arXiv:2403.05998v1 [math.DS] 9 Mar 2024

GENERALIZED GOTTSCHALK'S CONJECTURE FOR SOFIC GROUPS AND APPLICATIONS

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Abstract. We establish generalizations of the well-known surjunctivity theorem of Gromov and Weiss as well as the dual-surjunctivity theorem of Capobianco, Kari and Taati for cellular automata (CA) to local perturbations of CA over sofic group universes. We also extend the results to a class of non-uniform cellular automata (NUCA) consisting of global perturbations with uniformly bounded singularity of CA. As an application, we obtain the surjunctivity of algebraic NUCA with uniformly bounded singularity over sofic groups. Moreover, we prove the stable finiteness of twisted group rings over sofic groups to generalize known results on Kaplansky's stable finiteness conjecture for group rings.

1. Introduction

In computer science, CA is an interesting and powerful model of computation and of physical and biological simulation. It is well-known that the CA called Game of Life of Conway [22] is Turing complete. Notable recent mathematical theory of CA achieves a dynamical characterization of amenable groups by the Garden of Eden theorem for CA (cf. [29], [30], [17], [2]) and establishes the equivalence between Kaplansky's stable finiteness conjecture for group rings and the Gottschalk's surjunctivity conjecture for linear CA (cf. [39], [43], [40], [16]). In this paper, we study the generalized Gottschalk's surjunctivity conjecture for NUCA over the wide class of sofic groups and obtain several geometric and algebraic applications.

1.1. **Preliminaries.** For the main results, we recall briefly notions of symbolic dynamics. Given a discrete set A and a group G, a *configuration* $x \in A^G$ is simply a map $x\colon G \to A$. Two configurations $x, y \in A^G$ are *asymptotic* if $x|_{G\setminus E} = y|_{G\setminus E}$ for some finite subset $E \subset G$. The *Bernoulli shift* action $G \times A^G \to A^G$ is defined by $(g,x) \mapsto gx$, where $(gx)(h) = x(g^{-1}h)$ for $g, h \in G$, $x \in A^G$. The *full shift* A^G is equipped with the prodiscrete topology. For each $x \in A^G$, we define $\Sigma(x) = \overline{\{gx \colon g \in G\}} \subset A^G$ as the smallest closed subshift containing x.

Date: March 12, 2024.

2020 *Mathematics Subject Classification.* 05C25, 14A10, 16S34, 20C07, 20F69, 37B10, 37B15, 37B51, 68Q80.

Key words and phrases. sofic group, amenable group, surjunctivity, Gottschalk's conjecture, cellular automata, group rings, post-surjectivity, pre-injectivity, invertibility.

Following von Neumann and Ulam [32], we define a CA over the group G (the *universe*) and the set A (the *alphabet*) as a G-equivariant and uniformly continuous self-map $A^G \circlearrowleft$ (cf. [10], [25]). One refers to each element $g \in G$ as a cell of the universe. Then every CA is uniform in the sense that all the cells follow the same local transition map. More generally, when different cells can evolve according to different local transition maps to break down the above uniformity, we obtain NUCA (cf. [38, Definition 1.1], [18], [19]):

Definition 1.1. Given a group G and an alphabet A, let $M \subset G$ and let $S = A^{A^M}$ be the set of all maps $A^M \to A$. For $s \in S^G$, we define the NUCA $\sigma_s \colon A^G \to A^G$ by $\sigma_s(x)(g) = s(g)((g^{-1}x)|_M)$ for all $x \in A^G$ and $g \in G$.

The set M is called a *memory* and $s \in S^G$ the *configuration of local transition maps or local defining maps* of σ_s. Every CA is thus a NUCA with finite memory and constant configuration of local defining maps. Conversely, we regard NUCA as *global perturbations* of CA. When $s, t \in S^G$ are asymptotic, σ_s is a *local perturbation* of σ_t and vice versa. We observe also that NUCA with finite memory are precisely uniformly continuous selfmaps $A^G \circlearrowleft$. As for CA, such NUCA satisfy the closed image property [38, Theorem 4.4], several decidable properties [18], [44] and variants of the Garden of Eden theorem [41].

We will analyze in this paper the relations between the following fundamental stable dynamic properties of σ_s. We say that σ_s is *pre-injective* if $\sigma_s(x) = \sigma_s(y)$ implies $x = y$ whenever $x, y \in A^G$ are asymptotic. Similarly, σ_s is *post-surjective* if for all $x, y \in A^G$ with y asymptotic to $\sigma_s(x)$, then $y = \sigma_s(z)$ for some $z \in A^G$ asymptotic to x. We say that σ_s is *stably injective*, resp. *stably post-surjective*, if σ_p is injective, resp. post-surjective, for every configuration $p \in \Sigma(s)$.

The NUCA σ_s is said to be *invertible* if it is bijective and the inverse map σ_s^{-1} is a NUCA with finite memory [38]. We define also *stable invertibility* which is in general stronger than invertibility, namely, σ_s is stably invertible if there exists $N \subset G$ finite and $t \in T^G$ where $T = A^{A^N}$ such that for every $p \in \Sigma(s)$, we have $\sigma_p \circ \sigma_q = \sigma_q \circ \sigma_p = \mathrm{Id}$ for some $q \in \Sigma(t)$. In fact, the next lemma shows that every invertible NUCA with finite memory over a finite alphabet and a countable group universe is automatically stably invertible.

Lemma 1.2. *Let G be a countable group and let $M \subset G$ be a finite subset. Let A be a finite alphabet and let $s \in S^G$ where $S = A^{A^M}$. Suppose that σ_s is invertible. Then σ_s is also stably invertible.*

Proof. Since σ_s is invertible, there exists $N \subset G$ finite and $t \in T^G$ where $T = A^{A^N}$ such that $\sigma_s \circ \sigma_t = \sigma_t \circ \sigma_s = \mathrm{Id}$. For every $p \in \Sigma(s)$, we obtain from the relation $\sigma_s \circ \sigma_t = \mathrm{Id}$ and [38, Theorem 11.1] some $q \in \Sigma(t)$ such that $\sigma_p \circ \sigma_q = \mathrm{Id}$. Since $q \in \Sigma(t)$ and $\sigma_t \circ \sigma_s = \mathrm{Id}$, another application of [38, Theorem 11.1] shows that there exists $r \in \Sigma(s)$ such that $\sigma_q \circ \sigma_r = \mathrm{Id}$. It follows that σ_q is invertible and thus $\sigma_p \circ \sigma_q = \sigma_q \circ \sigma_p = \mathrm{Id}$. We conclude that σ_s is stably invertible. $\square$

Note that for CA, stable invertibility, resp. stable injectivity, resp. stable post-surjectivity, is equivalent to invertibility, resp. injectivity, resp. post-surjectivity since $\Sigma(s) = \{s\}$ if $s \in S^G$ is constant.

1.2. Main dynamical results.

1.2.1. *Stable surjunctivity for sofic groups.* The surjunctivity conjecture of Gottschalk [24] asserts that over any group universe, every injective CA with finite alphabet must be surjective. Over sofic group universes (Section 2), the surjunctivity conjecture was famously shown by Gromov and Weiss in [23], [45]. The vast class of sofic groups was first introduced by Gromov [23] and includes all amenable groups and all residually finite groups. The question of whether there exists a non-sofic group is still open.

