# THE EXPONENTIAL TRAPEZOIDAL METHOD FOR SEMILINEAR INTEGRO-DIFFERENTIAL EQUATIONS 

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#### Abstract

The exponential trapezoidal rule is proposed and analyzed for the numerical integration of semilinear integro-differential equations. Although the method is implicit, the numerical solution is easily obtained by standard fixed-point iteration, making its implementation straightforward. Second-order convergence in time is shown in an abstract Hilbert space framework under reasonable assumptions on the problem. Numerical experiments illustrate the proven order of convergence.


## 1. Introduction

In this paper we consider the full discretization of an abstract semilinear integrodifferential equation of the form

$$
\begin{equation*}
u^{\prime}(t)+\int_{0}^{t} K(t-s) A u(s) d s=f(u), \quad t \in[0, T], \quad u(0)=u_{0} \tag{1.1}
\end{equation*}
$$

where $-A$ is an elliptic differential operator and $K$ is a real-valued positive definite kernel, i.e., for any $T>0$, the kernel $K$ belongs to $L^{1}(0, T)$ and satisfies

$$
\int_{0}^{T} \varphi(t) \int_{0}^{t} K(t-s) \varphi(s) d s d t \geqslant 0 \quad \text { for all } \quad \varphi \in C[0, T]
$$

Equations of the above type and their linear versions are often used to model viscoelastic phenomena and heat conduction in materials with memory. We refer to the monograph [21] and references therein.

There is an extensive literature on the theoretical and numerical analysis of integrodifferential equations [2, 5, ,13, 14, 15, 20]. The proposed schemes use finite differences or finite element approximations in space, combined with standard time discretization schemes such as the backward Euler method, the Crank-Nicolson scheme, and other implicit Runge-Kutta or linear multistep methods. The integral term is discretized either by a standard quadrature rule or, in particular for the Riesz kernel, by a convolution quadrature formula [3, 4, 12].

For certain classes of ordinary and partial differential equations, exponential integrators have recently proven to be very efficient. For a survey of these integrators, see [7, 8, 9, 10]. Exponential integrators directly discretize the variation-of-constants formula, which for problem (1.1) has the form

$$
\begin{equation*}
u(t)=S(t) u_{0}+\int_{0}^{t} S(t-\sigma) f(u(\sigma)) d \sigma \tag{1.2}
\end{equation*}
$$

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where $S(t)$ is the solution operator of the linear problem with $f=0$. Exponential integrators can be used to solve this mild form of integro-differential equations. For example, the exponential Euler method applied to $\sqrt{1.2)}$ is given as

$$
\begin{equation*}
U_{m}=S\left(t_{m}\right) u_{0}+\sum_{j=0}^{m-1} \int_{t_{j}}^{t_{j+1}} S\left(t_{m}-\sigma\right) f\left(U_{j}\right) d \sigma, \quad 1 \leq m \leq M \tag{1.3}
\end{equation*}
$$

where $U_{m}$ is an approximation with step size $\tau$ to $u(t)$ at $t=t_{m}=m \tau$. Note that (1.3) is an explicit scheme that relies on computing the actions of certain operator functions. The method is efficient if the latter can be done efficiently.

In our previous work [16], we proposed explicit exponential Runge-Kutta methods for the time discretization of integro-differential equations. In the linear case, where $f$ is considered only as a function of time, we derived the order conditions for the general order $p$. The resulting exponential quadrature methods were shown to be also convergent of order $p$. In the semilinear case, however, we considered only orders 1 and 2. While the first-order exponential Euler method was simple, the order conditions for the second-order schemes already became involved due to the additional stage required. In this paper, we will consider the exponential trapezoidal integrator as an alternative second-order method for semilinear problems. The method does not require any stages and is easy to implement. Note that, unlike the methods in [16], it is implicit. However, since the stiffness of the operator $-A$ is no longer present in the variation-of-constants formula (1.2), the resulting system of nonlinear equations can be easily solved by standard methods without any time step restriction due to the stiffness induced by the operator $-A$ in (1.1).

The remainder of this paper is organized as follows. In section 2, we present the abstract framework and some preliminaries. In section 3, we introduce a second-order exponential time integrator for semilinear integro-differential equations along with a spectral Galerkin method for spatial discretization. The error analysis of the proposed integrator is presented in Theorem 3.2, which is the main result of the paper. Finally in section 4, we carry out some numerical experiments and illustrate the theoretical results obtained in the previous sections.

