VOICULESCU'S THEOREM IN PROPERLY INFINITE FACTORS

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Abstract. This paper investigates Voiculescu's theorem on approximate equivalence in separable properly infinite factors. We establish the norm-denseness of the set of all reducible operators and prove Voiculescu's bicommutant theorem. Additionally, we extend these results to the multiplier algebras within separable type III factors.

1. Introduction

A question regarding the norm-denseness of the set of all reducible operators on a separable complex Hilbert space was raised by P.R. Halmos [9, Problem 8]. In order to affirmatively answer this question, D. Voiculescu proved the noncommutative Weyl-von Neumann theorem in his groundbreaking paper [14]. Another significant consequence of Voiculescu's theorem is the relative bicommutant theorem in the Calkin algebra. In [3], W. Arveson provided an alternative proof of Voiculescu's theorem using quasicentral approximate units. Additionally, Arveson derived a distant formula for separable norm-closed algebras in the Calkin algebra. Numerous applications of Voiculescu's theorem can be found in Arveson's work [3].

Throughout this paper, $\mathcal{M}$ represents a separable properly infinite factor, and $\mathcal{K}_{\mathcal{M}}$ represents the norm-closed ideal generated by finite projections in $\mathcal{M}$. Our main focus is on Voiculescu's theorem in $\mathcal{M}$. In the paper, we will present a proof of the following theorem.

Theorem 4.3. *Let $\mathcal{M}$ be a separable properly infinite factor, $\mathcal{A}$ a separable unital C^*-subalgebra of $\mathcal{M}$, and $\mathcal{B}$ a type I_∞ unital subfactor of $\mathcal{M}$. If $\varphi\colon \mathcal{A}\to\mathcal{B}$ is a unital *-homomorphism with $\varphi|_{\mathcal{A}\cap\mathcal{K}_{\mathcal{M}}}=0$, then there exists a sequence $\{V_k\}_{k\in\mathbb{N}}$ of isometries in $\mathcal{M}\otimes M_2(\mathbb{C})$ such that*

$$\lim_{k\to\infty}\|(A\oplus\varphi(A))-V_k^*(A\oplus 0)V_k\|=0 \text{ for every } A\in\mathcal{A},$$

and $V_k^*V_k=I\oplus I, V_kV_k^*=I\oplus 0$ for every $k\in\mathbb{N}$. Furthermore, if $\mathcal{M}$ is semifinite, we can choose $\{V_k\}_{k\in\mathbb{N}}$ such that

$$(A\oplus\varphi(A))-V_k^*(A\oplus 0)V_k\in\mathcal{K}_{\mathcal{M}}\otimes M_2(\mathbb{C}) \text{ for every } k\in\mathbb{N} \text{ and } A\in\mathcal{A}.$$

Recall that an operator T is reducible in a separable properly infinite factor $\mathcal{M}$ if there exists a nontrivial projection P in $\mathcal{M}$ that commutes with T. A striking application of Voiculescu's noncommutative Weyl-von Neumann theorem shows that the set of reducible operators is norm-dense in a separable type I_∞ factor. In the paper, we obtain an extension of Voiculescu's result.

Theorem 5.1. *Let $\mathcal{M}$ be a separable properly infinite factor. Then the set of all reducible operators is norm-dense in $\mathcal{M}$.*

In [13], G.K. Pedersen posed the question of whether Voiculescu's bicommutant theorem can be extended to general corona algebras. In [6], T. Giordano and P.W. Ng provided an affirmative answer to Pedersen's question for the corona algebras of σ-unital stable simple and purely infinite C^*-algebras. Since $\mathcal{M}/\mathcal{K}_{\mathcal{M}}$ serves as the corona algebra of $\mathcal{K}_{\mathcal{M}}$ when $\mathcal{M}$ is semifinite, we affirmatively answer this question specifically for the case of $\mathcal{M}/\mathcal{K}_{\mathcal{M}}$ as an application of Theorem 4.3.

Theorem 5.3. *Let $\mathcal{M}$ be a separable properly infinite semifinite factor. Then every separable unital C^*-subalgebra of $\mathcal{M}/\mathcal{K}_{\mathcal{M}}$ equals its relative bicommutant.*

Let $\mathcal{M}$ be a separable properly infinite semifinite factor. In [10], S. Popa and F. Rădulescu proved that all derivations of a von Neumann subalgebra of $\mathcal{M}$ into $\mathcal{K}_{\mathcal{M}}$ are inner. When $\mathcal{M}$ is of type I_∞, J. Phillips and I. Raeburn [11] showed that for a separable infinite-dimensional C^*-subalgebra $\mathcal{A}$ of $\mathcal{M}$, not all derivations from $\mathcal{A}$ into $\mathcal{K}_{\mathcal{M}}$ are inner. Let $H^1(\mathcal{A}, \mathcal{K}_{\mathcal{M}})$ be the first cohomology group of $\mathcal{A}$ into $\mathcal{K}_{\mathcal{M}}$. As an application of Theorem 5.3, we obtained the following result.

Theorem 5.6. *Let $\mathcal{M}$ be a separable properly infinite semifinite factor, and $\mathcal{A}$ a separable unital C^*-subalgebra of $\mathcal{M}$. If $\pi(\mathcal{A}'')$ is infinite-dimensional, then $H^1(\mathcal{A}, \mathcal{K}_{\mathcal{M}}) \neq \{0\}$.*

This paper is structured as follows. In the next section, we present the fundamental definitions and results. The main theorems are provided in Section 3, and we establish the proof of Voiculescu's theorem in Section 4. Next, we discuss some applications in Section 5. Section 6 focuses on proving analogous results for the multiplier algebras within separable type III factors. Finally, in the last section, we introduce the concept of the nuclear length of C^*-algebras.

Acknowledgment. After our paper was typed up, we learned from our private communication with P. W. Ng that Giordano, Kaftal and Ng also obtained similar results with different proofs, including (i) noncommutative Weyl-von Neumann theorem, (ii) Double commutant theorem for type II_∞ factors, and (iii) Asymptotic double commutant theorem for type III factors.

2. Preliminaries

2.1. **Separable Properly Infinite Factors.** Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra consisting of all bounded operators on $\mathcal{H}$. A selfadjoint unital subalgebra of $\mathcal{B}(\mathcal{H})$ is said to be a *von Neumann algebra* if it is closed in the strong-operator topology. A *factor* is a von Neumann algebra whose center consisting of scalar multiples of the identity. A von Neumann algebra is considered *separable* if it has a separable predual space (see [12, Theorem 7.4.2]).

Factors are classified into *finite factors* and *properly infinite factors* determined by a relative dimension function of projections. Properly infinite factors can be further classified into properly infinite semifinite factors, namely type $\mathrm{I}_\infty, \mathrm{II}_\infty$ factors, and purely infinite factors, namely type III factors. For further details, please refer to [12].

Throughout, let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a separable properly infinite factor. We denote the identity of $\mathcal{M}$ as $I_{\mathcal{M}}$, or simply I. Two projections $P, Q \in \mathcal{M}$ are said to be *Murray-von Neumann equivalent*, denoted by $P \sim Q$, if $V^*V = P$ and $VV^* = Q$ for some partial isometry $V \in \mathcal{M}$. A projection P in $\mathcal{M}$ is said to be *infinite* if it is Murray-von Neumann equivalent to a proper subprojection in $\mathcal{M}$. Otherwise, P is said to be *finite*.

Let $\mathcal{K}_{\mathcal{M}}$ be the norm-closed ideal generated by finite projections in $\mathcal{M}$. Note that $\mathcal{K}_{\mathcal{M}} = \{0\}$ if $\mathcal{M}$ is of type III, and $\mathcal{K}_{\mathcal{M}}$ is strong-operator dense in $\mathcal{M}$ if $\mathcal{M}$ is semifinite. Note that, if $\mathcal{M}$ is of type I_∞, then $\mathcal{M}$ is *-isomorphic to $\mathcal{B}(\mathcal{H}_0)$, where $\mathcal{H}_0$ is a separable infinite-dimensional complex Hilbert space. In this case, $\mathcal{K}_{\mathcal{M}}$ is the set of compact operators in $\mathcal{M}$ and $\mathcal{M}/\mathcal{K}_{\mathcal{M}}$ is *-isomorphic the Calkin algebra.

2.2. **Factorable Maps with Respect to $\mathcal{K}_{\mathcal{M}}$.** Let $\mathcal{A}$ be a unital C^*-subalgebra contained in $\mathcal{M}$. Typically, a completely positive map $\psi\colon \mathcal{A} \to \mathcal{M}$ is called *factorable* if $\psi = \eta \circ \sigma$ for some completely positive maps $\sigma\colon \mathcal{A} \to M_n(\mathbb{C})$ and $\eta\colon M_n(\mathbb{C}) \to \mathcal{M}$. Furthermore, φ is said to be *nuclear* if it can be approximated in the pointwise-norm topology by factorable maps.

$$\begin{tikzcd} & M_n(\mathbb{C}) \arrow[dr, "\eta"] & \\ \mathcal{A} \arrow[ur, "\sigma"] \arrow[rr, "\psi"'] & & \mathcal{M} \end{tikzcd}$$

Definition 2.1. *Let $\psi\colon \mathcal{A} \to \mathcal{M}$ be a completely positive map with $\psi|_{\mathcal{A}\cap\mathcal{K}_{\mathcal{M}}} = 0$. If $\psi = \eta \circ \sigma$ for some completely positive maps $\sigma\colon \mathcal{A} \to M_n(\mathbb{C})$ and $\eta\colon M_n(\mathbb{C}) \to \mathcal{M}$ with $\sigma|_{\mathcal{A}\cap\mathcal{K}_{\mathcal{M}}} = 0$, then we say that ψ is* factorable with respect to $\mathcal{K}_{\mathcal{M}}$.

Let $\mathfrak{F} = \mathfrak{F}(\mathcal{A}, \mathcal{M}, \mathcal{K}_{\mathcal{M}})$ denote the set of all factorable maps with respect to $\mathcal{K}_{\mathcal{M}}$ from $\mathcal{A}$ into $\mathcal{M}$.

By definition, the set $\mathfrak{F}(\mathcal{A}, \mathcal{M}, \mathcal{K}_{\mathcal{M}})$ is a cone. Specifically, let $\psi_j = \eta_j \circ \sigma_j$, $j = 1, 2$ for some completely positive maps

$$\sigma_j\colon \mathcal{A} \to M_{n_j}(\mathbb{C}), \quad \eta_j\colon M_{n_j}(\mathbb{C}) \to \mathcal{M},$$

with $\sigma_j|_{\mathcal{A}\cap\mathcal{K}_{\mathcal{M}}} = 0$. We can define completely positive maps as follows:

$$\sigma\colon \mathcal{A} \to M_{n_1+n_2}(\mathbb{C}), \quad A \mapsto \sigma_1(A) \oplus \sigma_2(A),$$

and

$$\eta\colon M_{n_1+n_2}(\mathbb{C}) \to \mathcal{M}, \quad \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \mapsto \eta_1(X_{11}) + \eta_2(X_{22}).$$

Therefore, $\psi_1 + \psi_2 = \eta \circ \sigma \in \mathfrak{F}(\mathcal{A}, \mathcal{M}, \mathcal{K}_{\mathcal{M}})$.

Definition 2.2. *Let $\widehat{\mathfrak{F}} = \widehat{\mathfrak{F}}(\mathcal{A}, \mathcal{M}, \mathcal{K}_{\mathcal{M}})$ denote the closure of $\mathfrak{F}$ in the pointwise-norm topology. In other words, a map $\varphi\colon \mathcal{A} \to \mathcal{M}$ lies in $\widehat{\mathfrak{F}}$ if for any finite subset $\mathcal{F}$ of $\mathcal{A}$ and any $\varepsilon > 0$, there exists a map $\psi \in \mathfrak{F}$ such that $\|\varphi(A) - \psi(A)\| < \varepsilon$ for every $A \in \mathcal{F}$.*

It is clear that every map in $\widehat{\mathfrak{F}}$ is completely positive and vanishes on $\mathcal{A} \cap \mathcal{K}_{\mathcal{M}}$. Maps in $\widehat{\mathfrak{F}}$ are said to be *nuclear with respect to $\mathcal{K}_{\mathcal{M}}$*.