The situation for the surjunctivity of NUCA is more subtle. While stably injective NUCA with finite memory may fail to be surjective (see [38, Example 14.3]), results in [38] show that stably injective local perturbations of CA with finite memory over an amenable group or an residually finite group universe must be surjective. More generally, we establish the following generalization of the Gromov-Weiss surjunctivity theorem to cover all stably injective local perturbations of CA over sofic group universes.

Theorem A. *Let M be a finite subset of a finitely generated sofic group G. Let A be a finite alphabet and $S = A^{A^M}$. Suppose that σ_s is stably injective for some asymptotically constant $s \in S^G$. Then σ_s is stably invertible.*

Combining with the result in [38] where we can replace the stable injectivity by the weaker injectivity condition whenever G is an amenable group, we obtain the following general surjunctivity and invertibility result for NUCA which are local perturbations of classical CA.

Corollary 1.3. *Let M be a finite subset of a group G. Let A be a finite alphabet and $S = A^{A^M}$. Let $s \in S^G$ be an asymptotic constant configuration such that σ_s is injective. Then σ_s is stably invertible in the following cases:*

(i) *G is an amenable group;*
(ii) *G is a sofic group and σ_c is injective.*

To deal with global perturbations of CA, we introduce the following notion of uniformly bounded singularity.

Definition 1.4. A NUCA $\sigma_s \colon A^G \to A^G$ has *uniformly bounded singularity* if for every $E \subset G$ finite with $1_G \in E = E^{-1}$, we can find a finite subset $F \subset G$ containing E such that the restriction $s|_{FE \setminus F}$ is constant. In this case, we also say that the configuration s has *uniformly bounded singularity.*

It is clear that σ_s has uniformly bounded singularity if s is asymptotically constant. When the universe G is a residually finite group, our notion of uniformly bounded singularity is closely related but not equivalent to the notion of (periodically) bounded singularity for NUCA defined in [38, Definition 10.1]. The next result confirms that Theorem A can be generalized to NUCA with uniformly bounded singularity.

Theorem B. *Let M be a finite subset of a finitely generated sofic group G. Let A be a finite alphabet and let $S = A^{A^M}$. Suppose that $\sigma_s \colon A^G \to A^G$ is stably injective for some $s \in S^G$ with uniformly bounded singularity. Then σ_s is stably invertible.*

In particular, our result extends [38, Theorem C] to cover the class of NUCA with uniformly bounded singularity over sofic group universes. Note again that [38, Example 14.3] implies that we cannot remove the uniformly singularity condition in the above theorem.

1.2.2. *Stable dual-surjunctivity for sofic groups.* Our next results concern the dual-surjunctivity version of Gottschalk's conjecture was introduced recently by Capobianco, Kari, and Taati in [8] which states that every post-surjective CA over a group universe and a finite alphabet is also pre-injective. Moreover, the authors settled in the same paper [8] the dual-surjunctivity conjecture for CA over sofic universes. See also [37] for some extensions.

We establish the following extension of the dual-surjunctivity theorem of Capobianco, Kari, and Taati to the class of post-surjective local perturbations of CA over sofic groups.

Theorem C. *Let G be a finitely generated sofic group and let A be a finite alphabet. Let $M \subset G$ be finite and $S = A^{A^M}$. Let $s \in S^G$ be asymptotically constant such that σ_s is stably post-surjective. Then σ_s is stably invertible.*

Given a vector space V which is not necessarily finite, V^G is a vector space with component-wise operations and a NUCA $\tau \colon V^G \to V^G$ is said to be *linear* if it is also a linear map of vector spaces or equivalently, if its local transition maps are all linear. Linear NUCA with finite memory enjoy similar properties as linear CA such as the shadowing property [4], [6], [27], [5], [28], [7], [36], [34], [14], [42]. The recent result [42, Theorem D.(ii)] states that every stably post-surjective linear NUCA with finite vector space alphabet over a residually finite group universe is invertible whenever it is a local perturbation of a linear CA. When specialized to the class of linear NUCA, Theorem C thus generalizes [42, Theorem D.(ii)] because every residually finite group is sofic. As for Theorem B, we obtain the following extension of Theorem C to cover the class of NUCA with uniformly bounded singularity.

Theorem D. *Let G be a finitely generated sofic group and let A be a finite alphabet. Let $M \subset G$ be finite and let $S = A^{A^M}$. Suppose that $\sigma_s \colon A^G \to A^G$ is stably post-surjective for some $s \in S^G$ with uniformly bounded singularity. Then σ_s is stably invertible.*

1.2.3. *Conjectures.* Our main results naturally motivate the following surjunctivity conjectures for NUCA with uniformly bounded singularity which generalizes the surjunctivity and the dual surjunctivity conjectures for CA.

Conjecture 1.5 (Stable surjunctivity)**.** *Let G be a finitely generated group and let A be a finite alphabet. Let $M \subset G$ be finite and let $S = A^{A^M}$. Suppose that $\sigma_s \colon A^G \to A^G$ is stably injective for some $s \in S^G$ with uniformly bounded singularity. Then σ_s is stably invertible.*

Conjecture 1.6 (Stable dual-surjunctivity)**.** *Let G be a finitely generated group and let A be a finite alphabet. Let $M \subset G$ be finite and let $S = A^{A^M}$. Suppose that $\sigma_s \colon A^G \to A^G$ is stably post-surjective for some $s \in S^G$ with uniformly bounded singularity. Then σ_s is stably invertible.*

By Theorem B and Theorem D, the above conjectures hold true for finitely generated sofic groups. We observe that by results in [42] and Lemma 1.2, the stable surjunctivity conjecture and the stable dual-sujunctivity conjecture are in fact equivalent when restricted to the class of linear NUCA.

Proposition 1.7. *Let G be a finitely generated group and let A be a finite vector space alphabet. The stable surjunctivity conjecture holds true for all linear* NUCA *$A^G \to A^G$ of finite memory with uniformly bounded singularity if and only if so does the stable dual-surjunctivity conjecture.*

Proof. Let $M \subset G$ be a finite subset and let $s \in S^G$ where $S = A^{A^M}$. We identify the vector space A with its dual vector space A^*. Then we infer from [42] that σ_s admits some dual NUCA $\sigma_{s^*} \colon A^G \to A^G$ with finite memory where $s^* \in S^G$ such that

(i) σ_s is invertible $\Longleftrightarrow \sigma_{s^*}$ is invertible,
(ii) σ_s is stably injective $\Longleftrightarrow \sigma_{s^*}$ is stably post surjective,
(iii) σ_s is stably post-surjective $\Longleftrightarrow \sigma_{s^*}$ is stably injective.

Moreover, we infer from Lemma 1.2 that every invertible NUCA with finite memory over a finite alphabet and a countable group universe is automatically stably invertible. Finally, it is straightforward from the definition of s^* that s has uniformly bounded singularity if and only if so does s^*. The proof is thus complete. □

1.3. **Applications.** We will present below several algebraic and geometric applications of our main results.

1.3.1. *Direct finiteness of algebraic NUCA.* Our first application is the following geometric result which generalizes the direct finiteness property of algebraic CA over sofic groups [39] to the class of algebraic NUCA (see e.g. [23], [13], [12], [15], [35]) with uniformly bounded singularity over sofic groups. In this paper, an algebraic variety is a reduced separated scheme of finite type over an algebraically closed field and we identify such a variety with its closed points.