## 2. SEtting and preliminaries

Let $H$ be a real, separable Hilbert space with inner product $(\cdot, \cdot)_{H}$ and norm $\|v\|_{H}=\sqrt{(v, v)_{H}}$. The standard example is $H=L^{2}(\mathcal{D})$ for a bounded domain $\mathcal{D} \subset \mathbb{R}^{d}$, where

$$
(v, w)_{H}=\int_{\mathcal{D}} v(x) w(x) d x, \quad\|v\|_{H}=\left(\int_{\mathcal{D}} v(x)^{2} d x\right)^{\frac{1}{2}}, \quad v, w \in H
$$

Furthermore, we denote by $L(H)$ the space of all bounded linear operators on $H$ with the usual operator norm $\|\cdot\|_{L(H)}$. The following assumption will be used.
Assumption 2.1. Let $A$ be a densely defined, linear, self-adjoint, positive definite operator on $H$ with compact inverse, and let the kernel $K$ be positive definite.

A sufficient condition for $K$ to hold is $k$-monotonicity for some $k \geq 2$. For more details, we refer the reader to [21], Def. 3.4 and Prop. 3.3. Our prototypical example
will be the Riesz kernel

$$
\begin{equation*}
K(t)=\frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t>0, \quad 0<\beta<1 . \tag{2.1}
\end{equation*}
$$

However, the framework also includes kernels with less regularity. As an example, we mention a kernel with finite memory, as described in [1, p. 539].

An important example of $A$ is the negative Laplacian $A=-\Delta$ on a bounded domain $\mathcal{D} \subset \mathbb{R}^{d}$, subject to homogeneous Dirichlet boundary conditions. It is well known that the above assumptions on $A$ imply the existence of a sequence of nondecreasing positive real numbers $\left\{\lambda_{j}\right\}_{j=1}^{\infty}$ and an orthonormal basis $\left\{e_{j}\right\}_{j=1}^{\infty}$ of $H$ such that

$$
\begin{equation*}
A e_{j}=\lambda_{j} e_{j}, \quad \lim _{j \rightarrow \infty} \lambda_{j}=\infty \tag{2.2}
\end{equation*}
$$

For $\nu \in \mathbb{R}$, we consider the domain of $A^{\nu}$, which is a Hilbert space

$$
V=\mathcal{D}\left(A^{\nu}\right) \quad \text { with norm } \quad\|v\|_{V}=\left\|A^{\nu} v\right\|_{H}
$$

Our main assumptions on the nonlinearity $f$ are those of [6, 17].
Assumption 2.2. For some $0 \leq \nu<1$ and $V=\mathcal{D}\left(A^{\nu}\right)$, the nonlinearity $f: V \rightarrow H$ is locally Lipschitz continuous in a neighborhood of the exact solution, i.e. there exist constants $R>0$ and $L=L(R)$ such that

$$
\begin{equation*}
\|f(v)-f(w)\|_{H} \leqslant L\|v-w\|_{V} \tag{2.3}
\end{equation*}
$$

for all $0 \leq t \leq T$ and all $v, w \in V$ satisfying $\|v-u(t)\|_{V},\|w-u(t)\|_{V} \leqslant R$.
2.1. Solution operator. A family $\{S(t)\}_{t \geq 0}$ of bounded linear operators on $H$ is called a resolvent family for (1.1) whenever the solution operator $S(t)$ is strongly continuous on $\mathbb{R}_{+}$and the resolvent equation holds

$$
S(t) u_{0}+\int_{0}^{t} \int_{s}^{t} K(\xi-s) d \xi A S(s) u_{0} d s=u_{0}, \quad \text { for all } u_{0} \in H, t \geqslant 0
$$

If $t \rightarrow u(t)=S(t) u_{0}$ is differentiable for $t>0$, then $u$ is the unique solution of

$$
u^{\prime}(t)+\int_{0}^{t} K(t-s) A u(s) d s=0, \quad t>0, \quad u(0)=u_{0}
$$

We refer to the monograph [21] for a comprehensive theory of resolvent families for Volterra equations. Note that the operator family $\{S(t)\}_{t \geq 0}$ does not possess the semigroup property because of the presence of the memory term in (1.1). Nevertheless, it can be written explicitly using the spectral decomposition $(2.2)$ of $A$ as

$$
\begin{equation*}
S(t) v=\sum_{k=1}^{\infty} s_{k}(t)\left(v, e_{k}\right)_{H} e_{k}, \tag{2.4}
\end{equation*}
$$

where the functions $s_{k}(t), k=1,2, \ldots$ are the solutions of the scalar problems

$$
\begin{equation*}
s_{k}^{\prime}(t)+\lambda_{k} \int_{0}^{t} K(t-\sigma) s_{k}(\sigma) d \sigma=0, \quad s_{k}(0)=1, \quad t>0 . \tag{2.5}
\end{equation*}
$$

Our convergence analysis below will make use of certain smoothing properties of the solution operators.