Example 2.3. *Let $\varphi\colon \mathcal{A} \to \mathcal{M}$ be a unital *-homomorphism with $\varphi|_{\mathcal{A}\cap\mathcal{K}_{\mathcal{M}}} = 0$. If the inclusion $\mathrm{id}_{\varphi(\mathcal{A})}\colon \varphi(\mathcal{A}) \to \mathcal{M}$ is a nuclear map, then the composition $\varphi = \mathrm{id}_{\varphi(\mathcal{A})} \circ \varphi$ is a nuclear map with respect to $\mathcal{K}_{\mathcal{M}}$. There are two examples.*

(1) *Let $\mathcal{A}$ be a nuclear C^*-algebra. Since $\varphi(\mathcal{A})$ is a nuclear C^*-algebra, the inclusion map $\mathrm{id}_{\varphi(\mathcal{A})}\colon \varphi(\mathcal{A}) \to \mathcal{M}$ is automatically nuclear.*

(2) *Let $\mathcal{M}$ be an injective factor, and $\mathcal{A}$ an exact C^*-algebra. Since $\varphi(\mathcal{A})$ is an exact C^*-algebra, there exists an nuclear embedding $\pi\colon \varphi(\mathcal{A}) \to \mathcal{B}(\mathcal{H}_0)$ for some complex Hilbert space $\mathcal{H}_0$. By the injectivity of $\mathcal{M}$, $\mathrm{id}_{\varphi(\mathcal{A})} \circ \pi^{-1}\colon \pi(\varphi(\mathcal{A})) \to \mathcal{M}$ extends to a completely positive map $\psi\colon \mathcal{B}(\mathcal{H}_0) \to \mathcal{M}$. Therefore, $\mathrm{id}_{\varphi(\mathcal{A})} = \psi \circ \pi$ is nuclear.*

Lemma 2.4. *Let $\{\psi_n\}_{n\in\mathbb{N}}$ be a sequence of completely positive maps from $\mathcal{A}$ into $\mathcal{M}$. If the series $\sum_{n\in\mathbb{N}} \psi_n(I)$ converges in the strong-operator topology, then $\sum_{n\in\mathbb{N}} \psi_n(A)$ converges in the strong-operator topology for every $A \in \mathcal{A}$.*

Proof. Recall that $\mathcal{M}$ acts on a complex Hilbert space $\mathcal{H}$. Let $x_1, x_2, \ldots, x_k$ be vectors in $\mathcal{H}$, $A \in \mathcal{A}$, and $\varepsilon > 0$. Since $\sum_{n\in\mathbb{N}} \psi_n(I)$ converges in the strong-operator topology, $\|\sum_{n\in\mathbb{N}} \psi_n(I)\| < \infty$ by the uniform boundedness principle. Moreover, there exists a natural number N such that for any integer $s \geqslant r \geqslant N$, we have

$$\|A\|^2 \Big\| \sum_{n\in\mathbb{N}} \psi_n(I) \Big\| \Big\langle \sum_{n=r}^{s} \psi_n(I) x_j, x_j \Big\rangle < \varepsilon^2 \text{ for } 1 \leqslant j \leqslant k. \tag{2.1}$$

According to Stinespring's dilation theorem, for any completely positive map $\psi\colon \mathcal{A} \to \mathcal{M}$, we have

$$\|\psi(A)x_j\|^2 \leqslant \|A\|^2 \|\psi(I)\| \langle \psi(I)x_j, x_j \rangle. \tag{2.2}$$

Since the map $\sum_{n=r}^{s}\psi_n$ is completely positive, from (2.1) and (2.2), we obtain that

$$\Big\|\sum_{n=r}^{s}\psi_n(A)x_j\Big\| < \varepsilon \text{ for } 1 \leqslant j \leqslant k.$$

This completes the proof. □

By Lemma 2.4, we are able to define the infinite sum of a sequence of completely positive maps.

Definition 2.5. *Let $\mathfrak{S}\mathfrak{F} = \mathfrak{S}\mathfrak{F}(\mathcal{A},\mathcal{M},\mathcal{K}_{\mathcal{M}})$ denote the set of all maps of the form $\sum_{n\in\mathbb{N}}\psi_n$, where $\psi_n \in \mathfrak{F}(\mathcal{A},\mathcal{M},\mathcal{K}_{\mathcal{M}})$ for every $n\in\mathbb{N}$, and the series $\sum_{n\in\mathbb{N}}\psi_n(I)$ converges in the strong-operator topology.*

Let $\widehat{\mathfrak{S}\mathfrak{F}} = \widehat{\mathfrak{S}\mathfrak{F}}(\mathcal{A},\mathcal{M},\mathcal{K}_{\mathcal{M}})$ denote the closure of $\mathfrak{S}\mathfrak{F}$ in the pointwise-norm topology.

The definition of $\widehat{\mathfrak{S}\mathfrak{F}}$ is similar to $\widehat{\mathfrak{F}}$, see Definition 2.2. The following lemma shows that $\widehat{\mathfrak{S}\mathfrak{F}}$ is closed under countable addition.

Lemma 2.6. *If $\{\psi_n\}_{n\in\mathbb{N}}$ is a sequence in $\widehat{\mathfrak{S}\mathfrak{F}}$ such that $\sum_{n\in\mathbb{N}}\psi_n(I)$ converges in the strong-operator topology, then $\sum_{n\in\mathbb{N}}\psi_n \in \widehat{\mathfrak{S}\mathfrak{F}}$.*

Proof. Let $\mathcal{F}$ be a finite subset of $\mathcal{A}$ containing I, and $\varepsilon > 0$. For each $n\in\mathbb{N}$, there is a sequence $\{\psi_{n,m}\}_{m\in\mathbb{N}}$ in $\mathfrak{F}(\mathcal{A},\mathcal{M},\mathcal{K}_{\mathcal{M}})$ such that

$$\Big\|\psi_n(A) - \sum_m \psi_{n,m}(A)\Big\| < 2^{-n-1}\varepsilon \text{ for every } A\in\mathcal{F}.$$

It follows that

$$\Big\|\sum_n \psi_n(A) - \sum_{n,m}\psi_{n,m}(A)\Big\| < \varepsilon \text{ for every } A\in\mathcal{F}.$$

In particular, we have $\|\sum_n\psi_n(I) - \sum_{n,m}\psi_{n,m}(I)\| < \varepsilon$ since $I\in\mathcal{F}$. Thus the series $\sum_{n,m}\psi_{n,m}(I)$ converges in the strong-operator topology since $\sum_n\psi_n(I)$ converges in the strong-operator topology. Therefore, $\sum_{n,m}\psi_{n,m}\in\mathfrak{S}\mathfrak{F}$ and then $\sum_n\psi_n\in\widehat{\mathfrak{S}\mathfrak{F}}$. □

The following lemma is derived from [4, Lemma 3.4].

Lemma 2.7. *Let $\mathcal{M}$ be a separable properly infinite factor, $\mathcal{A}$ a unital C^*-subalgebra of $\mathcal{M}$, and $P\in\mathcal{K}_{\mathcal{M}}$ a finite projection. Then every map $\psi\in\mathfrak{F}(\mathcal{A},\mathcal{M},\mathcal{K}_{\mathcal{M}})$ can be approximated in the pointwise-norm topology by maps of the form*

$$A \mapsto V^*AV,$$

where $V\in\mathcal{M}$ and $PV = 0$. In particular, V can be selected as a partial isometry such that $V^*V = \psi(I)$ when $\psi(I)$ is a projection.

2.3. **Cutting down Projections.** In order to facilitate our discussion in subsequent sections, it is necessary to present a set of technical lemmas to cut down infinite projections.

Lemma 2.8. *Let P,Q be infinite projections in $\mathcal{M}$. Let $\delta_1,\ldots,\delta_n$ be positive numbers, and $\rho_1,\rho_2,\ldots,\rho_n$ normal states on $\mathcal{M}$ such that $\rho_j(Q) > \delta_j > 0$ for $1\leqslant j\leqslant n$. Then for any $A\in\mathcal{M}$, there exist infinite projections $P'\leqslant P$ and $Q'\leqslant Q$ in $\mathcal{M}$ such that $P'AQ' = 0$ and $\rho_j(Q') > \delta_j$ for $1\leqslant j\leqslant n$.*

Proof. Consider the polar decomposition $PAQ = VH$, in which V is a partial isometry and H is a positive operator in $\mathcal{M}$. Let $P_0 = VV^*\leqslant P$ and $Q_0 = V^*V\leqslant Q$. If P_0 is finite, then $P - P_0$ is infinite. In this case, we set $P' = P - P_0$ and $Q' = Q$.

Now assume $Q_0(\sim P_0)$ is infinite. Let $\mathcal{A}$ be a maximal abelian selfadjoint subalgebra of $\mathcal{M}$ that includes Q_0 and H. Then there exists a sequence $\{Q'_m\}_{m\in\mathbb{N}}$ of projections in $\mathcal{A}$ such that $Q_0 = \sum_m Q'_m$ and $Q'_m \sim Q_0$ for every m. Since ρ_j is normal, we have $\rho_j(Q'_m) < \rho_j(Q) - \delta_j$ for $1 \leqslant j \leqslant n$ when m is sufficiently large. We set

$$P' = VQ'_mV^* \leqslant P, \quad Q' = Q - Q'_m \leqslant Q.$$

Since $H = HQ = QH$ and $HQ'_m = Q'_mH$, we obtain

$$P'AQ' = VQ'_mV^*VH(Q - Q'_m) = VQ'_mQ_0(Q - Q'_m)H = 0.$$

It is evident that $\rho_j(Q') > \delta_j$ for $1 \leqslant j \leqslant n$. Furthermore, P' and Q' are infinite projections because $P' \sim Q'_m \sim Q_0$ and $Q' \geqslant Q'_{m+1} \sim Q_0$. □

Lemma 2.9. *Let P, Q be infinite projections in $\mathcal{M}$. Let $\delta_1, \ldots, \delta_n$ be positive numbers, and $\rho_1, \rho_2, \ldots, \rho_n$ normal states on $\mathcal{M}$ such that $\rho_j(Q) > \delta_j > 0$ for $1 \leqslant j \leqslant n$. Then for any finite subset $\mathcal{F}$ of $\mathcal{M}$, there exist infinite projections $P' \leqslant P$ and $Q' \leqslant Q$ in $\mathcal{M}$ such that $P'AQ' = 0$ for any $A \in \mathcal{F}$ and $\rho_j(Q') > \delta_j$ for $1 \leqslant j \leqslant n$.*

Proof. Let $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$. By Lemma 2.8, there exist infinite projections $P_1 \leqslant P$ and $Q_1 \leqslant Q$ such that $P_1A_1Q_1 = 0$ and $\rho_j(Q_1) > \delta_j$ for $1 \leqslant j \leqslant n$. Inductively, we can find infinite projections

$$P_m \leqslant P_{m-1} \leqslant \cdots \leqslant P_1 \leqslant P, \quad Q_m \leqslant Q_{m-1} \leqslant \cdots \leqslant Q_1 \leqslant Q$$

such that $P_kA_kQ_k = 0$ and $\rho_j(Q_k) > \delta_j$ for $1 \leqslant j \leqslant n$ and $1 \leqslant k \leqslant n$. We set $P' = P_m$ and $Q' = Q_m$. □

Lemma 2.10. *Let $\{P_n\}_{n\in\mathbb{N}}$ be a sequence of infinite projections in $\mathcal{M}$, and $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ a sequence of finite subsets of $\mathcal{M}$. Then there exists a sequence $\{Q_n\}_{n\in\mathbb{N}}$ of infinite projections in $\mathcal{M}$ such that $Q_n \leqslant P_n$ for $n \geqslant 0$, and $Q_nAQ_0 = 0$ for any A in $\mathcal{F}_n$ and any $n \geqslant 1$.*

Proof. Case 1. Suppose $\mathcal{M}$ is semifinite with a normal faithful tracial weight τ. We also assume that $\tau(E) = 1$ for every minimal projection E if $\mathcal{M}$ is of type I_∞. Let $E_1 \leqslant P_0$ be a projection with $\tau(E_1) = 2$. By Lemma 2.9, there exist infinite projections $P'_0 \leqslant P_0$ and $Q_1 \leqslant P_1$ such that $Q_1\mathcal{F}_1P'_0 = \{0\}$ and $\tau(P'_0E_1) > 1$. Let $E_2 \leqslant P'_0$ be a projection with $\tau(E_2) = 3$. Applying Lemma 2.9 once again, there exist infinite projections $P''_0 \leqslant P'_0$ and $Q_2 \leqslant P_2$ such that $Q_2\mathcal{F}_2P''_0 = \{0\}$ and $\tau(P''_0E_1) > 1$, $\tau(P''_0E_2) > 2$. Continuing this process, let $E_n \leqslant P_0^{(n-1)}$ be a projection with $\tau(E_n) = n + 1$. There are infinite projections $P_0^{(n)} \leqslant P_0^{(n-1)}$ and $Q_n \leqslant P_n$ such that $Q_n\mathcal{F}_nP_0^{(n)} = \{0\}$ and $\tau(P_0^{(n)}E_k) > k$ for $1 \leqslant k \leqslant n$. Now we set $Q_0 = \bigwedge P_0^{(n)}$. Since $\tau(Q_0) \geqslant \tau(Q_0E_k) > k$ for every $k \geqslant 1$, we conclude that Q_0 is infinite.

