd} & K^2 \arrow[dr, "(1\ 0)"] & \\ K^2 \arrow[ur, "id"] \arrow[dr, "id"'] & & K. \\ & K^2 \arrow[ur, "(0\ 1)"'] & \end{tikzcd}$$

By Lemma 1 dimension vector of $M/Soc\ M$ is [2,1,1,0]. *So the socle of $M/Soc\ M$ is necessarily thin and has only finitely many submodules. So it follows that all socles in the socle series have only finitely many submodules.*

We now directly show that representation M has infinitely many submodules. There are two arrows with source in 1, namely α and β. Each of them corresponds to one sinking subquiver. Namely, Q^2 is a full subquiver given by vertices 2 and 4 and Q^3 given by vertices 3 and 4. Their union Q' is a sinking subquiver given by vertices 2,3,4. Vertex 4 also gives one trivial sinking subquiver.

As in the proof we have

$$R_M(Q')\colon \begin{tikzcd} & K^2 \arrow[dr, "(1\ 0)"] & \\ 0 \arrow[ur, "0"] \arrow[dr, "0"'] & & K. \\ & K^2 \arrow[ur, "(0\ 1)"'] & \end{tikzcd}$$

Thus the factor $M/R_M(Q')$ is isomorphic to $S(1)^2$ a semisimple representation given by dimension vector [2,0,0,0]. *It has infinitely many submodules, and so does M.*

The following example shows that for quivers with oriented cycles, the above proposition does not hold, even when the given bounded path algebra is finite-dimensional.

Example 27. *For infinite filed K and a quiver* $\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\beta"] \end{tikzcd}$ *consider the Frobenius algebra $A := KQ/I$ where I is an admissible ideal $I = (\alpha\beta = 0 = \beta\alpha)$.*

Representation $\begin{tikzcd} K \arrow[r, bend left, "(1\ 0)"] & K^2 \arrow[l, bend left, "(0\ 1)"] \end{tikzcd} \in rep_K(Q, I)$ *is not thin, but it has only finitely many submodules by Corollary 14. Indeed, there is no subpresentation with dimension vector* [1,0], *so all non-trivial factors are thin.*

5.1 Calculation of the Möbius function

Let Q be a finite acyclic quiver, K a field, I an admissible ideal in KQ and let $M \in rep_k(Q, I)$ be a nonzero representation with finitely many submodules.

Following Lemma 1, $\mu(M) = 0$ if and only if M contains a nonzero structural map.

Now assume that M is semisimple, i.e., all structural maps are zero, and let $dim_K(M) = [a_1, \ldots, a_n]$. By Lemma 22

$$\mu(M) = \prod_{1 \leq i \leq n} (-1)^{a_i} q^{\frac{a_i(a_i-1)}{2}},$$

where $q = |K|$ if K is finite and $q = 1$ otherwise.

Aknowledgements: The author wishes to express his thanks and appreciation to Jan Žemlička for his guidance and valuable insights during the research.

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Bibliography

[1] Assem, I. Simson, D. Skowronski, A. (2006). Elements of the Representation Theory of Associative Algebras: Techniques of Representation Theory. *LMS Student Texts*. Cambridge: Cambridge University Press. vol. 65. https://doi.org/10.1017/CBO9780511614309

[2] Burnside, W. (2012). Theory of Groups of Finite Order. Cambridge: Cambridge University Press. https://doi.org/10.1017/CBO9781139237253

[3] Honold, T. (2001). Characterisation of finite Frobenius rings. *Arch. Math.* 76(6). 406–415. https://doi.org/10.1007/PL00000451

[4] Honold, T. Nechaev, A. A. (1999), Weighted modules and representations of codes. *Problems Inform. Transmission.* 35(3). 205 - 223.

[5] Leinster, T. Notions of Möbius inversion. *ArXiv*. (2012). https://doi.org/10.48550/arXiv.1201.0413

[6] Rota, G.C. (1964). On the foundations of combinatorial theory I. Theory of Möbius Functions. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*. 340–368. https://doi.org/10.1007/BF00531932