Assumption 2.3. There exist constants $C$ and $1 \leq \rho \leq 2$ such that for any $0 \leqslant \alpha \leqslant \frac{1}{\rho}$ the solution operator satisfies the bound

$$
\begin{equation*}
\left\|A^{\alpha} S(t)\right\|_{L(H)} \leq C t^{-\alpha \rho}, \quad t>0 \tag{2.6}
\end{equation*}
$$

For the Riesz kernel (2.1), the smoothing property with $\rho=\beta+1$ is verified in [14, Thm 5.5]. For 3-monotone kernels, estimate (2.6) is verified in [11, Prop. 2.2] for

$$
\rho=1+\frac{2}{\pi} \sup \{|\arg \widehat{K}(z)| ; \operatorname{Re} z>0\} \in(1,2)
$$

where $\widehat{K}$ denotes the Laplace transform of $K$; see also [1, Lem. A.4] and [21, Prop. 3.10].

## 3. Numerical scheme and main result

3.1. The numerical method. We are now in a position to construct a fully discrete scheme for the numerical solution of problem (1.1). For the spatial discretization we will use a spectral Galerkin method and for the temporal discretization the exponential trapezoidal rule.

For $M \in \mathbb{N}$ we consider the uniform mesh $0=t_{0}<t_{1}<\cdots<t_{M}=T$ on the time interval $[0, T]$ with time step $h=t_{m+1}-t_{m}, m=0,1, \ldots, M-1$. Then, by using the variation-of-constants formula, we consider the mild formulation of (1.1), viz.

$$
\begin{equation*}
u\left(t_{m}\right)=S\left(t_{m}\right) u_{0}+\int_{0}^{t_{m}} S\left(t_{m}-\sigma\right) f(u(\sigma)) d \sigma \tag{3.1}
\end{equation*}
$$

Obviously, this can also be written as

$$
\begin{equation*}
u\left(t_{m}\right)=S\left(t_{m}\right) u_{0}+\sum_{j=0}^{m-1} \int_{t_{j}}^{t_{j+1}} S\left(t_{m}-\sigma\right) f(u(\sigma)) d \sigma \tag{3.2}
\end{equation*}
$$

Here, the operator $S(t)$ denotes the solution operator of the linear problem (i.e. for the case $f=0$ ). We recall that the operator $S(t)$ does not enjoy the semigroup property due to the non-locality of the kernel in (1.1).

For the time discretization of (3.2), we employ the exponential trapezoidal rule, i.e.

$$
\begin{equation*}
U_{m}=S\left(t_{m}\right) U_{0}+\frac{1}{2} \sum_{j=0}^{m-1} \int_{t_{j}}^{t_{j+1}} S\left(t_{m}-\sigma\right) d \sigma\left\{f\left(U_{j}\right)+f\left(U_{j+1}\right)\right\} \tag{3.3}
\end{equation*}
$$

where $U_{m}(1 \leq m \leq M)$ denotes the numerical approximation to the exact solution $u(t)$ at time $t=t_{m}$; for notational convenience we set $U_{0}=u_{0}$.

For the spatial discretization, we choose $N \in \mathbb{N}$ and consider the finite dimensional subspace $H_{N} \subseteq H$, given by $H_{N} \equiv \operatorname{span}\left\{e_{1}, e_{2}, \cdots, e_{N}\right\}$, where $\left\{e_{k}\right\}_{k=1}^{\infty}$ are the eigenfunctions of $A$, i.e., $A e_{k}=\lambda_{k} e_{k}, k \in \mathbb{N}$. Further we use the projectors $\mathcal{P}_{N}: H \rightarrow H_{N}$ given by

$$
\mathcal{P}_{N} v=\sum_{k=1}^{N}\left(v, e_{k}\right) e_{k}
$$

for $v \in H$ and the projected operator $A_{N}: H_{N} \rightarrow H_{N}, A_{N}=A \mathcal{P}_{N}$ which generates a family of resolvent operators $\left\{S_{N}(t)\right\}_{t \geqslant 0}$ in $H_{N}$. It is clear that

$$
\begin{equation*}
S_{N}(t) \mathcal{P}_{N}=S(t) \mathcal{P}_{N} \tag{3.4}
\end{equation*}
$$

and also

$$
\begin{equation*}
\left\|A^{-\nu}\left(I-\mathcal{P}_{N}\right)\right\|_{L(H)}=\sup _{k \geqslant N+1} \lambda_{k}^{-\nu}=\lambda_{N+1}^{-\nu}, \quad \nu \geqslant 0 . \tag{3.5}
\end{equation*}
$$

A representation of $S_{N}$ is given by

$$
\begin{equation*}
S_{N}(t) v=\sum_{k=1}^{N} s_{k}(t)\left(v, e_{k}\right) e_{k} \tag{3.6}
\end{equation*}
$$

This motivates us to consider the following fully discrete scheme

$$
\begin{equation*}
U_{m}^{N}=S_{N}\left(t_{m}\right) \mathcal{P}_{N} u_{0}+\frac{1}{2} \sum_{j=0}^{m-1} \int_{t_{j}}^{t_{j+1}} S_{N}\left(t_{m}-\sigma\right) d \sigma \mathcal{P}_{N}\left\{f\left(U_{j}^{N}\right)+f\left(U_{j+1}^{N}\right)\right\}, \tag{3.7}
\end{equation*}
$$

which we propose for the numerical solution of (3.1).
In order to get a solution in $V$, we assume that the initial data satisfies $u_{0} \in V$. More regularity, however, improves the spatial convergence result. To elaborate this, we make the following regularity assumption.

