Automorphisms and derivations on algebras endowed with formal infinite sums

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Abstract

We establish a correspondence between automorphisms and derivations on certain algebras of generalised power series. In particular, we describe a Lie algebra of derivations on a field $k((G))$ of generalised power series, exploiting our knowledge of its group of valuation preserving automorphisms. The correspondence is given by the formal Taylor expansion of the exponential. In order to define the exponential map, we develop an appropriate notion of summability of infinite families in algebras. We show that there is a large class of algebras in which the exponential induces the above correspondence.

Introduction

Let k be a field of characteristic 0. The automorphism group of the valued field $k((t))$ of Laurent series was studied in Schilling's classical paper [22]. A derivation on $k((t))$ is a k-linear map $\partial\colon k((t)) \longrightarrow k((t))$ satisfying the Leibniz product rule. The k-vector space of derivations on $k((t))$ becomes a Lie algebra once endowed with the Lie bracket $[\partial_1, \partial_2] := \partial_1 \circ \partial_2 - \partial_2 \circ \partial_1$. Let σ and ∂ denote respectively an automorphism and a derivation. Via the Taylor series of the logarithm and of the exponential,

$$\log(\sigma) = \sum_{n\geqslant 1} \frac{(-1)^{n+1}}{n} (\mathrm{Id} - \sigma)^{[n]} \text{ and } \exp(\partial) = \sum_{n\geqslant 0} \frac{1}{n!} \partial^{[n]} \tag{1}$$

(where $\varphi^{[n]}$ denotes the n^{th} iterate of a map φ), one obtains the fundamental relation between the group of automorphisms and the Lie algebra of derivations of $k((t))$ (cf. [21, Chapter 3]).

Fields of generalized power series $k(\!(G)\!)$ with coefficients in k and exponents in an arbitrary ordered abelian group G are instrumental to the valuation and model theory of fields (for example, when $G = \mathbb{Z}$, then $k(\!(G)\!)$ is just the Laurent series field). In [18], the authors described the group of valuation preserving automorphisms of $k(\!(G)\!)$, as a semidirect product of four distinct factors.

Our initial motivation for the present work was to describe the Lie algebra of derivations on $k(\!(G)\!)$, exploiting our knowledge of its group of valuation preserving automorphisms. To this end, we cast the problem into the following more general setting. Given a k-algebra A (not necessarily commutative), we study the interplay between the Lie algebra of its derivations and the group of its automorphisms. Here, the essential issue is to give appropriate conditions under which the infinite sums in (1) are indeed well-defined. To deal with this issue, we explore a general notion of summability in algebras, and of strongly linear maps (that is, linear maps that commute with infinite sums). We show that there is a large class of algebras in which the exponential and logarithm induce the desired correspondence for strongly linear ∂ and σ.

In Section 1 we introduce the axiomatic notion of vector spaces and algebras with a structure of formal summability, called summability spaces and summability algebras (see Definitions 1.1 and 1.27). A related notion has been presented independently in a categorical framework by Freni [6]. We show that the algebra of strongly linear maps on a summability space inherits a natural structure of summability algebra (Proposition 1.30). In Section 1.4, we first recall the construction of the algebra $k\langle\!\langle J\rangle\!\rangle$ of formal series with coefficients in k and non-commuting variables $X_j, j \in J$. We show in Proposition 1.34 that this algebra has a natural structure of summability algebra.

In Section 2, we introduce the notion of summability algebras with evaluations (see Definition 2.3). This allows any power series in $k\langle\!\langle J\rangle\!\rangle$ to be evaluated in A for each set J (see Remark 2.4). More precisely, such an algebra A is local (Proposition 2.8), and each summable family $\mathbf{a} = (\mathbf{a}_j)_{j\in J}$ ranging in its maximal ideal $\mathfrak{m}$ defines a unique strongly linear algebra morphism $\mathrm{ev}_{\mathbf{a}} \colon k\langle\!\langle J\rangle\!\rangle \longrightarrow A$ that maps each variable X_j to $\mathbf{a}_j$.

In this paper, we will apply evaluation morphisms for the univariate and bivariate cases only. In the univariate case (when J is a singleton), we can identify $k\langle\!\langle J\rangle\!\rangle$ with $k[\![X]\!]$ (the algebra of formal power series in the variable X). In particular, the formal power series

$$\log(1+X) := \sum_{n\geqslant 1} \frac{(-1)^{n+1}}{n} X^n \text{ and } \exp(X) := \sum_{n\geqslant 0} \frac{1}{n!} X^n$$

can be evaluated at each $\varepsilon \in \mathfrak{m}$. Furthermore, the relations

$$\exp(\log(1+\varepsilon)) = 1+\varepsilon \quad \text{and} \quad \log(\exp(\varepsilon)) = \varepsilon$$

follow (see Corollary 2.10) from evaluating the corresponding identities [24, Section 1.7, Theorem 7.2] in $k[\![X]\!]$ at ε. In the bivariate case, we likewise obtain (see Corollary 2.12) the Baker–Campbell–Hausdorff formula (see [8, Section 1.3.2] or [24, Section 1.4.7])

$$\log(\exp(\varepsilon)\cdot\exp(\delta)) = \varepsilon+\delta+\frac{1}{2}[\varepsilon,\delta]+\frac{1}{12}([\varepsilon,[\varepsilon,\delta]]-[\delta,[\varepsilon,\delta]])+\cdots$$

for $\varepsilon,\delta\in\mathfrak{m}$. Our first main result is a correspondence between automorphisms and derivations (Theorem 2.14) for all summability subalgebras (of strongly linear maps) with evaluations.

The purpose of Section 3 is to provide a large class of algebras to which Theorem 2.14 applies. To this end, we introduce algebras of Noetherian series in Section 3.3. Those include algebras of polynomials, algebras of formal series in commuting or non-commuting variables, and fields of generalised power series $k(\!(G)\!)$. Given an algebra $\mathbb{A}$ of Noetherian series, we consider the algebra $k\,\mathrm{Id}_{\mathbb{A}}+\mathfrak{m}$ where $\mathfrak{m}$ is the closed ideal of contracting strongly linear maps (see Definition 3.12). We show (Theorem 3.16) that this is an algebra with evaluations. Our second main result Theorem 3.17 is that the exponential is a bijection between the subset of $\mathfrak{m}$ consisting of derivations and the subset of $\mathrm{Id}_{\mathbb{A}}+\mathfrak{m}$ consisting of automorphisms of $\mathbb{A}$. In the case when $\mathbb{A}=k(\!(G)\!)$ is a field of Hahn series, this group is one of the three factors in the decomposition [18] of strongly linear valuation preserving automorphisms of $k(\!(G)\!)$. It follows in particular (Corollary 3.20) that this group is divisible and torsion-free. We derive from Theorem 3.17 our third and final main result Theorem 3.19, which moreover takes into account the Lie structure on the corresponding algebras. More precisely, we prove a formal analog of the Lie homomorphism theorem (cf. [8, Theorem 3.7]) for contracting strongly linear derivations and strongly linear automorphisms.

In Section 4, we focus on the case where k is an ordered exponential field [14], and we discuss the possibility of extending our correspondence to the other two factors in the semidirect decomposition of the group of valuation preserving automorphisms of $k(\!(G)\!)$.

Conventions and notations

We denote by $\mathbb{N}$ the set of natural numbers *with* 0 and by $\mathbb{N}^{>0}$ *without* 0. The power set of a set X is denoted by $\mathcal{P}(X)$ and its subset consisting of all finite subsets of X is denoted by $\mathcal{P}_{\mathrm{fin}}(X)$.

Given sets A,B, a function $f\colon A\longrightarrow B$ is a subset of $A\times B$ with the functional property. We often also identify the function f with the corresponding family $(f(a))_{a\in A}$. The set of functions from A to B is denoted by B^A. If $B\subseteq C$, then $B^A\subseteq C^A$. Note also that $A^{\emptyset}=\{\emptyset\}$.

Recall that an **ordering** on a set Ω is a binary relation $<$ on Ω such that for all $p,q,r\in\Omega$, we have

$$p\not< p \text{ and } [(p<q\wedge q<r)\Longrightarrow p<r].$$

We then say that $(\Omega,<)$ is an ordered set and for $p,q\in\Omega$, we write $p\leqslant q$ if $p=q$ or $p<q$. If $u\colon\mathbb{N}\longrightarrow\Omega$ is a sequence, then we say that u is **increasing** (resp. **strictly increasing**) if for all $m,n\in\mathbb{N}$ with $m<n$, we have $u(m)\leqslant u(n)$

(resp. $u(m) < u(n)$). A **subsequence** of u is a sequence $v = u \circ \varphi \colon \mathbb{N} \longrightarrow \Omega$ where $\varphi \colon \mathbb{N} \longrightarrow \mathbb{N}$ is strictly increasing.

If I is a set and $(M, +, 0)$ is a **monoid** (i.e. an associative, unital magma), then the **support** of an $\mathbf{f} \in M^I$ is the subset

$$\operatorname{supp} \mathbf{f} := \{i \in I : \mathbf{f}(i) \neq 0\}$$

of I. This object will be ubiquitous in the paper.

Throughout the paper, we fix a field k. All vector spaces and algebras considered below are over k. All considered algebras are associative, but not be necessarily unital, nor commutative. We will sometimes assume that k has characteristic 0. Given a vector space V and a set I, the set V^I of maps $\mathbf{v} \colon I \longrightarrow V$ is equipped with its natural vector space structure (pointwise operations). Note that the subset $V^{(I)}$ of maps with finite support is a subspace of V^I. Given vector spaces V_1, V_2, we write $\operatorname{Lin}(V_1, V_2)$ for the vector space of linear maps $V_1 \longrightarrow V_2$, and we set $\operatorname{Lin}(V_1) := \operatorname{Lin}(V_1, V_1)$. A *unital* algebra A is called **local** if one of the following equivalent assertions is satisfied [19, Theorem 19.1]:

- A has a unique maximal left ideal;
- A has a unique maximal right ideal;
- the set $A \setminus U(A)$ of non-units in A is an ideal of A.

Then the maximal left and right ideals are equal to $A \setminus U(A)$. Given an algebra $(A, +, \cdot)$ and $a, b \in A$, we write

$$[a, b] := a \cdot b - b \cdot a \in A,$$

and we recall that $(A, +, [\cdot, \cdot])$ is a Lie algebra.

1 Formal summability in algebras

1.1 Summability spaces

Freni [6] introduced a category $\Sigma \operatorname{Vect}$ whose objects are vector spaces equipped with a notion of formal sums. We propose an axiomatic description of such spaces, in the vein of [10, Section 6.2], that is more tailored to our purposes. It was shown by Freni to be equivalent to his [6, Proposition 3.27].

Let V be a vector space. We generalise the notion of finite summation operators

$$\Sigma_n \colon V^n \longrightarrow V,\ (v_0, \ldots, v_{n-1}) \mapsto v_0 + \cdots + v_{n-1}$$

to abstract sums $\Sigma_I \mathbf{v} \in V$ of families $\mathbf{v} \colon I \longrightarrow V$ indexed by possibly infinite sets I. Not all families $\mathbf{v}$ in the space V^I can be summed in a consistent way: if I is infinite and there is a $v \in V \setminus \{0\}$, then the constant family $(v)_{i \in I}$ cannot be summed. So we are to introduce axioms specifying the intended properties of sets $\operatorname{dom}\Sigma_I \subseteq V^I$ of summable families in conjunction with properties of operators Σ_I. It is very convenient for application to be able to do this for arbitrary sets I. This leads us to the following notion of summability spaces.

For a set I, let Σ_I be a k-linear function

$$\Sigma_I \colon \operatorname{dom}\Sigma_I \longrightarrow V$$

whose domain $\operatorname{dom}\Sigma_I$ is a subspace of the vector space V^I. Given sets I, J, consider the following axioms:

SS1 $V^{(I)} \subseteq \operatorname{dom}\Sigma_I$ and $\Sigma_I \mathbf{v} = \sum_{i\in \operatorname{supp}\mathbf{v}} \mathbf{v}(i)$ for all $\mathbf{v} \in V^{(I)}$.

SS2 if $\varphi\colon I \longrightarrow J$ is bijective and $\mathbf{v} \in \operatorname{dom}\Sigma_J$, then $\mathbf{v}\circ\varphi \in \operatorname{dom}\Sigma_I$, and $\Sigma_I(\mathbf{v}\circ\varphi) = \Sigma_J v$.

SS3 if $I = \bigsqcup_{j\in J} I_j$ and $\mathbf{v} \in \operatorname{dom}\Sigma_I$, then writing $\mathbf{v}_j := \mathbf{v}\restriction_{I_j}$ for each $j \in J$, we have

SS3a $\mathbf{v}_j \in \operatorname{dom}\Sigma_{I_j}$ for all $j \in J$,

SS3b $(\Sigma_{I_j}\mathbf{v}_j)_{j\in J} \in \operatorname{dom}\Sigma_J$, and

SS3c $\Sigma_I \mathbf{v} = \Sigma_J\left((\Sigma_{I_j}\mathbf{v}_j)_{j\in J}\right)$.

SS4 if $I = I_1 \sqcup I_2$ and $(\mathbf{v}, \mathbf{w}) \in \operatorname{dom}\Sigma_{I_1} \times \operatorname{dom}\Sigma_{I_2}$, then the function $(\mathbf{v} \sqcup \mathbf{w})\colon I \longrightarrow V$ given by $(\mathbf{v}\sqcup\mathbf{w})(i_1) := \mathbf{v}(i_1)$ and $(\mathbf{v}\sqcup\mathbf{w})(i_2) = \mathbf{w}(i_2)$ for all $i_1 \in I_1$ and $i_2 \in I_2$ lies in $\operatorname{dom}\Sigma_I$.

UF For all $\mathbf{v} \in \operatorname{dom}\Sigma_I$ and all families $f = (f_i)_{i\in I}$ of k-valued functions $f_i : X_i \longrightarrow k$ with *finite* domains X_i, writing $If := \{(i,x) : i \in I \wedge x \in X_i\}$, the family $f\mathbf{v} := (f_i(x)\mathbf{v}(i))_{(i,x)\in If}$ lies in $\operatorname{dom}\Sigma_{If}$.

The reader can see that these axioms generalise various properties of finite summation operators Σ_n for $n \in \mathbb{N}$, including associativity and commutativity (or invariance under reindexing) of the sum.

Definition 1.1. We say that $\Sigma = (\Sigma_I)_I$ (where I ranges in the class of all sets) is a **summability structure** on V, or that (V, Σ) is a **summability space**, if (V, Σ) satisfies the axioms **SS1** to **SS4** above for all sets I, J. If, moreover, axiom **UF** is satisfied for all I, then the summability space (V, Σ) is called **ultrafinite**.

Remark 1.2. In the ultrafinite case, one can bound the cardinality of supports of summable families by that of the underlying vector space. Thus we could consider families Σ indexed by I in the powerset of V. Our choice of indexing by all sets is for practical purposes. □

Throughout the rest of the section (V, Σ) denotes a summability space.

Definition 1.3. For any set I, any element of $\operatorname{dom}\Sigma_I$ is called a **summable family**.

In the following, we establish some basic properties of summability spaces.

Notation 1.4. We shall write $\sum_{i\in I}\mathbf{v}(i)$ for $\Sigma_I\mathbf{v}$.

Lemma 1.5. *Let (V,Σ) be a summability space, let I, I_1, I_2 be sets and let $f=(f_i)_{i\in I}$ is a family of k-valued functions with finite domains $X_i := \operatorname{dom} f_i$. Then the following hold:*

(i) If I is finite and of the form $I=\{i_1,\ldots,i_n\}$ for some $n\in\mathbb{N}$, then $\operatorname{dom}\Sigma_I = V^I = V^{(I)}$ and for any $\mathbf{v}\in V^I$ we have $\Sigma_I = \sum_{i\in I}\mathbf{v}(i) = \sum_{j=1}^n \mathbf{v}(i_j)$.

(ii) If $I = I_1 \sqcup I_2$, then for any $(\mathbf{v},\mathbf{w})\in\operatorname{dom}\Sigma_{I_1}\times\operatorname{dom}\Sigma_{I_2}$ we have $\Sigma_I(\mathbf{v}\sqcup\mathbf{w}) = \Sigma_{I_1}\mathbf{v}+\Sigma_{I_2}\mathbf{w}$.

(iii) If (V,Σ) is ultrafinite, then for any $\mathbf{v}\in\Sigma_I$ also $\left((\sum_{x\in X_i} f_i(x))\mathbf{v}(i)\right)_{i\in I}\in\Sigma_I$ and

$$\Sigma_{If}(f\mathbf{v}) = \sum_{i\in I}\left(\sum_{x\in X_i} f_i(x)\right)\mathbf{v}(i).$$

Proof.

(i) This follows immediately from **SS1**.