Case 2. Suppose $\mathcal{M}$ is of type III with a normal faithful state ρ such that $\rho(P_0) > \delta > 0$. By Lemma 2.9, there exist infinite projections $P'_0 \leqslant P_0$ and $Q_1 \leqslant P_1$ such that $Q_1\mathcal{F}_1P'_0 = \{0\}$ and $\rho(P'_0) > \delta$. Similarly, there exist infinite projections $P''_0 \leqslant P'_0$ and $Q_2 \leqslant P_2$ such that $Q_2\mathcal{F}_2P''_0 = \{0\}$ and $\rho(P''_0) > \delta$. Inductively, we can find infinite projections $P_0^{(n)} \leqslant P_0^{(n-1)}$ and $Q_n \leqslant P_n$ such that $Q_n\mathcal{F}_nP_0^{(n)} = \{0\}$ and $\rho(P_0^{(n)}) > \delta$. Let $Q_0 = \bigwedge P_0^{(n)}$. Since $\rho(Q_0) \geqslant \delta > 0$, we conclude that $Q_0 \neq 0$. Therefore, Q_0 is an infinite projection. □

Lemma 2.11. *Let $\{P_n\}_{n\in\mathbb{N}}$ be a sequence of infinite projections in $\mathcal{M}$, and $\{\mathcal{F}_{m,n}\}_{m,n\in\mathbb{N}}$ a family of finite subsets of $\mathcal{M}$. Then there exists a sequence $\{Q_n\}_{n\in\mathbb{N}}$ of infinite projections in $\mathcal{M}$ such that $Q_n \leqslant P_n$ for $n \geqslant 0$, and $Q_mAQ_n = \{0\}$ for all $A \in \mathcal{F}_{m,n}$ when $m \neq n$.*

Proof. We can assume that $\mathcal{F}^*_{n,m} = \mathcal{F}_{m,n}$ by replacing $\mathcal{F}_{m,n}$ with $\mathcal{F}_{m,n} \cup \mathcal{F}^*_{n,m}$. By Lemma 2.10, there exist infinite projections $Q_0 \leqslant P_0$ and $P'_m \leqslant P_m$ such that $P'_m\mathcal{F}_{m,0}Q_0 = \{0\}$ for $m \geqslant 1$. There exist infinite projections $Q_1 \leqslant P'_1$ and $P''_m \leqslant P'_m$ such that $P''_m\mathcal{F}_{m,1}Q_1 = \{0\}$ for $m \geqslant 2$. Inductively, there exist infinite projections $Q_n \leqslant P_n^{(n)}$ and $P_m^{(n+1)} \leqslant P_m^{(n)}$ such that $P_m^{(n+1)}\mathcal{F}_{m,n}Q_n = \{0\}$ for $m \geqslant n + 1$. It is obvious that $Q_m\mathcal{F}_{m,n}Q_n = \{0\}$ when $m > n$. Furthermore, since $\mathcal{F}_{n,m}^* = \mathcal{F}_{m,n}$, we have $Q_m\mathcal{F}_{m,n}Q_n = \{0\}$ when $m \neq n$. □

3. Main Theorems

The following result relies on the concept of *quasicentral approximate units* (see [3]), which states that a significant number of completely positive maps from $\mathcal{A}$ into $\mathcal{M}$ lie in the set $\widehat{\mathfrak{S}\mathfrak{F}} = \widehat{\mathfrak{S}\mathfrak{F}}(\mathcal{A}, \mathcal{M}, \mathcal{K}_{\mathcal{M}})$ as defined in Definition 2.5.

Proposition 3.1. *Let $\mathcal{M}$ be a separable properly infinite factor, $\mathcal{A}$ a unital C^*-subalgebra of $\mathcal{M}$, and $\mathcal{B}$ a type I_∞ unital subfactor of $\mathcal{M}$. Then $\psi \in \widehat{\mathfrak{S}\mathfrak{F}}$ for every completely positive map $\psi\colon \mathcal{A} \to \mathcal{B}$ with $\psi|_{\mathcal{A}\cap\mathcal{K}_{\mathcal{M}}} = 0$.*

Proof. Let $\mathcal{F}$ be a finite subset of $\mathcal{A}$ containing I, and $\varepsilon > 0$. According to [3, Theorem 2], there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ of finite rank operators in $\mathcal{B}$ such that $\sum_n E_n^2 = I$ and

$$\Big\|\psi(A) - \sum_n E_n\psi(A)E_n\Big\| < \varepsilon \text{ for every } A \in \mathcal{F}.$$

For $n \in \mathbb{N}$, let P_n denote the finite rank projection $R(E_n)$ in $\mathcal{B}$. Since $P_n\mathcal{B}P_n$ is a matrix algebra, we can construct a map $\psi \in \mathfrak{F}(\mathcal{A}, \mathcal{M}, \mathcal{K}_{\mathcal{M}})$ by

$$\psi_n\colon \mathcal{A} \to P_n\mathcal{B}P_n, \quad A \mapsto E_n\psi(A)E_n.$$

It is clear that $\|\psi(I) - \sum_n \psi_n(I)\| < \varepsilon$ since $I \in \mathcal{F}$. Consequently, the series $\sum_n \psi_n(I)$ converges in the strong-operator topology. Therefore, $\sum_n \psi_n \in \mathfrak{S}\mathfrak{F}$ and it follows that $\psi \in \widehat{\mathfrak{S}\mathfrak{F}}$. □

In [7], U. Haagerup proved that every completely positive map from a finite-dimensional unital subfactor of $\mathcal{M}$ into $\mathcal{M}$ can be expressed in the form $B \mapsto T^*BT$. Utilizing Haagerup's result, we are now able to demonstrate our main theorem.

Theorem 3.2. *Let $\mathcal{M}$ be a separable properly infinite factor, $\mathcal{A}$ a unital C^*-subalgebra of $\mathcal{M}$, and $P \in \mathcal{K}_{\mathcal{M}}$ a finite projection. Then any $\psi \in \widehat{\mathfrak{S}\mathfrak{F}}$ can be approximated in the pointwise-norm topology by maps of the form*

$$A \mapsto V^*AV,$$

where $V \in \mathcal{M}$ and $PV = 0$. In particular, V can be selected as a partial isometry such that $V^*V = \psi(I)$ when $\psi(I)$ is a projection.

Proof. Let $\mathcal{F}$ be a finite subset of $\mathcal{A}$ containing I, and $\varepsilon > 0$. Then there exists a sequence $\{\psi_n\}_{n\in\mathbb{N}}$ in $\mathfrak{F}(\mathcal{A}, \mathcal{M}, \mathcal{K}_{\mathcal{M}})$ such that

$$\Big\|\psi(A) - \sum_n \psi_n(A)\Big\| < \frac{\varepsilon}{2} \text{ for every } A \in \mathcal{F}.$$

Suppose $\psi_n = \eta_n \circ \sigma_n$ for some completely positive maps $\sigma_n\colon \mathcal{A} \to \mathcal{B}_n$ and $\eta_n\colon \mathcal{B}_n \to \mathcal{M}$ with $\sigma_n|_{\mathcal{A}\cap\mathcal{K}_{\mathcal{M}}} = 0$, where $\mathcal{B}_n$ is a type $\mathrm{I}_{r(n)}$ unital subfactor of $\mathcal{M}$ with a system of matrix units $\{E_{st}^{(n)}\}_{1\leqslant s,t\leqslant r(n)}$.

According to [7, Proposition 2.1], there exists an operator $T_n \in \mathcal{M}$ such that $\eta_n(B) = T_n^*BT_n$ for every $B \in \mathcal{B}_n$. By Lemma 2.7, there is an operator $V_n \in \mathcal{M}$ such that

$$r(n)^2\|T_n\|^2\|\sigma_n(A) - V_n^*AV_n\| < 2^{-n-2}\varepsilon \text{ for every } A \in \mathcal{F}, \tag{3.1}$$

and $PV_n = 0$. For $m, n \geqslant 0$, we define a finite subset of $\mathcal{M}$ by

$$\mathcal{F}_{m,n} = \{E_{1s}^{(m)}V_m^*AV_nE_{t1}^{(n)}\colon 1 \leqslant s \leqslant r(m), 1 \leqslant t \leqslant r(n), A \in \mathcal{F}\}.$$

Based on Lemma 2.11, we can find a sequence $\{Q_n\}_{n\in\mathbb{N}}$ of infinite projections in $\mathcal{M}$ such that $Q_n \leqslant E_{11}^{(n)}$ for $n \geqslant 0$, and $Q_m\mathcal{F}_{m,n}Q_n = \{0\}$ when $m \neq n$. Since Q_n and $E_{11}^{(n)}$ are infinite projections, there is a partial isometry W_n in $\mathcal{M}$ such that

$$W_n^*W_n = E_{11}^{(n)}, \quad W_nW_n^* = Q_n.$$

Since $E_{1s}^{(n)}\sigma_n(A)E_{t1}^{(n)} \in \mathbb{C}E_{11}^{(n)}$ and $Q_n \leqslant E_{11}^{(n)}$, it is straightforward to deduce

$$E_{s1}^{(n)}W_n^*Q_nE_{1s}^{(n)}\sigma_n(A)E_{t1}^{(n)}Q_nW_nE_{1t}^{(n)} = E_{s1}^{(n)}E_{1s}^{(n)}\sigma_n(A)E_{t1}^{(n)}E_{1t}^{(n)} = E_{ss}^{(n)}\sigma_n(A)E_{tt}^{(n)}.$$

Consequently, $\sigma_n(A) = \sum_{s,t} E_{s1}^{(n)}W_n^*Q_nE_{1s}^{(n)}\sigma_n(A)E_{t1}^{(n)}Q_nW_nE_{1t}^{(n)}$ and then

$$\sum_n \psi_n(A) = \sum_{n,s,t} T_n^*E_{s1}^{(n)}W_n^*Q_nE_{1s}^{(n)}\sigma_n(A)E_{t1}^{(n)}Q_nW_nE_{1t}^{(n)}T_n. \tag{3.2}$$

Since $E_{1s}^{(m)}V_m^*AV_nE_{t1}^{(n)} \in \mathcal{F}_{m,n}$ for every $A \in \mathcal{F}$, we have $Q_mE_{1s}^{(m)}V_m^*AV_nE_{t1}^{(n)}Q_n = 0$ when $m \neq n$. Specifically, the operators $\{\sum_t V_nE_{t1}^{(n)}Q_nW_nE_{1t}^{(n)}T_n\}_{n\in\mathbb{N}}$ have orthogonal ranges when considering $A = I$. Based on this, we can define an operator $V = \sum_{n,t} V_nE_{t1}^{(n)}Q_nW_nE_{1t}^{(n)}T_n$, and then

$$V^*AV = \sum_{n,s,t} T_n^*E_{s1}^{(n)}W_n^*Q_nE_{1s}^{(n)}V_n^*AV_nE_{t1}^{(n)}Q_nW_nE_{1t}^{(n)}T_n \text{ for every } A \in \mathcal{F}, \tag{3.3}$$

and $PV = 0$. From (3.1), (3.2) and (3.3), it follows that

$$\Big\|\sum_n \psi_n(A) - V^*AV\Big\| < \frac{\varepsilon}{2} \text{ for every } A \in \mathcal{F}.$$

Consequently, $\|\psi(A) - V^*AV\| < \varepsilon$ for every $A \in \mathcal{F}$. In particular, V^*V is a bounded operator if we take $A = I$. We can conclude that V belongs to $\mathcal{M}$. Furthermore, due to $\|\psi(I) - V^*V\| < \varepsilon$, we can choose V as a partial isometry such that $V^*V = \psi(I)$ when $\psi(I)$ is a projection. □

We now establish an enhanced version of our main theorem for separable C^*-subalgebras in semifinite factors.

Theorem 3.3. *Let $\mathcal{M}$ be a separable properly infinite semifinite factor, $\mathcal{A}$ a separable unital C^*-subalgebra of $\mathcal{M}$, and $P \in \mathcal{K}_{\mathcal{M}}$ a finite projection. Then for any $\psi \in \widehat{\mathfrak{S}\mathfrak{F}}$, there is a sequence $\{V_k\}_{k\in\mathbb{N}}$ in $\mathcal{M}$ such that*

1. *$PV_k = 0$ for every $k \in \mathbb{N}$.*
2. *$\lim_{k\to\infty}\|\psi(A) - V_k^*AV_k\| = 0$ for every $A \in \mathcal{A}$.*
3. *$\psi(A) - V_k^*AV_k \in \mathcal{K}_{\mathcal{M}}$ for every $k \in \mathbb{N}$ and $A \in \mathcal{A}$.*

In particular, V_k can be selected as a partial isometry such that $V_k^*V_k = \psi(I)$ when $\psi(I)$ is a projection.

Proof. Let $\{Q_n\}_{n\in\mathbb{N}}$ be a sequence of finite projections in $\mathcal{K}_{\mathcal{M}}$ with $\bigvee_{n\in\mathbb{N}} Q_n = I$, and $\mathscr{A}$ the separable unital C^*-subalgebra of $\mathcal{M}$ generated by $\psi(\mathcal{A}) \cup \{Q_n\}_{n\in\mathbb{N}}$. It is clear that

$$\mathcal{I} = \{B \in \mathscr{A} \colon R(B) \in \mathcal{K}_{\mathcal{M}}\}$$

is a non-degenerate ideal of $\mathscr{A}$. Additionally, let $\{A_j\}_{j\in\mathbb{N}}$ be a norm-dense sequence in $\mathcal{A}^{\text{s.a.}}$ with $A_0 = I$, where $\mathcal{A}^{\text{s.a.}}$ is defined as $\{A \in \mathcal{A} \colon A^* = A\}$.