Assumption 3.1. Let $g:[0, T] \rightarrow H: t \mapsto g(t)=f(u(t))$ be twice differentiable, let $\nu$ be given by Assumption 2.2 and assume that the following conditions hold:
(a) $\nu \rho<1$ for $\rho$ given by Assumption 2.3;
(b) $u_{0}=u(0) \in \mathcal{D}\left(A^{\nu+\beta}\right)$ for some $\beta \geq 0$;
(c) $A^{\gamma} g \in L^{\infty}(0, T ; H)$ for some $\gamma \geq 0$;
(d) $A^{\eta} g^{\prime} \in L^{\infty}(0, T ; H)$ for some $0 \leq \eta \leq \nu$;
(e) $A^{-\delta} g^{\prime \prime} \in L^{\infty}(0, T ; H)$ for some $0 \leq \delta \leq \frac{1}{\rho}-\nu$.

Note that the properties (b)-(e) can also be seen as the definition of the four nonnegative parameters $\beta, \gamma, \delta$, and $\eta$.

Under this assumption, we have the following convergence result.
Theorem 3.2. For the solution of (1.1) in the mild form (3.1), consider the exponential integrator (3.7). If the Assumptions 2.1, 2.2, 2.3, and 3.1 hold, then there exist constants $h_{0}>0$ and $C>0$ such that for all step sizes $0<h \leq h_{0}$ and all $N \in \mathbb{N}$, the global error satisfies for $0<t_{m}=m h \leq T$ and $0 \leq \alpha<\frac{1}{\rho}$ the bound

$$
\begin{aligned}
\left\|u\left(t_{m}\right)-U_{m}^{N}\right\|_{V} \leqslant C( & t_{m}^{-\alpha \rho} \lambda_{N+1}^{-\alpha-\beta}+h t_{m}^{-\nu \rho} \lambda_{N+1}^{-\beta}+\lambda_{N+1}^{\nu-\alpha-\gamma} \\
& \left.+h^{2-(\nu-\eta) \rho} \sup _{0 \leqslant t \leqslant T}\left\|A^{\eta} g^{\prime}(t)\right\|_{H}+h^{2} \sup _{0 \leqslant t \leqslant T}\left\|A^{-\delta} g^{\prime \prime}(t)\right\|_{H}\right),
\end{aligned}
$$

where the constant $C$ depends on $T$, but it is independent of $N, m$, and $h$.
In particular, if $g^{\prime}$ is uniformly bounded in $V$, we can choose $\eta=\nu$ and the scheme turns out to be second-order convergent in time.

## 4. Proof of Theorem 3.2

First recall that $g(t)=f(u(t))$ and that the operators $A, S$, and $P_{N}$ commute. By subtracting the numerical solution (3.7) from the exact solution (3.2) we have

$$
\begin{aligned}
u\left(t_{m}\right)-U_{m}^{N}=S\left(t_{m}\right) u_{0}-S_{N}\left(t_{m}\right) \mathcal{P}_{N} u_{0} & +\sum_{j=0}^{m-1} \int_{t_{j}}^{t_{j+1}}\left\{S\left(t_{m}-\sigma\right) f(u(\sigma))\right. \\
& \left.-S_{N}\left(t_{m}-\sigma\right) \mathcal{P}_{N}\left(\frac{1}{2}\left(f\left(U_{j}^{N}\right)+f\left(U_{j+1}^{N}\right)\right)\right)\right\} d \sigma
\end{aligned}
$$

which, by (3.4), can be written as

$$
\begin{aligned}
u\left(t_{m}\right)-U_{m}^{N}= & S\left(t_{m}\right)\left(I-\mathcal{P}_{N}\right) u_{0} \\
& +\int_{0}^{t_{m}} S\left(t_{m}-\sigma\right)\left(g(\sigma)-\mathcal{P}_{N} g(\sigma)\right) d \sigma \\
& +\sum_{j=0}^{m-1} \int_{t_{j}}^{t_{j+1}} S\left(t_{m}-\sigma\right) \mathcal{P}_{N}\left\{g(\sigma)-\frac{1}{2}\left(f\left(U_{j}^{N}\right)+f\left(U_{j+1}^{N}\right)\right)\right\} d \sigma .
\end{aligned}
$$