(ii) Let $\mathbf{u}=(\mathbf{v}\sqcup\mathbf{w})$. Then $\mathbf{u}_1=\mathbf{v}$ and $\mathbf{u}_2=\mathbf{w}$. By (i), **SS3** and **SS4**,

$$\Sigma_I\mathbf{u} = \Sigma_{\{1,2\}}(\Sigma_{I_j}\mathbf{u}_j)_{j\in\{1,2\}} = \Sigma_{I_1}\mathbf{u}_1+\Sigma_{I_2}\mathbf{u}_2 = \Sigma_{I_1}\mathbf{v}+\Sigma_{I_2}\mathbf{w}.$$

(iii) For any $i\in I$, let $I_j=\{i\}\times X_i$. Then $If=\bigsqcup_{i\in I} I_i$. Applying **SS3**, **UF** and (i), we obtain

$$\begin{aligned}\Sigma_{If}(f\mathbf{v}) &= \Sigma_I\left((\Sigma_{I_i}(f_i(x)\mathbf{v}(i))_{(i,x)\in\{i\}\times X_i})_{i\in I}\right) = \sum_{i\in I}\left(\sum_{x\in X_i} f_i(x)\mathbf{v}(i)\right)\\ &= \sum_{i\in I}\left(\sum_{x\in X_i} f_i(x)\right)\mathbf{v}(i).\end{aligned}$$

□

We now show that any summability space satisfies a version of Dirichlet's rearrangement theorem.

Proposition 1.6. *Let (V,Σ) be a summability space. Let I,J be sets and let $\mathbf{v}\in\operatorname{dom}\Sigma_{I\times J}$. Then for each $i_0\in I$ and for each $j_0\in J$, we have $\mathbf{v}(i_0,\cdot)\in\operatorname{dom}\Sigma_J$ and $\mathbf{v}(\cdot,j_0)\in\operatorname{dom}\Sigma_I$. Moreover, $\left(\sum_{j\in J}\mathbf{v}(i,j)\right)_{i\in I}\in\operatorname{dom}\Sigma_I$ and $\left(\sum_{i\in I}\mathbf{v}(i,j)\right)_{j\in J}\in\operatorname{dom}\Sigma_J$, with*

$$\sum_{i\in I}\left(\sum_{j\in J}\mathbf{v}(i,j)\right)_{j\in J} = \sum_{(i,j)\in I\times J}\mathbf{v}(i,j) = \sum_{j\in J}\left(\sum_{i\in I}\mathbf{v}(i,j)\right)_{i\in I}.$$

Proof. Apply **SS3** for $I \times J$ both with $(I \times J)_j := I \times \{j\}$ for each $j \in J$ and with $(I \times J)_i := \{i\} \times J$ for each $i \in I$. $\square$

Let $W \subseteq V$ be a subspace. Let $\Sigma' = (\Sigma'_I)_{I \text{ is a set}}$ be a family of partial functions $\Sigma'_I \colon W^I \longrightarrow W$. We say that Σ' is a **restriction of** Σ if for each I, the partial function Σ_I extends Σ'_I (that is $\operatorname{dom} \Sigma'_I \subseteq \operatorname{dom} \Sigma_I$ and Σ_I restricts to Σ'_I on $\operatorname{dom} \Sigma'_I$).

Definition 1.7. Let $W \subseteq V$ be a subspace.

(i) We say that W is a **closed subspace** if for any set I and any $\mathbf{w} \in \operatorname{dom} \Sigma_I \cap W^I$, we have $\Sigma_I \mathbf{w} \in W$.

(ii) If W is a closed subspace of V, then Σ naturally restricts to a summability structure $\mathbf{\Sigma^W}$ on W by setting $\operatorname{dom} \Sigma^W_I := \operatorname{dom} \Sigma_I \cap W^I$ and $\Sigma^W_I \mathbf{w} := \Sigma_I \mathbf{w}$ for any set I and $\mathbf{w} \in \operatorname{dom} \Sigma^W_I$.

We leave it to the reader to check that (W, Σ^W) in Definition 1.7 is indeed a summability space, and that (W, Σ^W) is ultrafinite if (V, Σ) is ultrafinite. We next give some simple examples of summability spaces.

Example 1.8. Any vector space V has a summability structure called the **minimal summability structure**, where for each set I, we have

$$\operatorname{dom} \Sigma_I = V^{(I)} \text{ and } \forall \mathbf{v} \in V^{(I)}, \Sigma_I \mathbf{v} = \sum_{i \in \operatorname{supp} \mathbf{v}} \mathbf{v}(i).$$

This structure is ultrafinite. $\square$

Example 1.9. Let $(\Gamma, <)$ be a non-empty linearly ordered set. Let $\mathbf{H}_{\gamma \in \Gamma} k$ denote the Hahn product of the constant family $(k)_{\gamma \in \Gamma}$, i.e. the vector space of functions $\Gamma \to k$ with well-ordered support, under pointwise sum and scalar product. A natural summability structure Σ on $V := \mathbf{H}_{\gamma \in \Gamma} k$ is obtained as follows. Given a set I, a function $\mathbf{v} \colon I \to V$ is **formally summable** if the subset $S_{\mathbf{v}} := \bigcup\limits_{i \in I} \operatorname{supp} \mathbf{v}(i)$ of Γ is well-ordered, and for each $\gamma \in \Gamma$, the set $I_{\mathbf{v},\gamma} := \{i \in I \ : \ \gamma \in \operatorname{supp} \mathbf{v}(i)\}$ is finite. Define $\operatorname{dom} \Sigma_I := \{\mathbf{v} \colon I \longrightarrow V \ : \ \mathbf{v} \text{ is formally summable }\}$ as the set of formally summable families $I \longrightarrow V$. For $\mathbf{v} \in \operatorname{dom} \Sigma_I$, we define $\Sigma_I \mathbf{v}$ to be the function

$$\Gamma \longrightarrow k, \quad \gamma \mapsto \sum_{i \in I_{\mathbf{v},\gamma}} \mathbf{v}(i),$$

whose support is indeed well-ordered, as it is contained in $S_{\mathbf{v}}$. This summability space is ultrafinite. $\square$

Example 1.10. Assume that $k = \mathbb{R}$ or $k = \mathbb{C}$. Let $(V, | \ \ |)$ be a Banach space with absolute value $| \ \ | : V \longrightarrow \mathbb{R}^{\geqslant 0}$. For any set I, define $\operatorname{dom} \Sigma_I$ to be the set of families $\mathbf{f} \colon I \longrightarrow V$ with countable or finite support, such that given a bijection $i \colon \lambda \longrightarrow \operatorname{supp} \mathbf{f}$ where $\lambda \leqslant \omega$, the real-valued sequence $\left(\sum_{p=0}^{n} |\mathbf{f}(i(p))|\right)_{n \in \lambda}$

converges. We then define $\sum_I \mathbf{f} = \sum_{n\in\lambda} \mathbf{f}(i(n))$. Note that this does not depend on the choice of bijection.

This summability structure is *not* ultrafinite in general. Indeed if V is non-trivial, then for any non-zero $v \in V$, the family $\left(\frac{v}{n!}\right)_{n\in\mathbb{N}}$ is summable, whereas $(vn!)_{n\in\mathbb{N}}$ is not. $\square$

Bornological spaces

We now consider an important class of summability spaces called bornological spaces (see also [6, Section 2.1]). We will appeal to this in Section 2 and Section 3.

Definition 1.11. Let Ω be a set. A **bornology** on Ω is a set $\mathcal{B}$ of subsets of Ω which contains all finite subsets of Ω and which is closed under finite unions and subsets, i.e. such that for all $A, B \subseteq \Omega$, we have

$$(A \in \mathcal{B} \wedge B \in \mathcal{B}) \Longrightarrow A \cup B \in \mathcal{B} \qquad \text{and} \qquad (A \subseteq B \wedge B \in \mathcal{B}) \Longrightarrow A \in \mathcal{B}.$$

Given a bornology $\mathcal{B}$ on Ω and a vector space V, we write $V^{[\mathcal{B}]}$ for the set of functions $\Omega \longrightarrow V$ whose support lies in $\mathcal{B}$. Assume that (V, Σ) is a summability space and let Ω be a non-empty set. We define a structure of summability space on $V^{[\mathcal{B}]}$ as follows. If I is a set and $\mathbf{f} \colon I \longrightarrow V^{[\mathcal{B}]}$ is a function, then we say that $\mathbf{f}$ is **$\mathcal{B}$-summable** if

- the set $\bigcup_{i\in I} \operatorname{supp} \mathbf{f}(i)$ lies in $\mathcal{B}$, and
- for all $p \in \Omega$, the family $(\mathbf{f}(i)(p))_{i\in I}$ is summable in (V, Σ).

If $\mathbf{f}$ is $\mathcal{B}$-summable, then we define $\sum_I^{\mathcal{B}} \mathbf{f} \in V^{[\mathcal{B}]}$ to be the function $p \mapsto \sum_{i\in I} \mathbf{f}(i)(p)$.

In the particular case when $\mathcal{B} = \mathcal{P}(\Omega)$, we have $V^{[\mathcal{B}]} = V^{\Omega}$. We say that a family $\mathbf{f} \colon I \longrightarrow V^{\Omega}$ is **pointwise summable with respect to** Σ if it is $\mathcal{P}(\Omega)$-summable, i.e. if each family $(\mathbf{f}(i)(p))_{i\in I}$ for $p \in \Omega$ is summable in (V, Σ).

Proposition 1.12. *Let (V, Σ) be a summability space. Then $(V^{[\mathcal{B}]}, \Sigma^{\mathcal{B}})$ is a summability space. Moreover, it is ultrafinite if (V, Σ) is ultrafinite.*

Proof. Throughout the proof, we fix sets I, J and a generic element $p \in \Omega$.

Let $\mathbf{f}, \mathbf{g} : I \longrightarrow V^{[\mathcal{B}]}$ be $\mathcal{B}$-summable families and let $c \in k$. For all $i \in I$, we have

$$\operatorname{supp}(\mathbf{f}(i) + c\mathbf{g}(i)) \subseteq (\operatorname{supp} \mathbf{f}(i)) \cup (\operatorname{supp} \mathbf{g}(i))$$

so $\bigcup_{i\in I} \operatorname{supp}((\mathbf{f} + c\mathbf{g})(i)) \subseteq \left(\bigcup_{i\in I} \operatorname{supp} \mathbf{f}(i)\right) \cup \left(\bigcup_{i\in I} \operatorname{supp} \mathbf{g}(i)\right)$.

Since $\bigcup_{i\in I} \operatorname{supp} \mathbf{f}(i), \bigcup_{i\in I} \operatorname{supp} \mathbf{g}(i) \in \mathcal{B}$ and $\mathcal{B}$ is closed under finite unions we deduce that $\bigcup_{i\in I} \operatorname{supp}((\mathbf{f} + c\mathbf{g})(i))$. Since $\operatorname{dom} \Sigma_I$ is a vector subspace of V^I, the family $(\mathbf{f}(i)(p) + c\mathbf{g}(i)(p))_{i\in I}$ is summable in (V, Σ). This shows that $\mathbf{f} + c\mathbf{g}$ is $\mathcal{B}$-summable. Moreover

$$\sum_I (\mathbf{f}(i)(p) + c\mathbf{g}(i)(p))_{i\in I} = \sum_I (\mathbf{f}(i)(p))_{i\in I} + c \sum_I (\mathbf{g}(i)(p))_{i\in I},$$

so $\sum_{i\in I}(\mathbf{f}+c\mathbf{g})=\sum_{i\in I}\mathbf{f}+c\sum_{i\in I}\mathbf{g}$.

If $\mathbf{f}$ has finite support, then $\bigcup_{i\in I}\operatorname{supp}\mathbf{f}(i)$ is a finite union of sets in $\mathcal{B}$, whence an element of $\mathcal{B}$. The family $F=(\mathbf{f}(i)(p))_{i\in I}$ has finite support, so it is summable with sum $\sum_{i\in\operatorname{supp}F}\mathbf{f}(i)(p)$. We deduce that $\mathbf{f}$ is $\mathcal{B}$-summable, with $\sum_{i\in I}\mathbf{f}=\sum_{i\in\operatorname{supp}F}\mathbf{f}(i)\in\mathcal{B}$. So **SS1** holds for $\mathcal{B}$-summability.

Let $\varphi : J\longrightarrow I$ be a bijection. Then we have

$$\bigcup_{j\in J}\operatorname{supp}\mathbf{f}(\varphi(j))=\bigcup_{i\in I}\operatorname{supp}\mathbf{f}(i)$$

which lies in $\mathcal{B}$. The family $(\mathbf{f}(\varphi(j))(p))_{j\in J}$ lies in $\operatorname{dom}\Sigma_L$ by **SS2** in (V,Σ), and we have

$$\left(\sum_{j\in J}\mathbf{f}(\varphi(j))\right)(p)=\sum_{j\in J}\mathbf{f}(\varphi(j))(p)=\sum_{i\in I}\mathbf{f}(i)(p)=\left(\sum_{i\in I}\mathbf{f}(i)\right)(p).$$

So $\sum_J\mathbf{f}\circ\varphi=\sum_I\mathbf{f}$, whence **SS2** holds for $\mathcal{B}$-summability.

Now assume that $I=\bigsqcup_{j\in J}I_j$. For $j\in J$, we have $\bigcup_{i\in I_j}\operatorname{supp}\mathbf{f}(i)\subseteq\bigcup_{i\in I}\operatorname{supp}\mathbf{f}(i)$ so $\bigcup_{i\in I_j}\operatorname{supp}\mathbf{f}(i)\in\mathcal{B}$ by closedness of $\mathcal{B}$ under subsets. The family $(\mathbf{f}(i)(p))_{i\in I_j}$ is also summable by **SS3a** in (Σ,V). So $(\mathbf{f}(i))_{i\in I_j}$ is $\mathcal{B}$-summable. We set $\mathbf{f}_j:=\sum_{i\in I_j}\mathbf{f}(i)$.

We have

$$\bigcup_{j\in J}\operatorname{supp}\mathbf{f}_j\subseteq\bigcup_{j\in J}\left(\bigcup_{i\in I_j}\operatorname{supp}\mathbf{f}(i)\right)\subseteq\bigcup_{i\in I}\operatorname{supp}\mathbf{f}(i),$$

so $\bigcup_{j\in J}\operatorname{supp}\mathbf{f}_l\in\mathcal{B}$. By **SS3b** and 1.1 in (V,Σ), the family $(\mathbf{f}_j(p))_{j\in J}$ is summable with sum $\sum_{i\in I}\mathbf{f}(i)(p)$. It follows that $(\mathbf{f}_j)_{j\in J}$ is $\mathcal{B}$-summable with sum $\sum_{i\in I}\mathbf{f}(i)$. So **SS3** holds.

Let I_1,I_2 be sets with $I=I_1\sqcup I_2$, and let $\mathbf{f}_1 : I_1\longrightarrow V^{[\mathcal{B}]}$ and $\mathbf{f}_2 : I_2\longrightarrow V^{[\mathcal{B}]}$ be $\mathcal{B}$-summable. We have

$$\bigcup_{i\in I}\operatorname{supp}(\mathbf{f}_1\sqcup\mathbf{f}_2)(i)=(\bigcup_{i\in I_1}\operatorname{supp}\mathbf{f}_1(i))\cup(\bigcup_{i\in I_2}\operatorname{supp}\mathbf{f}_2(i)),$$

which lies in $\mathcal{B}$ since it is closed under unions. The family $((\mathbf{f}_1\sqcup\mathbf{f}_2)(i)(p))_{i\in I}=((\mathbf{f}_1(i)(p))_{i\in I_1}\sqcup(\mathbf{f}_2(i)(p)))_{i\in I_2}$ is summable in (V,Σ) by **SS4** in (V,Σ), so $\mathbf{f}_1\sqcup\mathbf{f}_2$ is $\mathcal{B}$-summable.

Assume finally that (V,Σ) is ultrafinite. Let $(h_i)_{i\in I}$ be a family indexed by I of k-valued functions h_i with finite domains X_i. We have

$$\bigcup_{i\in I}\bigcup_{x\in X_i}\operatorname{supp}h_i(x)\mathbf{f}(i)\subseteq\bigcup_{i\in I}\operatorname{supp}\mathbf{f}(i),$$

which lies in $\mathcal{B}$. By ultrafiniteness, the family $(h_i(x)\mathbf{f}(i)(p))_{i\in I\wedge x\in X_i}$ is summable, so $(h_i(x)\mathbf{f}(i))_{i\in I\wedge x\in X_i}$ is $\mathcal{B}$-summable. So **UF** holds. □

We say that $(V^{[\mathcal{B}]}, \Sigma^{\mathcal{B}})$ is a **bornological space**. For $v \in V$ and $p \in \Omega$, write $\mathbb{1}_{\{p\}}v$ for the function $\Omega \longrightarrow V$ with $(\mathbb{1}_{\{p\}}v)(q) = 0$ if $q \neq p$ and $(\mathbb{1}_{\{p\}}v)(p) = v$. For $f \in V^{[\mathcal{B}]}$, the family $(\mathbb{1}_{\{p\}}f(p))_{p\in\Omega}$ is $\mathcal{B}$-summable with sum f. So any element f of $V^{[\mathcal{B}]}$ is a sum

$$f = \sum_{p\in\Omega} \mathbb{1}_{\{p\}} f(p), \tag{2}$$

which can be considered as a formal series with coefficients in V and with support in $\mathcal{B}$.

Example 1.13. On the Hahn space $\mathbf{H}_{n\in\mathbb{N}}k = k^{\mathbb{N}}$ of formal power series, pointwise summability with respect to the minimal summability structure on k coincides withformal summability as described in Example 1.9.