Fix $k \in \mathbb{N}$. According to [3, Theorem 2], there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ of positive operators in $\mathcal{I}$ such that $\sum_n E_n^2 = I$, $\psi(A) - \sum_n E_n\psi(A)E_n \in \mathcal{K}_{\mathcal{M}}$ for every $A \in \mathcal{A} \subseteq \mathscr{A}$, and

$$\Big\|\psi(A_j) - \sum_n E_n\psi(A_j)E_n\Big\| < 2^{-k-1} \text{ for } 0 \leqslant j \leqslant k.$$

We will define U_n inductively. Let $P_0 = P$ and

$$P_n = \bigvee\{P, R(A_jU_mE_m) \colon 0 \leqslant j \leqslant n+k, 0 \leqslant m \leqslant n-1\} \text{ for } n \geqslant 1.$$

Since P and $R(E_m)$ are finite, P_n is also finite. By Theorem 3.2, there exists an operator U_n in $\mathcal{M}$ such that

$$\|\psi(A_j) - U_n^*A_jU_n\| < 2^{-n-k-2} \text{ for } 0 \leqslant j \leqslant n+k, \tag{3.4}$$

and $P_nU_n = 0$. For $0 \leqslant j \leqslant n+k$ and $0 \leqslant m \leqslant n-1$, we have $P_nA_jU_mE_m = A_jU_mE_m$ and then $E_mU_m^*A_jU_nE_n = E_mU_m^*A_jP_nU_nE_n = 0$. It follows that

$$E_mU_m^*A_jU_nE_n = 0 \text{ when } 0 \leqslant j \leqslant \max\{m,n\}+k, m \neq n. \tag{3.5}$$

Specifically, the operators $\{U_nE_n\}_{n\in\mathbb{N}}$ have orthogonal ranges when considering $A_0 = I$. Based on this, we can define an operator $V = \sum_n U_nE_n$, and then

$$\sum_n E_n\psi(A_j)E_n - V^*A_jV = \sum_n E_n\big(\psi(A_j) - U_n^*A_jU_n\big)E_n - \sum_{m\neq n} E_mU_m^*A_jU_nE_n$$

for every $j \geqslant 0$, and $PV = 0$. The first sum is norm-convergent by (3.4) and the second sum is a finite sum by (3.5). It follows that $\sum_n E_n\psi(A_j)E_n - V^*A_jV \in \mathcal{K}_{\mathcal{M}}$ for $j \geqslant 0$ since each summand lies in $\mathcal{K}_{\mathcal{M}}$. We also have the estimation

$$\Big\| \sum_n E_n\psi(A_j)E_n - V^*A_jV \Big\| < 2^{-k-1} \text{ for } 0 \leqslant j \leqslant k.$$

Therefore, $\psi(A_j) - V^*A_jV \in \mathcal{K}_{\mathcal{M}}$ for $j \geqslant 0$, and $\|\psi(A_j) - V^*A_jV\| < 2^{-k}$ for $0 \leqslant j \leqslant k$. In particular, V^*V is a bounded operator if we consider $A_0 = I$. We can conclude that V belongs to $\mathcal{M}$. Now we set $V_k = V$. □

4. Voiculescu's Theorem

In this section, we focus on unital *-homomorphisms in $\widehat{\mathfrak{S}\mathfrak{F}} = \widehat{\mathfrak{S}\mathfrak{F}}(\mathcal{A}, \mathcal{M}, \mathcal{K}_{\mathcal{M}})$ as defined in Definition 2.5.

Lemma 4.1. *Let $\mathcal{M}$ be a separable properly infinite factor, and $\mathcal{A}$ a separable unital C^*-subalgebra of $\mathcal{M}$. If $\varphi \in \widehat{\mathfrak{S}\mathfrak{F}}$ is a unital *-homomorphism, then there is a sequence $\{V_k\}_{k\in\mathbb{N}}$ of isometries in $\mathcal{M}$ such that*

$$\lim_{k\to\infty} \|V_k\varphi(A) - AV_k\| = 0 \text{ for every } A \in \mathcal{A}.$$

Furthermore, if $\mathcal{M}$ is semifinite, we can choose $\{V_k\}_{k\in\mathbb{N}}$ such that

$$V_k\varphi(A) - AV_k \in \mathcal{K}_{\mathcal{M}} \text{ for every } k \in \mathbb{N} \text{ and } A \in \mathcal{A}.$$

Proof. By Theorem 3.2, there exists a sequence $\{V_k\}_{k\in\mathbb{N}}$ of isometries in $\mathcal{M}$ such that

$$\lim_{k\to\infty} \|\varphi(A) - V_k^*AV_k\| = 0 \text{ for every } A \in \mathcal{A}.$$

Since φ is a unital *-homomorphism, we have

$$\begin{aligned} &\big(V_k\varphi(A) - AV_k\big)^*\big(V_k\varphi(A) - AV_k\big) \\ =\ &\varphi(A^*)\big(\varphi(A) - V_k^*AV_k\big) + \big(\varphi(A^*) - V_k^*A^*V_k\big)\varphi(A) - \big(\varphi(A^*A) - V_kA^*AV_k\big). \end{aligned} \tag{4.1}$$

It follows that $\lim_{k\to\infty}\|V_k\varphi(A) - AV_k\| = 0$ for every $A \in \mathcal{A}$. Furthermore, if $\mathcal{M}$ is semifinite, then we can assume that $\varphi(A) - V_k^*AV_k \in \mathcal{K}_{\mathcal{M}}$ by Theorem 3.3. As a result, we can deduce $V_k\varphi(A) - AV_k \in \mathcal{K}_{\mathcal{M}}$ from (4.1). □

The following theorem is known as Voiculescu's theorem [14]. We will employ the notation $P^{\perp} = I - P$ for a projection $P \in \mathcal{M}$.

Theorem 4.2. *Let $\mathcal{M}$ be a separable properly infinite factor, and $\mathcal{A}$ a separable unital C^*-subalgebra of $\mathcal{M}$. If $\varphi \in \widehat{\mathfrak{S}\mathfrak{F}}$ is a unital *-homomorphism, then there is a sequence $\{V_k\}_{k\in\mathbb{N}}$ of isometries in $\mathcal{M} \otimes M_2(\mathbb{C})$ such that*

$$\lim_{k\to\infty} \|(A \oplus \varphi(A)) - V_k^*(A \oplus 0)V_k\| = 0 \text{ for every } A \in \mathcal{A},$$

and $V_k^*V_k = I \oplus I, V_kV_k^* = I \oplus 0$ for every $k \in \mathbb{N}$. Furthermore, if $\mathcal{M}$ is semifinite, we can choose $\{V_k\}_{k\in\mathbb{N}}$ such that

$$(A \oplus \varphi(A)) - V_k^*(A \oplus 0)V_k \in \mathcal{K}_{\mathcal{M}} \otimes M_2(\mathbb{C}) \text{ for every } k \in \mathbb{N} \text{ and } A \in \mathcal{A}.$$

Proof. Let $\{E_{mn}\}_{m,n\in\mathbb{N}}$ be a system of matrix units in $\mathcal{M}$ such that $\sum_n E_{nn} = I$ and $E_{00} \sim I$. Let T be an isometry in $\mathcal{M}$ with $TT^* = E_{00}$, and let S denote the isometry $\sum_n E_{n+1,n} \in \mathcal{M}$. We define a map

$$\psi\colon \mathcal{A} \to \mathcal{M}, \quad A \mapsto \sum_n E_{n0} T\varphi(A)T^* E_{0n}.$$

Clearly, ψ is a unital *-homomorphism and lies in $\widehat{\mathfrak{S}\mathfrak{F}}$ by Lemma 2.6. By Lemma 4.1, we can find a sequence $\{U_k\}_{k\in\mathbb{N}}$ of isometries in $\mathcal{M}$ such that

$$\lim_{k\to\infty} \|U_k\psi(A) - AU_k\| = 0 \text{ for every } A \in \mathcal{A}. \tag{4.2}$$

Furthermore, if $\mathcal{M}$ is semifinite, then we can assume that

$$U_k\varphi(A) - AU_k \in \mathcal{K}_{\mathcal{M}} \text{ for every } k \in \mathbb{N} \text{ and } A \in \mathcal{A}.$$

Let $P_k = U_kU_k^*$ be a projection in $\mathcal{M}$, let W_k be an isometry in $\mathcal{M}$ with $W_kW_k^* = I - TP_k^\perp T^*$, and let $F_k = P_k^\perp T^* + U_kW_k^*$ be a unitary operator in $\mathcal{M}$. Then

$$F_k^*AF_k = TP_k^\perp AP_k^\perp T^* + TP_k^\perp AU_kW_k^* + W_kU_k^*AP_k^\perp T^* + W_kU_k^*AU_kW_k^*.$$

Since $P_k^\perp AU_k = \big(AU_k - U_k\psi(A)\big) + U_k\big(\psi(A)U_k^* - U_k^*A\big)U_k$, we deduce from (4.2) that

$$\lim_{k\to\infty} \|P_k^\perp AU_k\| = 0 \text{ for every } A \in \mathcal{A}.$$

It follows that

$$\lim_{k\to\infty} \|F_k^*AF_k - (TP_k^\perp AP_k^\perp T^* + W_k\psi(A)W_k^*)\| = 0 \text{ for every } A \in \mathcal{A}. \tag{4.3}$$

Let $X_k = \begin{pmatrix} TP_k^\perp T^* + W_kSW_k^* & W_kT \\ 0 & 0 \end{pmatrix}$ be an isometry in $\mathcal{M} \otimes M_2(\mathbb{C})$. One computes that $X_kX_k^* = I \oplus 0$. Since $S^*\psi(A)S = \psi(A)$ and $T^*\psi(A)T = \varphi(A)$, we have

$$X_k^* \begin{pmatrix} TP_k^\perp AP_k^\perp T^* + W_k\psi(A)W_k^* & 0 \\ 0 & 0 \end{pmatrix} X_k = \begin{pmatrix} TP_k^\perp AP_k^\perp T^* + W_k\psi(A)W_k^* & 0 \\ 0 & \varphi(A) \end{pmatrix}.$$

Then (4.3) implies that

$$\lim_{n\to\infty} \|X_k^*(F_k^*AF_k \oplus 0)X_k - (F_k^*AF_k \oplus \varphi(A))\| = 0 \text{ for every } A \in \mathcal{A}.$$

Now we set $V_k = (F_k \oplus I)X_k(F_k^* \oplus I)$. ∎

According to Proposition 3.1, Theorem 4.3 is a special case of Theorem 4.2.

Theorem 4.3. *Let $\mathcal{M}$ be a separable properly infinite factor, $\mathcal{A}$ a separable unital C^*-subalgebra of $\mathcal{M}$, and $\mathcal{B}$ a type I_∞ unital subfactor of $\mathcal{M}$. If $\varphi\colon \mathcal{A} \to \mathcal{B}$ is a unital *-homomorphism with $\varphi|_{\mathcal{A}\cap\mathcal{K}_{\mathcal{M}}} = 0$, then there exists a sequence $\{V_k\}_{k\in\mathbb{N}}$ of isometries in $\mathcal{M} \otimes M_2(\mathbb{C})$ such that*

$$\lim_{k\to\infty} \|(A \oplus \varphi(A)) - V_k^*(A \oplus 0)V_k\| = 0 \textit{ for every } A \in \mathcal{A},$$

and $V_k^*V_k = I \oplus I, V_kV_k^* = I \oplus 0$ for every $k \in \mathbb{N}$. Furthermore, if $\mathcal{M}$ is semifinite, we can choose $\{V_k\}_{k\in\mathbb{N}}$ such that

$$(A \oplus \varphi(A)) - V_k^*(A \oplus 0)V_k \in \mathcal{K}_{\mathcal{M}} \otimes M_2(\mathbb{C}) \textit{ for every } k \in \mathbb{N} \textit{ and } A \in \mathcal{A}.$$

5. Applications

We provide two applications of Voiculescu's theorem in this section.

5.1. **Reducible Operators.** Let $\mathcal{M}$ be a separable properly infinite factor, and T an operator in $\mathcal{M}$. We say that T is *reducible* in $\mathcal{M}$ if there is a projection $P \in \mathcal{M}$ such that $PT = TP$ and $P \neq 0, I$.

Theorem 5.1. *Let $\mathcal{M}$ be a separable properly infinite factor. Then the set of all reducible operators is norm-dense in $\mathcal{M}$.*

Proof. Let $\mathcal{B}$ be a type I_∞ unital subfactor of $\mathcal{M}$, and $T \in \mathcal{M}$. Let $\mathcal{A}$ be the separable unital C^*-algebra generated by T, and $\mathcal{I} = \mathcal{A} \cap \mathcal{K}_\mathcal{M}$.

Let $\psi\colon \mathcal{A}/\mathcal{I} \to \mathcal{B}$ be a unital $*$-homomorphism, $\pi\colon \mathcal{A} \to \mathcal{A}/\mathcal{I}$ the quotient map, and $\varphi = \psi \circ \pi\colon \mathcal{A} \to \mathcal{B}$. By Proposition 3.1 and Theorem 4.2, there is a sequence $\{V_k\}_{k\in\mathbb{N}}$ of isometries in $\mathcal{M} \otimes M_2(\mathbb{C})$ such that

$$\lim_{k\to\infty} \|(A \oplus \varphi(A)) - V_k^*(A \oplus 0)V_k\| = 0 \text{ for every } A \in \mathcal{A},$$

and $V_kV_k^* = I \oplus 0$ for every $k \in \mathbb{N}$. We can write

$$V_k(T \oplus \varphi(T))V_k^* = T_k \oplus 0, \quad V_k(I \oplus 0)V_k^* = P_k \oplus 0.$$

It is clear that $P_kT_k = T_kP_k$ and $P_k \neq 0, I$. Therefore, T_k is reducible in $\mathcal{M}$. Moreover, we have $\lim_{k\to\infty}\|T_k - T\| = 0$. This completes the proof. □

5.2. **Voiculescu's Bicommutant Theorem.** Let $\mathcal{M}$ be a separable properly infinite semifinite factor, and $\mathscr{A}$ a separable unital subalgebra of $\mathcal{M}/\mathcal{K}_\mathcal{M}$. As defined in [3, Page 334], the *essential lattice* $Lat_e(\mathscr{A})$ of $\mathscr{A}$ is the set of all projections $p \in \mathcal{M}/\mathcal{K}_\mathcal{M}$ such that $p^\perp ap = 0$ for every $a \in \mathscr{A}$. If $t \in \mathcal{M}/\mathcal{K}_\mathcal{M}$, then $\|p^\perp tp\| = \|p^\perp(t-a)p\| \leqslant \|t-a\|$ for every $a \in \mathscr{A}$. It follows that

$$\sup_p \|p^\perp tp\| \leqslant d(t, \mathscr{A}).$$

The subsequent result is commonly referred to as Arveson's distance formula.