Theorem E. *Let G be a finitely generated sofic group and let X be an algebraic variety. Let $M \subset G$ be finite and let S be the set of morphisms of algebraic varieties $X^M \to X$. Suppose that $s, t \in S^G$ have uniformly bounded singularity and $\sigma_s \circ \sigma_t = \mathrm{Id}_{X^G}$. Then $\sigma_t \circ \sigma_s = \mathrm{Id}_{X^G}$.*

Proof. By using Lemma 6.1, the proof follows closely, *mutatis mutandis*, the proof of [39, Theorem C] which reduces the surjunctivity of σ_s to the surjunctivity of *all* NUCA $\sigma_q \colon A^G \to A^G$ where A is a finite alphabet, $N \subset G$ is a finite subset, $Q = A^{A^N}$, and $q \in Q^G$ is a configuration with uniformly bounded singularity. Consequently, Theorem E from Theorem B by the above reduction. The proof is thus complete. □

We remark that the same proof above actually shows that Theorem E still holds true when we replace the group G by any stable surjunctive group, namely, a group which satisfies Conjecture 1.5 on the stable surjunctivity.

1.3.2. *Stable finiteness of twisted group rings.* Our next application contributes to Kaplansky's stable finiteness conjecture [26] which states that for every group G and every field k, the group ring $k[G]$ is stably finite, that is, every one-sided invertible element of the ring $M_n(k[G])$ of square matrices of size $n \times n$ with coefficients in $k[G]$ must be a two-sided unit. Kaplansky's stable finiteness holds true for all sofic groups [1], [21], [9] and when $k = \mathbb{C}$ regardless of the group G [26], [20].

Given a group G and a vector space V over a field k, the k-algebra $\mathrm{LNUCA}_c(G, V)$ consists of linear NUCA $V^G \to V^G$ of finite memory which are local perturbations of a linear CA. The multiplication in $\mathrm{LNUCA}_c(G, V)$ is induced by the composition of maps and the addition is component-wise.

By results in [9], there exists an isomorphism $M_n(k[G]) \simeq \mathrm{LCA}(G, k^n)$ of k-algebras where $\mathrm{LCA}(G, k^n) \subset \mathrm{LNUCA}_c(G, k^n)$ is the subalgebra consisting of linear CA. When G is infinite, we can extend the above isomorphism to an isomorphism $M_n(D^1(k[G])) \simeq \mathrm{LNUCA}_c(G, k^n)$ (see [40]) where

$$D^1(k[G]) = k[G] \times (k[G])[G]$$

is the twisted group ring with component-wise addition and with multiplication given by $(\alpha_1, \beta_1) * (\alpha_2, \beta_2) = (\alpha_1\alpha_2, \alpha_1\beta_2 + \beta_1\alpha_2 + \beta_1\beta_2)$. Here, $\alpha_1\alpha_2$ is computed with the multiplication rule in the group ring $k[G]$ so that $k[G]$ is naturally a subring of $D^1(k[G])$ via the map $\alpha \mapsto (\alpha, 0)$. However, $\alpha_1\beta_2$, $\beta_1\alpha_2$, $\beta_1\beta_2$ are twisted products and defined for $g, h \in G$ by $(\alpha\beta)(g)(h) = \sum_{t \in G} \alpha(t)\beta(gt)(t^{-1}h)$, $(\beta\alpha)(g)(h) = \sum_{t \in G} \beta(g)(t)\alpha(t^{-1}h)$, $(\beta\gamma)(g)(h) = \sum_{t \in G} \beta(g)(t)\gamma(gt)(t^{-1}h)$. Twisted products are different from the multiplication rule of the group ring $(k[G])[G]$ with coefficients in $k[G]$.

It was shown that all sofic groups are L-surjunctive [23], [9], that is, for every finite-dimensional vector space V, every injective $\tau \in \mathrm{LCA}(G, V)$ is also surjective. Moreover, a group G is L-surjunctive if and only if $k[G]$ is stably finite for every field k [9]. In general, we say that a group G is L^1-surjunctive, resp. dual L^1-surjunctive, if for every finite-dimensional vector space V and $\tau \in \mathrm{LNUCA}_c(G,V)$ that is stably injective, resp. stably post-surjective, then τ must be surjective, resp. pre-injective. [40, Theorem C] shows that initially subamenable groups and residually finite groups are both L^1-surjunctive and dual L^1-surjunctive. As an application of Theorem A, we obtain a generalization of [40, Theorem C] which also confirms the stable finiteness of twisted group rings over sofic groups.

Theorem F. *Let G be a finitely generated sofic group. Then we have*
- (i) *G is L^1-surjunctive,*
- (ii) *G is dual L^1-surjunctive,*
- (iii) *$D^1(k[G])$ is stably finite for every field k.*

Proof. The theorem is trivial if G is finite so we can suppose that G is infinite. We infer from Theorem A that for every *finite* vector space V, every stably injective $\tau \in \mathrm{LNUCA}_c(G,V)$ must be invertible. Since invertibility implies surjectivity, we deduce that G is *finitely* L^1-surjunctive (see [40, Definition 1.3]). Consequently, we conclude from [40, Theorem B] that G is both L^1-surjunctive and dual L^1-surjunctive and that $D^1(k[G])$ is stably finite for every field k. The proof is thus complete. □

1.4. **Organization of the paper.** We concisely collect the definition and basic properties of finitely generated sofic groups in Section 2. We then fix the notations and recall the construction of induced local maps of NUCA in Section 3. The proof of Theorem A is given in Section 4 where we need to relate several stable dynamic properties of NUCA (Lemma 4.1) and analyze the global-local structures of stably injective NUCA with uniformly bounded singularity in the proof of Theorem 4.2. Then in Section 5, we give the proof of Theorem C and Theorem D which is similar to but less involved than the proof of Theorem A. Finally, the proof of Theorem B is given in Section 6 which reduces Theorem B to Theorem A by a technical lemma (Lemma 6.1).

2. Sofic groups

The important class of sofic groups was introduced by Gromov [23] as a common generalization of residually finite groups and amenable groups. Many conjectures for groups have been established for the sofic ones such as Gottschalk's surjunctivity conjecture and Kaplansky's stable finiteness conjecture [23], [45], [21], [11], [39], [43]. We recall below a useful geometric characterization of finitely generated sofic groups.

Given a finite set Δ, a Δ-labeled graph is a pair $\mathcal{G} = (V, E)$, where V is the set of vertices, and $E \subset V \times \Delta \times V$ is the set of Δ-labeled edges. The length of a path ρ in $\mathcal{G}$ is denoted by $l(\rho)$. For $v, w \in V$, we set $d_{\mathcal{G}}(v,w) = \min\{l(\rho) : \rho \text{ is a path from } v \text{ to } w\}$ if v and w are connected by a path, and $d_{\mathcal{G}}(v,w) = \infty$ otherwise. For $v \in V$ and $r \geqslant 0$, we have a Δ-labeled subgraph $B_{\mathcal{G}}(v,r)$ of $\mathcal{G}$ defined by:

$$B_{\mathcal{G}}(v,r) = \{w \in V : d_{\mathcal{G}}(v,w) \leqslant r\}.$$

Let (V, E) and (W, F) be Δ-labeled graphs. A map $\phi\colon V \to W$ is an Δ-labeled graph homomorphism from (V, E) to (W, F) if $(\phi(v), s, \phi(w)) \in F$ for all $(v, s, w) \in E$. A bijective Δ-labeled graph homomorphism $\phi\colon V \to W$ is an Δ-labeled graph isomorphism if its inverse $\phi^{-1}\colon W \to V$ is an Δ-labeled graph homomorphism.