Now taking the norm in $V$, we obtain

$$
\begin{aligned}
\left\|u\left(t_{m}\right)-U_{m}^{N}\right\|_{V} \leqslant & \left\|S\left(t_{m}\right)\left(I-\mathcal{P}_{N}\right) u_{0}\right\|_{V} \\
& +\int_{0}^{t_{m}}\left\|S\left(t_{m}-\sigma\right)\left(g(\sigma)-\mathcal{P}_{N} g(\sigma)\right)\right\|_{V} d \sigma \\
& +\sum_{j=0}^{m-1} \int_{t_{j}}^{t_{j+1}}\left\|S\left(t_{m}-\sigma\right) \mathcal{P}_{N}\left\{g(\sigma)-\frac{1}{2}\left(f\left(U_{j}^{N}\right)+f\left(U_{j+1}^{N}\right)\right)\right\}\right\|_{V} d \sigma \\
= & I_{1}+I_{2}+I_{3}
\end{aligned}
$$

where $I_{1}, I_{2}$ and $I_{3}$ correspond to the spatial and temporal discretization errors respectively.

First, using (2.6) and the fact (3.5) enable us to bound $I_{1}$ as follows

$$
\begin{aligned}
I_{1} & =\left\|A^{\alpha} S\left(t_{m}\right) A^{-\alpha-\beta}\left(I-\mathcal{P}_{N}\right) A^{\beta} u_{0}\right\|_{V} \\
& \left.\leqslant\left\|A^{\alpha} S\left(t_{m}\right)\right\|_{L(H)}\right)\left\|A^{-\alpha-\beta}\left(I-\mathcal{P}_{N}\right) A^{\nu+\beta} u_{0}\right\|_{H} \\
& \leqslant C t_{m}^{-\alpha \rho} \lambda_{N+1}^{-\alpha-\beta}\left\|A^{\nu+\beta} u_{0}\right\|_{H} \\
& \leqslant C t_{m}^{-\alpha \rho} \lambda_{N+1}^{-\alpha-\beta} .
\end{aligned}
$$

To estimate $I_{2}$, we again employ (2.6) and (3.5) to obtain

$$
\begin{aligned}
I_{2} & =\int_{0}^{t_{m}}\left\|A^{\alpha} S\left(t_{m}-\sigma\right) A^{\nu-\alpha-\gamma}\left(I-\mathcal{P}_{N}\right) A^{\gamma} g(\sigma)\right\|_{H} d \sigma \\
& \leqslant C \int_{0}^{t_{m}}\left(t_{m}-\sigma\right)^{-\alpha \rho}\left\|A^{\nu-\alpha-\gamma}\left(I-\mathcal{P}_{N}\right)\right\|_{L(H)}\left\|A^{\gamma} g(\sigma)\right\|_{H} d \sigma \\
& \leqslant C \lambda_{N+1}^{\nu-\alpha-\gamma} .
\end{aligned}
$$

It remains to estimate the term

$$
\begin{equation*}
I_{3}=\sum_{j=0}^{m-1} \int_{t_{j}}^{t_{j+1}}\left\|S\left(t_{m}-\sigma\right) \mathcal{P}_{N}\left\{g(\sigma)-\frac{1}{2}\left(f\left(U_{j}^{N}\right)+f\left(U_{j+1}^{N}\right)\right)\right\}\right\|_{V} d \sigma . \tag{4.1}
\end{equation*}
$$

We put

$$
g(\sigma)=\frac{1}{2}\left(g\left(t_{j}\right)+g\left(t_{j+1}\right)\right)+g(\sigma)-\frac{1}{2}\left(g\left(t_{j}\right)+g\left(t_{j+1}\right)\right)
$$

on the right-hand side of (4.1) to get

$$
\begin{aligned}
I_{3} \leqslant & \frac{1}{2} \sum_{j=0}^{m-1} \int_{t_{j}}^{t_{j+1}}\left\|S\left(t_{m}-\sigma\right) \mathcal{P}_{N}\left(g\left(t_{j}\right)-f\left(U_{j}^{N}\right)\right)\right\|_{V} d \sigma \\
& +\frac{1}{2} \sum_{j=0}^{m-1} \int_{t_{j}}^{t_{j+1}}\left\|S\left(t_{m}-\sigma\right) \mathcal{P}_{N}\left(g\left(t_{j+1}\right)-f\left(U_{j+1}^{N}\right)\right)\right\|_{V} d \sigma \\
& +\int_{t_{m-1}}^{t_{m}}\left\|S\left(t_{m}-\sigma\right) \mathcal{P}_{N}\left[g(\sigma)-\frac{1}{2}\left(g\left(t_{m-1}\right)+g\left(t_{m}\right)\right)\right]\right\|_{V} d \sigma \\
& +\sum_{j=0}^{m-2} \int_{t_{j}}^{t_{j+1}}\left\|S\left(t_{m}-\sigma\right) \mathcal{P}_{N}\left[g(\sigma)-\frac{1}{2}\left(g\left(t_{j}\right)+g\left(t_{j+1}\right)\right)\right]\right\|_{V} d \sigma \\
= & I_{3,1}+I_{3,2}+I_{3,3}+I_{3,4} .
\end{aligned}
$$