Lemma 5.2. *Let $\mathcal{M}$ be a separable properly infinite semifinite factor, and $\mathscr{A}$ a separable unital subalgebra of $\mathcal{M}/\mathcal{K}_\mathcal{M}$. Then for any $t \in \mathcal{M}/\mathcal{K}_\mathcal{M}$, there is a projection q in $Lat_e(\mathscr{A})$ such that*

$$\|q^\perp tq\| = d(t, \mathscr{A}).$$

Proof. Let $\pi\colon \mathcal{M} \to \mathcal{M}/\mathcal{K}_\mathcal{M}$ be the quotient map, $\mathscr{A}_t$ the separable unital C^*-algebra generated by t and $\mathscr{A}$, and $\mathcal{B}$ a separable type I_∞ unital subfactor of $\mathcal{M}$. By GNS construction, there is a unital $*$-homomorphism $\sigma\colon \mathscr{A}_t \to \mathcal{B}$ and a $\sigma(\mathscr{A})$-invariant projection $P \in \mathcal{B}$ such that

$$\|P^\perp\sigma(t)P\|_e \geqslant d(t, \mathscr{A}),$$

where $\|A\|_e = \|\pi(A)\|$ for $A \in \mathcal{M}$.

Let $\mathcal{A}_t$ be a separable unital C^*-subalgebra of $\mathcal{M}$ such that $\pi(\mathcal{A}_t) = \mathscr{A}_t$, and let $\varphi = \sigma \circ \pi\colon \mathcal{A}_t \to \mathcal{B}$ be a unital $*$-homomorphism with $\varphi|_{\mathcal{A}_t\cap\mathcal{K}_\mathcal{M}} = 0$. By Theorem 4.2, there is an isometry $V \in \mathcal{M} \times M_2(\mathbb{C})$ such that

$$(A \oplus \varphi(A)) - V^*(A \oplus 0)V \in \mathcal{K}_\mathcal{M} \otimes M_2(\mathbb{C}) \text{ for every } A \in \mathcal{A}_t,$$

and $VV^* = I \oplus 0$.

Let $\mathcal{A} = \{A \in \mathcal{A}_t \colon \pi(A) \in \mathscr{A}\}$, and $Q \oplus 0 = V(0 \oplus P)V^*$. Since $\pi(\mathcal{A}) = \mathscr{A}$ and $\varphi(\mathcal{A}) = \sigma(\mathscr{A})$, P is $\varphi(\mathcal{A})$-invariant. We conclude that $Q^\perp AQ \in \mathcal{K}_\mathcal{M}$ for every $A \in \mathcal{A}$. This implies that $q = \pi(Q)$ belongs to $Lat_e(\mathscr{A})$. Choose $T \in \mathcal{A}_t$ such that $\pi(T) = t$. Then

$$(Q^\perp TQ \oplus 0) - V(0 \oplus P^\perp\varphi(T)P)V^* \in \mathcal{K}_\mathcal{M} \otimes M_2(\mathbb{C}).$$

It follows that $\|q^\perp tq\| = \|Q^\perp TQ\|_e = \|P^\perp\varphi(T)P\|_e = \|P^\perp\sigma(t)P\|_e \geqslant d(t, \mathscr{A})$. □

Lemma 5.2 implies that every separable norm-closed unital subalgebra of $\mathcal{M}/\mathcal{K}_\mathcal{M}$ is reflexive. In particular, Voiculescu's relative bicommutant theorem holds.

Theorem 5.3. *Let $\mathcal{M}$ be a separable properly infinite semifinite factor. Then every separable unital C^*-subalgebra of $\mathcal{M}/\mathcal{K}_{\mathcal{M}}$ equals its relative bicommutant.*

It is worth noting that $\mathcal{K}_{\mathcal{M}} = \{0\}$ if $\mathcal{M}$ is a separable type III factor. From this fact and Theorem 4.2, we can obtain the following approximate result.

Lemma 5.4. *Let $\mathcal{M}$ be a separable type* III *factor, and $\mathscr{A}$ a separable unital subalgebra of $\mathcal{M}$. Then for any $T \in \mathcal{M}$, there is a sequence $\{Q_k\}_{k\in\mathbb{N}}$ of projections in $\mathcal{M}$ such that*

$$\lim_{k\to\infty} \|Q_k^\perp A Q_k\| = 0 \text{ for every } A \in \mathscr{A},$$

and

$$\lim_{k\to\infty} \|Q_k^\perp T Q_k\| = d(T, \mathscr{A}).$$

In [8], D. Hadwin proved that every separable unital C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ equals its approximate bicommutant, where $\mathcal{H}$ is a separable infinite-dimensional complex Hilbert space. If $\mathcal{M}$ is a separable type III factor, then Lemma 5.4 implies that every separable unital C^*-subalgebra of $\mathcal{M}$ is equal to its approximate bicommutant.

5.3. **The First Cohomology Group.** Let $\mathcal{M}$ be a separable properly infinite semifinite factor, and $\mathcal{A}$ a unital C^*-subalgebra of $\mathcal{M}$.

Definition 5.5. *A linear map $\delta\colon \mathcal{A} \to \mathcal{K}_{\mathcal{M}}$ is said to be a* derivation *if it satisfies the Leibniz rule $\delta(AB) = \delta(A)B + A\delta(B)$. The set of all derivations of $\mathcal{A}$ into $\mathcal{K}_{\mathcal{M}}$ is denoted by* $\operatorname{Der}(\mathcal{A}, \mathcal{K}_{\mathcal{M}})$.

For any $K \in \mathcal{K}_{\mathcal{M}}$, the inner derivation *$\delta_K\colon \mathcal{A} \to \mathcal{K}_{\mathcal{M}}$ is given by $\delta_K(A) = KA - AK$. The set of all inner derivations of $\mathcal{A}$ into $\mathcal{K}_{\mathcal{M}}$ is denoted by* $\operatorname{Inn}(\mathcal{A}, \mathcal{K}_{\mathcal{M}})$.

The quotient space $H^1(\mathcal{A}, \mathcal{K}_{\mathcal{M}}) = \operatorname{Der}(\mathcal{A}, \mathcal{K}_{\mathcal{M}})/\operatorname{Inn}(\mathcal{A}, \mathcal{K}_{\mathcal{M}})$ is called the first cohomology group *of $\mathcal{A}$ with coefficients in $\mathcal{K}_{\mathcal{M}}$.*

We introduce some notation. If $\mathcal{A}$ is a unital C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, then the *commutant* $\mathcal{A}'$ is the set of all bounded operators on $\mathcal{H}$ commuting with all operators in $\mathcal{A}$. The von Neumann bicommutant theorem asserts that the bicommutant $\mathcal{A}''$ is the von Neumann algebra generated by $\mathcal{A}$.

If $\mathcal{A}$ is a unital C^*-subalgebra of $\mathcal{M}$, then the *relative commutant* of $\mathcal{A}$ in $\mathcal{M}$ is denoted by

$$\mathcal{A}^c = \{T \in \mathcal{M}\colon TA = AT \text{ for all } A \in \mathcal{A}\}.$$

Since $\mathcal{A}^c = \mathcal{A}' \cap \mathcal{M} \subseteq \mathcal{A}'$, we have $\mathcal{A}^{cc} = (\mathcal{A}^c)' \cap \mathcal{M} \supseteq (\mathcal{A}')' \cap \mathcal{M} = \mathcal{A}'' \supseteq \mathcal{A}$. Hence the relative bicommutant $\mathcal{A}^{cc}$ contains $\mathcal{A}$. Similarly, the *relative commutant* of a unital C^*-subalgebra $\mathscr{A}$ of $\mathcal{M}/\mathcal{K}_{\mathcal{M}}$ is denoted by

$$\mathscr{A}^c = \{t \in \mathcal{M}/\mathcal{K}_{\mathcal{M}}\colon ta = at \text{ for all } a \in \mathscr{A}\}.$$

Let $\pi\colon \mathcal{M} \to \mathcal{M}/\mathcal{K}_{\mathcal{M}}$ be the quotient map. It is clear that $\pi(\mathcal{A})^c \supseteq \pi(\mathcal{A}^c)$.

The following theorem is similar to [11, Theorem 2.2], which states that not all derivations of $\mathcal{A}$ with coefficients in $\mathcal{K}_{\mathcal{M}}$ are inner under certain conditions.

Theorem 5.6. *Let $\mathcal{M}$ be a separable properly infinite semifinite factor, and $\mathcal{A}$ a separable unital C^*-subalgebra of $\mathcal{M}$. If $\pi(\mathcal{A}'')$ is infinite-dimensional, then $H^1(\mathcal{A}, \mathcal{K}_{\mathcal{M}}) \neq \{0\}$.*

Proof. Let $\pi\colon \mathcal{M} \to \mathcal{M}/\mathcal{K}_{\mathcal{M}}$ be the quotient map. If $\pi(T) \in \pi(\mathcal{A})^c$, then $TA - AT \in \mathcal{K}_{\mathcal{M}}$ for every $A \in \mathcal{A}$, which gives a derivation $\delta_T \in \operatorname{Der}(\mathcal{A}, \mathcal{K}_{\mathcal{M}})$. If $\pi(T) = \pi(S)$, then $T - S \in \mathcal{K}_{\mathcal{M}}$. It follows that $\delta_T - \delta_S = \delta_{T-S} \in \operatorname{Inn}(\mathcal{A}, \mathcal{K}_{\mathcal{M}})$. Thus, we have a well-defined homomorphism

$$\varphi\colon \pi(\mathcal{A})^c \to H^1(\mathcal{A}, \mathcal{K}_{\mathcal{M}}), \quad \pi(T) \mapsto \delta_T + \operatorname{Inn}(\mathcal{A}, \mathcal{K}_{\mathcal{M}}).$$

If $\pi(T) \in \ker \varphi$, then there is an operator $K \in \mathcal{K}_{\mathcal{M}}$ such that $\delta_T = \delta_K$. It follows that $T - K \in \mathcal{A}^c$, and then $\pi(T) \in \pi(\mathcal{A}^c)$. Therefore, the induced map

$$\widetilde{\varphi}\colon \pi(\mathcal{A})^c/\pi(\mathcal{A}^c) \to H^1(\mathcal{A}, \mathcal{K}_{\mathcal{M}})$$

is injective. It suffices to show that $\pi(\mathcal{A})^c \neq \pi(\mathcal{A}^c)$.

Suppose on the contrary, that $\pi(\mathcal{A})^c = \pi(\mathcal{A}^c)$. Since $\pi(\mathcal{A})$ is a separable unital C^*-subalgebra of $\mathcal{M}/\mathcal{K}_{\mathcal{M}}$, we have $\pi(\mathcal{A}) = \pi(\mathcal{A})^{cc}$ by Theorem 5.3. It follows that

$$\pi(\mathcal{A}^{cc}) \supseteq \pi(\mathcal{A}'') \supseteq \pi(\mathcal{A}) = \pi(\mathcal{A})^{cc} = \pi(\mathcal{A}^c)^c \supseteq \pi(\mathcal{A}^{cc}).$$

Therefore, $\pi(\mathcal{A}^{cc}) = \pi(\mathcal{A}'') = \pi(\mathcal{A})$, which is an infinite-dimensional separable C^*-algebra. This contradicts the next result, Proposition 5.8. □

Example 5.7. *We give two examples.*

1. *Let $\mathcal{M}$ be a separable I_∞ factor, and $\mathcal{A} = \mathbb{C}I + \mathcal{K}_{\mathcal{M}}$. Then $H^1(\mathcal{A}, \mathcal{K}_{\mathcal{M}}) \neq \{0\}$.*
2. *Let $\mathcal{M}$ be the II_∞ factor $\mathcal{N} \mathbin{\overline{\otimes}} \mathcal{B}(L^2(\mathbb{T}, \mu))$, where $\mathcal{N}$ is a separable II_1-factor, and μ is the Lebesgue measure on the unit circle $\mathbb{T}$. Suppose that $C(\mathbb{T})$ acts on $L^2(\mathbb{T}, \mu)$ by left multiplication. If $\mathcal{A} = I_{\mathcal{N}} \otimes C(\mathbb{T})$, then $H^1(\mathcal{A}, \mathcal{K}_{\mathcal{M}}) \neq \{0\}$.*

The following proposition is well-known to experts.

Proposition 5.8. *Let $\mathcal{M}$ be a separable properly infinite semifinite factor, $\pi\colon \mathcal{M} \to \mathcal{M}/\mathcal{K}_{\mathcal{M}}$ the quotient map, and $\mathcal{N}$ a unital von Neumann subalgebra of $\mathcal{M}$. Then the C^*-algebra $\pi(\mathcal{N})$ is either finite-dimensional or non-separable.*

Proof. Suppose that $\pi(\mathcal{N})$ is an infinite-dimensional C^*-algebra. According to [12, Exercise 4.6.13], there is a positive element $A \in \mathcal{N}$ such that $\pi(A)$ has infinite spectrum. We can find a sequence $\{[a_n, b_n]\}_{n\in\mathbb{N}}$ of disjoint intervals such that each interval contains a spectral point of $\pi(A)$. Let f_n be a continuous function on $\mathbb{R}$, which is positive within the interval (a_n, b_n), and zero elsewhere. Then $f_n(\pi(A)) \neq 0$.

Let χ_n be the characteristic function of the interval $[a_n, b_n]$, and P_n the spectral projection $\chi_n(A)$. Let $\mathcal{N}_1$ be the von Neumann algebra generated by $\{P_n\}_{n\in\mathbb{N}}$. For any nonzero bounded complex sequence $\{c_n\}_{n\in\mathbb{N}}$, say $c_m \neq 0$, we have

$$\pi\left(\sum_{n\in\mathbb{N}} c_n P_n\right)\pi(f_m(A)) = \pi(c_m f_m(A)) = c_m f_m(\pi(A)) \neq 0.$$

It follows that $\pi|_{\mathcal{N}_1}$ is injective. Therefore, $\pi(\mathcal{N}_1)$ is non-separable in the norm topology. This completes the proof. □

6. Multiplier Algebras

In this section, let $\mathcal{M}$ be a separable type III factor. Note that $\mathcal{K}_{\mathcal{M}} = \{0\}$.