Let G be a finitely generated group and let $\Delta \subset G$ be a finite symmetric generating subset, i.e., $\Delta = \Delta^{-1}$. The Cayley graph of G with respect to Δ is the connected Δ-labeled graph $C_\Delta(G) = (V, E)$, where $V = G$ and $E = \{(g, s, gs) : g \in G \text{ and } s \in \Delta)\}$. For $g \in G$ and $r \geqslant 0$, we denote

$$B_\Delta(r) = B_{C_\Delta(G)}(1_G, r)$$

We have the following characterization of sofic groups ([11, Theorem 7.7.1]).

Theorem 2.1. *Let G be a finitely generated group. Let $\Delta \subset G$ be a finite symmetric generating subset. Then the following are equivalent:*

(a) *the group G is sofic;*

(b) *for all $r, \varepsilon > 0$, there exists a finite Δ-labeled graph $\mathcal{G} = (V, E)$ satisfying*

$$|V(r)| \geqslant (1 - \varepsilon)|V|,$$

where $V(r) \subset V$ consists of $v \in V$ such that there exists a unique S-labeled graph isomorphism $\psi_{v,r}\colon B_{\mathcal{G}}(v, r) \to B_\Delta(r)$ with $\psi_{v,r}(v) = 1_G$.

If $0 \leqslant r \leqslant s$ then $V(s) \subset V(r)$ as every S-labeled graph isomorphism $\psi_{v,s}\colon B_{\mathcal{G}}(v, s) \to B_\Delta(s)$ induces by restriction an Δ-labeled graph isomorphism $B_{\mathcal{G}}(v, r) \to B_\Delta(r)$. We will also need the following Packing lemma (cf. [45], [11, Lemma 7.7.2], see also [33] for (ii)).

Lemma 2.2. *With the notation as in Theorem 2.1, the following hold*

(i) *$B_{\mathcal{G}}(v, r) \subset V(kr)$ for all $v \in V((k + 1)r)$ and $k \geqslant 0$;*

(ii) *There exists a finite subset $W \subset V(3r)$ such that the balls $B_{\mathcal{G}}(w, r)$ are pairwise disjoint for all $w \in W$ and that $V(3r) \subset \bigcup_{w \in W} B_{\mathcal{G}}(w, 2r)$.*

3. Induced local maps of NUCA

Let G be a group and let A be an alphabet. For every subset $E \subset G$ and $x \in A^E$ we define $gx \in A^{gE}$ by $gx(gh) = x(h)$ for all $h \in E$. In particular, $gA^E = A^{gE}$. Let $M \subset G$ and let $S = A^{A^M}$ be the collection of all maps $A^M \to A$. For every finite subset $E \subset G$ and $w \in S^E$, we define a map $f^{+M}_{E,w}\colon A^{EM} \to A^E$ as follows. For every $x \in A^{EM}$ and $g \in E$, we set:

$$f^{+M}_{E,w}(x)(g) = w(g)((g^{-1}x)|_M). \tag{3.1}$$

In the above formula, note that $g^{-1}x \in A^{g^{-1}EM}$ and $M \subset g^{-1}EM$ since $1_G \in g^{-1}E$ for $g \in E$. Therefore, the map $f^{+M}_{E,w}\colon A^{EM} \to A^E$ is well defined. Consequently, for every $s \in S^G$, we have a well-defined induced local map $f^{+M}_{E,s|_E}\colon A^{EM} \to A^E$ for every finite subset $E \subset G$ which satisfies:

$$\sigma_s(x)(g) = f^{+M}_{E,s|_E}(x|_{EM})(g) \tag{3.2}$$

for every $x \in A^G$ and $g \in E$. Equivalently, we have for all $x \in A^G$ that:

$$\sigma_s(x)|_E = f^{+M}_{E,s|_E}(x|_{EM}). \tag{3.3}$$

For every $g \in G$, we have a canonical bijection $\gamma_g \colon G \mapsto G$ induced by the translation $a \mapsto g^{-1}a$. For each subset $K \subset G$, we denote by $\gamma_{g,K} \colon gK \to K$ the restriction to gK of γ_g. Now let $N \subset G$ and $T = A^{A^N}$. Let $t \in T^G$. With the above notations, we have the following auxiliary lemma.

Lemma 3.1. *Suppose that $1_G \in M \cap N$. Then for every $g \in G$, the condition $\sigma_t(\sigma_s(x))(g) = x(g)$ for all $x \in A^G$ is equivalent to the condition*

$$t(g) \circ \gamma_{g,N} \circ f^{+M}_{gN,s|_{gN}} \circ \gamma^{-1}_{g,NM} = \pi,$$

where $\pi \colon A^{NM} \to A$ is the projection $z \mapsto z(1_G)$.

Proof. For every $g \in G$ and $x \in A^G$, we deduce from Definition 1.1 and the relation (3.3) that

$$\begin{aligned}
\sigma_t(\sigma_s(x))(g) &= t(g)\left((g^{-1}\sigma_s(x))|_N\right) \\
&= t(g)\left(\gamma_{g,N}\left((\sigma_s(x))|_{gN}\right)\right) \\
&= t(g)\left(\gamma_{g,N} \circ f^{+M}_{gN,s|_{gN}}(x|_{gNM})\right) \\
&= t(g) \circ \gamma_{g,N} \circ f^{+M}_{gN,s|_{gN}}(x|_{gNM}) \\
&= t(g) \circ \gamma_{g,N} \circ f^{+M}_{gN,s|_{gN}} \circ \gamma^{-1}_{g,NM}\left((g^{-1}x)|_{NM}\right)
\end{aligned}$$

whence the conclusion as $x(g) = (g^{-1}x)(1_G)$. □

4. Invertibility of stably injective local perturbations of CA

For the proof of Theorem A, we need the following auxiliary result.

Lemma 4.1. *Let M be a finite subset of a group G. Let A be a finite alphabet and let $S = A^{A^M}$. Suppose that $\sigma_s \colon A^G \to A^G$ is stably injective for some asymptotically constant $s \in S^G$. Then there exists a finite subset $N \subset G$ and an asymptotically constant configuration $t \in T^G$ where $T = A^{A^N}$ such that $\sigma_t \circ \sigma_s = \mathrm{Id}$.*

Proof. As $s \in S^G$ is asymptotically constant, we can find a finite subset $E \subset G$ and a constant configuration $c \in S^G$ such that $s|_{G\setminus E} = c|_{G\setminus E}$. Since σ_s is stably injective, we infer from [38] that σ_s is reversible, that is, there exists a finite subset $N \subset G$ and $r \in T^G$ where $T = A^{A^N}$ such that $\sigma_r \circ \sigma_s = \mathrm{Id}$. Up to enlarging M, N if necessary, we can suppose without loss of generality that $N = N^{-1}$ and $1_G \in M \cap N$. Hence, $1_G \in N^2$ and we obtain a canonical projection map $\pi \colon A^{N^2} \to A$ given by $x \mapsto x(1_G)$. For $g \in G\setminus EN$, we have $gN \cap E = \varnothing$ and thus $s|_{gN} = c|_{gN}$. Consequently, for every $g \in G\setminus EN$, the condition $\sigma_r(\sigma_s(x))(g) = x(g)$ for all $x \in A^G$ is equivalent to the condition $r(g) \circ f^{+N}_{N,c|_N} = \pi$ by Lemma 3.1. We fix $g_0 \in G\setminus EN$ and define an asymptotically constant configuration $t \in T^G$ by setting $t(g) = r(g_0)$ if $g \in G \backslash EN$ and $t(g) = r(g)$ if $g \in EN$. It follows from our construction that $\sigma_t(\sigma_s(x))(g) = x(g)$ for all $x \in A^G$ and $g \in G \backslash EN$. As $t|_{EN} = r|_{EN}$, we infer from $\sigma_r \circ \sigma_s = \mathrm{Id}$ that $\sigma_t(\sigma_s(x))(g) = x(g)$ for all $x \in A^G$ and $g \in EN$. Therefore, $\sigma_t \circ \sigma_s = \mathrm{Id}$ and the proof is complete. □

We are now in position to give the proof of the following result which together with Lemma 1.2 prove Theorem A.