□

Example 1.14. If $(G, +, <)$ is a totally ordered Abelian group, then the notion of Rayner field family gives a bornology on G contained in that of well-ordered subsets of G. The resulting Rayner field [12] is a bornological space. These fields were among those for which we sought to understand the derivation-automorphism correspondence. Our results apply in particular to them. □

1.2 Strongly linear functions

Let (V, Σ), (V_1, Σ_1) and (V_2, Σ_2) be summability spaces.

Definition 1.15. A function $\phi : V_1 \longrightarrow V_2$ is said **strongly linear** if it is linear, and if, for all sets I and families $\mathbf{v} \in \operatorname{dom}\Sigma_{1,I}$, the family $\phi \circ \mathbf{v}$ lies in $\operatorname{dom}\Sigma_{2,I}$ and

$$\Sigma_{2,I}(\phi \circ \mathbf{v}) = \phi(\Sigma_{1,I}\mathbf{v}).$$

We write $\operatorname{Lin}^+(V_1, V_2)$ for the set of strongly linear functions $V_1 \longrightarrow V_2$, and $\operatorname{Lin}^+(V) := \operatorname{Lin}^+(V, V)$.

Example 1.16. If Σ_1 is the minimal summability structure on V_1, then we have $\operatorname{Lin}^+(V_1, V_2) = \operatorname{Lin}(V_1, V_2)$. □

Example 1.17. Consider the Hahn space $V = \mathbf{H}_{\gamma\in\Gamma}k$ of Example 1.9. For all order-preserving maps $u\colon \Gamma \longrightarrow \Gamma$, the function $V \longrightarrow V \;:\; f \mapsto f \circ u$ is strongly linear. □

In general, linear maps between summability spaces that are not defined using non-constructive methods often turn out to be strongly linear. As the next example shows, using the axiom of choice allows one to define non-strongly linear but linear functions.

Example 1.18. Let $k^{\mathbb{N}} = k[\![X]\!]$ be the space of formal series with coefficients in k, together with its summability structure of Example 1.9. Let W be a complement to the subspace $k[X]$ of $k[\![X]\!]$ of functions $\mathbb{N} \to k$ with finite support. Let μ be the unique linear map $k[\![X]\!] \to k[\![X]\!]$ with $\mu(W) = \{0\}$ and

$\mu(X^n) = (1+X)^n$ for each $n \in \mathbb{N}$. Then μ is not strongly linear. Indeed, the family $(X^n)_{n\in\mathbb{N}}$ is summable in $k[\![X]\!]$ but $((1+X)^n)_{n\in\mathbb{N}}$ is not summable, since 0 lies in the support of each of its elements. $\square$

Example 1.19. If $V = \mathbf{H}_{\gamma\in\Gamma} k$ is the Hahn space of Example 1.9, then given a strongly linear map $\mu\colon V \longrightarrow V$ and $a \in V$, we have

$$\mu(a) = \mu(\sum_{\gamma\in\Gamma} a(\gamma)\mathbb{1}_\gamma) = \sum_{\gamma\in\Gamma} a(\gamma)\mu(\mathbb{1}_\gamma).$$

In particular, the function μ is determined by the family $(\mu(\mathbb{1}_\gamma))_{\gamma\in\Gamma}$. $\square$

Proposition 1.20. $\mathrm{Lin}^+(V_1, V_2)$ *is a subspace of* $\mathrm{Lin}(V_1, V_2)$*,*

Proof. Let $\phi, \psi \in \mathrm{Lin}^+(V_1, V_2)$, $c \in k$, let I be a set and let $\mathbf{v} \in \operatorname{dom}\Sigma_{1,I}$. Since $\operatorname{dom}\Sigma_{2,I}$ is a vector subspace of V_2^I and $\Sigma_{2,I}$ is a linear map, the family $\phi\circ\mathbf{v} + c\psi\circ\mathbf{v}$ is summable with sum

$$\Sigma_{2,I}(\phi\circ\mathbf{v}) + c\Sigma_{2,I}(\psi\circ\mathbf{v}) = \phi(\Sigma_{1,I}\mathbf{v}) + c\psi(\Sigma_{2,I}\mathbf{v}) = (\phi + c\psi)(\Sigma_{1,I}\mathbf{v}).$$

This shows that $\phi + c\psi \in \mathrm{Lin}^+(V_1, V_2)$. $\square$

Proposition 1.21. $\mathrm{Lin}^+(V_1)$ *is a subalgebra of* $(\mathrm{Lin}(V_1), +, \circ)$*.*

Proof. Let $\phi, \psi \in \mathrm{Lin}^+(V_1)$, let I be a set and let $\mathbf{v} \in \operatorname{dom}\Sigma_{1,I}$. Then $\psi\circ\mathbf{v}$ is summable with sum $\sum_{2,I}\psi\circ\mathbf{v} = \psi\left(\sum_{1,I}\mathbf{v}\right)$ by strong linearity of ψ. So $(\phi\circ(\psi\circ\mathbf{v}))_{i\in I}$ is summable by strong linearity of ϕ, with sum $\sum_{2,I}\phi\circ(\psi\circ\mathbf{v}) = \phi\left(\psi\left(\sum_{1,I}\mathbf{v}\right)\right)$. Since $\phi\circ\psi$ is linear, this shows that $\phi\circ\psi \in \mathrm{Lin}^+(V_1)$. $\square$

Corollary 1.22. $\mathrm{Lin}^+(V_1)$ *is a Lie subalgebra of* $(\mathrm{Lin}(V_1), +, [\quad], .)$*.*

We next equip $\mathrm{Lin}^+(V_1, V_2)$ with a structure of summability space.

Definition 1.23. Let J be a set, and let $\boldsymbol{\phi} : J \longrightarrow \mathrm{Lin}^+(V_1, V_2)$ be a function.

(i) We say that $\boldsymbol{\phi}$ is Lin-**summable** if for all sets I and all $\mathbf{v} \in \operatorname{dom}\Sigma_{1,I}$, the family $\boldsymbol{\phi}(\mathbf{v}) := (\boldsymbol{\phi}(j)(\mathbf{v}(i)))_{(i,j)\in I\times J}$ lies in $\operatorname{dom}\Sigma_{2,I\times J}$.

(ii) If $\boldsymbol{\phi}$ is Lin-summable, then we define a function $\sum_{j\in J}\boldsymbol{\phi}(j) : V_1 \longrightarrow V_2$ as follows. For $v \in V_1$, define

$$\left(\sum_{j\in J}\boldsymbol{\phi}(j)\right)(v) := \Sigma_{2,J}(\boldsymbol{\phi}(j)(v))_{j\in J}.$$

Lemma 1.24. *If J is a set and $\boldsymbol{\phi} : J \longrightarrow \mathrm{Lin}^+(V_1, V_2)$ is* Lin*-summable, then its sum $\phi := \sum_{j\in J}\boldsymbol{\phi}(j)$ is strongly linear.*

Proof. Let $u_0, v_0 \in V$ and $c \in k$. The linearity of each $\boldsymbol{\phi}(j)$ for $j \in J$ gives that $\boldsymbol{\phi}(u_0+cv_0) = \boldsymbol{\phi}(u_0)+c\boldsymbol{\phi}(v_0)$. Now $\Sigma_{2,J}$ is linear, so $\phi(u_0+cv_0) = \phi(u_0)+c\phi(v_0)$, i.e. ϕ is linear.

We next prove that ϕ is strongly linear. Let I be a set, let $\mathbf{v} \in \operatorname{dom} \Sigma_{1,I}$ and set

$$v_1 := \Sigma_{1,I}\mathbf{v} \in V_1.$$

The family $\boldsymbol{\phi}(\mathbf{v})$ is summable. By Proposition 1.6, both $\left(\sum_{j\in J} \boldsymbol{\phi}(j)(\mathbf{v}(i))\right)_{i\in I}$ and $\left(\sum_{i\in I} \boldsymbol{\phi}(j)(\mathbf{v}(i))\right)_{j\in J}$ are summable with

$$\Sigma_{2,J}(\Sigma_{2,I}(\boldsymbol{\phi}(j) \circ \mathbf{v}))_{j\in J} = \Sigma_{2,I}(\Sigma_{2,J}\boldsymbol{\phi}(\mathbf{v}(i)))_{i\in I} = \Sigma_{2,I}(\phi \circ \mathbf{v}).$$

By strong linearity of each $\boldsymbol{\phi}(j)$, we have $\Sigma_{2,I}(\boldsymbol{\phi}(j) \circ \mathbf{v}) = \boldsymbol{\phi}(j)(v_0)$ for all $j \in J$, whence $\phi(v_0) = \Sigma_{2,I}(\phi \circ \mathbf{v})$. This shows that ϕ is strongly linear. □

For each set I, we define a function Σ_I^{Lin} as follows. The domain of Σ_I^{Lin} is the set of Lin-summable families $V_1 \longrightarrow V_2$ indexed by I, and for such a family ϕ, we define $\Sigma_I^{\mathrm{Lin}} = \Sigma_{i\in I}\phi(i) \in \mathrm{Lin}^+(V_1, V_2)$.

Proposition 1.25. *The structure* $(\mathrm{Lin}^+(V_1, V_2), \Sigma^{\mathrm{Lin}})$ *is a summability space. Moreover, it is ultrafinite if* (V_2, Σ_2) *is ultrafinite.*

Proof. Let J be a set, let $(\boldsymbol{\phi}(j))_{j\in J}$ and $(\boldsymbol{\psi}(j))_{j\in J}$ be Lin-summable, and let $c \in k$. We also fix once and for all a set I and a summable family $\mathbf{v} \in \operatorname{dom} \Sigma_{1,I}$.

Note that for each set L and Lin-summable family $\boldsymbol{\varphi}$ indexed by L, the function $\Sigma_{l\in L}\boldsymbol{\varphi}(l)$ the pointwise sum $\Sigma_L^{\mathcal{P}(V_1)}\boldsymbol{\varphi}$ in the bornological space $(V_2)^{(\mathcal{P}(V_1))}$. So the equalities in **SS1**, **SS2** and **SS3c** are automatically satisfied, and we only need to show the summability of families involved in the axioms **SS1**–**SS4** and **UF**.

The family

$$(\boldsymbol{\phi}(j)(\mathbf{v}(i)) + c\boldsymbol{\psi}(j)(\mathbf{v}(i)))_{(i,j)\in I\times J}$$

is summable by **SS2** in (V_2, Σ_2) and definition of Lin-summability. So $\boldsymbol{\phi} + c\boldsymbol{\psi}$ is Lin-summable. The axiom **SS1** follows trivially from the validity of **SS1** in (V_2, Σ_2), and likewise **SS4** follows from the validity of **SS4** in (V_2, Σ_2).

If J_1, J_2 are sets and $\varphi : J_1 \longrightarrow J_2$ is a bijection, then for all Lin-summable $\boldsymbol{\phi} : J_2 \longrightarrow \mathrm{Lin}^+(V_1, V_2)$, the function

$$\begin{array}{rcl} (\varphi, \mathrm{Id}) : J_1 \times I & \longrightarrow & J_2 \times I \\ (j, i) & \longmapsto & (\varphi(j), i) \end{array}$$

is bijective, so by **SS2** in (V_2, Σ), the family $(\boldsymbol{\phi}(\varphi(j))(\mathbf{v}(i)))_{(j,i)\in J_1\times I}$ is summable. Therefore $\boldsymbol{\phi} \circ \varphi$ is Lin-summable, i.e. **SS2** holds.

Let us now show that **SS3** holds. Suppose that $J = \bigsqcup_{l\in L} J_l$ for a set L. For $l \in L$, we write $\boldsymbol{\phi}_l = \phi \upharpoonright J_l$. Note that

$$I \times J = \bigsqcup_{l\in L} I \times J_l.$$

It follows that and from **SS3** in (V_2, Σ_2) that each family $\boldsymbol{\phi}_l(\mathbf{v})$ for $l \in L$ is summable in (V_2, Σ_2). So each $\boldsymbol{\phi}_l$ is Lin-summable. Write $\boldsymbol{\sigma}(l) = \sum_{j \in J_l} \phi_j$ for each $l \in L$. We claim that $\boldsymbol{\sigma} := (\boldsymbol{\sigma}(l))_{l \in L}$ is Lin-summable. Indeed, the family

$$\boldsymbol{\sigma}(\mathbf{v}) = \left(\sum_{j \in J_l} \phi_j(\mathbf{v}(i)) \right)_{(i,l) \in I \times L}$$

is summable by **SS3** in (V_2, Σ_2). Therefore **SS3** holds.

Let us next prove that **SS4** holds. Suppose that $J = J_1 \sqcup J_2$, that $\boldsymbol{\phi}_1 : J_1 \longrightarrow \mathrm{Lin}^+(V_1, V_2)$ and $\boldsymbol{\phi}_2 : J_2 \longrightarrow \mathrm{Lin}^+(V_1, V_2)$ are Lin-summable and write $\boldsymbol{\phi}_1 \sqcup \boldsymbol{\phi}_2 = \boldsymbol{\varphi}$. Then we have $\boldsymbol{\varphi}(\mathbf{v}) = \boldsymbol{\phi}_1(\mathbf{v}) \sqcup \boldsymbol{\phi}_2(\mathbf{v})$, so $\boldsymbol{\varphi}(\mathbf{v})$ is summable by **SS4** in (V_2, Σ_2).

Assume now that (V_2, Σ) is ultrafinite. Let $(f(j))_{j \in J}$ be a family of k-valued functions with finite domains $\operatorname{dom} f_j = X_j$ for each $j \in J$. The family $(f_j(x)\boldsymbol{\phi}(j)(\mathbf{v}(i)))_{(i,j) \in I \times J \wedge x \in X_j}$ is summable by ultrafiniteness of (V_2, Σ_2), so $f\boldsymbol{\phi}$ is Lin-summable. Thus **UF** holds. □

We now give a criterion for the summability of families of strongly linear maps on bornological spaces.

Proposition 1.26. *Assume that (V, Σ) is ultrafinite. Suppose that k is equipped with the minimal summability structure. Let Ω be a set and let $\mathcal{B}$ be a bornology on Ω. Let J be a set and let $\boldsymbol{\phi}\colon J \longrightarrow \mathrm{Lin}^+(k^{[\mathcal{B}]}, V)$ be a function. Then $\boldsymbol{\phi}$ is* Lin*-summable if and only if for all $S \in \mathcal{B}$, the family $(\boldsymbol{\phi}(j)(\mathbb{1}_{\{p\}}))_{(j,p) \in J \times S}$ is summable in (V, Σ).*

Proof. We need only prove the "if" direction of the equivalence. So let I be a set and let $\mathbf{f} : I \longrightarrow k^{[\mathcal{B}]}$ be $\mathcal{B}$-summable. Write

$$I_p := \{i \in I : p \in \operatorname{supp} \mathbf{f}(i)\}$$

for each $p \in \Omega$, and set

$$S := \bigcup_{i \in I} \operatorname{supp} \mathbf{f}(i) \in \mathcal{B}.$$

Since $S \in \mathcal{B}$, the family $(\boldsymbol{\phi}(j)(\mathbb{1}_{\{p\}}))_{j \in J \wedge p \in S}$ is summable in (V, Σ). For $p \in \Omega$, the family $(\mathbf{f}(i)(p))_{i \in I}$ is summable in k, so I_p is finite. We deduce by ultrafiniteness that the family $(\boldsymbol{\phi}(j)(\mathbb{1}_{\{p\}}\mathbf{f}(i)(p)))_{j \in J \wedge p \in \Omega \wedge i \in I_p}$ is summable, whence $(\boldsymbol{\phi}(j)(\mathbb{1}_{\{p\}}\mathbf{f}(i)(p)))_{j \in J \wedge p \in \Omega \wedge i \in I}$ is summable by **SS4** in (V_1, Σ_1). It follows from Proposition 1.6 that the family $\left(\sum_{p \in S} \boldsymbol{\phi}(j)(\mathbb{1}_{\{p\}}\mathbf{f}(i)(p))\right)_{j \in J \wedge i \in I}$ is summable. Since each $\boldsymbol{\phi}(j)$ is strongly linear, we have $(\boldsymbol{\phi}(j)(\mathbf{f}(i)))_{j \in J \wedge i \in I} = \left(\sum_{p \in S} \boldsymbol{\phi}(j)(\mathbb{1}_{\{p\}}\mathbf{f}(i)(p))\right)_{j \in J \wedge i \in I}$. So $(\boldsymbol{\phi}(j)(\mathbf{f}(i)))_{j \in J \wedge i \in I}$ is summable, i.e. $\boldsymbol{\phi}$ is Lin-summable. □

1.3 Summability algebras

Let $(A, +, \cdot)$ be a algebra, where $(A, +, \cdot)$ is a possibly non-commutative, possibly non-unital ring. Let Σ be a summability structure on the underlying vector space of A.

Definition 1.27. We say that (A, Σ) is a **summability algebra** if the following axiom is satisfied.