6.1. **Multiplier Algebras.** Let $\mathcal{B}$ be a type I_∞ unital subfactor of $\mathcal{M}$, $\mathcal{K}_{\mathcal{B}}$ the ideal of compact operators in $\mathcal{B}$, and $\mathcal{N} = \mathcal{B}' \cap \mathcal{M}$ the relative commutant of $\mathcal{B}$ in $\mathcal{M}$. Then $\mathcal{M}$ is generated by $\mathcal{N} \cup \mathcal{B}$ as a von Neumann algebra, and

$$\mathcal{M} \cong \mathcal{N} \mathbin{\overline{\otimes}} \mathcal{B}.$$

Let $\mathcal{J}$ be the C^*-subalgebra of $\mathcal{M}$ generated by $\mathcal{N}\mathcal{K}_{\mathcal{B}} = \{NK\colon N \in \mathcal{N}, K \in \mathcal{K}_{\mathcal{B}}\}$, then we have

$$\mathcal{J} \cong \mathcal{N} \otimes \mathcal{K}_{\mathcal{B}}.$$

The *multiplier algebra* of $\mathcal{J}$ is defined as

$$\mathcal{M}(\mathcal{J}) = \{T \in \mathcal{M}\colon T\mathcal{J} \subseteq \mathcal{J}, \mathcal{J}T \subseteq \mathcal{J}\}.$$

For more details about multiplier algebras, please refer to [15, Chapter 2].

Lemma 6.1. *$\mathcal{J} = \mathcal{J}\mathcal{M}\mathcal{J}$.*

Proof. Since $\mathcal{J}$ is a C^*-algebra and $I \in \mathcal{M}$, it is evident that $\mathcal{J} \subseteq \mathcal{J}\mathcal{M}\mathcal{J}$.

Let $\{E_{mn}\}_{m,n\in\mathbb{N}}$ be a system of matrix units in $\mathcal{B}$ such that E_{00} is a minimal projection in $\mathcal{B}$ and $\sum_n E_{nn} = I$. For $A \in \mathcal{M}$, we set

$$A_{ij} = \sum_n E_{ni} A E_{jn}, \quad i, j \in \mathbb{N}.$$

Then $A_{ij} \in \mathcal{B}' \cap \mathcal{M} = \mathcal{N}$ because $E_{mn}A_{ij} = E_{mi}AE_{jn} = A_{ij}E_{mn}$ for all $m, n \in \mathbb{N}$. It is clear that $E_{ij} \in \mathcal{K}_{\mathcal{B}}$, and therefore, $E_{ii}AE_{jj} = A_{ij}E_{ij} \in \mathcal{J}$. Consequently,

$$E_{ii}\mathcal{M}E_{jj} \subseteq \mathcal{J}, \quad i, j \in \mathbb{N}.$$

Let $P_n = E_{00} + E_{11} + \cdots + E_{nn}$. For any $A \in \mathcal{M}$ and $J_1, J_2 \in \mathcal{J}$, we have

$$P_nJ_1AJ_2P_n \in P_n\mathcal{M}P_n \subseteq \mathcal{J}.$$

Since $J_1 = \lim_{n\to\infty} P_nJ_1$ and $J_2 = \lim_{n\to\infty} J_2P_n$, we conclude that

$$J_1AJ_2 = \lim_{n\to\infty} P_nJ_1AJ_2P_n \in \mathcal{J}.$$

Thus, we have shown that $\mathcal{J}\mathcal{M}\mathcal{J} \subseteq \mathcal{J}$. □

The following result suggests that it is reasonable to consider C^*-subalgebras within $\mathcal{M}(\mathcal{J})$.

Proposition 6.2. *Let $\mathcal{M}$ be a separable type* III *factor, and $\mathcal{A}$ a separable unital C^*-subalgebra of $\mathcal{M}$. Then there is a unitary operator $U \in \mathcal{M}$ such that $U^*\mathcal{A}U \subseteq \mathcal{M}(\mathcal{J})$.*

Proof. Let $\{A_j\}_{j\in\mathbb{N}}$ be a norm-dense sequence in $\mathcal{A}$, $\{X_j\}_{j\in\mathbb{N}}$ a strong-operator dense sequence in $\mathcal{M}$, $\{Y_j\}_{j\in\mathbb{N}}$ the set of all noncommutative *-monomials generated by $\{A_j\}_{j\in\mathbb{N}} \cup \{X_j\}_{j\in\mathbb{N}}$, and $\mathcal{F}_n = \{Y_0, Y_1, \ldots, Y_n\}$ for $n \in \mathbb{N}$. By Lemma 2.10, there exists a sequence $\{Q_n\}_{n\in\mathbb{N}}$ of infinite projections in $\mathcal{M}$ such that $Q_n\mathcal{F}_nQ_0 = \{0\}$ for every $n \geqslant 1$. Let

$$P_n = \bigvee\{R(YQ_0) \colon Y \in \mathcal{F}_n\} \leqslant I - Q_n, \quad n \in \mathbb{N}. \tag{6.1}$$

Then $\bigvee_{n\in\mathbb{N}} P_n = I$ since the sequence $\{X_j\}_{j\in\mathbb{N}}$ generates $\mathcal{M}$ as a von Neumann algebra. Let $E_0 = P_0$, and $E_n = P_n - P_{n-1}$ for $n \geqslant 1$. Since $P_n \neq I$, we may assume that $E_n \neq 0$ for every $n \in \mathbb{N}$ if we consider a subsequence of $\{P_n\}_{n\in\mathbb{N}}$.

Let $\mathcal{B}_1$ be a type I_∞ unital subfactor of $\mathcal{M}$ with a system of matrix units $\{E_{mn}\}_{m,n\in\mathbb{N}}$ such that $E_{nn} = E_n$ for $n \in \mathbb{N}$. Let $\mathcal{K}_{\mathcal{B}_1}$ be the ideal of compact operators in $\mathcal{B}_1$, $\mathcal{N}_1 = \mathcal{B}_1' \cap \mathcal{M}$ the relative commutant of $\mathcal{B}_1$ in $\mathcal{M}$, and $\mathcal{J}_1$ the C^*-subalgebra of $\mathcal{M}$ generated by $\mathcal{N}_1\mathcal{K}_{\mathcal{B}_1}$. For any $j, n \in \mathbb{N}$, (6.1) shows that

$$R(Y_jP_n) \leqslant \bigvee\{R(Y_jYQ_0) \colon Y \in \mathcal{F}_n\} \leqslant \bigvee\{R(YQ_0) \colon Y \in \mathcal{F}_m\} = P_m$$

for all sufficiently large $m \in \mathbb{N}$. Then $Y_jP_n = P_mY_jP_n \in \mathcal{J}_1$ by Lemma 6.1. It follows that $Y_j \in \mathcal{M}(\mathcal{J}_1)$ since $\{P_n\}_{n\in\mathbb{N}}$ is an approximate unit of $\mathcal{J}_1$. In particular, $A_j \in \mathcal{M}(\mathcal{J}_1)$ for $j \in \mathbb{N}$, and therefore, $\mathcal{A} \subseteq \mathcal{M}(\mathcal{J}_1)$. Let U be a unitary operator in $\mathcal{M}$ such that $U^*\mathcal{B}_1U = \mathcal{B}$. Then $U^*\mathcal{A}U \subseteq \mathcal{M}(\mathcal{J})$. □

6.2. **Main Results in $\mathcal{M}(\mathcal{J})$.** The result presented below can be derived from the proof of [7, Proposition 2.1]. We will use it to prove a comparable version of Lemma 2.7 in the context of $\mathcal{M}(\mathcal{J})$.

Proposition 6.3. *Let $\mathcal{B}_0$ be a finite-dimensional unital subfactor of $\mathcal{M}(\mathcal{J})$, and $\eta \colon \mathcal{B}_0 \to \mathcal{M}(\mathcal{J})$ a completely positive map. Then there exists a single operator $T \in \mathcal{M}(\mathcal{J})$ such that $\eta(B) = T^*BT$ for every $B \in \mathcal{B}_0$.*

Lemma 6.4. *Let $\mathcal{M}$ be a separable type* III *factor, $\mathcal{A}$ a unital C^*-subalgebra of $\mathcal{M}(\mathcal{J})$, and $P \in \mathcal{J}$ a projection. Suppose $\psi \colon \mathcal{A} \to \mathcal{J}$ is a completely positive map, and there exist completely positive maps $\sigma \colon \mathcal{A} \to M_n(\mathbb{C})$ and $\eta \colon M_n(\mathbb{C}) \to \mathcal{J}$ such that*

1. *$\psi = \eta \circ \sigma$.*
2. *$\sigma|_{\mathcal{A}\cap\mathcal{J}} = \psi|_{\mathcal{A}\cap\mathcal{J}} = 0$.*

Then ψ can be approximated in the pointwise-norm topology by maps of the form

$$A \mapsto V^*AV,$$

where $V \in \mathcal{J}$ and $PV = 0$. In particular, V can be selected as a partial isometry such that $V^*V = \psi(I)$ when $\psi(I)$ is a projection.

Proof. Let $\mathcal{B}_0$ be a finite-dimensional unital subfactor of $\mathcal{B}$ with a system of matrix units $\{E_{ij}\}_{1\leqslant i,j\leqslant n}$. We can assume that $\sigma\colon \mathcal{A}\to\mathcal{B}_0$ and $\eta\colon \mathcal{B}_0\to\mathcal{J}$. By Proposition 6.3, there is an operator $T\in\mathcal{M}(\mathcal{J})$ such that

$$\eta(B)=T^*BT \text{ for every } B\in\mathcal{B}_0.$$

Let $T=U|T|$ be the polar decomposition. Then $|T|\in\mathcal{J}$ since $\eta(I)=T^*T\in\mathcal{J}$.

Let $\mathcal{F}$ be a finite subset of $\mathcal{A}$ containing I, and $\varepsilon>0$. We may assume that $P\in\mathcal{A}$ and $P\in\mathcal{F}$. According to [2, Lemma 4.4], there are pure states $\rho^1,\rho^2,\ldots,\rho^k$ on $\mathcal{A}$ with $\rho^t|_{\mathcal{A}\cap\mathcal{J}}=0$ for $1\leqslant t\leqslant k$, and operators $A_{t,j}, 1\leqslant t\leqslant k, 1\leqslant j\leqslant n$ in $\mathcal{A}$, such that

$$\|T\|^2\Big\|\sigma(A)-\sum_{t,i,j}\rho^t(A_{t,i}^*AA_{t,j})E_{ij}\Big\|<\frac{\varepsilon}{2} \text{ for every } A\in\mathcal{F}.$$

It follows that

$$\Big\|\psi(A)-\sum_{t,i,j}\rho^t(A_{t,i}^*AA_{t,j})T^*E_{ij}T\Big\|<\frac{\varepsilon}{2} \text{ for every } A\in\mathcal{F}.$$

According to [1, Proposition 2.2], let C_t be a positive operator in $\mathcal{A}$ with $\|C_t\|=1$ and $\rho^t(C_t)=1$ such that

$$\|T\|^2\|C_t(X-\rho^t(X))C_t\|<\frac{\varepsilon}{4kn^2} \tag{6.2}$$

for every $X\in\{A_{t,i}^*AA_{t,j}\colon 1\leqslant t\leqslant k, 1\leqslant i,j\leqslant n, A\in\mathcal{F}\}$. Since $\mathcal{K}_{\mathcal{B}}$ is non-degenerate, there exists a projection $Q\in\mathcal{K}_{\mathcal{B}}$ such that $\|QC_t^2Q\|>\frac{1}{2}$ for $1\leqslant t\leqslant k$. Then there exists a nonzero spectral projection $P_t\leqslant Q$ of QC_t^2Q such that

$$P_t\geqslant P_tC_t^2P_t\geqslant\frac{1}{2}P_t \text{ for } 1\leqslant t\leqslant k. \tag{6.3}$$

Let $\mathcal{G}=\{C_tA_{t,i}^*AA_{t,j}C_t\colon 1\leqslant t\leqslant k, 1\leqslant i,j\leqslant n, A\in\mathcal{F}\}$. By Lemma 2.10, there exist infinite projections $\{Q_t\}_{1\leqslant t\leqslant k}$ such that $Q_t\leqslant P_t$ for $1\leqslant t\leqslant k$, and $Q_s\mathcal{G}Q_t=\{0\}$ when $s\neq t$. Let U_t be a partial isometry in $\mathcal{M}$ such that

$$U_t^*U_t=E_{11},\quad U_tU_t^*=Q_t.$$

Since (6.3) implies that

$$E_{11}\geqslant U_t^*Q_tC_t^2Q_tU_t\geqslant\frac{1}{2}U_t^*Q_tU_t=\frac{1}{2}E_{11},$$

there exists a positive operator $X_t\in E_{11}\mathcal{M}(\mathcal{J})E_{11}\subseteq\mathcal{J}$ with $\|X_t\|^2\leqslant 2$ such that

$$X_t^2(U_t^*Q_tC_t^2Q_tU_t)=(U_t^*Q_tC_t^2Q_tU_t)X_t^2=E_{11} \text{ for } 1\leqslant t\leqslant k.$$