Theorem 4.2. *Let M be a finite subset of a finitely generated sofic group G. Let A be a finite set and let $S = A^{A^M}$. Suppose that $\sigma_s \colon A^G \to A^G$ is stably injective for some asymptotically constant $s \in S^G$. Then σ_s is invertible.*

Proof. Let us denote $\Gamma = \sigma_s(A^G) \subset A^G$. Let $\Delta \subset G$ be a finite symmetric generating subset of G such that $1_G \in \Delta$. For $n \in \mathbb{N}$, we denote by $B_\Delta(n)$ the ball centered at 1_G of radius s in the Cayley graph $C_\Delta(G)$.

Since σ_s is stably injective and $s \in S^G$ is asymptotically constant, we infer from Lemma 4.1 that there exists a nonempty finite subset $N \subset G$ and an asymptotically constant configuration $t \in T^G$, where $T = A^{A^N}$, such that $\sigma_t \circ \sigma_s = \mathrm{Id}$. We can thus find constant configurations $c \in S^G$ and $d \in T^G$ and a finite subset $F \subset G$ such that $s|_{G \backslash F} = c|_{G \backslash F}$ and $t|_{G \backslash F} = d|_{G \backslash F}$.

We suppose on the contrary that σ_s is not surjective, i.e., $\Gamma \subsetneq A^G$. It follows from [38, Theorem 4.4] that Γ is closed in A^G with respect to the prodiscrete topology. Hence, there exists a finite subset $E \subset G$ such that $\Gamma|_E \subsetneq A^E$ since otherwise, Γ would be a dense closed subset of A^G and thus $\Gamma = A^G$ which is a contradiction.

We fix $r \in \mathbb{N}$ large enough such that $r \geqslant 1$ and $B_\Delta(r) \supset M \cup N \cup E \cup F$. Up to enlarging M, N, E, and F if necessary, we can suppose without loss of generality that

$$M = N = E = F = B_\Delta(r)$$

and thus in particular $S = T$ and

$$\Gamma|_{B_S(r)} \subsetneq A^{B_S(r)}. \tag{4.1}$$

Let $R = 4r$. If $|A| \in \{0, 1\}$ then the theorem is trivial. Hence, we suppose in the rest of the proof that $|A| \geqslant 2$. Therefore, $\log\left(1 - |A|^{-|B_\Delta(R)|}\right) < 0$ and we can thus fix $\varepsilon \in \mathbb{R}$ which satisfies

$$0 < \varepsilon < 1 - \frac{|B_\Delta(2R)| \log |A|}{|B_\Delta(2R)| \log |A| - \log\left(1 - |A|^{-|B_\Delta(R)|}\right)}. \tag{4.2}$$

Since the group G is sofic, Theorem 2.1 implies that there exists a finite S-labeled graph $\mathcal{G} = (V, E)$ associated to the pair $(3R, \varepsilon)$ such that

$$|V(3R)| \geqslant (1 - \varepsilon)|V|, \tag{4.3}$$

where for every $n \in \mathbb{N}$, the subset $V(n) \subset V$ consists of all $v \in V$ such that there exists a Δ-labeled graph isomorphism $\psi_{v,n} \colon B_\Delta(n) \to B(v, n)$ which satisfies $\psi_{v,n}(1_G) = v$ (cf. Theorem 2.1) where we denote $B(v, n) = B_{\mathcal{G}}(v, n)$ for all $v \in V$ and $n \in \mathbb{N}$. Thus, for every $g \in G$, we obtain the following

Δ-labeled graph isomorphism where we recall that $\gamma_{g,K}\colon gK \to K$, for every subset $K \subset G$, is the restriction to gK of the translation map $a \mapsto g^{-1}a$:

$$\psi_{v,n} \circ \gamma_{g,B_\Delta(n)}\colon gB_\Delta(n) \to B(v,n).$$

Note that $V(m) \subset V(n) \subset V$ for all $m \geqslant n \geqslant 0$. We obtain from Lemma 2.2.(ii) a subset $W \subset V(3R)$ such that $B(w,R)$ are pairwise disjoint for all $w \in W$ and that $V(3R) \subset \bigcup_{w\in W} B(w,2R)$. Since $W \subset V(3R)$, we have $|B(w,2R)| = |B_\Delta(2R)|$ for every $w \in W$ and thus

$$|W||B_\Delta(2R)| \geqslant \left|\bigcup_{w\in W} B(w,2R)\right| \geqslant |V(3R)|. \tag{4.4}$$

Let us denote $\overline{W} = \coprod_{w\in W} B(w,R)$ then $\overline{W} \subset V(2R)$ by Lemma 2.2.(i). Since $|B(w,R)| = |B_\Delta(R)|$ for all $w \in W$, we find that

$$|V(2R)| = |W||B_\Delta(R)| + |V(2R)\backslash\overline{W}|. \tag{4.5}$$

We construct a map $\Phi\colon A^{V(R)} \to A^{V(2R)}$ defined by the following formula for all $x \in A^{V(R)}$ and $v \in V(2R)$ (note that $B(v,r) \subset B(v,R) \subset V(R)$ by Lemma 2.2.(i)):

$$\Phi(x)(v) = \begin{cases} s(\psi_{w,R}^{-1}(v))(x|_{B(v,r)} \circ \psi_{v,r}) & \text{if } v \in B(w,R) \text{ for some } w \in W \\ c(1_G)\left(x|_{B(v,r)} \circ \psi_{v,r}\right) & \text{if } v \in V(2R)\backslash\overline{W}. \end{cases} \tag{4.6}$$

By the choice of $B_\Delta(r) = M = F$ and $R = 4r$, observe that for every $u \in B(v,r) \cap B(w,R)$ where $v \in V(2R)\backslash\overline{W}$ and $w \in W$, we have

$$s(\psi_{w,R}^{-1}(u)) = c(1_G). \tag{4.7}$$

Let $v \in B(w,R) \cap V(3R)$ for some $w \in W$. Then $B(v,R) \subset V(2R)$. Let $g = \psi_{w,R}^{-1}(v) \in B_\Delta(R)$. We infer from (4.6) and the Δ-labeled graph isomorphism $\psi_{v,2r} \circ \gamma_{g,B_\Delta(r)}\colon gB_\Delta(2r) \to B(v,2r)$ the following relation for all $x \in A^{V(R)}$:

$$\Phi(x)|_{B(v,r)} \circ \psi_{v,r} \circ \gamma_{g,B_\Delta(r)} = f^{+B_\Delta(r)}_{gB_\Delta(r),s|_{gB_\Delta(r)}}(x|_{B(v,2r)} \circ \psi_{v,2r} \circ \gamma_{g,B_\Delta(2r)}). \tag{4.8}$$