Next we handle these four terms separately. We first note that

$$
\int_{t_{j}}^{t_{j+1}}\left(t_{m}-\sigma\right)^{-\nu \rho} d \sigma= \begin{cases}C h\left(t_{m}-t_{j}\right)^{-\nu \rho}, & j \leq m-1 \\ h\left(t_{m}-t_{j+1}\right)^{-\nu \rho}, & j<m-1 .\end{cases}
$$

Using the local Lipschitz continuity of $f$ and (2.6), we infer that

$$
\begin{aligned}
I_{3,1} & =\frac{1}{2} \sum_{j=0}^{m-1} \int_{t_{j}}^{t_{j+1}}\left\|A^{\nu} S\left(t_{m}-\sigma\right) \mathcal{P}_{N}\left(f\left(u\left(t_{j}\right)\right)-f\left(U_{j}^{N}\right)\right)\right\|_{H} d \sigma \\
& \leqslant C \sum_{j=0}^{m-1} \int_{t_{j}}^{t_{j+1}}\left\|A^{\nu} S\left(t_{m}-\sigma\right)\right\|_{L(H)}\left\|u\left(t_{j}\right)-U_{j}^{N}\right\|_{V} d \sigma \\
& \leqslant C h \sum_{j=1}^{m-1}\left(t_{m}-t_{j}\right)^{-\nu \rho}\left\|u\left(t_{j}\right)-U_{j}^{N}\right\|_{V}+C h t_{m}^{-\nu \rho} \lambda_{N+1}^{-\beta} .
\end{aligned}
$$

In the same way, we get

$$
\left.\left.I_{3,2} \leqslant C h \sum_{j=0}^{m-2}\left(t_{m}-t_{j+1}\right)^{-\nu \rho} \| u\left(t_{j+1}\right)\right)-U_{j+1}^{N}\left\|_{V}+C h\left(t_{m}-t_{m-1}\right)^{-\nu \rho}\right\| u\left(t_{m}\right)\right)-U_{m}^{N} \|_{V}
$$

In order to bound the term

$$
I_{3,3}=\int_{t_{m-1}}^{t_{m}}\left\|S\left(t_{m}-\sigma\right) \mathcal{P}_{N}\left[g(\sigma)-\frac{1}{2}\left(g\left(t_{m-1}\right)+g\left(t_{m}\right)\right)\right]\right\|_{V} d \sigma,
$$

we need to expand $g$ in a Taylor series with integral remainder as follows

$$
\begin{aligned}
g(\sigma)= & \frac{1}{2}\left(g\left(t_{m-1}\right)+g\left(t_{m}\right)\right) \\
& +\frac{1}{2}\left(g^{\prime}\left(t_{m-1}\right)\left(\sigma-t_{m-1}\right)+g^{\prime}\left(t_{m}\right)\left(\sigma-t_{m}\right)\right) \\
& +\frac{1}{2} \int_{t_{m-1}}^{\sigma}\left(\sigma-\xi_{1}\right) g^{\prime \prime}\left(\xi_{1}\right) d \xi_{1}-\frac{1}{2} \int_{\sigma}^{t_{m}}\left(\sigma-\xi_{2}\right) g^{\prime \prime}\left(\xi_{2}\right) d \xi_{2} .
\end{aligned}
$$

Using this in $I_{3,3}$ we arrive at

$$
\begin{aligned}
I_{3,3} \leqslant & \frac{1}{2} \int_{t_{m-1}}^{t_{m}}\left(\sigma-t_{m-1}\right)\left\|A^{\nu-\eta} S\left(t_{m}-\sigma\right) \mathcal{P}_{N} A^{\eta} g^{\prime}\left(t_{m-1}\right)\right\|_{H} d \sigma \\
& +\frac{1}{2} \int_{t_{m-1}}^{t_{m}}\left(t_{m}-\sigma\right)\left\|A^{\nu-\eta} S\left(t_{m}-\sigma\right) \mathcal{P}_{N} A^{\eta} g^{\prime}\left(t_{m}\right)\right\|_{H} d \sigma \\
& +\frac{1}{2} \int_{t_{m-1}}^{t_{m}}\left\|A^{\nu+\delta} S\left(t_{m}-\sigma\right) \mathcal{P}_{N} \int_{t_{m-1}}^{\sigma}\left(\sigma-\xi_{1}\right) A^{-\delta} g^{\prime \prime}\left(\xi_{1}\right) d \xi_{1} d \sigma\right\|_{H} \\
& +\frac{1}{2} \int_{t_{m-1}}^{t_{m}}\left\|A^{\nu+\delta} S\left(t_{m}-\sigma\right) \mathcal{P}_{N} \int_{\sigma}^{t_{m}}\left(\sigma-\xi_{2}\right) A^{-\delta} g^{\prime \prime}\left(\xi_{2}\right) d \xi_{2} d \sigma\right\|_{H}
\end{aligned}
$$

which then yields

$$
I_{3,3} \leq C h^{2-(\nu-\eta) \rho} \sup _{0 \leqslant t \leqslant T}\left\|A^{\eta} g^{\prime}(t)\right\|_{H}+C h^{2} \sup _{0 \leqslant t \leqslant T}\left\|A^{-\delta} g^{\prime \prime}(t)\right\|_{H}
$$