SA For all sets I, J and all $(\mathbf{f}, \mathbf{g}) \in \operatorname{dom} \Sigma_I \times \operatorname{dom} \Sigma_J$, the family $\mathbf{f} \cdot \mathbf{g} := (\mathbf{f}(i) \cdot \mathbf{g}(j))_{(i,j) \in I \times J}$ lies in $\operatorname{dom} \Sigma_{I \times J}$, and we have

$$\sum_{I \times J} (\mathbf{f} \cdot \mathbf{g}) = \left(\sum_I \mathbf{f} \right) \cdot \left(\sum_J \mathbf{g} \right).$$

Proposition 1.28. *Let (A, Σ) be a summability algebra. Then for $a \in A$, the left and right product functions $a \cdot : A \longrightarrow A; b \mapsto a \cdot b$ and $\cdot\, a : A \longrightarrow A; b \mapsto b \cdot a$ are strongly linear.*

Proof. By **SA**, for all summable families $(b_i)_{i \in I}$ in A, the families $(a \cdot b_i)_{i \in I}$ and $(b_i \cdot a)_{i \in I}$ are summable, with $\sum_{i \in I} a \cdot b_i = a \cdot \left(\sum_{i \in I} b_i \right)$ and $\sum_{i \in I} b_i \cdot a = \left(\sum_{i \in I} b_i \right) \cdot a$. □

Example 1.29. Let $(\Gamma, +, 0, <)$ be a linearly ordered Abelian group, and consider the summability space $A := \mathbf{H}_{\gamma \in \Gamma} k$ of example 1.9. Using the group structure on Γ, one can define an algebra operation

$$\forall f, g \in A, \forall \gamma \in \Gamma, (f \cdot g)(\gamma) := \sum_{\alpha + \beta = \gamma} f(\alpha) g(\beta).$$

The definition is due to Hahn [7]. This algebra is a summability algebra. See section 3 for more details. □

Proposition 1.30. *Let (V, Σ) be a summability space. Then $(\mathrm{Lin}^+(V), \Sigma^{\mathrm{Lin}})$ is a summability algebra.*

Proof. By Proposition 1.25, we need only prove that **SA** holds in $\mathrm{Lin}^+(V)$. Let I and J be sets, let $\boldsymbol{\phi} \colon I \longrightarrow \mathrm{Lin}^+(V)$ and $\boldsymbol{\psi} \colon J \longrightarrow \mathrm{Lin}^+(V)$ be Lin-summable with respective sums ϕ and ψ. Let L be a set and let $\mathbf{v} \colon L \longrightarrow V$ be summable in (V, Σ).

Since $\boldsymbol{\psi}$ is Lin-summable, the family $(\boldsymbol{\psi}(j)(\mathbf{v}(l)))_{(j,l) \in J \times L}$ is summable in (V, Σ). Since $\boldsymbol{\phi}$ is Lin-summable, the family $((\boldsymbol{\phi}(i) \circ \boldsymbol{\psi}(j))(\mathbf{v}(l)))_{(i,j,l) \in I \times J \times L}$ is summable in (V, Σ). This shows that $(\boldsymbol{\phi}(i) \circ \boldsymbol{\psi}(j))_{(i,j) \in I \times J}$ is Lin-summable. Let $v_0 \in V$. As above, the family

$$F := ((\boldsymbol{\phi}(i) \circ \boldsymbol{\psi}(j))(v_0))_{(i,j,l) \in I \times J}$$

is summable in (V, Σ). We have

$$\begin{aligned}
(\phi \circ \psi)(v_0) &= \left(\sum_{i \in I} \boldsymbol{\phi}(i)\right) \left(\sum_{j \in J} \boldsymbol{\psi}(j)(v_0)\right) \\
&= \sum_{i \in I} \boldsymbol{\phi}(i) \left(\sum_{j \in J} \boldsymbol{\psi}(j)(v_0)\right) \\
&= \sum_{i \in I} \sum_{j \in J} \boldsymbol{\phi}(i)(\boldsymbol{\psi}(j)(v_0)) && \text{(each } \boldsymbol{\phi}(i) \text{ is strongly linear)} \\
&= \sum_{(i,j) \in I \times J} \boldsymbol{\phi}(i)(\boldsymbol{\psi}(j)(v_0)). && \text{(Proposition 1.6)}
\end{aligned}$$

So $\phi \circ \psi = \sum_{I \times J} \boldsymbol{\phi}(i) \circ \boldsymbol{\psi}(j)$. Therefore **SA** holds in $\mathrm{Lin}^+(V)$. □

A **closed ideal** of a summability algebra (A, Σ) is a *two-sided* ideal of A which is also a closed subspace of (A, Σ).

1.4 Algebras of formal power series

Let J be a set. Write

$$J^\star := \bigcup_{n \in \mathbb{N}} J^n$$

where $J^0 = \{\emptyset\}$. We see elements of $J^\star$ as **finite words** with letters in J.

For $m, n \in \mathbb{N}$, if $\beta = (\beta_1, \ldots, \beta_m), \gamma = (\gamma_1, \ldots, \gamma_n) \in J^\star$, we define

$$\beta\gamma := (\beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_n) \in J^{m+n} \subseteq J^\star,$$

where it is implied that $\emptyset\theta = \theta\emptyset = \theta$ for all $\theta \in J^\star$. This concatenation operation endows $J^\star$ with a structure of cancellative monoid, with the important property that given $\theta \in J^\star$, the set

$$\{(\beta, \gamma) \in J^\star \times J^\star : \theta = \beta\gamma\} \tag{3}$$

is finite (it has exactly $n+1$ elements when $\theta \in J^n$).

The following construction, also considered in [21, Chapter 0, p 17], is a particular case of Bourbaki's notion of total algebra [2, Chapter III, Section 2.10].

Proposition 1.31. *The vector space $k^{J^\star}$ is a unital algebra under the Cauchy product*

$$(P \cdot Q)(\theta) := \sum_{\theta = \beta\gamma} P(\beta) Q(\gamma), \tag{4}$$

for any $P, Q \in k^{J^\star}$ and any $\theta \in J^\star$.

Notation 1.32. We write $k\langle\langle J \rangle\rangle := k^{J^\star}$.

We recall some properties of $k\langle\langle J\rangle\rangle$. For $P \in k\langle\langle J\rangle\rangle$ and $n \in \mathbb{N}$, we write

$$\operatorname{supp}_n P := (\operatorname{supp} P) \cap J^n = \{\theta \in J^n : P(\theta) \neq \emptyset\}.$$

Note that we have

$$\operatorname{supp}_n P \cdot Q \subseteq \bigcup_{m+p=n} (\operatorname{supp}_m P)\ (\operatorname{supp}_p Q), \tag{5}$$

where $AB = \{ab : (a,b) \in A \times B\}$ for all subsets $A, B \subseteq J^\star$. Concatenation of subsets of $J^\star$ is associative. An easy induction gives:

Lemma 1.33. *For $m, n \in \mathbb{N}$ and $P_1, \ldots, P_n \in k\langle\langle J\rangle\rangle$, we have*

$$\operatorname{supp}_m P_1 \cdots P_n \subseteq \bigcup_{m_1+\cdots+m_n=m} (\operatorname{supp}_{m_1} P_1) \cdots (\operatorname{supp}_{m_n} P_n).$$

The set

$$k\langle\langle J\rangle\rangle_0 := \{P \in k\langle\langle J\rangle\rangle : \emptyset \notin \operatorname{supp} P\} = \{P - P(\emptyset) : P \in k\langle\langle J\rangle\rangle\}$$

is a two-sided ideal in $k\langle\langle J\rangle\rangle$.

Given $\theta \in J^\star$, we write X_θ for the function $\mathbb{1}_{\{\theta\}} : J^\star \longrightarrow k$ with support $\{\theta\}$ and $X_\theta(\theta) = 1$. So $X_\emptyset = 1$, and writing $\theta = (\theta_1, \ldots, \theta_n)$, we have

$$X_\theta = X_{\theta_1} \cdots X_{\theta_n}.$$

We will write $k\langle\langle m\rangle\rangle := k\langle\langle J\rangle\rangle$ for $m \in \mathbb{N}$ and $J = \{0, \ldots, m-1\}$. Note that $k\langle\langle 0\rangle\rangle = k$ and that $k\langle\langle 1\rangle\rangle$ is the commutative algebra of power series in one variable X_0 and with coefficients in k.

By Proposition 1.12, pointwise summability with respect to the minimal summability structure on k gives an ultrafinite summability structure on $k\langle\langle J\rangle\rangle = k^{J^\star}$, which we denote by Σ. Recall that a family $\mathbf{P} : I \longrightarrow k\langle\langle J\rangle\rangle$ is pointwise summable here if and only if for all $\theta \in J^\star$, the family $(\mathbf{P}(i)(\theta))_{i\in I}$ has finite support, i.e. if and only if the set

$$I_\theta := \{i \in I : \theta \in \operatorname{supp} \mathbf{P}(i)\}$$

is finite. As in (2), the series representation of a $P \in k\langle\langle J\rangle\rangle$ is

$$P = \sum_{\theta \in J^\star} P(\theta) X_\theta.$$

Proposition 1.34. *The structure $(k\langle\langle J\rangle\rangle, \Sigma)$ is a summability algebra.*

Proof. We need only show that **SA** holds. Let I and L be sets, let $\mathbf{P} : I \longrightarrow k\langle\langle J\rangle$ and $\mathbf{Q} : L \longrightarrow k\langle\langle J\rangle\rangle$ be pointwise summable. Let $n \in \mathbb{N}$ and $\theta \in J^n$. We have

$$\begin{aligned}(I \times L)_\theta &= \{(i,l) \in I \times L : \theta \in \operatorname{supp} \mathbf{P}(i) \cdot \mathbf{Q}(j)\} \\ &= \{(i,l) \in I \times L : \exists \gamma, \beta \in J^\star, i \in I_\beta \wedge l \in L_\gamma\}\end{aligned}$$

where for $\beta, \gamma \in J^\star$ the sets $I_\beta := \{i \in I : \beta \in \operatorname{supp} \mathbf{P}(i)\}$ and $L_\gamma := \{l \in L : \gamma \in \operatorname{supp} \mathbf{Q}(l)\}$ are finite. Since $\{(\beta, \gamma) \in J^\star \times J^\star : \beta : \gamma = \theta\}$ is finite, we deduce that $(I \times L)_\theta$ is finite. So $(\mathbf{P}(i)\mathbf{Q}(l))_{(i,l)\in I\times L}$ is pointwise summable.

For $n \in \mathbb{N}$ and $\theta \in J^n$, the previous arguments give

$$\begin{aligned}
\left(\sum_{(i,l)\in I\times L} \mathbf{P}(i)\cdot \mathbf{Q}(l)\right)(\theta) &= \sum_{\beta:\gamma=\theta} \sum_{(i,l)\in I_\beta \times L_\gamma} \mathbf{P}(i)(\beta)\mathbf{Q}(l)(\gamma) \\
&= \sum_{\beta:\gamma=\theta} \left(\sum_{i\in I_\beta} \mathbf{P}(i)(\beta)\right)\left(\sum_{l\in L_\gamma} \mathbf{Q}(l)(\gamma)\right) \\
&= \sum_{\beta:\gamma=\theta} \left(\sum_{i\in I} \mathbf{P}(i)\right)(\beta)\left(\sum_{l\in L} \mathbf{Q}(l)\right)(\gamma) \\
&= \left(\left(\sum_{i\in I} \mathbf{P}(i)\right)\cdot\left(\sum_{l\in L} \mathbf{Q}(l)\right)\right)(\theta).
\end{aligned}$$

This shows that $\sum_{(i,l)\in I\times L} \mathbf{P}(i)\cdot \mathbf{Q}(l) = \left(\sum_{i\in I} \mathbf{P}(i)\right)\cdot\left(\sum_{l\in L} \mathbf{Q}(l)\right)$, hence that **SA** holds. □

2 Derivations and endomorphisms

2.1 Strongly linear derivations

Let (A, Σ) be a summability algebra. We write $\mathrm{End}^+(A)$ for the set of strongly linear endomorphisms of algebra of A. Note by Proposition 1.21 that $\mathrm{End}^+(A)$ is closed under composition.

Definition 2.1. A **strongly linear derivation** on A is a strongly linear function $\partial : A \longrightarrow A$ which satisfies the Leibniz product rule

$$\forall a, b \in A, \partial(a \cdot b) = \partial(a) \cdot b + a \cdot \partial(b).$$

We write $\mathrm{Der}^+(A)$ for the set of strong derivations on A.

It follows from Proposition 1.21 that $\left(\mathrm{Der}^+(A), +, [\quad], .\right)$ is a Lie algebra.

Proposition 2.2. $\mathrm{Der}^+(A)$ *is a closed subspace of* $\mathrm{Lin}^+(A)$*.*

Proof. Let I be a set, let $(\partial_i)_{i\in I}$ be a family in $\mathrm{Der}^+(A)$ which is Lin-summable, and write $\partial := \sum_{i\in I} \partial_i$. Let $a, b \in A$. We have

$$\begin{aligned}
\partial(a \cdot b) &= \sum_{i\in I} \partial_i(a \cdot b) \\
&= \sum_{i\in I} (\partial_i(a) \cdot b + a \cdot \partial_i(b)) \\
&= \sum_{i\in I} \partial_i(a) \cdot b + \sum_{i\in I} a \cdot \partial_i(b) \\
&= \left(\sum_{i\in I} \partial_i(a)\right) \cdot b + a \cdot \left(\sum_{i\in I} \partial_i(b)\right) && (\mathbf{SA}) \\
&= \partial(a) \cdot b + a \cdot \partial(b).
\end{aligned}$$

So $\partial \in \mathrm{Der}^+(A)$. □

2.2 Evaluating formal power series

Given a set J, the summability algebra $k\langle\langle J\rangle\rangle$ extends the completion $k\langle J\rangle$ of the free associative algebra on J. It is well-known [21, Chapter 0, p 17–18] that elements of $k\langle J\rangle$ can be evaluated at tuples of elements in its maximal ideal. This allows for the development of a formal Lie correspondence for formal power series, based on the evaluation of the Taylor series of the exponential and logarithm.

It is very convenient to extend these results to more general summability algebras. Then one can see $k\langle\langle J\rangle\rangle$ as a universal and free summability algebra, acting by evaluation on summability algebras, so that universal identities (such as $\exp \circ \log = \mathrm{id}$) that can be stated in summability algebras could be proved once in $k\langle\langle J\rangle\rangle$ and then obtained in general by evaluating into summability algebras.

Definition 2.3. Let (A, Σ) be a unital ultrafinite summability algebra. Then we say that (A, Σ) **has evaluations** if A is of the form $A = k+\mathfrak{m}$ where $\mathfrak{m}$ is a closed ideal, and for all sets J, all $\mathbf{f} \in \operatorname{dom} \Sigma_J^{\mathfrak{m}}$, the family $(\mathbf{f}(\theta_1) \cdots \mathbf{f}(\theta_n))_{\theta=(\theta_1,\ldots,\theta_n)\in J^\star}$ is summable in (A, Σ).

Remark 2.4. Let us justify this definition.

Given a summability algebra A and a family $\mathbf{f}\colon J \longrightarrow A$, the evaluation $\mathrm{ev}_{\mathbf{f}}(P)$ of a formal power series $P \in k\langle\langle J\rangle\rangle$ at $\mathbf{f}$ ought to be the sum of the family $(P(j_1, \ldots, j_n)\mathbf{f}(j_1) \cdots \mathbf{f}(j_n))_{(j_1,\ldots,j_n)\in J^\star}$. Indeed, if all such families are summable, then $\mathrm{ev}_{\mathbf{f}}$ will be the only strongly linear morphism of algebras between $k\langle\langle J\rangle\rangle$ and A that sends X_j to $\mathbf{f}(j)$ for each $j \in J$.

For $P = \sum\limits_{j\in J} X_j$, this entails that $\mathbf{f}$ be summable. Furthermore, if $\mathbf{f}(j) \in k^\times$ for a certain $j \in J$, then for $P = \sum\limits_{k\in\mathbb{N}} (\mathbf{f}(j))^{-k} X_j^k$, we see that the family $(P(j_1, \ldots, j_n)\mathbf{f}(j_1) \cdots \mathbf{f}(j_n))_{(j_1,\ldots,j_n)\in J^\star}$ has $(1)_{\theta\in\{j\}^\star}$ as a subfamily, hence is not summable. So if evaluations are to be defined in the case when $A = k\langle\langle J\rangle\rangle$, then $\mathbf{f}$ should range in the maximal ideal $k\langle\langle J\rangle\rangle_0$ of $k\langle\langle J\rangle\rangle$.

In general, writing $\mathfrak{m}$ for the set of elements $\mathrm{ev}_{\mathbf{a}}(P) \in A$ where J ranges among all sets, P ranges among all elements in $k\langle\langle J\rangle\rangle_0$ and $\mathbf{f}$ ranges among summable families for which $(P(j_1, ..., j_n)\mathbf{f}(j_1) \cdots \mathbf{f}(j_n))_{(j_1,...,j_n)\in J^\star}$ is summable, then $\mathfrak{m}$ is a two-sided ideal of the algebra $A' := k + \mathfrak{m}$.

Imposing furthermore that $\mathfrak{m}$ be a closed subspace of A, we might as well take $A = A'$ and thus work with summability algebras of the form $A = k + \mathfrak{m}$ for a two-sided ideal $\mathfrak{m}$ of A. □

We will see (Proposition 2.8) that such a summaility algebra $k + \mathfrak{m}$ is local with maximal ideal $\mathfrak{m}$. Let J be set. Let us check that $k\langle\langle J\rangle\rangle = k + k\langle\langle J\rangle\rangle_0$ itself has evaluations.