Similarly, let $v \in V(3R)\backslash\overline{W}$ and $x \in A^{V(R)}$. We infer from (4.6), (4.7), and the Δ-labeled graph isomorphism $\psi_{v,2r}\colon B_\Delta(2r) \to B(v,2r)$ that

$$\Phi(x)|_{B(v,r)} \circ \psi_{v,r} = f^{+B_\Delta(r)}_{B_\Delta(r),s|_{B_\Delta(r)}}(x|_{B(v,2r)} \circ \psi_{v,2r}) \tag{4.9}$$

Let us denote

$$Z = \Phi(A^{V(R)}) \subset A^{V(2R)}.$$

We consider also the map $\Psi\colon Z \to A^{V(3R)}$ defined for each $z \in Z \subset A^{V(2R)}$ and $v \in V(3R)$ by the formula:

$$\Psi(z)(v) = \begin{cases} t(\psi_{w,R}^{-1}(v))(x|_{B(v,r)} \circ \psi_{v,r}) & \text{if } v \in B(w,R) \text{ for some } w \in W \\ d(1_G)\left(x|_{B(v,r)} \circ \psi_{v,r}\right) & \text{if } v \in V(3R) \backslash \overline{W}. \end{cases} \tag{4.10}$$

Because of the inclusions $V(3R) \subset V(R) \subset V$, we obtain a canonical projection map $\pi\colon A^{V(R)} \to A^{V(3R)}$ given by $x \mapsto x|_{V3R}$ for every $x \in A^{V(R)}$.

Claim: we have

$$\Psi \circ \Phi = \pi. \tag{4.11}$$

Indeed, let $x \in A^{V(R)}$ and $v \in V(3R)$. Recall that $M = N = B_\Delta(r)$ and $R = 4r$ so that we have

$$B_\Delta(r)M = B_\Delta(r)N = B_\Delta(2r) = B_\Delta(R/2).$$

On the other hand, we have by Lemma 2.2.(i) that $B(v,R) \subset V(2R)$ and $B(v,2R) \subset V(R)$ so that in particular $x|_{B(v,R)}$ is well-defined. We distinguish two cases according to whether $v \in \overline{W}$. Let us define

$$g = \begin{cases} \psi_{w,R}^{-1}(v) \in B_\Delta(R) & \text{if } v \in B(w,R) \cap V(3R) \text{ for some } w \in W \\ 1_G & \text{if } v \in V(3R) \backslash \overline{W}. \end{cases}$$

and fix some $z \in A^G$ extending $x|_{B(v,2R)} \circ \psi_{v,2R} \circ \gamma_{g,B_\Delta(2R)} \in A^{gB_\Delta(2R)}$.

Case 1: $v \in B(w,R) \cap V(3R)$ for some $w \in W$. By (4.8), (4.10), and the relation $\sigma_t \circ \sigma_s = \mathrm{Id}$, we can thus compute:

$$\begin{aligned} \Psi(\Phi(x))(v) &= t(g)(\Phi(x)|_{B(v,r)} \circ \psi_{v,r}) \\ &= t(g)\left(\gamma_{g,B_\Delta(r)}\left(f^{+B_\Delta(r)}_{gB_\Delta(r),s|_{gB_\Delta(r)}}\left(x|_{B(v,2r)} \circ \psi_{v,2r} \circ \gamma_{g,B_\Delta(2r)}\right)\right)\right) \\ &= \sigma_t(\sigma_s(z))(g) \\ &= z(g) \\ &= \left(x|_{B(v,2r)} \circ \psi_{v,2r} \circ \gamma_{g,B_\Delta(2r)}\right)(g) \\ &= x(\psi_{v,2r}(1_G)) \\ &= x(v). \end{aligned}$$

Case 2: $v \in V(3R) \backslash \overline{W}$. Similarly, we deduce from (4.9) and (4.10) and the relation $\sigma_t \circ \sigma_s = \mathrm{Id}$ that

$$\begin{aligned}
\Psi(\Phi(x))(v) &= d(1_G)\left(\Phi(x)|_{B(v,r)} \circ \psi_{v,r}\right) \\
&= d(1_G)\left(f^{+B_\Delta(r)}_{B_\Delta(r),s|_{B_\Delta(r)}}(x|_{B(v,2r)} \circ \psi_{v,2r})\right) \\
&= \sigma_t(\sigma_s(z))(1_G) \\
&= z(1_G) \\
&= \left(x|_{B(v,2r)} \circ \psi_{v,2r}\right)(1_G) \\
&= x(\psi_{v,2r}(1_G)) \\
&= x(v).
\end{aligned}$$

Therefore, $\Psi(\Phi)(x) = \pi(x)$ for all $x \in A^{V(R)}$ and Claim (4.11) is proved. Consequently, we deduce that

$$\Psi(Z) = \Psi(\Phi(A^{V(R)})) = \pi(A^{V(R)}) = A^{V(3R)}. \tag{4.12}$$

It follows that

$$\log|Z| \geqslant \log|A^{V(3R)}| = \log|A|^{|V(3R)|} = |V(3R)|\log|A|. \tag{4.13}$$

For all $w \in W$, we have by the construction of Z and Φ an isomorphism:

$$Z_{B(w,R)} \to \Gamma_{B_\Delta(R)}, \quad c \mapsto c \circ \psi_{w,R}.$$

Since $B_\Delta(r) \subset B_\Delta(R)$, we infer from (4.1) that $\Gamma_{B_\Delta(R)} \subsetneq A^{B_\Delta(R)}$ is a proper subset. It follows that for all $w \in W$, we have:

$$|Z_{B(w,R)}| = |\Gamma_{B_\Delta(R)}| \leqslant |A^{B_\Delta(R)}| - 1. \tag{4.14}$$

Consequently, we find that:

$$\begin{aligned}
\log|Z| &\leqslant \log\left(|Z_{\overline{W}}||A^{V(2R)\backslash\overline{W}}|\right) && (\text{as } Z \subset Z_{\overline{W}} \times A^{V(2R)}) \\
&= \log|Z_{\overline{W}}| + \log|A^{V(2R)\backslash\overline{W}}| \\
&\leqslant \log\Big|\prod_{w\in W} Z_{B(w,R)}\Big| + \log|A^{V(2R)\backslash\overline{W}}| && (\text{as } Z_{\overline{W}} \subset \prod_{w\in W} Z_{B(w,R)}) \\
&= \sum_{w\in W}\log|Z_{B(w,R)}| + |V(2R)\backslash\overline{W}|\log|A| \\
&\leqslant |W|\log\left(|A|^{|B_\Delta(R)|} - 1\right) + |V(2R)\backslash\overline{W}|\log|A| && (\text{by (4.14)}).
\end{aligned}$$

Therefore, we deduce that:

$$\begin{aligned}
\log|Z| &\leqslant \left(|W||B_\Delta(R)| + |V(2R)\backslash\overline{W}|\right)\log|A| + |W|\log\left(1 - |A|^{-|B_\Delta(R)|}\right) \\
&= |V(2R)|\log|A| + |W|\log\left(1 - |A|^{-|B_\Delta(R)|}\right)
\end{aligned}$$

where the last equality follows from (4.5). Combining the above equality with the relation (4.13), we obtain:

$$|V(3R)|\log|A| \leqslant |V(2R)|\log|A| + |W|\log\left(1 - |A|^{-|B_\Delta(R)|}\right).$$