Finally, for estimating

$$
\begin{aligned}
I_{3,4} & =\sum_{j=0}^{m-2} \int_{t_{j}}^{t_{j+1}}\left\|A^{\nu} S\left(t_{m}-\sigma\right) \mathcal{P}_{N}\left[g(\sigma)-\frac{1}{2}\left(g\left(t_{j}\right)+g\left(t_{j+1}\right)\right)\right]\right\|_{H} d \sigma \\
& \leq \sum_{j=0}^{m-2}\left(t_{m}-t_{j+1}\right)^{-(\nu+\delta) \rho}\left\|\int_{t_{j}}^{t_{j+1}} A^{-\delta}\left[g(\sigma)-\frac{1}{2}\left(g\left(t_{j}\right)+g\left(t_{j+1}\right)\right)\right]\right\|_{H} d \sigma
\end{aligned}
$$

we use the following formula (Peano kernel of the trapezoidal rule):

$$
\begin{aligned}
\int_{t_{j}}^{t_{j+1}}[g(\sigma) & \left.-\frac{1}{2}\left(g\left(t_{j}\right)+g\left(t_{j+1}\right)\right)\right] d \sigma \\
& =\int_{t_{j}}^{t_{j+1}} g(\sigma) d \sigma-\frac{h}{2}\left(g\left(t_{j}\right)+g\left(t_{j+1}\right)\right) \\
& =\frac{1}{2} \int_{t_{j}}^{t_{j+1}}\left(\xi-t_{j}\right)\left(\xi-t_{j+1}\right) g^{\prime \prime}(\xi) d \xi
\end{aligned}
$$

This immediately leads us to

$$
\begin{aligned}
I_{3,4} & \leq \frac{1}{2} \sum_{j=0}^{m-2}\left(t_{m}-t_{j+1}\right)^{-(\nu+\delta) \rho} \int_{t_{j}}^{t_{j+1}}\left(\xi-t_{j}\right)\left(t_{j+1}-\xi\right)\left\|A^{-\delta} g^{\prime \prime}(\xi)\right\|_{H} d \xi \\
& \leqslant C h^{2} \sup _{0 \leqslant t \leqslant T}\left\|A^{-\delta} g^{\prime \prime}(t)\right\|_{H} h \sum_{j=0}^{m-2}\left(t_{m}-t_{j+1}\right)^{-(\nu+\delta) \rho}
\end{aligned}
$$

$$
\leqslant C h^{2} \sup _{0 \leqslant t \leqslant T}\left\|A^{-\delta} g^{\prime \prime}(t)\right\|_{H}
$$

Putting the above estimates together implies (for $h$ sufficiently small)

$$
\begin{aligned}
\left\|u\left(t_{m}\right)-U_{m}^{N}\right\|_{V} & \leq C\left(h \sum_{j=1}^{m-1}\left(t_{m}-t_{j}\right)^{-\nu \rho}\left\|u\left(t_{j}\right)-U_{j}^{N}\right\|_{V}+t_{m}^{-\alpha \rho} \lambda_{N+1}^{-\alpha-\beta}+\lambda_{N+1}^{\nu-\alpha-\gamma}\right. \\
& \left.+h t_{m}^{-\nu \rho} \lambda_{N+1}^{-\beta}+h^{2-(\nu-\eta) \rho} \sup _{0 \leqslant t \leqslant T}\left\|A^{\eta} g^{\prime}(t)\right\|_{H}+h^{2} \sup _{0 \leqslant t \leqslant T}\left\|A^{-\delta} g^{\prime \prime}(t)\right\|_{H}\right) .
\end{aligned}
$$

Applying a discrete Gronwall lemma finally shows the desired bound.

## 5. Implementation and numerical experiments

In this section we present some numerical experiments to illustrate the error bounds obtained in Theorem 3.2. We carry out the experiments in one space dimension, choosing $\mathcal{D}=(0,1)$ and $A=-\frac{d^{2}}{d x^{2}}$, subject to homogeneous Dirichlet boundary conditions. Thus, the eigenvalues and (normalised) eigenfunctions of $A$ are

$$
\lambda_{k}=k^{2} \pi^{2} \quad \text { and } \quad e_{k}(x)=\sqrt{2} \sin (k \pi x), \quad k \geqslant 1
$$

5.1. Explicit representation of the solution. We recall form section 2.1 that

$$
\left(S_{N}(t)(v)\right)(x)=\sum_{k=1}^{N} 2 s_{k}(t) \sin (k \pi x) \int_{0}^{1} \sin (k \pi \xi) v(\xi) d \xi
$$

for $x \in(0,1)$. The functions $s_{k}$ are the solutions of (2.5). We consider the following two problems.