Proposition 2.5. *The set $k\langle\langle J\rangle\rangle_0$ is a closed ideal of $k\langle\langle J\rangle\rangle$, and $(k\langle\langle J\rangle\rangle, \Sigma)$ has evaluations.*

Proof. Let $\mathbf{Q}\colon I \longrightarrow k\langle\langle J\rangle\rangle_0$ be pointwise summable. We have $\left(\sum_{i\in I} \mathbf{Q}(i)\right)(\emptyset) = \sum_{i\in\emptyset} \mathbf{Q}(i)(\emptyset) = 0$, so $\sum_{i\in I} \mathbf{Q}(i) \in k\langle\langle J\rangle\rangle_0$, which is thus a closed ideal.

We next want to show that the family $(\mathbf{Q}(i_1) \cdots \mathbf{Q}(i_n))_{(i_1,...,i_n)\in I^\star}$ is pointwise summable. Let $m \in \mathbb{N}$ and $\theta = (\theta_1, \ldots, \theta_m) \in J^m$. Writing $I_\beta := \{i \in I : \beta \in \operatorname{supp} \mathbf{Q}(i)\}$ for all $\beta \in J^\star$, we have

$$\begin{aligned} I_\theta^\star &= \{i \in I^n : n \in \mathbb{N} \wedge \theta \in \operatorname{supp}_m \mathbf{Q}(i_1) \cdots \mathbf{Q}(i_n)\} \\ &\subseteq \bigcup_{n\in\mathbb{N}} \bigcup_{\beta_1\cdots\beta_n=\theta} I_{\beta_1} \times \cdots \times I_{\beta_n}, \end{aligned}$$

Since each $\mathbf{Q}(i)$ lies in $k\langle\langle J\rangle\rangle_0$, we have $\operatorname{supp}_0 \mathbf{Q}(i) = \emptyset$ for all $i \in I$, so we have in fact

$$I_\theta^\star \subseteq \bigcup_{n\leqslant m} \bigcup_{\beta_1\cdots\beta_n=\theta} I_{\beta_1} \times \cdots \times I_{\beta_n}.$$

Now since each set $X_n := \{(\beta_1, \ldots, \beta_n) \in (J^\star)^n : \beta_1 \cdots \beta_n = \theta\}$ is finite and each $I_\beta, \beta \in J^\star$ is finite, we deduce that $I_\theta^\star$ is finite. Therefore the family $(\mathbf{Q}(i_1) \cdots \mathbf{Q}(i_n))_{i=(i_1,...,i_n)\in I^\star}$ is pointwise summable. □

If $(A, \mathfrak{m}, \Sigma)$ are as in Definition 2.3, then for all sets J, all $\mathbf{f} \in \operatorname{dom} \Sigma_J^{\mathfrak{m}}$ and all $P \in k\langle\langle J\rangle\rangle$, we define the **evaluation** of P at $\mathbf{f}$ as

$$\mathrm{ev}_{\mathbf{f}}(P) := \sum_{\theta=(\theta_1,...,\theta_n)\in J^\star} P(\theta)\mathbf{f}(\theta_1) \cdots \mathbf{f}(\theta_n) \in A.$$

If $J = \{0, \ldots, m-1\}$ for an $m \in \mathbb{N}$, then we simply write $\mathrm{ev}_{\mathbf{f}(0),...,\mathbf{f}(m-1)}(P) := \mathrm{ev}_{\mathbf{f}}(P)$.

Theorem 2.6. *Assume that $A = k + \mathfrak{m}$ has evaluations. For all sets J and all $\mathbf{f} \in \operatorname{dom} \Sigma_J^{\mathfrak{m}}$, the evaluation map $\mathrm{ev}_{\mathbf{f}} : k\langle\langle J\rangle\rangle \longrightarrow A$ is a strongly linear morphism of algebras.*

Proof. That $\mathrm{ev}_{\mathbf{f}}$ is a linear map is a direct consequence of the fact that $\Sigma^{\mathfrak{m}}_{\{(n,\theta):n\in\mathbb{N}\wedge\theta\in J^n\}}$ is linear. For $P, Q \in k\langle\langle J\rangle\rangle$, **SA** gives

$$\begin{aligned}\mathrm{ev}_{\mathbf{f}}(P)\cdot\mathrm{ev}_{\mathbf{f}}(Q) &= \sum_{m,p\in\mathbb{N}\wedge\beta\in J^m\wedge\gamma\in J^p} P(\beta)Q(\gamma)\mathbf{f}(\beta_1)\cdots\mathbf{f}(\beta_m)\cdot\mathbf{f}(\gamma_1)\cdots\mathbf{f}(\gamma_p)\\ &= \sum_{n\in\mathbb{N}\wedge\theta\in J^n}(P\cdot Q)(\theta)\mathbf{f}(\theta_1)\cdots\mathbf{f}(\theta_n)\\ &= \mathrm{ev}_{\mathbf{f}}(P\cdot Q).\end{aligned}$$

So $\mathrm{ev}_{\mathbf{f}}$ is a morphism of algebras.

Let $\mathbf{P}\colon I \longrightarrow k\langle\langle J\rangle\rangle$ be pointwise summable and set $P := \sum_{i\in I}\mathbf{P}(i)$. Consider the element

$$P' := \sum_{\theta\in J^\star} X_\theta$$

of $k\langle\langle J\rangle\rangle$. Since (A,Σ) has evaluations, we have a pointwise summable family $(P'(\theta)\mathbf{f}(\theta_1)\cdots\mathbf{f}(\theta_n))_{\theta=(\theta_1,\ldots,\theta_n)\in J^\star}$. Consider the family of finite subsets of k

$$(C_{(n,\theta)})_{\theta=(\theta_1,\ldots,\theta_n)\in J^\star} = (\{\mathbf{P}(i)(\theta) : \mathbf{P}(i)(\theta)\neq 0\})_{\theta=(\theta_1,\ldots,\theta_n)\in J^\star}.$$

Recall that (A,Σ) is ultrafinite, so $(cP'(\theta)\mathbf{f}(\theta_1)\cdots\mathbf{f}(\theta_n))_{\theta=(\theta_1,\ldots,\theta_n)\in J^\star\wedge c\in C_{(n,\theta)}}$ is summable in (A,Σ). By **SS2**, so is the family

$$\mathbf{g} := (\mathbf{P}(i)(\theta)\mathbf{f}(\theta_1)\cdots\mathbf{f}(\theta_n))_{\theta=(\theta_1,\ldots,\theta_n)\in J^\star\wedge i\in I_\theta}.$$

By **SS4**, the family

$$(\mathrm{ev}_{\mathbf{f}}(\mathbf{P}(i)))_{i\in I} = (\mathbf{P}(i)(\theta)\mathbf{f}(\theta_1)\cdots\mathbf{f}(\theta_n))_{\theta=(\theta_1,\ldots,\theta_n)\in J^\star\wedge i\in I}$$

is summable as the union of the families $\mathbf{g}$ and $(0)_{\theta=(\theta_1,\ldots,\theta_n)\in J^\star\wedge i\in I\setminus I_\theta}$. Again by ultrafiniteness of (A,Σ), we have

$$\sum_{i\in I}\mathrm{ev}_{\mathbf{f}}(\mathbf{P}(i)) = \sum_{n\in\mathbb{N}\wedge\theta=(\theta_1,\ldots,\theta_n)}\left(\sum_{i\in I_\theta}\mathbf{P}(i)(\theta)\right)\mathbf{f}(\theta_1)\cdots\mathbf{f}(\theta_n) = \mathrm{ev}_{\mathbf{f}}(P).$$

This shows that $\mathrm{ev}_{\mathbf{f}}$ is strongly linear. □

The previous theorem allows us to derive identities in a sumability algebra $A = k+\mathfrak{m}$ with evaluations from universal identities in $k\langle\langle J\rangle\rangle$ which only involve finite products and infinite sums. We apply this in the next two results.

The next proposition extends this to identities involving composition of formal power series (i.e. evaluations of formal power series at formal power series). We will apply this to the exponential and logarithmic series in the next subsection.

Proposition 2.7. *Let $A = k+\mathfrak{m}$ be a summability algebra that has evaluations. Let I, J be sets. Consider two families $\mathbf{Q} : I \longrightarrow k\langle\langle J\rangle\rangle_0$ and $\mathbf{f} : J \longrightarrow \mathfrak{m}$ which are summable in their respective algebras. For $P \in k\langle\langle I\rangle\rangle$, we have*

$$\mathrm{ev}_{\mathbf{f}}(\mathrm{ev}_{\mathbf{Q}}(P)) = \mathrm{ev}_{(\mathrm{ev}_{\mathbf{f}}(\mathbf{Q}(i)))_{i\in I}}(P).$$

Proof. Let $i_0 \in I$. We have $\mathrm{ev}_{\mathbf{Q}}(X_{i_0}) = \mathbf{Q}(i_0)$, so

$$\mathrm{ev}_{\mathbf{f}}(\mathrm{ev}_{\mathbf{Q}}(X_{i_0})) = \mathrm{ev}_{\mathbf{f}}(\mathbf{Q}(i_0)) = \mathrm{ev}_{(\mathrm{ev}_{\mathbf{f}}(\mathbf{Q}(i)))_{i \in I}}(X_{i_0}).$$

Since $\mathrm{ev}_{\mathbf{f}} \circ \mathrm{ev}_{\mathbf{Q}}$ and $\mathrm{ev}_{(\mathrm{ev}_{\mathbf{f}}(\mathbf{Q}(i)))_{i \in I}}$ are strongly linear morphisms of algebras, we deduce that the identity holds for all $P \in k\langle\langle I \rangle\rangle$. □

Proposition 2.8. *Let $A = k + \mathfrak{m}$ have evaluations. Then A is local algebra with maximal ideal $\mathfrak{m}$.*

Proof. We first show that each element of $1 + \mathfrak{m}$ is invertible in A. Let $a = 1 + \varepsilon \in 1 + \mathfrak{m}$. Since (A, Σ) has evaluations, writing

$$P := 1 - X_0 + X_0^2 - X_0^3 + \cdots \in k\langle\langle 1 \rangle\rangle,$$

we may consider the evaluation $\mathrm{ev}_\varepsilon(P) \in A$. We have $P(1+X_0) = (1+X_0)P = 1$ in $k\langle\langle 1 \rangle\rangle$, so Theorem 2.6 gives $\mathrm{ev}_\varepsilon(P) \cdot (1 + \varepsilon) = (1 + \varepsilon) \cdot \mathrm{ev}_\varepsilon(P) = 1$, i.e. $1 + \varepsilon$ is a unit in A with inverse $\mathrm{ev}_\varepsilon(P)$.

We deduce that $k^\times(1 + \mathfrak{m}) = k^\times + \mathfrak{m}$ is contained in $U(A)$. Since $\mathfrak{m}$ is a proper ideal, it follows that $A \setminus U(A) = \mathfrak{m}$. So $A \setminus U(A)$ is an ideal, whence A is local with maximal ideal $\mathfrak{m}$. □

2.3 Exponential and logarithm

In the sequel of Section 2, we assume that k has characteristic zero. It is known [24] that given a finite set J, the algebra $k\langle\langle J \rangle\rangle$ is equipped with an exponential $\exp\colon k\langle\langle J \rangle\rangle_0 \longrightarrow 1 + k\langle\langle J \rangle\rangle_0$ and a logarithm $\log\colon 1 + k\langle\langle J \rangle\rangle_0 \longrightarrow k\langle\langle J \rangle\rangle_0$, which are inverses of one another, and are given by evaluating the usual formal series. Using our previous results, we will recover a number of known identities known in that case for all summability algebras with evaluations.

We consider two particular elements of $k\langle\langle 1 \rangle\rangle$ defined as follows

$$E_0 := \sum_{n \geqslant 0} \frac{1}{n!} X_0^n \qquad \text{and} \qquad L_0 := \sum_{n \geqslant 1} \frac{(-1)^{n+1}}{n} X_0^n.$$

Given a summability algebra (A, Σ) which has evaluations, and writing $\mathfrak{m}$ for its maximal ideal, we have two functions

$$\begin{aligned} \exp : \mathfrak{m} &\longrightarrow 1 + \mathfrak{m} \\ \delta &\longmapsto \mathrm{ev}_\delta(E_0) \end{aligned}$$

and

$$\begin{aligned} \log : 1 + \mathfrak{m} &\longrightarrow \mathfrak{m} \\ \delta &\longmapsto \mathrm{ev}_{\delta - 1}(L_0). \end{aligned}$$

Proposition 2.9. [24, Section 1.7, Theorem 7.2] *We have*

$$\mathrm{ev}_{L_0}(E_0 - 1) = \mathrm{ev}_{E_0 - 1}(L_0) = X_0.$$

Corollary 2.10. *Let (A, Σ) be a summability algebra with evaluations, with maximal ideal $\mathfrak{m}$. Then $\exp : \mathfrak{m} \longrightarrow 1 + \mathfrak{m}$ and $\log : 1 + \mathfrak{m} \longrightarrow \mathfrak{m}$ are bijective, and are functional inverses of one another.*

Proof. This follows from Proposition 2.7. For instance, for $\varepsilon \in \mathfrak{m}$, we have

$$\log(\exp(\varepsilon)) = \mathrm{ev}_{\mathrm{ev}_\varepsilon(E_0-1)}(L_0) = \mathrm{ev}_\varepsilon(\mathrm{ev}_{L_0}(E_0 - 1)) = \mathrm{ev}_\varepsilon(X_0) = \varepsilon.$$

The other identity follows similarly. □

Next consider the following elements of $k\langle\langle 2\rangle\rangle$ for $n \in \mathbb{N}$:

$$\begin{aligned} E_1 &:= \exp(X_1) = \sum_{n\in\mathbb{N}} \frac{1}{n!} X_1^n \\ K_n &:= \sum_{\substack{m_1+p_1,\cdots,m_i+p_i\geqslant 1\\ m_1+p_1+\ldots+m_i+p_i=n}} \frac{1}{m_1!p_1!\cdots m_n!p_n!} X_0^{m_1} X_1^{p_1} \cdots X_0^{m_n} X_1^{p_n} \quad (6) \\ X_0 * X_1 &:= \sum_{n>0} \frac{(-1)^{n+1}}{n} K_n. \end{aligned}$$

We have the formal Baker–Campbell–Hausdorff Theorem:

Proposition 2.11. [24, Section 1.8] *We have $\mathrm{ev}_{E_0\cdot E_1-1}(L_0) = X_0 * X_1$. Moreover, $K_0 = X_0 + X_1$ and each $K_n, n > 0$ lies in the Lie subalgebra of $(k\langle\langle 2\rangle\rangle, +, 0, [\cdot,\cdot])$ generated by commutators of X_0 and X_1.*

Let $A = k + \mathfrak{m}$ have evaluations. For all δ_1, δ_2 in $\mathfrak{m}$, we define

$$\delta_1 * \delta_2 \quad := \quad \mathrm{ev}_{\delta_1,\delta_2}(X_0 * X_1) \in \mathfrak{m}. \tag{7}$$

As a consequence of Proposition 2.7, we have:

Corollary 2.12. *Let (A, Σ) be a summability algebra with evaluations, and let $\mathfrak{m}$ denote its maximal ideal. Then for all $\delta_1, \delta_2 \in \mathfrak{m}$, we have $\exp(\delta_1) \cdot \exp(\delta_2) = \exp(\delta_1 * \delta_2)$.*

2.4 A group isomorphism between derivations and automorphisms

Let (A, Σ) be an ultrafinite, unital summability algebra. Let $\mathfrak{m} \subseteq \mathrm{Lin}^+(A)$ be a closed subalgebra such that $k\,\mathrm{Id}_A + \mathfrak{m}$ has evaluations.