By dividing both sides by $\log|A|$ and rearranging the terms, we have:

$$|V(2R)| \geqslant |V(3R)| - |W|\frac{\log\left(1-|A|^{-|B_\Delta(R)|}\right)}{\log|A|}. \tag{4.15}$$

Since $\log\left(1-|A|^{-|B_\Delta(R)|}\right) < 0$ and $V(2R) \subset V$, we obtain

$$\begin{aligned} |V| \geqslant |V(2R)| &\geqslant |V(3R)| - |W|\frac{\log\left(1-|A|^{-|B_\Delta(R)|}\right)}{\log|A|} && \text{(by (4.15))}\\ &\geqslant |V(3R)| - \frac{|V(3R)|\log\left(1-|A|^{-|B_\Delta(R)|}\right)}{|B_\Delta(2R)|\log|A|} && \text{(by (4.4))}\\ &> |V(3R)|(1-\varepsilon)^{-1}. && \text{(by (4.2))} \end{aligned}$$

It follows that $|V(3R)| < (1-\varepsilon)|V|$ which contradicts (4.3). Therefore, σ_s must be surjective. Since $\sigma_t \circ \sigma_s = \mathrm{Id}$, we conclude that σ_s is invertible and the proof is thus complete. □

5. Invertibility of stably post-surjective NUCA over sofic groups with uniformly bounded singularity

By Lemma 1.2, Theorem C reduces to the following generalization of the dual surjunctivity result [8, Theorem 2] for post-surjective CA to the class of stably post-surjective NUCA over sofic group universes.

Theorem 5.1. *Let M be a finite subset of a finitely generated sofic group G. Let A be a finite set and let $S = A^{A^M}$. Let $s \in S^G$ be asymptotically constant such that σ_s is stably post-surjective. Then σ_s is invertible.*

Proof. Let Δ be a symmetric generating set of G, i.e., $\Delta = \Delta^{-1}$, such that $1_G \in \Delta$. Recall the balls $B_\Delta(n) \subset G$ of radius $n \geqslant 0$ in the corresponding Cayley graph $C_\Delta(G)$ of G. By hypothesis, we can find a constant configuration $c \in S^G$ and a finite subset $F \subset G$ such that $s|_{G\setminus F} = c|_{G\setminus F}$. Since σ_s is stably post-surjective, we infer from [38, Lemma 13.2] a finite subset $E \subset G$ such that for all $g \in G$ and $x, y \in A^G$ with $y|_{G\setminus\{g\}} = \sigma_s(x)|_{G\setminus\{g\}}$, there exists $z \in A^G$ such that $\sigma_s(z) = y$ and $z|_{G\setminus gE} = x|_{G\setminus gE}$. We claim that σ_s is pre-injective. Indeed, there exists on the contrary distinct asymptotic configurations $p, q \in A^G$ with $\sigma_s(p) = \sigma_s(q)$. Up to enlarging M and F, we can suppose that $p|_{FM} \neq q|_{FM}$. We fix $r \in \mathbb{N}$ such that $r \geqslant 1$ and $B_\Delta(r) \supset M \cup E \cup F$. Up to enlarging M, E, and F again if necessary, we can suppose that

$$M = E = F = B_\Delta(r).$$

We infer from (3.1), (3.2), and the relation $\sigma_s(p) = \sigma_s(q)$ that:

$$f^{+M}_{F,s|_F}(p|_{FM}) = f^{+M}_{F,s|_F}(q|_{FM}). \tag{5.1}$$

Let $R = 4r$. The theorem is trivial when $|A| \leqslant 1$ so we suppose from now on that $|A| \geqslant 2$. Let us fix $0 < \varepsilon < \frac{1}{2}$ small enough so that

$$|A|^{\varepsilon}\left(1 - |A|^{-|B_\Delta(R)|}\right)^{\frac{1}{2|B_\Delta(2R)|}} < 1, \tag{5.2}$$

By Theorem 2.1, there exists a finite S-labeled graph $\mathcal{G} = (V, E)$ associated to the pair $(3R, \varepsilon)$ such that

$$|V(3R)| \geqslant (1-\varepsilon)|V| \geqslant (1-\varepsilon)|V(R)| \tag{5.3}$$

where $V(n)$ consists of $v \in V$ such that there exists a Δ-labeled graph isomorphism $\psi_{v,n} \colon B(v,n) \to B_\Delta(n)$ mapping v to 1_G and $B(v,n) = B_{\mathcal{G}}(v,n)$.

We recall briefly the map $\Phi \colon A^{V(R)} \to A^{V(3R)}$ constructed in the proof of Theorem 4.2. From Lemma 2.2.(ii), we have a subset $W \subset V(3R)$ such that the balls $B(w,R)$, for all $w \in W$, are pairwise disjoint and $V(3R) \subset \bigcup_{w \in W} B(w, 2R)$. In particular,

$$|V(3R)| \leqslant |W||B_\Delta(2R)|. \tag{5.4}$$

Let $\overline{W} = \coprod_{w\in W} B(w,R) \subset V(2R)$. For $x \in A^{V(R)}$ and $v \in V(3R)$, let

(5.5)

$$\Phi(x)(v) = \begin{cases} s(\psi_{w,R}^{-1}(v))(x|_{B(v,r)} \circ \psi_{v,r}) & \text{if } v \in B(w,R) \text{ for some } w \in W \\ c(1_G)\left(x|_{B(v,r)} \circ \psi_{v,r}\right) & \text{if } v \in V(3R) \setminus \overline{W}. \end{cases}$$

For $v \in B(w,R) \cap V(3R)$ where $w \in W$, we have for $g = \psi_{w,R}^{-1}(v) \in B_\Delta(R)$ and for all $x \in A^{V(R)}$ that:

(5.6)

$$\Phi(x)|_{B(v,r)} \circ \psi_{v,r} \circ \gamma_{g,B_\Delta(r)} = f^{+B_\Delta(r)}_{gB_\Delta(r), s|_{gB_\Delta(r)}}(x|_{B(v,2r)} \circ \psi_{v,2r} \circ \gamma_{g,B_\Delta(2r)}).$$

For $v \in V(3R) \setminus \overline{W}$ and $x \in A^{V(R)}$, we have

$$\Phi(x)|_{B(v,r)} \circ \psi_{v,r} = f^{+B_\Delta(r)}_{B_\Delta(r), s|_{B_\Delta(r)}}(x|_{B(v,2r)} \circ \psi_{v,2r}) \tag{5.7}$$

Let $y \in A^{V(3R)}$ and fix an arbitrary $x \in A^{V(R)}$. Let $v_1, \dots, v_n$ be elements of $V(3R)$. We describe inductively below a procedure on $1 \leqslant i \leqslant n$ which allows us to obtain a sequence $x_0 = x, x_1, \dots, x_n$ such that

$$\Phi(x_i)|_{\{v_1,\dots,v_i\}} = y|_{\{v_1,\dots,v_i\}}, \quad i = 1, \dots, n.$$