Problem 5.1. Consider the Riesz kernel $K(t)=t^{\rho-2} / \Gamma(\rho-1)$ with $1<\rho<2$. Using the Laplace transform in (2.5), we get

$$
s_{k}(t)=E_{\rho}\left(-\lambda_{k} t^{\rho}\right),
$$

where $E_{a}(z)$ is the one-parameter Mittag-Leffler function, defined as $E_{a}(z)=E_{a, 1}(z)$, where

$$
E_{a, b}(z)=\sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(a k+b)}, \quad z \in \mathbb{C}, \quad a, b>0 .
$$

The numerical approximation $U_{m}^{N}$ at time $t_{m}=m \tau$ can be written as

$$
\begin{aligned}
U_{m}^{N}= & \sum_{k=1}^{N}\left(E_{\rho}\left(-\lambda_{k} t^{\rho}\right)\left(u_{0}, e_{k}\right)_{H} e_{k}\right. \\
& \left.+\frac{1}{2} \sum_{j=0}^{m-1} \int_{t_{j}}^{t_{j+1}} E_{\rho}\left(-\lambda_{k}\left(t_{m}-\sigma\right)^{\rho}\right) d \sigma\left\{\left(f\left(U_{j}^{N}\right), e_{k}\right)_{H}+\left(f\left(U_{j+1}^{N}\right), e_{k}\right)_{H}\right\} e_{k}\right) .
\end{aligned}
$$

Fixed-point iteration is used to solve this nonlinear problem. Note that the integrals of the Mittag-Leffler function can be computed by a simple quadrature such as the trapezoidal rule. However, the integrals can also be computed exactly as

$$
\int_{t_{j}}^{t_{j+1}} E_{\rho}\left(-\lambda_{k}\left(t_{m}-\sigma\right)^{\rho}\right) d \sigma=\int_{t_{m-j-1}}^{t_{m-j}} E_{\rho}\left(-\lambda_{k} \sigma^{\rho}\right) d \sigma
$$

$$
=t_{m-j} E_{\rho, 2}\left(-\lambda_{k} t_{m-j}^{\rho}\right)-t_{m-j-1} E_{\rho, 2}\left(-\lambda_{k} t_{m-j-1}^{\rho}\right)
$$

which is proved in [19, Eq. (1.100)]. For evaluating the Mittag-Leffler function we use the routine from [18.

Problem 5.2. Let $K$ be the smooth kernel

$$
K(t)=e^{-a t} \quad \text { with } \quad 0<a \leqslant 2 \text { for } t \geqslant 0 .
$$

Since $K^{\prime}=-a K$, it is easy to see that the ordinary integro-differential equation (2.5) is equivalent to

$$
s_{k}^{\prime \prime}+a s_{k}^{\prime}+\lambda_{k} s_{k}=0, \quad s_{k}(0)=1, \quad s_{k}^{\prime}(0)=0 .
$$

It has the solution

$$
s_{k}(t)=e^{-\frac{a}{2} t}\left\{\cos \sqrt{\frac{4 \lambda_{k}-a^{2}}{4}} t+\frac{a}{\sqrt{4 \lambda_{k}-a^{2}}} \sin \sqrt{\frac{4 \lambda_{k}-a^{2}}{4}} t\right\} .
$$

The integrals of $s_{k}$ can be computed exactly.
5.2. Numerical experiments. In all experiments, we chose the nonlinearity $f(u)=$ $\sin u$, the initial data $u_{0}=4 x(1-x), x \in[0,1]$ and $N=100$ frequencies. The problems were integrated with various time step sizes $h$ up to time $T=M h=1$ and the errors were calculated in a discrete $L^{2}$-norm using the difference between the numerical solution $U_{M}^{N}$ and a reference solution $U_{\text {ref }}^{N}$ at time $T=1$ :

$$
\text { error }=\left(\Delta x \sum_{j=1}^{N}\left(U_{M}^{N}\left(x_{j}\right)-U_{\text {ref }}^{N}\left(x_{j}\right)\right)^{2}\right)^{1 / 2}, \quad x_{j}=j \Delta x, \quad \Delta x=\frac{1}{N+1} .
$$

The reference solution was computed with the second-order explicit exponential integrator from [16] using sufficiently small time steps.

In the experiments, we considered two different values of $\rho$ for the Riesz kernel and the value $a=2$ for the exponential kernel. Figure 1 presents a double logarithmic plot of the errors as a function of the time step. The figure confirms the proven theoretical results.


Figure 1. Temporal rate of convergence of the exponential trapezoidal method for three different problems (see text).

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