Proposition 2.13. *The exponential induces a bijection between $\mathrm{Der}^+(A) \cap \mathfrak{m}$ and $\mathrm{End}^+(A) \cap \mathrm{Id}_A + \mathfrak{m}$.*

Proof. Our proof is a slight adaptation of [20, Theorem 4] to our formal context. Let $\partial \in \mathrm{Der}^+(A) \cap \mathfrak{m}$ and let $a, b \in A$. For $n \in \mathbb{N}$, an easy induction using the Leibniz product rule shows that

$$\partial^{[n]}(a\cdot b)=\sum_{i=0}^{n}\binom{n}{i}\partial^{[i]}(a)\cdot\partial^{[n-i]}(b).$$

We have

$$\begin{aligned}
\exp(\partial)(a)\cdot\exp(\partial)(b) &= \left(\sum_{m\in\mathbb{N}}\frac{1}{m!}\partial^{[m]}(a)\right)\cdot\left(\sum_{p\in\mathbb{N}}\frac{1}{p!}\partial^{[p]}(b)\right)\\
&= \sum_{m,p\in\mathbb{N}}\frac{1}{m!p!}\partial^{[m]}(a)\cdot\partial^{[p]}(b) && (\textbf{SA})\\
&= \sum_{n\in\mathbb{N}}\sum_{m+p=n}\frac{1}{m!p!}\partial^{[m]}(a)\cdot\partial^{[p]}(b) && (\textbf{SS3})\\
&= \sum_{n\in\mathbb{N}}\sum_{i=0}^{n}\frac{1}{i!(n-i)!}\partial^{[i]}(a)\cdot\partial^{[n-i]}(b)\\
&= \sum_{n\in\mathbb{N}}\frac{\partial^{[n]}(a\cdot b)}{k!}\\
&= \exp(\partial)(a\cdot b).
\end{aligned}$$

So $\exp(\partial)\in\mathrm{End}^{+}(A)$. Conversely, let $\sigma\in\mathrm{End}^{+}(A)\cap\mathrm{Id}_A+\mathfrak{m}$ and write $\varepsilon:=\mathrm{Id}_A-\sigma\in\mathfrak{m}$. Let $a,b\in A$. As in the proof of [20, Theorem 4], there is a family $(c_{k,l,n})_{k,l,n\in\mathbb{N}}\in\mathbb{Q}^{\mathbb{N}^3}$ such that for all $n>0$, we have both

$$\begin{aligned}
\varepsilon^{[n]}(a\cdot b) &= \sum_{l=0}^{n}\sum_{m=0}^{l}c_{n,l,m}\varepsilon^{[m]}(a)\cdot\varepsilon^{[l-m]}(b) && \text{in } A\text{, and}\\
(z_1+z_2-z_1z_2)^n &= \sum_{l=0}^{n}\sum_{m=0}^{l}c_{n,l,m}z_1^m z_2^{l-m} && \text{in } \mathbb{Q}[[z_1,z_2]].
\end{aligned}$$

Note that given $l,m\in\mathbb{N}$ we have

$$\forall n>l+m, c_{n,l,m}=0. \tag{8}$$

So the sum $S_{l,m}:=\sum_{n=1}^{+\infty}\frac{1}{n}c_{n,l,m}$ has finite support. We have

$$\log(1-(z_1+z_2-z_1z_2))=\log(1-z_1)+\log(1-z_2)$$

in $\mathbb{Q}[[z_1,z_2]]$. Identifying in the left and right hand terms the coefficients of $z_1^p z_2^q$ for $p\neq 0$ and $q\neq 0$, we deduce ithat $S_{l,m}=0$ if $m\notin\{0,l\}$ or $l=0$. Considering the coefficients of z_1^l and z_2^l for $l\geqslant 1$, we see that $S_{l,0}=S_{l,1}=\frac{1}{l}$ otherwise.

Now

$$
\begin{aligned}
\log(\sigma)(a\cdot b) &= \sum_{n\geqslant 1}\frac{1}{n}\sum_{l=0}^{n}\sum_{m=0}^{l} c_{n,l,m}\varepsilon^{[m]}(a)\cdot\varepsilon^{[l-m]}(b) \\
&= \sum_{n\geqslant 1}\sum_{m\leqslant l\leqslant n}^{n}\frac{c_{n,l,m}}{n}\varepsilon^{[m]}(a)\cdot\varepsilon^{[l-m]}(b) \\
&= \sum_{n\geqslant 1\wedge m\leqslant l\leqslant n}\frac{c_{n,l,m}}{n}\varepsilon^{[m]}(a)\cdot\varepsilon^{[l-m]}(b) && \text{(Proposition 1.6)} \\
&= \sum_{l\geqslant 0\wedge m\leqslant l\wedge n\geqslant l\wedge n\geqslant 1}\frac{c_{n,l,m}}{n}\varepsilon^{[m]}(a)\cdot\varepsilon^{[l-m]}(b) && (\textbf{SS2}) \\
&= \sum_{l\geqslant 0}\sum_{n\geqslant 1\wedge n\geqslant l\geqslant m}\frac{c_{n,l,m}}{n}\varepsilon^{[m]}(a)\cdot\varepsilon^{[l-m]}(b) && \text{(Proposition 1.6)} \\
&= \sum_{l\geqslant 0}\sum_{m\leqslant l}\sum_{n\geqslant 1\wedge n\geqslant l\wedge n\leqslant l+m}\frac{c_{n,l,m}}{n}\varepsilon^{[m]}(a)\cdot\varepsilon^{[l-m]}(b) && ((8)) \\
&= \sum_{l\geqslant 0}\sum_{m\leqslant l} S_{l,m}\varepsilon^{[m]}(a)\cdot\varepsilon^{[l-m]}(b) \\
&= \sum_{l\geqslant 1}\frac{1}{l}\sum_{m\in\{0,l\}}\varepsilon^{[m]}(a)\cdot\varepsilon^{[l-m]}(b) \\
&= \sum_{l\geqslant 1}\frac{1}{l}(a\cdot\varepsilon^{[l]}(b)+\varepsilon^{[l]}(a)\cdot b) \\
&= a\cdot\log(1-\varepsilon)(b)+\log(1-\varepsilon)(a)\cdot b \\
&= a\cdot\log(\sigma)(b)+\log(\sigma)(a)\cdot b.
\end{aligned}
$$

Therefore $\log(\sigma)$ is a derivation. □

Theorem 2.14. *Consider the operation $*$ of section 2.3 on $\mathfrak{m}$. The structures $(\mathrm{Der}^{+}(A)\cap\mathfrak{m},*)$ and $(\mathrm{End}^{+}(A)\cap\mathrm{Id}_A+\mathfrak{m},\circ)$ are groups, and the exponential map of Proposition 2.13 is a group isomorphism.*

Proof. By Proposition 2.8, each $\sigma\in\mathrm{Id}_A+\mathfrak{m}$ is invertible in $\mathrm{Id}_A+\mathfrak{m}$, and its inverse is obviously a morphism of algebra of A, so $\mathrm{End}^{+}(A)\cap\mathrm{Id}_A+\mathfrak{m}$ is a group under composition. We conclude with Corollary 2.12 and Proposition 2.13. □

Corollary 2.15. *The group $(\mathrm{End}^{+}(A)\cap\mathrm{Id}_A+\mathfrak{m},\circ)$ is divisible and torsion-free.*

Proof. Let $n\in\mathbb{N}^{>0}$ and $a\in\mathrm{Der}^{+}(A)\cap\mathfrak{m}$. Note that the n-fold iterate of a in $(\mathrm{Der}^{+}(A),*)$ is $a*a\cdot *a=na$. It follows since k has haracteristic zero that $(\mathrm{Der}^{+}(A)\cap\mathfrak{m},*)$ is torsion-free. Furthermore, we see that the n-fold iterate of $\frac{1}{n}a$ in $(\mathrm{Der}^{+}(A),*)$ is a, whence $(\mathrm{Der}^{+}(A)\cap\mathfrak{m},*)$ is divisible. We conclude with Theorem 2.14. □

Proposition 2.16. *For all* $\sigma \in \mathrm{Id}_A + \mathfrak{m}$*, writing* $\mathcal{C}(\sigma) = \{\mu \in \mathrm{End}^+(A) \cap \mathrm{Id}_A + \mathfrak{m} : \mu \circ \sigma = \sigma \circ \mu\}$*, we have a group morphism*

$$\begin{array}{rcl} {}^{[\cdot]} : (k, +, 0) & \longrightarrow & (\mathcal{C}(\sigma), \circ, \mathrm{Id}_A) \\ c & \longmapsto & \sigma^{[c]} := \exp(c \log(\sigma)), \end{array}$$

with $\sigma^{[1]} = \sigma$ *and* $(\sigma^{[c]})^{[c']} = \sigma^{[cc']}$ *for all* $c, c' \in k$*. It is injective if* $\sigma \neq \mathrm{Id}_A$*.*

Proof. Write $\partial := \log(\sigma) \in \mathrm{Der}^+_{\prec}(\mathbb{A})$ and let $c \in k$. Recall by Proposition 2.11 that for $n \in \mathbb{N}^{>0}$, the terms $\mathrm{ev}_{(\partial, c\partial)}(K_n)$ and $\mathrm{ev}_{(c\partial,\partial)}(K_n)$, where K_n is as in (6), lie in the Lie algebra generated by commutators in ∂ and $c\partial$. All such commutators are zero, so $\mathrm{ev}_{(\partial,c\partial)}(K_n) = \mathrm{ev}_{(c\partial,\partial)}(K_n) = 0$. It follows since $\mathrm{ev}_{(\partial,c\partial)}(K_0) = \partial + c\partial = \mathrm{ev}_{(c\partial,\partial)}(K_0)$ that $(c\partial) * \partial = (c+1)\partial = \partial * (c\partial)$, so $c\partial$ commutes with ∂, whence $\exp(c\partial) = \sigma^{[c]} \in \mathcal{C}(\sigma)$.

For $c, c' \in k$, we have $\sigma^{[c+c']} = \exp(c\partial + c'\partial) = \exp((c\partial) * (c'\partial))$ as above. So $\sigma^{[c+c']} = \exp(c\partial) \circ \exp(c'\partial) = \sigma^{[c]} \circ \sigma^{[c']}$. Thus ${}^{[\cdot]}$ is a group morphism. We also have $\sigma^{[cc']} = \exp(c' \log(\exp(c \log(\sigma)))) = \exp(cc' \log(\sigma)) = \sigma^{[cc']}$.

Assume that $\sigma \neq \mathrm{Id}_A$, so $\log(\sigma) \neq 0$. The kernel of the morphism is

$$\{c \in k : \exp(c \log(\sigma)) = \mathrm{Id}_A\} = \{c \in k : c \log(\sigma) = 0\} = \{0\}.$$

So this morphism is injective. □

3 Application to Noetherian series

3.1 Noetherian orderings

Definition 3.1. Let $(\Omega, <)$ be a partially ordered set. A **chain** in $(\Omega, <)$ is a subset of Ω which is linearly ordered by the induced ordering. A **decreasing chain** in $(\Omega, <)$ is chain $Y \subseteq \Omega$ without minimal element, i.e. with

$$\forall y \in Y, \exists z \in Y, (z < y).$$

An **antichain** in $(\Omega, <)$ is a subset $Y \subseteq \Omega$, no two distinct elements of which are comparable, i.e. with

$$\forall y, z \in Y, (y < z \vee y = z) \Longrightarrow y = z.$$

We say that $(\Omega, <)$ is **Noetherian**, or that $<$ is a Noetherian ordering on Ω, if there are no infinite decreasing chains and no infinite antichains in $(\Omega, <)$.

Note that linear Noetherian orderings are exactly well-orderings.

Proposition 3.2. [10, Proposition A.1] *An ordered set* $(\Omega, <)$ *is Noetherian if and only if every sequence* $u : \mathbb{N} \longrightarrow \Omega$ *has an increasing subsequence.*

If $(X, <_X)$ is a partially ordered set, then a **bad sequence** in X is a sequence $u : \mathbb{N} \longrightarrow X$ such that there are no numbers $i, j \in \mathbb{N}$ with $i < j$ and $u_i \leqslant_X u_j$. Given a function $f : X \longrightarrow \mathbb{N}$, a bad sequence u in **X** is said **minimal** for f if for all $i \in \mathbb{N}$, there are no bad sequences v in X with $(v_0, \ldots, v_{i-1}) = (u_0, \ldots, u_{i-1})$ and $f(v_i) < f(u_i)$. If there is a bad sequence in X, then there is a minimal one for f (see [10, p 307]).

Lemma 3.3. [9, Theorem 2.1] *Let $(X, <)$ be a partially ordered set. The following statements are equivalent*

a) $(X, <)$ *is Noetherian.*

b) *There is no bad sequence in* $(X, <)$.

See [9, Theorem 2.1] for other characterizations of Noetherian orderings.

Lemma 3.4. [9, Theorem 2.3] *Let $(\Omega_1, <_1)$ and $(\Omega_2, <_2)$ be Noetherian ordered sets. Then their product $\Omega_1 \times \Omega_2$ is Noetherian for the ordering*

$$(p_1, p_2) < (q_1, q_2) \Longleftrightarrow ((p_1, p_2) \neq (q_1, q_2) \wedge p_1 \leqslant p_2 \wedge q_1 \leqslant q_2).$$

Proposition 3.5. *Let $(\Omega, <)$ be an ordered set. Then the set $\mathcal{N}$ of Noetherian subsets of Ω is a bornology on Ω.*

Proof. If $X \subseteq Y$ are subsets of Ω, then a decreasing chain (resp. an antichain) in X is a decreasing chain (resp. an antichain) in Y. So X is Noetherian if Y is Noetherian. Let X_1, X_2 be Noetherian subsets of Ω. If C were an infinite decreasing chain in $X_1 \cup X_2$, then $C \cap X_1$ or $C \cap X_2$ would be a decreasing chain in X_1 or X_2 respectively, which cannot be. If A is an antichain in $X_1 \cup X_2$, then $A \cap X_1$ and $A \cap X_2$ are antichains in X_1 and X_2 respectively, so A must be finite. So $X_1 \cup X_2$ is Noetherian. Thus $\mathcal{N}$ is a bornology on Ω. □

We next state a weaker and simplified version of van der Hoeven's theorem [10, Appendix A.4]. A function ϑ sending each $p \in \Omega$ to a subset $\vartheta(p)$ of Ω is called a **choice operator** on Ω. The choice operator ϑ is said **Noetherian** if for all Noetherian subsets $Y \subseteq \Omega$, the set

$$Y_\vartheta := \bigcup_{y \in Y} \vartheta(y) \subseteq \Omega$$

is Noetherian. It is said **strictly extensive** if for all $p \in \Omega$, we have

$$p < \vartheta(p), \quad \text{that is,} \quad \forall y \in \vartheta(p), p < y.$$

For any non-empty word $w = (w_0, \ldots, w_m) \in \Omega^\star \setminus \{\emptyset\}$, we write $w_\bullet := w_m \in \Omega$ for the last "letter" of w. Let $Y \subseteq \Omega$ be a subset. Let $\vartheta^+(Y)$ denote the set of non-empty words $(w_0, \ldots, w_n) \in \Omega^\star \setminus \{\emptyset\}$ where $w_0 \in Y$, and for each $i < n$, we have $w_{i+1} \in \vartheta(w_i)$. We endow $\vartheta^+(Y)$ with the ordering $<_\vartheta$ defined by

$$w <_\vartheta w' \Longleftrightarrow w_\bullet < w'_\bullet.$$

Proposition 3.6. *Let ϑ be a Noetherian and strictly extensive choice operator on Ω. Then for all Noetherian subsets Y of Ω, the set $\vartheta^+(Y)$ is Noetherian for $<_\vartheta$.*

Proof. This version follows from an application of [10, Theorem A.4] to a simple case. Nonetheless, let us adapt van der Hoeven's proof to the present simplified setting.

Assume for contradiction that $\vartheta^+(Y)$ is not Noetherian. So there is a minimal bad sequence $(w_i)_{i\in\mathbb{N}} \in \vartheta^+(Y)^{\mathbb{N}}$ for the length function $w \mapsto |w|$. Assume that there is an infinite set $I \subseteq \mathbb{N}$ with $|w_i| = 2$ for all $i \in I$. Then $\mathcal{Y} := \{w_{i,0} : i \in I\} \subseteq Y$ is Noetherian. Since ϑ is a Noetherian choice operator, the set

$$\mathcal{Y}_\vartheta := \bigcup_{i\in I} \vartheta(x_{i,0})$$

is Noetherian. But then $\{w_i : i \in I\}$ is Noetherian for $<_\vartheta$: a contradiction.

So there is an $m \in \mathbb{N}$ with $|w_j| > 2$ for all $j \geqslant m$. For $j \geqslant m$, we write $z_j := (w_{j,0}, \ldots, w_{j,|w_j|-2}) \in \vartheta^+(Y)$. We claim that the set $\mathcal{Z} := \{z_j : j \geqslant m\}$, is Noetherian for $<_\vartheta$. Indeed, assume for contradiction that $(z_{j_i})_{i\in\mathbb{N}}$ is a bad sequence in $\mathcal{Z}$ with $j_0 \leqslant j_1 \leqslant \cdots$. We further claim that

$$u := (w_0, \ldots, w_{j_0-1}, z_{j_0}, z_{j_1}, \ldots)$$

is a bad sequence, contradicting the minimality of $(w_i)_{i\in\mathbb{N}}$. Assume for contradiction that u is not bad. Since $(z_{j_i})_{i\in\mathbb{N}}$ is bad, there must exist $i < j_0$ and $p \in \mathbb{N}$ with $w_i \leqslant_\vartheta z_{j_p}$. Since ϑ is strictly extensive, we have

$$(w_{j_p})_\bullet \in \vartheta((z_{j_p})_\bullet) > (z_{j_p})_\bullet,$$

so $w_i <_\vartheta w_m$: a contradiction. Therefore $\mathcal{Z}$ is Noetherian. It follows since ϑ is Noetherian that $\{w_i : i \geqslant m\}$ is Noetherian: a contradiction. □

3.2 Spaces of Noetherian series

We fix an ordered set $(\Omega, <)$ and a summability space V with the minimal summability structure. We have an ultrafinite summability space $(V^{[\mathcal{N}]}, \Sigma^{\mathcal{N}})$ where $\mathcal{N}$ is the bornology of Noetherian subsets of Ω. We say that $(V^{[\mathcal{N}]}, \Sigma^{\mathcal{N}})$ is a **space of Noetherian series**.

Lemma 3.7. *Let I be a* set *and let $\mathbf{f} : I \longrightarrow V^{[\mathcal{N}]}$ be a function. Consider the set*

$$N_{\mathbf{f}} := \{(i, p) \in I \times \Omega : p \in \operatorname{supp} \mathbf{f}(i)\},$$

ordered by $(i, p) <_{\mathbf{f}} (j, q) \iff p < q$. Then $\mathbf{f}$ is $\mathcal{N}$-summable if and only if $(N_{\mathbf{f}}, <_{\mathbf{f}})$ is Noetherian.