Consequently, $\rho^t(A_{t,i}^*AA_{t,j})E_{ij}=E_{i1}X_tU_t^*Q_tC_t\rho^t(A_{t,i}^*AA_{t,j})C_tQ_tU_tX_tE_{1j}$, and then

$$\sum_{t,i,j}\rho^t(A_{t,i}^*AA_{t,j})T^*E_{ij}T=\sum_{t,i,j}T^*E_{i1}X_tU_t^*Q_tC_t\rho^t(A_{t,i}^*AA_{t,j})C_tQ_tU_tX_tE_{1j}T. \tag{6.4}$$

Moreover, $Q_tU_tX_tE_{1j}T=QQ_tU_tX_tE_{1j}U|T|\in\mathcal{J}$ by Lemma 6.1 because $Q,|T|\in\mathcal{J}$. Let $Y=\sum_{t,j}A_{t,j}C_tQ_tU_tX_tE_{1j}T\in\mathcal{J}$. Since $Q_s\mathcal{F}Q_t=\{0\}$ when $s\neq t$, we have

$$Y^*AY=\sum_{t,i,j}T^*E_{i1}X_tU_t^*Q_tC_tA_{t,i}^*AA_{t,j}C_tQ_tU_tX_tE_{1j}T. \tag{6.5}$$

From (6.2), (6.4) and (6.5), it follows that

$$\Big\|\sum_{t,i,j}\rho^t(A_{t,i}^*AA_{t,j})T^*E_{ij}T-Y^*AY\Big\|<\frac{\varepsilon}{2} \text{ for every } A\in\mathcal{F}.$$

Consequently, $\|\psi(A)-Y^*AY\|<\varepsilon$ for every $A\in\mathcal{F}$. In particular, $\|Y^*PY\|<\varepsilon$ by the assumption $P\in\mathcal{F}$, and then we can replace Y with $V=(1-P)Y\in\mathcal{J}$. Furthermore, since $\|\psi(I)-V^*V\|<\varepsilon$, we can choose V as a partial isometry such that $V^*V=\psi(I)$ when $\psi(I)$ is a projection. $\square$

Now, we present the main theorem for this section. Similar conclusion can be found in [5, Lemma 11].

Theorem 6.5. *Let $\mathcal{M}$ be a separable type* III *factor, $\mathcal{A}$ a separable unital C^*-subalgebra of $\mathcal{M}(\mathcal{J})$, and $P \in \mathcal{J}$ a projection. For any completely positive map $\psi\colon \mathcal{A} \to \mathcal{B}$ with $\psi|_{\mathcal{A}\cap\mathcal{J}} = 0$, there is a sequence $\{V_k\}_{k\in\mathbb{N}}$ in $\mathcal{M}(\mathcal{J})$ such that*

(1) *$PV_k = 0$ for every $k \in \mathbb{N}$.*
(2) *$\lim_{k\to\infty}\|\psi(A) - V_k^* A V_k\| = 0$ for every $A \in \mathcal{A}$.*
(3) *$\psi(A) - V_k^* A V_k \in \mathcal{J}$ for every $k \in \mathbb{N}$ and $A \in \mathcal{A}$.*

In particular, V_k can be selected as a partial isometry such that $V_k^* V_k = \psi(I)$ when $\psi(I)$ is a projection.

Proof. Let $\{Q_n\}_{n\in\mathbb{N}}$ be a sequence of finite projections in $\mathcal{K}_{\mathcal{M}}$ with $\bigvee_{n\in\mathbb{N}} Q_n = I$, and $\{A_j\}_{j\in\mathbb{N}}$ a norm-dense sequence in $\mathcal{A}^{\text{s.a.}}$ with $A_0 = I$.

Fix $k \in \mathbb{N}$. According to [3, Theorem 2], there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ of finite rank operators in $\mathcal{K}_{\mathcal{B}}$ such that

(1) $\sum_n E_n^2 = I$ and $\|E_n Q_m\| < 2^{-n}$ for $0 \leqslant m \leqslant n-1$.
(2) $\|\psi(A_j) - \sum_n E_n\psi(A_j)E_n\| < 2^{-k-1}$ for $0 \leqslant j \leqslant k$.
(3) $\psi(A) - \sum_n E_n\psi(A)E_n \in \mathcal{K}_{\mathcal{B}}$ for every $A \in \mathcal{A}$.

Let P_n denote the finite rank projection $R(E_n)$ in $\mathcal{K}_{\mathcal{B}}$. We define a completely positive map

$$\psi_n\colon \mathcal{A} \to \mathcal{J}, \quad A \mapsto P_n\psi(A)P_n.$$

By Lemma 6.4, we can choose a sequence $\{U_n\}_{n\in\mathbb{N}}$ in $\mathcal{J}$ inductively such that

(1) $PU_n = 0$ for $n \geqslant 0$, and $\|Q_m U_n\| < 2^{-n}$ for $0 \leqslant m \leqslant n-1$.
(2) $\|U_m^* A_j U_n\| < 2^{-2n-k-4}$ for $0 \leqslant j \leqslant n+k$ and $0 \leqslant m \leqslant n-1$.
(3) $\|\psi_n(A_j) - U_n^* A_j U_n\| < 2^{-n-k-3}$ for $0 \leqslant j \leqslant n+k$.

Then $\|U_m^* A_j U_n\| < 2^{-2\max\{m,n\}-k-4}$ when $0 \leqslant j \leqslant \max\{m,n\}+k$ and $m \neq n$. Let $V = \sum_n U_n E_n$. Then

$$\sum_n E_n\psi(A_j)E_n - V^*A_jV = \sum_n E_n\big(\psi(A_j) - U_n^*A_jU_n\big)E_n - \sum_{m\neq n} E_m U_m^* A_j U_n E_n$$

for every $j \geqslant 0$, and $PV = 0$. The above sums are norm-convergent and each summand lies in $\mathcal{J}$. It follows that $\sum_n E_n\psi(A_j)E_n - V^*A_jV \in \mathcal{J}$ for $j \geqslant 0$. We also have the estimation

$$\Big\|\sum_n E_n\psi(A_j)E_n - V^*A_jV\Big\| < 2^{-k-1} \text{ for } 0 \leqslant j \leqslant k.$$

Therefore, $\psi(A_j) - V^*A_jV \in \mathcal{J}$ for $j \geqslant 0$, and $\|\psi(A_j) - V^*A_jV\| < 2^{-k}$ for $0 \leqslant j \leqslant k$. In particular, V^*V is a bounded operator if we consider $A_0 = I$. We can conclude that V belongs to $\mathcal{M}$. Furthermore, since $\|E_nQ_m\| < 2^{-n}$ and $\|Q_mU_n\| < 2^{-n}$ for $n > m$, we have

$$VQ_m = \sum_n U_nE_nQ_m \in \mathcal{J}, \quad Q_mV = \sum_n Q_mU_nE_n \in \mathcal{J},$$

for every $m \geqslant 0$. It follows that $V \in \mathcal{M}(\mathcal{J})$. Now we set $V_k = V$. $\square$

6.3. **Voiculescu's Theorem in $\mathcal{M}(\mathcal{J})$.** We now prove Voiculescu's theorem for $\mathcal{M}(\mathcal{J})$, whose proof is similar to the proof of Theorem 4.2.

Theorem 6.6. *Let $\mathcal{M}$ be a separable type* III *factor, and $\mathcal{A}$ a separable unital C^*-subalgebra of $\mathcal{M}(\mathcal{J})$. If $\varphi\colon \mathcal{A} \to \mathcal{B}$ is a unital $*$-homomorphism with $\varphi|_{\mathcal{A}\cap\mathcal{J}} = 0$, then there is a sequence $\{V_k\}_{k\in\mathbb{N}}$ of isometries in $\mathcal{M}(\mathcal{J}) \otimes M_2(\mathbb{C})$ such that*

(1) *$V_k^*V_k = I \oplus I, V_kV_k^* = I \oplus 0$ for every $k \in \mathbb{N}$.*
(2) *$\lim_{k\to\infty}\|(A \oplus \varphi(A)) - V_k^*(A \oplus 0)V_k\| = 0$ for every $A \in \mathcal{A}$.*
(3) *$(A \oplus \varphi(A)) - V_k^*(A \oplus 0)V_k \in \mathcal{J} \otimes M_2(\mathbb{C})$ for every $k \in \mathbb{N}$ and $A \in \mathcal{A}$.*

Proof. Let $\{E_{mn}\}_{m,n\in\mathbb{N}}$ be a system of matrix units in $\mathcal{B}$ such that $\sum_n E_{nn} = I$ and $E_{00} \sim I$ in $\mathcal{B}$. Let T be an isometry in $\mathcal{B}$ with $TT^* = E_{00}$, and let S denote the isometry $\sum_n E_{n+1,n} \in \mathcal{B}$. We define

$$\psi\colon \mathcal{A} \to \mathcal{B}, \quad A \mapsto \sum_n E_{n0} T\varphi(A)T^* E_{0n}.$$

By Theorem 6.5, there is a sequence $\{U_k\}_{k\in\mathbb{N}}$ of isometries in $\mathcal{M}(\mathcal{J})$ such that

$$\lim_{k\to\infty} \|U_k\psi(A) - AU_k\| = 0 \text{ for every } A \in \mathcal{A},$$

and $U_k\psi(A) - AU_k \in \mathcal{J}$ for every $k \in \mathbb{N}$ and $A \in \mathcal{A}$. Let $P_k = U_kU_k^* \in \mathcal{M}(\mathcal{J})$. There is a partial isometry $Y_k \in \mathcal{J}$ such that

$$Y_k^*Y_k = E_{00} + E_{11}, \quad Y_kY_k^* = (E_{00} + E_{11}) - TP_k^{\perp}T^*.$$

Let $W_k = Y_k + I - (E_{00} + E_{11})$ be an isometry in $\mathcal{M}(\mathcal{J})$ with $W_kW_k^* = I - TP_k^{\perp}T^*$, and $F_k = P_k^{\perp}T^* + U_kW_k^*$ a unitary operator in $\mathcal{M}(\mathcal{J})$. The rest of the proof is the same as Theorem 4.2. ☐

6.4. **Applications in $\mathcal{M}(\mathcal{J})$.** Let $T \in \mathcal{M}(\mathcal{J})$. We say that T is *reducible* in $\mathcal{M}(\mathcal{J})$ if there is a projection $P \in \mathcal{M}(\mathcal{J})$ such that $PT = TP$ and $P \neq 0, I$. Similar to Theorem 5.1, Theorem 6.6 implies the following denseness result.

Theorem 6.7. *Let $\mathcal{M}$ be a separable type* III *factor. Then the set of all reducible operators is norm-dense in $\mathcal{M}(\mathcal{J})$.*

If $\mathscr{A}$ a separable unital subalgebra of $\mathcal{M}(\mathcal{J})/\mathcal{J}$, then the *essential lattice $Lat_e(\mathscr{A})$* of $\mathscr{A}$ is the set of all projections $p \in \mathcal{M}(\mathcal{J})/\mathcal{J}$ such that $p^{\perp}ap = 0$ for every $a \in \mathscr{A}$. Similar to Lemma 5.2, Theorem 6.6 implies the following distance formula.

Lemma 6.8. *Let $\mathcal{M}$ be a separable type* III *factor, and $\mathscr{A}$ a separable unital subalgebra of $\mathcal{M}(\mathcal{J})/\mathcal{J}$. Then for any $t \in \mathcal{M}(\mathcal{J})/\mathcal{J}$, there is a projection q in $Lat_e(\mathscr{A})$ such that*

$$\|q^{\perp}tq\| = d(t, \mathscr{A}).$$

Note that every separable norm-closed unital subalgebra of $\mathcal{M}(\mathcal{J})/\mathcal{J}$ is reflexive by Lemma 6.8. In particular, Voiculescu's relative bicommutant theorem holds.

Theorem 6.9. *Let $\mathcal{M}$ be a separable type* III *factor. Then every separable unital C^*-subalgebra of $\mathcal{M}(\mathcal{J})/\mathcal{J}$ equals its relative bicommutant.*

7. Nuclear Length

7.1. **Nuclear Length.** Let $\mathcal{M}$ be a separable properly infinite factor, and $\mathcal{B}$ a C^*-subalgebra of $\mathcal{M}$. Inspired by quasicentral approximate units, we introduce the *nuclear length* of $\mathcal{B}$ in $\mathcal{M}$.

Definition 7.1. *We set $L_{\text{nuc}}(\mathcal{B}, \mathcal{M}) = 0$ if $\mathcal{B}$ is nuclear. Inductively, we set*

$$L_{\text{nuc}}(\mathcal{B}, \mathcal{M}) = m,$$

if $L_{\text{nuc}}(\mathcal{B}, \mathcal{M}) \neq k$ for $0 \leqslant k \leqslant m-1$, and for any finite subset $\mathcal{F}$ of $\mathcal{B}$ and any $\varepsilon > 0$, there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ of positive operators in $\mathcal{M}$ and a sequence $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$ of C^*-subalgebras of $\mathcal{M}$ such that

1. $\sum_n E_n^2 = I$*, and* $L_{\text{nuc}}(\mathcal{B}_n, \mathcal{M}) \leqslant m-1$ *for every* $n \in \mathbb{N}$.
2. $E_n\mathcal{B}E_n \subseteq \mathcal{B}_n$ *for every* $n \in \mathbb{N}$.
3. $\|B - \sum_n E_nBE_n\| < \varepsilon$ *for every* $B \in \mathcal{F}$.