Proof. Consider a non-empty chain C for $(N_{\mathbf{f}}, <_{\mathbf{f}})$. Given $(i, p) \in C$, we have $p \in \bigcup_{i\in I} \operatorname{supp} \mathbf{f}(i)$, and (i, p) is $<_{\mathbf{f}}$ minimal in C if and only if p is minimal in

$\bigcup_{i\in I}\operatorname{supp}\mathbf{f}(i)$. So $N_{\mathbf{f}}$ as infinite decreasing chains if and only if $\bigcup_{i\in I}\operatorname{supp}\mathbf{f}(i)$ has infinite decreasing chains.

Consider an antichain A in $(N_{\mathbf{f}},<_{\mathbf{f}})$. For $(i,p),(j,q)\in A$, either $p=q$ and $i\neq j$ or p and q are not comparable in $(\Omega,<)$. So $N_{\mathbf{f}}$ has an infinite antichain if and only if there is an infinite antichain in $\bigcup_{i\in I}\operatorname{supp}\mathbf{f}(i)$ or there is an $p\in\bigcup_{i\in I}\operatorname{supp}\mathbf{f}(i)$ such that the set $I_p=\{i\in I : p\in\operatorname{supp}\mathbf{f}(i)\}$ is infinite.

In view of the definitions of $\mathcal{N}$-summability and Noetherian orderings, we deduce that $\mathbf{f}$ is $\mathcal{N}$-summable if and only if $(N_{\mathbf{f}},<_{\mathbf{f}})$ is Noetherian. □

3.3 Algebras of Noetherian series

We fix an ordered monoid $(M,+,0,<)$, i.e. a monoid $(M,+,0)$ together with an ordering $<$ on M with

$$\forall f,g,h\in M, f<g\Longrightarrow (f+h<g+h\wedge h+f<h+g). \tag{9}$$

Let $\mathcal{N}$ denote the bornology of Noetherian subsets of $(M,<)$. We write $k(\!(M)\!):=k^{[\mathcal{N}]}$, with its pointwise summmability structure $\Sigma^{\mathcal{N}}$ with respect to the minimal summmability structure on k. As in [23, 18], given $g\in M$, we write t^g for the function $M\longrightarrow k$ with support $\{g\}$ and value 1 at g, i.e. $t^g=\mathbb{1}_{\{g\}}$. Recall by (2) that for each $a\in k(\!(M)\!)$, the family $(a(g)t^g)_{g\in M}$ is $\mathcal{N}$-summable, with sum

$$a=\sum_{g\in M}a(g)t^g.$$

The vector space $k(\!(M)\!)$ is equipped with the Cauchy product

$$\forall g\in M, (a\cdot b)(g):=\sum_{f+h=g}a(g_1)b(g_2). \tag{10}$$

Lemma 3.8. *The function $a\cdot b$ is well-defined and lies in $k(\!(M)\!)$.*

Proof. We first show that for $g\in M$, the set

$$I:=\{(f,h)\in(\operatorname{supp}a)\times(\operatorname{supp}b) : f+h=g\}$$

is finite. Assume for contradiction that it is infinite. If its projection I_1 on the first variable is infinite, then we find an injective sequence $f:\mathbb{N}\longrightarrow I_1$. Let $h:\mathbb{N}\longrightarrow\operatorname{supp}b$ be a sequence with $(f(n),h(n))\in I$ for all $n\in\mathbb{N}$. Since $I_1\subseteq\operatorname{supp}a$ is Noetherian, by Proposition 3.2, there is an increasing subsequence $f\circ\varphi$ of f. Likewise, there is an increasing subsequence $h\circ\varphi\circ\psi$ of $h\circ\varphi$. We have $g=f\circ\varphi\circ\psi(1)+h\circ\varphi\circ\psi(1)>f\circ\varphi\circ\psi(0)+h\circ\varphi\circ\psi(1)$ and $f\circ\varphi\circ\psi(0)+h\circ\varphi\circ\psi(1)\geqslant f\circ\varphi\circ\psi(0)+h\circ\varphi\circ\psi(0)=g$ by (9), so $g>g$: a contradiction.

If I_1 is finite, then the projection of I on the second variable must be infinite, and we obtain a symmetric contradiction. Therefore I is finite. So $a\cdot b$ is well-defined.

We next show that the set

$$S := \{g \in M : \exists (g_1, g_2) \in (\operatorname{supp} a) \times (\operatorname{supp} b), g = g_1 + g_2\}$$

is Noetherian. It will then follow since $\operatorname{supp}(a \cdot b) \subseteq S$ that $a \cdot b \in k(\!(M)\!)$. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in S and let $f \in (\operatorname{supp} a)^{\mathbb{N}}$ and $g \in (\operatorname{supp} b)^{\mathbb{N}}$ with $g_n = f(n) + h(n)$ for all $n \in \mathbb{N}$. By Proposition 3.2, there is an increasing subsequence $f \circ \varphi$ of f, and an increasing subsequence $h \circ \varphi \circ \psi$ of $h \circ \varphi$. Thus $(g_{\varphi(\psi(n))})_{n\in\mathbb{N}} = f \circ \varphi \circ \psi + h \circ \varphi \circ \psi$ is an increasing subsequence of $(g_n)_{n\in\mathbb{N}}$. We deduce with Proposition 3.2 that S is Noetherian. □

Proposition 3.9. *For the Cauchy product, the structure $k(\!(M)\!)$ is a unital algebra with multiplicative identity $1 := t^0$.*

Proof. Let $a, b, c \in k(\!(M)\!)$ and $g \in M$. We have

$$\begin{aligned}(a \cdot b) \cdot c &= \sum_{g_1+g_2=g} (a \cdot b)(g_1)c(g_2)\\ &= \sum_{g_1+g_2=g} \left(\sum_{g_3+g_4=g_1} a(g_3)b(g_4)\right) c(g_2)\\ &= \sum_{g_1+g_3+g_4=g} a(g_3)b(g_4)c(g_2). \qquad ((M,+) \text{ is associative})\end{aligned}$$

Likewise $a \cdot (b \cdot c) = \sum_{g_1+g_3+g_4=g} a(g_3)b(g_4)c(g_2) = (a \cdot b) \cdot c$. So the product is associative. It is clearly bilinear. It is easy to see that

$$(t^0 \cdot a)(g) = (a \cdot t^0)(g) = a(g),$$

so 1 is the multiplicative identity in $k(\!(M)\!)$. □

Proposition 3.10. *The algebra $k(\!(M)\!)$ with its structure of bornological space is a summability algebra.*

Proof. We have to show that **SA** holds. This follows as in [11, Proposition 3.3], where the commutativity of the monoid does not play a role. □

In particular, for $a, b \in k(\!(M)\!)$, the family $(a(g_1)b(g_2)t^{g_1+g_2})_{g_1,g_2\in M}$ is $\mathcal{N}$-summable, with sum

$$a \cdot b = \sum_{g_1,g_2\in M} a(g_1)b(g_2)t^{g_1+g_2}.$$

We call $k(\!(M)\!)$ the **algebra of Noetherian series** (with coefficients in k and exponents in M).

3.4 Contracting linear maps

In this subsection and the next one, we fix an ordered set Ω, we consider again the bornology $\mathcal{N}$ of Noetherian subsets of Ω, and we write $\mathbb{V} := k^{[\mathcal{N}]}$ for the corresponding bornological space. Given $v, w \in \mathbb{V}$, we write

$$v \prec w$$

if $w \neq 0$ and for all $p \in \operatorname{supp} v$, there is a $q \in \operatorname{supp} w$ with $p > q$.

Proposition 3.11. [11, Proposition 4.6] *The relation $\prec$ is an ordering on $\mathbb{V}$. Moreover, for $u, v, w \in \mathbb{V}$, we have $u \prec w \wedge v \prec w \Longrightarrow u + v \prec w$.*

Definition 3.12. A function $\phi : \mathbb{V} \longrightarrow \mathbb{V}$ is said **contracting** if for all $a, b \in \mathbb{V}$, we have

$$v \neq w \Longrightarrow \phi(w) - \phi(v) \prec w - v.$$

We write $\operatorname{Lin}_{\prec}(\mathbb{V})$ (resp. $\operatorname{Lin}^+_{\prec}(\mathbb{V})$) for the sets of contracting linear (resp. strongly linear) maps $\mathbb{V} \longrightarrow \mathbb{V}$. Note that

$$\operatorname{Lin}^+_{\prec}(\mathbb{V}) = \operatorname{Lin}_{\prec}(\mathbb{V}) \cap \operatorname{Lin}^+(\mathbb{V}). \tag{11}$$

Lemma 3.13. $\operatorname{Lin}_{\prec}(\mathbb{V})$ *is a subalgebra of* $\operatorname{Lin}(\mathbb{V})$*.*

Proof. If $\phi, \psi \in \operatorname{Lin}_{\prec}(\mathbb{V})$ and $c \in k$, then for $v \in \mathbb{V} \setminus \{0\}$, we have $\operatorname{supp} c\psi(v) \subseteq \operatorname{supp} \psi(v)$ so $c\psi(v) \prec v$. We deduce with Proposition 3.11 that $\phi(v) + c\psi(v) \prec v$, whence $\phi + c\psi \in \operatorname{Lin}_{\prec}(\mathbb{V})$. If $\psi(v) = 0$, then $\phi(\psi(v)) = 0 \prec v$. Otherwise $\phi(\psi(v)) \prec \psi(v)$ where $\psi(v) \prec v$ so $\phi \circ \psi(v) \prec v$ by Proposition 3.11, whence $\phi \circ \psi \in \operatorname{Lin}_{\prec}(\mathbb{V})$. □

Lemma 3.14. *If $\phi : \mathbb{V} \longrightarrow \mathbb{V}$ is strongly linear, then it is contracting if and only if* $\operatorname{supp} \phi(\mathbb{1}_{\{p\}}) > p$ *for all* $p \in \Omega$*.*

Proof. Assume that ϕ satisfies the condition above, and let $v \in \mathbb{V}$ be non-zero. By strong linearity, for each $p \in \operatorname{supp} \phi(v)$, there is a $q \in \operatorname{supp} v$ with $p \in \operatorname{supp} \phi(\mathbb{1}_{\{q\}})$. Thus there is an $r \in \operatorname{supp} \mathbb{1}_{\{q\}}$ with $p > u$. But $\operatorname{supp} \mathbb{1}_{\{q\}} = \{q\}$ so $r = q$, whence $p > q \in \operatorname{supp} v$. This shows that $\phi(v) \prec v$. Since ϕ is linear, it follows that ϕ is contracting. The converse is immediate since $\phi(\mathbb{1}_{\{p\}}) \prec \mathbb{1}_{\{p\}}$ is equivalent to $\operatorname{supp} \phi(\mathbb{1}_{\{p\}}) > p$. □

Corollary 3.15. $\operatorname{Lin}^+_{\prec}(\mathbb{V})$ *is a closed subalgebra of* $\operatorname{Lin}^+(\mathbb{V})$ *and a closed ideal of* $k \operatorname{Id}_{\mathbb{V}} + \operatorname{Lin}^+_{\prec}(\mathbb{V})$*.*

Proof. That $\operatorname{Lin}^+_{\prec}(\mathbb{V})$ is a subalgebra follows from Proposition 1.21, (11) and Lemma 3.13. Let $\boldsymbol{\phi}\colon J \longrightarrow \operatorname{Lin}^+_{\prec}(\mathbb{V})$ be Lin-summable and set

$$\boldsymbol{\sigma} := \sum_{j \in J} \boldsymbol{\phi}(j) \in \operatorname{Lin}^+(\mathbb{V}).$$

Let $p \in \Omega$. We have $\operatorname{supp} \boldsymbol{\sigma}(\mathbb{1}_{\{p\}}) \subseteq \bigcup_{j \in J} \operatorname{supp} \boldsymbol{\phi}(j)(\mathbb{1}_{\{p\}})$. Lemma 3.14 gives $\operatorname{supp} \boldsymbol{\phi}(j)(\mathbb{1}_{\{p\}}) > p$ for each $j \in J$, so $\operatorname{supp} \boldsymbol{\sigma}(\mathbb{1}_{\{p\}}) > p$, so $\boldsymbol{\sigma}$ is contracting, again by Lemma 3.14. □

Theorem 3.16. *The algebra $k\,\mathrm{Id}_{\mathbb{V}} + \mathrm{Lin}^+_{\prec}(\mathbb{V})$ has evaluations.*

Proof. Let J be a set and let $(\phi_j)_{j\in J} \subseteq (\mathrm{Lin}^+_{\prec}(\mathbb{V}))^J$ be Lin-summable. Let us show that the family

$$(\phi_{\theta_1} \circ \cdots \circ \phi_{\theta_n})_{(\theta_1,\ldots,\theta_n)\in J^\star}$$

is Lin-summable. We may assume that J is non-empty, and it suffices to show that

$$(\phi_{\theta_0} \circ \cdots \circ \phi_{\theta_n})_{(\theta_0,\ldots,\theta_n)\in J^\star}$$

is Lin-summable.

Let $\pi : \Omega \times J \longrightarrow \Omega$ denote the projection on the first variable. We consider the ordering $<$ on $\Omega \times J$ given by $(p, i) < (q, j) \iff p < q$. Consider the choice operator ϑ on $\Omega \times J$ given by

$$\forall p \in \Omega, \vartheta(p, i) := \{(q, j) : j \in J \wedge q \in \operatorname{supp} \phi_j(\mathbb{1}_{\{p\}})\}.$$

Since each ϕ_j for $j \in J$ is contracting, this is a strictly extensive choice operator. Let $Y \subseteq \Omega \times J$ be Noetherian. Let us show that the set

$$Y_\vartheta = \{(q, j) : j \in J \wedge (\exists p \in \pi(Y), (q \in \operatorname{supp} \phi_j(\mathbb{1}_{\{p\}})))\}$$

is Noetherian by showing that each sequence in Y_ϑ has an increasing subsequence. Let $(q_n, j_n)_{n\in\mathbb{N}}$ be a sequence in Y_ϑ, and pick for each $n \in \mathbb{N}$ a $(p_n, j'_n) \in Y$ with $q_n \in \operatorname{supp} \phi_{j_n}(\mathbb{1}_{\{p_n\}})$. Since Y is Noetherian, we may assume by Proposition 3.2 that $(p_n, j'_n)_{n\in\mathbb{N}}$ is increasing.

Assume that $(p_n)_{n\in\mathbb{N}}$ has a constant subsequence. Without loss of generality, we may assume that it is constant itself. Assume that $(j_n)_{n\in\mathbb{N}}$ has no injective subsequence. So it has a constant subsequence $(j_{\psi(n)})_{n\in\mathbb{N}}$. The sequence $(q_{\psi(n)})_{n\in\mathbb{N}}$ in the Noetherian set $\operatorname{supp} \phi_{j_{\psi(0)}}(\mathbb{1}_{\{p_{\psi(0)}\}})$ has an increasing subsequence $(q_{\psi\circ\mu(n)})_{n\in\mathbb{N}}$, and $(q_{\psi\circ\mu(n)}, j_{\psi(0)})_{n\in\mathbb{N}}$ is an increasing subsequence of $(q_n, j_n)_{n\in\mathbb{N}}$. Assume now that $(j_n)_{n\in\mathbb{N}}$ has an injective subsequence. Since $(\phi_{j_n})_{n\in\mathbb{N}}$ is Lin-summable, the family $(\phi_{j_n}(\mathbb{1}_{\{p_0\}}))_{n\in\mathbb{N}}$ is $\mathcal{N}$-summable. Therefore $(q_n)_{n\in\mathbb{N}}$ must have a strictly increasing subsequence, whence $(q_n, j_n)_{n\in\mathbb{N}}$ has a strictly increasing subsequence.

Assume now that $(p_n)_{n\in\mathbb{N}}$ has no constant subsequence. So $(p_n)_{n\in\mathbb{N}}$ has a strictly increasing subsequence $(p_{\varphi(n)})_{n\in\mathbb{N}}$. The family $(\phi_{j_{\varphi(n)}}(\mathbb{1}_{\{p_{\varphi(n)}\}}))_{n\in\mathbb{N}}$ is $\mathcal{N}$-summable. In particular $(q_{\varphi(n)})_{n\in\mathbb{N}}$ has a strictly increasing subsequence, whence again $(q_n, j_n)_{n\in\mathbb{N}}$ has a strictly increasing subsequence.

This shows that ϑ is Noetherian. Let $S \subseteq \Omega$ be a Noetherian subset and pick an arbitrary $j \in J$. By Proposition 3.6, the set $\vartheta^+(S \times \{j\})$ is Noetherian for $<_\vartheta$. This means by Lemma 3.7 that the family $(p_\bullet)_{p\in\vartheta^+(S\times\{j\})}$ is $\mathcal{N}$-summable. Write

$$I := \{(p, n, \theta_0, \ldots, \theta_n) : p \in S \wedge n \in \mathbb{N} \wedge \theta_0, \ldots, \theta_n \in J\}.$$

For $i = (p, n, \theta_0, \ldots, \theta_n) \in I$, let W_i denote the set of words

$$w = (w_0, \ldots, w_{n+1}) \in \vartheta^+(S \times \{j\})$$

where $w_0 = (p, j)$ and $w_{k+1} = (p_{k+1}, \phi_{\theta_k})$ for a $p_{k+1} \in \operatorname{supp} \phi_{\theta_k}(\mathbb{1}_{\{\pi(w_k)\}})$. Note that for each $i = (p, n, \phi_0, \ldots, \phi_n) \in I$, there is a family $c \in k^{W_i}$ such that $\phi_n \circ \cdots \circ \phi_0(p) = \sum_{w \in W_i} c(w) \mathbb{1}_{\{\pi(w_\bullet)\}}$.

The sets $W_i, i \in I$ are pairwise disjoint, so by **SS4**, Proposition 1.6 and **UF**, the family $\left(\sum_{w \in W_i} c(w) \mathbb{1}_{\{\pi(w_\bullet)\}}\right)_{i \in I} = (\phi_{\theta_0} \circ \cdots \circ \phi_{\theta_n}(p))_{p \in S \wedge n \in \mathbb{N} \wedge \theta_0, \ldots, \theta_n \in J}$ is $\mathcal{N}$-summable. We conclude with Proposition 1.26. □

3.5 The Der-Aut correspondence for Noetherian series

Let $\mathbb{V} = k^{[\mathcal{N}]}$ be a space of Noetherian series. By Theorem 3.16, the summability algebra $k \operatorname{Id}_{\mathbb{V}} + \operatorname{Lin}^+_{\prec}(\mathbb{V})$ has evaluations. As a consequence of Corollaries 2.10 and 2.12, we have a group operation

$$\begin{aligned} * : \operatorname{Lin}^+_{\prec}(\mathbb{V}) \times \operatorname{Lin}^+_{\prec}(\mathbb{V}) &\longrightarrow \operatorname{Lin}^+_{\prec}(\mathbb{V}) \\ (\phi, \psi) &\longmapsto \phi + \psi + \frac{1}{2}(\phi \circ \psi - \psi \circ \phi) + \cdots, \end{aligned}$$

and a group isomorphism

$$\begin{aligned} \exp : (\operatorname{Lin}^+_{\prec}(\mathbb{V}), *) &\longrightarrow (\operatorname{Id}_{\mathbb{V}} + \operatorname{Lin}^+_{\prec}(\mathbb{V}), \circ) \\ \phi &\longmapsto \sum_{n \in \mathbb{N}} \frac{1}{n!} \phi^{[n]}. \end{aligned} \tag{12}$$

Let $\mathbb{A} = k(\!(M)\!)$ be an algebra of Noetherian series, where M is an ordered monoid. Let $1\text{-}\operatorname{Aut}^+_k(\mathbb{A})$ denote the group of bijective strongly linear morphisms of algebra $\sigma : \mathbb{A} \longrightarrow \mathbb{A}$ which preserve products, and with $\sigma(a) - a \prec a$ for all $a \in \mathbb{A} \setminus \{0\}$. In the case when M is a linearly ordered group, our notation $1\text{-}\operatorname{Aut}^+_k(\mathbb{A})$ is compatible with that of [18]. Applying Theorem 2.14, we obtain:

Theorem 3.17. *The structures* $(\operatorname{Der}^+_{\prec}(\mathbb{A}), *)$ *and* $\left(1\text{-}\operatorname{Aut}^+_k(\mathbb{A}), \circ\right)$ *are groups, and we have an isomorphism*

$$\begin{aligned} \exp : (\operatorname{Der}^+_{\prec}(\mathbb{A}), *) &\longrightarrow \left(1\text{-}\operatorname{Aut}^+_k(\mathbb{A}), \circ\right) \\ \partial &\longmapsto \sum_{n \in \mathbb{N}} \frac{\partial^{[n]}}{k!}. \end{aligned}$$

We finish with a formal analog of the Lie homomorphism theorem.

Theorem 3.18. *Let W be an ordered set, let $\mathcal{N}_W$ be its bornology of Noetherian subsets and consider the space of Noetherian series $\mathbb{W} = k^{[\mathcal{N}_W]}$. Let $\Phi : \operatorname{Lin}^+_{\prec}(\mathbb{V}) \longrightarrow \operatorname{Lin}^+_{\prec}(\mathbb{W})$ be a strongly linear morphism of Lie algebras. Then there exists a unique group morphism $\Psi : \operatorname{Id}_{\mathbb{V}} + \operatorname{Lin}^+_{\prec}(\mathbb{V}) \longrightarrow \operatorname{Id}_{\mathbb{W}} + \operatorname{Lin}^+_{\prec}(\mathbb{W})$ with*

$$\Psi(\exp(\phi)) = \exp(\Phi(\phi))$$

for all $\phi \in \operatorname{Lin}^+_{\prec}(\mathbb{V})$.

Proof. We have $\mathrm{Id}_{\mathbb{V}} + \mathrm{Lin}^+_{\prec}(\mathbb{V}) = \exp(\mathrm{Lin}^+_{\prec}(\mathbb{V}))$ by (12), so the function Ψ si uniquely determined by

$$\Psi(\sigma) := \exp(\Phi(\log(\sigma)))$$

for all $\sigma \in \mathrm{Id}_{\mathbb{V}} + \mathrm{Lin}^+_{\prec}(\mathbb{V})$. Let $\sigma_1, \sigma_2 \in \mathrm{Id}_{\mathbb{V}} + \mathrm{Lin}^+_{\prec}(\mathbb{V})$ and write $(\phi_1, \phi_2) := (\log(\sigma_1), \log(\sigma_2)) \in \mathrm{Der}^+_{\prec}(\mathbb{A})$. Since Φ is a morphism of Lie algebras, we have $\Phi(\mathrm{ev}_{(\sigma_1,\sigma_2)}(Z_n)) = \mathrm{ev}_{(\Phi(\sigma_1),\Phi(\sigma_2))}(K_n)$ for all $n \in \mathbb{N}$, where $K_n \in k\langle\langle 2\rangle\rangle$ is as described in (6). Since Φ is strongly linear, we deduce that $\Phi(\mathrm{ev}_{(\phi_1,\phi_2)}(X_0 * X_1)) = \mathrm{ev}_{(\Phi(\phi_1),\Phi(\phi_2))}(X_0 * X_1)$, i.e.

$$\Phi(\phi_1 * \phi_2) = \Phi(\phi_1) * \Phi(\phi_2). \tag{13}$$

It follows that

$$\begin{aligned}
\Psi(\sigma_1 \circ \sigma_2) &= \Psi(\exp(\phi_1 * \phi_2)) && \text{(Corollary 2.12)}\\
&= \exp(\Phi(\phi_1 * \phi_2))\\
&= \exp(\Phi(\phi_1) * \Phi(\phi_2)) && (13)\\
&= \exp(\Phi(\phi_1)) \circ \exp(\Phi(\phi_2)) && \text{(Corollary 2.12)}\\
&= \Psi(\sigma_1) \circ \Psi(\sigma_2).
\end{aligned}$$

Therefore Ψ is a morphism. □

The same arguments using the identity $1\text{-}\mathrm{Aut}^+_k(\mathbb{A}) = \exp(\mathrm{Der}^+_{\prec}(\mathbb{A}))$ give:

Theorem 3.19. *Let $\mathbb{B}$ be an angelra of Noetherian series. Let $\Phi : \mathrm{Der}^+_{\prec}(\mathbb{A}) \longrightarrow \mathrm{Der}^+_{\prec}(\mathbb{B})$ be a strongly linear morphism of Lie algebras. Then there exists a unique group morphism $\Psi : 1\text{-}\mathrm{Aut}^+_k(\mathbb{A}) \longrightarrow 1\text{-}\mathrm{Aut}^+_k(\mathbb{B})$ with*

$$\Psi(\exp(\partial)) = \exp(\Phi(\partial))$$

for all $\partial \in \mathrm{Der}^+_{\prec}(\mathbb{A})$.

As a consequence of Corollary 2.15, we have:

Corollary 3.20. *The group $1\text{-}\mathrm{Aut}^+_k(\mathbb{A})$ is divisible and torsion-free.*

4 Toward a full correspondence

4.1 Decomposing valuation preserving automorphisms

Let k be an ordered field and let G be a linearly ordered Abelian group. There is a natural ordering on the field $\mathbb{K} := k(\!(G)\!)$ for which it is an ordered field [14]. The field $\mathbb{K}$ is a Hahn field as per [23, 18].

Let $v\text{-}\mathrm{Aut}^+_k(\mathbb{K})$ denote the group of strongly linear automorphisms of the algebra $\mathbb{K}$ with $a \asymp b \Rightarrow \sigma(a) \asymp \sigma(b)$ for all $a, b \in U(\mathbb{K})$ where $a \asymp b$ if and only if $a \not\prec b$ and $b \not\prec a$. By [18, Theorem 3.7.1], the group $v\text{-}\mathrm{Aut}^+_k(\mathbb{K})$ is the semi-direct product of the following three subgroups:

a) The group $1\text{-Aut}_k^+(\mathbb{K})$.

b) The group $G\text{-Exp}(\mathbb{K})$ of functions

$$\Psi_x : a \mapsto \sum_{g \in G} x(g) t^g$$

where $x \in \operatorname{Hom}((G,+),(k^\times,\cdot))$.

c) The group $o\text{-Aut}(G)$ of functions

$$a \mapsto \sum_{g \in G} a(g) t^{\mu(g)}$$

where $\mu \in \operatorname{Aut}(G,+,<)$.

4.2 Prelogarithms

Write $\mathbb{K}_\succ := \{a \in \mathbb{K} \ : \ \operatorname{supp} a < 0\}$. The ordered group $(\mathbb{K},+,0)$ has [14, Theorems 1.4 and 1.8] an additive lexicographic decomposition

$$\mathbb{K} = \mathbb{K}_\succ + k + \mathbb{K}^\prec,$$

whereas $(\mathbb{K}^{>0},\cdot,1,<)$ has a multipicative decomposition

$$\mathbb{K}^{>0} = t^G \cdot k^{>0} \cdot (1+\mathbb{K}^\prec).$$

A prelogarithm is an embedding $\log\colon (\mathbb{K}^{>0},\cdot,1,<) \longrightarrow (\mathbb{K},+,0,<)$. It is compatible with the valuation v on $\mathbb{K}$ if for all $a > 0$, we have $v(\log(a)) \geq 0 \iff v(a) \geq 0$ and $v(\log(a)) > 0 \iff v(a-1) > 0$. A logarithm is a surjective prelogarithm. The existence of a compatible logarithm is equivalent [14, Lemma 1.21] to the existence of the three following isomorphisms of ordered groups.

a) A right logarithm, i.e. an isomorphism $(1+\mathbb{K}^\prec,\cdot,1,<) \longrightarrow (\mathbb{K}^\prec,+,0,<)$.

b) A middle logarithm, i.e. an isomorphism $(k^{>0},\cdot,1,<) \longrightarrow (k,+,0,<)$.

c) A left logarithm, i.e. an isomorphism $(t^G,\cdot,1,<) \longrightarrow (\mathbb{K}_\succ,+,0,<)$.

This is illustrated in the following picture

$$\begin{tikzcd}
\mathbb{K} \arrow[d, leftrightarrow] & = & \mathbb{K}_\succ \arrow[d, leftrightarrow, red] & + & k \arrow[d, leftrightarrow, blue] & + & \mathbb{K}^\prec \arrow[d, leftrightarrow, green] \\
\mathbb{K}^{>0} & = & t^G & \cdot & k^{>0} & \cdot & 1+\mathbb{K}^\prec
\end{tikzcd}$$

Exponentiation along the additive and multiplicative decompositions

Although prelogarithms always exist, no right logarithm exsits if G is non-trivial [17, Theorem 1]. This obstruction can be circumvented by considering directed unions of fields of Hahn series, such as log-exp series, or fields of EL-series [14, Chap 5, Section 2].

One can interpret the formal exp-log correspondence between each $\mathrm{Der}^+_{\prec}(\mathbb{K})$ and $1\text{-}\mathrm{Aut}^+_k(\mathbb{K})$ as a non-commutative generalisation of the natural right logarithm on $\mathbb{K}$, which is the isomorphism

$$(1+\mathbb{K}^{\prec},\cdot,<) \longrightarrow (\mathbb{K}^{\prec},+,<)\ ;\ 1+\varepsilon \mapsto \sum_{m\in\mathbb{N}} \frac{(-1)^m\varepsilon^{m+1}}{m}.$$

We want to investigate how this correspondence extends to the group v-$\mathrm{Aut}^+_k(\mathbb{K})$, mapping it to an appropriate Lie subalgebra of $\mathrm{Der}^+(\mathbb{K})$. Let us first extend the correspondence to the subgroup $\mathrm{IntAut}^+_k(\mathbb{K}) := G\text{-}\mathrm{Exp}(\mathbb{K}) \ltimes 1\text{-}\mathrm{Aut}^+_k(\mathbb{K})$ on $\mathbb{K}$.

4.3 The middle correspondence

Let G-$\mathrm{Der}(\mathbb{K})$ denote the commutative group, under pointwise sum, of (strongly linear) derivations d_α of the form

$$\forall a\in\mathbb{K}, \mathrm{d}_\alpha(\sum_{g\in G} a(g)t^g) = \sum_{g\in G}\alpha(g)a(g)t^g$$

where $\alpha \in \mathrm{Hom}((G,+),(k,+))$.

Note that $\mathrm{Hom}((G,+),(k,+))$ and G-$\mathrm{Der}(\mathbb{K})$ are vector spaces and $\mathrm{d}.$ is a k-linear isomorphism. Let $\mathrm{e} : (k,+,<) \longrightarrow (k^{>0},\cdot,<)$ be an isomorphism of ordered groups, i.e. the inverse of a middle logarithm on $\mathbb{K}$. Writing $k^\times$ as the direct product $k^\times \simeq \{1,-1\}\times k^{>0}$, we obtain

$$\begin{aligned}\mathrm{Hom}((G,+),(k^\times,\cdot)) &\simeq \mathrm{Hom}((G,+),(\{1,-1\},\cdot))\times \mathrm{Hom}((G,+),(k^{>0},\cdot))\\ &\simeq \mathrm{Hom}((G,+),(\mathbb{Z}_2,+))\times\mathrm{Hom}((G,+),(k,+))\end{aligned}$$

We have $\mathrm{Hom}((G,+),(\mathbb{Z}_2,+)) \simeq \mathrm{Hom}((G/2G,+),(\mathbb{Z}_2,+))$ which is trivial if and only if G is 2-divisile. In that case, we have an isomorphism

$$\mathrm{Hom}((G,+),(k,+)) \longrightarrow \mathrm{Hom}((G,+),(k^\times,\cdot))\ ;\ \alpha\mapsto \mathrm{e}\circ\alpha.$$

which yields a Der-Aut correspondence $G\text{-}\mathrm{Der}(\mathbb{T}) \longrightarrow G\text{-}\mathrm{Exp}(\mathbb{T})\ ;\ \mathrm{d}_\alpha \mapsto \psi_{\mathrm{e}\circ\alpha}$.

4.4 Toward a left correspondence

Consider the Lie algebra $\mathrm{Der}^+_{\preceq}(\mathbb{K})$ of strongly linear derivations $\partial\colon \mathbb{K}\longrightarrow\mathbb{K}$ such that $v(\partial(a)) \geq v(a)$ for all $a\in\mathbb{K}$. This is the semi-direct product of the Lie algebra G-$\mathrm{Der}(\mathbb{K})$ with the Lie ideal $\mathrm{Der}^+_{\prec}(\mathbb{K})$. Combining the right and middle Der-Aut correspondences, we obtain an isomorphism between $\mathrm{Der}^+_{\preceq}(\mathbb{K})$ and the group $\mathrm{IntAut}^+_k(\mathbb{K})$ of internal strongly linear v-automorphisms of $\mathbb{K}$. This is summed up in the following picture, where the upper (resp. lower) decomposition is given by semi-direct internal products of groups (resp Lie algebras).

$$
\begin{tikzcd}
\mathrm{Der}^{+}_{\precsim}(\mathbb{K}) \arrow[d, leftrightarrow] & = & G\text{-}\mathrm{Der}(k) \arrow[d, leftrightarrow] & \ltimes & \mathrm{Der}^{+}_{\prec}(\mathbb{K}) \arrow[d, leftrightarrow] \\
\mathrm{IntAut}^{+}_{k}(\mathbb{K}) & = & G\text{-}\mathrm{Exp}(k) & \circ & 1\text{-}\mathrm{Aut}^{+}_{k}(\mathbb{K})
\end{tikzcd}
$$

Der-Aut correspondence along middle and left logarithms

What of the left part of the decomposition of v-Aut$(\mathbb{K})$? Under what conditions on $\mathbb{K}$ is there a corresponding Lie algebra $\mathbf{D}$ of strongly linear derivations such that $\mathrm{Der}^{+}_{\precsim}(\mathbb{K})$ is a Lie ideal of $\mathbf{D}+\mathrm{Der}^{+}_{\precsim}(\mathbb{K})$ and that there exists a bijective correspondence between $\mathbf{D}$ and o-Aut(G)?

In some cases, automorphisms of $(G, +, <)$ have been shown [15, Proposition 4.9] to induce strongly linear derivations on $\mathbb{K}$. Furthermore, the derivations can be chosen compatible with a specific prelogarithm on $\mathbb{K}$ [16, Section 3].

Future work. We plan to investigate this construction method in order to obtain further derivations coming from automorphisms on G.

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