It is evident from the above definition that $L_{\text{nuc}}(U^*\mathcal{B}U, \mathcal{M}) = L_{\text{nuc}}(\mathcal{B}, \mathcal{M})$ for every unitary operator U in $\mathcal{M}$. Consequently, the nuclear length is unitarily invariant.

Let $P_{\mathcal{B}} = \bigvee_{B\in\mathcal{B}} R(B)$, where $R(B)$ is the range projection of B. The *multiplier algebra* of $\mathcal{B}$ is then defined as

$$\mathcal{M}(\mathcal{B}) = \{T \in P_{\mathcal{B}}\mathcal{M}P_{\mathcal{B}} \colon T\mathcal{B} \subseteq \mathcal{B}, \mathcal{B}T \subseteq \mathcal{B}\}.$$

Note that $\mathcal{B}$ is an ideal of $\mathcal{M}(\mathcal{B})$ and $P_{\mathcal{B}}$ is the identity of $\mathcal{M}(\mathcal{B})$.

Lemma 7.2. *If $L_{\mathrm{nuc}}(\mathcal{B},\mathcal{M})<\infty$, then $L_{\mathrm{nuc}}(\mathcal{M}(\mathcal{B}),\mathcal{M})\leqslant 1+L_{\mathrm{nuc}}(\mathcal{B},\mathcal{M})$.*

Proof. Let $\mathcal{F}$ be a finite subset of $\mathcal{M}(\mathcal{B})$, and $\varepsilon>0$. According to [3, Theorem 2], there is a sequence $\{E_n\}_{n=1}^{\infty}$ in $\mathcal{B}$ such that $\sum_{n=1}^{\infty}E_n^2=P_{\mathcal{B}}$ and

$$\Big\|B-\sum_{n=1}^{\infty}E_nBE_n\Big\|<\varepsilon \text{ for every } B\in\mathcal{F}.$$

We set $E_0=I-P_{\mathcal{B}}$, and $\mathcal{B}_n=\mathcal{B}$ for every $n\in\mathbb{N}$. ☐

Let $\mathcal{B}$ be a type I_∞ subfactor of $\mathcal{M}$, and $\mathcal{K}_{\mathcal{B}}$ the ideal generated by finite rank projections in $\mathcal{B}$. It is well-known that $\mathcal{K}_{\mathcal{B}}$ is nuclear while $\mathcal{B}$ is not. Since $\mathcal{B}$ is the multiplier algebra of $\mathcal{K}_{\mathcal{B}}$, we have $L_{\mathrm{nuc}}(\mathcal{B},\mathcal{M})=1$ by Lemma 7.2.

Example 7.3. *If $\mathcal{B}$ is a von Neumann algebra of type* I, *then $L_{\mathrm{nuc}}(\mathcal{B},\mathcal{M})\leqslant 1$.*

Proof. There is a sequence $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ of abelian von Neumann algebras such that

$$\mathcal{B}=\big(\mathcal{A}_0\mathbin{\overline{\otimes}}\mathcal{B}(l^2)\big)\bigoplus\prod_{n=1}^{\infty}{}^{\oplus}\mathcal{A}_n\otimes M_n(\mathbb{C}).$$

Let

$$\mathcal{B}_0=\big(\mathcal{A}_0\otimes\mathcal{K}(l^2)\big)\bigoplus\sum_{n=1}^{\infty}{}^{\oplus}\mathcal{A}_n\otimes M_n(\mathbb{C}).$$

Since $\mathcal{B}_0$ is nuclear and $\mathcal{B}=\mathcal{M}(\mathcal{B}_0)$, we get $L_{\mathrm{nuc}}(\mathcal{B},\mathcal{M})\leqslant 1$ by Lemma 7.2. ☐

The following theorem is a generalization of Proposition 3.1.

Theorem 7.4. *Let $\mathcal{M}$ be a separable properly infinite factor, $\mathcal{A}$ a unital C^*-subalgebra of $\mathcal{M}$, and $\mathcal{B}$ a C^*-subalgebra of $\mathcal{M}$ with $L_{\mathrm{nuc}}(\mathcal{B},\mathcal{M})<\infty$. Assume that $\psi\colon\mathcal{A}\to\mathcal{B}$ is a $*$-homomorphism such that $\psi|_{\mathcal{A}\cap\mathcal{K}_{\mathcal{M}}}=0$. Then $\psi\in\widehat{\mathfrak{S}\mathfrak{F}}$ and there is a sequence $\{V_k\}_{k\in\mathbb{N}}$ of isometries in $\mathcal{M}$ such that*

$$\lim_{k\to\infty}\|V_k\varphi(A)-AV_k\|=0 \text{ for every } A\in\mathcal{A}.$$

Furthermore, if $\mathcal{M}$ is semifinite, we can choose $\{V_k\}_{k\in\mathbb{N}}$ such that

$$V_k\varphi(A)-AV_k\in\mathcal{K}_{\mathcal{M}} \text{ for every } k\in\mathbb{N} \text{ and } A\in\mathcal{A}.$$

Proof. Induction on $L_{\mathrm{nuc}}(\mathcal{B},\mathcal{M})=m$ is performed. If $\mathcal{B}$ is nuclear, then the inclusion map $\mathrm{id}_{\mathcal{B}}\colon\mathcal{B}\to\mathcal{M}$ is nuclear. Therefore, the composition $\psi=\mathrm{id}_{\mathcal{B}}\circ\psi$ is a nuclear map with respect to $\mathcal{K}_{\mathcal{M}}$, and thus $\psi\in\widehat{\mathfrak{F}}\subseteq\widehat{\mathfrak{S}\mathfrak{F}}$.

Assume $m\geqslant 1$. Let $\mathcal{F}$ be a finite subset of $\mathcal{A}$ containing I, and $\varepsilon>0$. We can find $\{E_n\}_{n\in\mathbb{N}}$ and $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$ such that

(1) $\sum_n E_n^2=I$, and $L_{\mathrm{nuc}}(\mathcal{B}_n,\mathcal{M})\leqslant m-1$ for every $n\in\mathbb{N}$.
(2) $E_n\mathcal{B}E_n\subseteq\mathcal{B}_n$ for every $n\in\mathbb{N}$.
(3) $\|\psi(A)-\sum_n E_n\psi(A)E_n\|<\varepsilon$ for every $A\in\mathcal{F}$.

By induction, the completely positive map $\psi_n\colon\mathcal{A}\to\mathcal{B}_n$ defined by $A\mapsto E_n\psi(A)E_n$ lies in $\widehat{\mathfrak{S}\mathfrak{F}}$, and

$$\Big\|\psi(A)-\sum_n\psi_n(A)\Big\|<\varepsilon \text{ for every } A\in\mathcal{F}.$$

Then $\sum_n\psi_n(I)$ converges in the strong-operator topology since $I\in\mathcal{F}$. Hence $\psi\in\widehat{\mathfrak{S}\mathfrak{F}}$ by Lemma 2.6. Now the result follows from Lemma 4.1. ☐

7.2. **Approximate Nuclear Length.** At last, we introduce the *approximate nuclear length*. Let $\mathcal{M}$ be a separable properly infinite factor, and $\mathcal{B}$ a C^*-subalgebra of $\mathcal{M}$.

Definition 7.5. *We set $AL_{\text{nuc}}(\mathcal{B},\mathcal{M}) = 0$ if the inclusion map $\text{id}_{\mathcal{B}}\colon \mathcal{B} \to \mathcal{M}$ is nuclear. Inductively, we set*

$$AL_{\text{nuc}}(\mathcal{B},\mathcal{M}) = m,$$

if $AL_{\text{nuc}}(\mathcal{B},\mathcal{M}) \neq k$ for $0 \leqslant k \leqslant m-1$, and for any finite subset $\mathcal{F}$ of $\mathcal{B}$ and any $\varepsilon > 0$, there is a sequence $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$ of C^*-subalgebras of $\mathcal{M}$, and a sequence $\{\psi_n\colon \mathcal{B} \to \mathcal{B}_n\}_{n\in\mathbb{N}}$ of completely positive maps such that

(1) *$AL_{\text{nuc}}(\mathcal{B}_n,\mathcal{M}) \leqslant m-1$ for every $n \in \mathbb{N}$.*
(2) *$\|B - \sum_n \psi_n(B)\| < \varepsilon$ for every $B \in \mathcal{F}$.*

It is clear that $AL_{\text{nuc}}(\mathcal{B},\mathcal{M}) \leqslant L_{\text{nuc}}(\mathcal{B},\mathcal{M})$ and the approximate nuclear length is unitarily invariant. Let $\pi_1, \pi_2\colon \mathcal{B} \to \mathcal{M}$ be *-homomorphisms. We say that π_1 and π_2 are *approximately unitarily equivalent* (denoted by $\pi_1 \sim_a \pi_2$) if for any finite subset $\mathcal{F}$ of $\mathcal{B}$ and any $\varepsilon > 0$, there is a unitary operator U in $\mathcal{M}$ such that

$$\|\pi_1(A) - U^*\pi_2(A)U\| < \varepsilon \text{ for every } A \in \mathcal{F}.$$

Obviously, $\pi_1 \sim_a \pi_2$ implies that $\ker\pi_1 = \ker\pi_2$. The following result shows that the approximate nuclear length is approximately unitarily invariant.

Lemma 7.6. *Let $\mathcal{M}$ be a separable properly infinite factor, and $\mathcal{B}$ a C^*-subalgebra of $\mathcal{M}$. If $\pi\colon \mathcal{B} \to \mathcal{M}$ is a *-homomorphism with $\pi \sim_a \text{id}_{\mathcal{B}}$, then*

$$AL_{\text{nuc}}(\pi(\mathcal{B}),\mathcal{M}) = AL_{\text{nuc}}(\mathcal{B},\mathcal{M}).$$

Proof. Note that π is faithful since $\text{id}_{\mathcal{B}}$ is. Let $\mathcal{F}$ be a finite subset of $\mathcal{B}$, and $\varepsilon > 0$. There is a unitary operator U in $\mathcal{M}$ such that

$$\|\pi(B) - U^*BU\| < \frac{\varepsilon}{2} \text{ for every } B \in \mathcal{F}.$$

If the inclusion map $\text{id}_{\mathcal{B}}\colon \mathcal{B} \to \mathcal{M}$ is nuclear, then there is a factorable map $\psi\colon \mathcal{B} \to \mathcal{M}$ such that $\|B - \psi(B)\| < \frac{\varepsilon}{2}$ for every $B \in \mathcal{F}$. It follows that

$$\|\pi(B) - U^*\psi(B)U\| < \varepsilon \text{ for every } B \in \mathcal{F}.$$

Let $\varphi\colon \pi(\mathcal{B}) \to \mathcal{M}, \pi(B) \mapsto U^*\psi(B)U$ be a factorable map. Then

$$\|\pi(B) - \varphi(\pi(B))\| < \varepsilon \text{ for every } B \in \mathcal{F}.$$

Hence $\text{id}_{\pi(\mathcal{B})}$ is nuclear.

If $AL_{\text{nuc}}(\mathcal{B}) = m \geqslant 1$, then we can find $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$ and $\{\psi_n\colon \mathcal{B} \to \mathcal{B}_n\}_{n\in\mathcal{M}}$ such that

(1) $AL_{\text{nuc}}(\mathcal{B}_n,\mathcal{M}) \leqslant m-1$ for every $n \in \mathbb{N}$.
(2) $\|B - \sum_n \psi_n(B)\| < \frac{\varepsilon}{2}$ for every $B \in \mathcal{F}$.

Let $\mathcal{A}_n = U^*\mathcal{B}_nU$, and $\varphi_n\colon \pi(\mathcal{B}) \to \mathcal{A}_n, \pi(B) \mapsto U^*\psi_n(B)U$. Then

(1) $AL_{\text{nuc}}(\mathcal{A}_n,\mathcal{M}) \leqslant m-1$ for every $n \in \mathbb{N}$.
(2) $\|\pi(B) - \sum_n \varphi_n(\pi(B))\| < \varepsilon$ for every $B \in \mathcal{F}$.

Hence $AL_{\text{nuc}}(\pi(\mathcal{B}),\mathcal{M}) \leqslant AL_{\text{nuc}}(\mathcal{B},\mathcal{M})$. Conversely, $AL_{\text{nuc}}(\mathcal{B},\mathcal{M}) \leqslant AL_{\text{nuc}}(\pi(\mathcal{B}),\mathcal{M})$ since $\pi^{-1} \sim_a \text{id}_{\pi(\mathcal{B})}$. This completes the proof. □

Similar to Theorem 7.4, we have the following result.

Theorem 7.7. *Let $\mathcal{M}$ be a separable properly infinite factor, $\mathcal{A}$ a unital C^*-subalgebra of $\mathcal{M}$, and $\mathcal{B}$ a C^*-subalgebra of $\mathcal{M}$ with $AL_{\text{nuc}}(\mathcal{B},\mathcal{M}) < \infty$. Assume that $\psi\colon \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism such that $\psi|_{\mathcal{A}\cap\mathcal{K}_{\mathcal{M}}} = 0$. Then $\psi \in \widehat{\mathfrak{S}\mathfrak{F}}$ and there is a sequence $\{V_k\}_{k\in\mathbb{N}}$ of isometries in $\mathcal{M}$ such that*

$$\lim_{k\to\infty} \|V_k\varphi(A) - AV_k\| = 0 \textit{ for every } A \in \mathcal{A}.$$

Furthermore, if $\mathcal{M}$ is semifinite, we can choose $\{V_k\}_{k\in\mathbb{N}}$ such that

$$V_k\varphi(A) - AV_k \in \mathcal{K}_{\mathcal{M}} \textit{ for every } k \in \mathbb{N} \textit{ and } A \in \mathcal{A}.$$

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