For $i = 1, \dots, n$, let $g_i = \psi_{w,R}^{-1}(v_i)$ if $v_i \in B(w,R) \cap V(3R)$ for some $w \in W$ and $g_i = 1_G$ otherwise. By (5.6), (5.7), and the choice of E, we can modify, via the isomorphism $\psi_{v,r} \circ \gamma_{g_i, B_\Delta(r)}$, the restriction $x_{i-1}|_{B(v_i,r)}$ to obtain a new configuration $x_i \in A^{V(R)}$ such that $x_i|_{V(R)\setminus B(v_i,r)} = x_{i-1}|_{V(R)\setminus B(v_i,r)}$ and

$$\Phi(x_i)|_{V(3R)\setminus\{v_i\}} = \Phi(x_{i-1})|_{V(3R)\setminus\{v_i\}}, \qquad \Phi(x_i)(v_i) = y(v_i).$$

Consequently, $\Phi(x_n) = y$ and we deduce that Φ is surjective. In particular,

$$|\Phi(A^{V(R)})| = |A^{V(3R)}|. \tag{5.8}$$

By (5.1), we have $f^{+B_\Delta(r)}_{B_\Delta(r), s|_{B_\Delta(r)}}(p|_{B_\Delta(2r)}) = f^{+B_\Delta(r)}_{B_\Delta(r), s|_{B_\Delta(r)}}(q|_{B_\Delta(2r)})$. Let

$$\Lambda = \prod_{w \in W} \psi^{-1}_{w,R}(\{p|_{B_\Delta(R)}, q|_{B_\Delta(R)}\}) \subset A^{\overline{W}}.$$

Then for every $e \in A^{V(R)\setminus\overline{W}}$ and all $x, y \in \Lambda$, we deduce from (5.5) and $R = 4r$ that $\Phi(e \times x) = \Phi(e \times y)$. Consequently,

$$\Phi(A^{V(R)}) = \Phi\left(A^{V(R)\setminus\overline{W}} \times \prod_{w \in W}\left(A^{B(w,R)} \setminus \psi^{-1}_{w,R}(q|_{B_\Delta(R)})\right)\right).$$

Therefore, as $|B(w,R)| = |B_\Delta(R)|$ for all $w \in W$, we obtain the estimation

$$\begin{aligned}|\Phi(A^{V(R)})| &\leqslant \left|A^{V(R)\setminus\overline{W}} \times \prod_{w \in W}\left(A^{B(w,R)} \setminus \psi^{-1}_{w,R}(q|_{B_\Delta(R)})\right)\right| \\ &\leqslant |A|^{|V(R)\setminus\overline{W}|}(|A|^{|B_\Delta(R)|} - 1)^{|W|}.\end{aligned} \tag{5.9}$$

Finally, we find that

$$\begin{aligned}|A|^{|V(3R)|} &= |\Phi(A^{V(R)})| && \text{(by (5.8))} \\ &\leqslant |A|^{|V(R)| - |\overline{W}|}(|A|^{|B_\Delta(R)|} - 1)^{|W|} && \text{(by (5.9))} \\ &= |A|^{|V(R)|}(1 - |A|^{-|B_\Delta(R)|})^{|W|} && \text{(as } |\overline{W}| = |B_\Delta(R)||W|) \\ &\leqslant |A|^{|V(R)|}(1 - |A|^{-|B_\Delta(R)|})^{\frac{|V(3R)|}{|B_\Delta(2R)|}} && \text{(by (5.4))} \\ &\leqslant |A|^{|V(R)|}(1 - |A|^{-|B_\Delta(R)|})^{\frac{|V(R)|}{2|B_\Delta(2R)|}} && \text{(by (5.3))} \\ &< |A|^{|V(R)|}|A|^{-\varepsilon|V(R)|} && \text{(by (5.2))} \\ &\leqslant |A|^{|V(3R)|} && \text{(by (5.3))}\end{aligned}$$

which is a contradiction. Hence, σ_s must be pre-injective. Since σ_s is also stably post-surjective by hypothesis, we conclude from [38, Theorem 13.4] that σ_s is invertible and the proof is complete. $\square$

We can generalize in a direct manner the above proof of Theorem 5.1 to obtain Theorem D as follows.

Proof of Theorem D. Let Δ be a symmetric generating set of G, i.e., $\Delta = \Delta^{-1}$, such that $1_G \in \Delta$. As σ_s is stably post-surjective, [38, Lemma 13.2] provides a finite subset $E \subset G$ such that for all $g \in G$ and $x, y \in A^G$ with $y|_{G\setminus\{g\}} = \sigma_s(x)|_{G\setminus\{g\}}$, there exists $z \in A^G$ such that $\sigma_s(z) = y$ and $z|_{G\setminus gE} = x|_{G\setminus gE}$. As in the proof of Theorem D, we only need to show that σ_s is pre-injective. Suppose on the contrary that there exists a finite subset $K \subset G$ and configurations $p, q \in A^G$ such that $\sigma_s(p) = \sigma_s(q)$, $p|_{G\setminus K} = q|_{G\setminus K}$, and $p|_K \neq q|_K$. Up to enlarging M, E, and K if necessary, we can suppose without loss of generality that $1 \in M = M^{-1}$ and $M = E = K$. Since s has uniformly bounded singularity, we can find a constant configuration $c \in S^G$ and finite subset $H \subset G$ such that $s|_{HM\setminus H} = c|_{HM\setminus H}$ and $M \subset H$. We fix $r \geqslant 1$ large enough so that $H \subset B_\Delta(r)$. Let $R = 4r$ and we define an asymptotically constant configuration $t \in S^G$ by

$$t|_{B_\Delta(r)} = s|_{B_\Delta(r)}, \quad t|_{G\setminus B_\Delta(r)} = c|_{G\setminus B_\Delta(r)}.$$

Up to enlarging M, E, K again, we can assume that $M = E = K = B_\Delta(r)$ and let $F = M$. From this point, it suffices to apply the rest of the proof of Theorem 5.1 for t instead of s to obtain a contradiction. The proof of Theorem D is thus complete. □

6. Stable surjunctivity of NUCA over sofic groups with uniformly bounded singularity

We prove the following key technical lemma which is useful for the proof of Theorem F and allows us to reduce Theorem B to Theorem A.

Lemma 6.1. *Let A be an alphabet and let G be a group. Let $M \subset G$ be a finite subset and $S = A^{A^M}$. Suppose that $\sigma_t \circ \sigma_s = \mathrm{Id}$ for some $s, t \in S^G$ and s has uniformly bounded singularity. Then for each $E \subset G$ finite, there exists asymptotically constant configurations $p, q \in S^G$ such that $p|_E = s|_E$, $q|_E = t|_E$, and $\sigma_q \circ \sigma_p = \mathrm{Id}$.*

Proof. Up to enlarging M, we can clearly assume that $1_G \in M = M^{-1}$. Let $E \subset G$ be a finite subset. Up to enlarging E, we can suppose without loss of generality that $M \subset E$ and $E = E^{-1}$. As s has uniformly bounded singularity, there exists a constant configuration $c \in A^{A^M}$ and a finite subset $F \subset G$ containing E^3 such that $s|_{FE^3\setminus F} = c|_{FE^3\setminus F}$. We define an asymptotically constant configuration $p \in S^G$ by setting $p|_{FE} = s|_{FE}$ and $p|_{G\setminus FE} = c|_{G\setminus FE}$. We fix $g_0 \in FE^2\setminus FE$ and define $q \in S^G$ by $q(g) = t(g_0)$ if $g \in G\setminus FE$ and $q(g) = t(g)$ if $g \in FE$. Then q is asymptotic to the constant configuration $d \in S^G$ defined by $d(g) = t(g_0)$ for all $g \in G$. Since $1_G \in M$, we have a projection $\pi\colon A^{M^2} \to A$ given by $x \mapsto x(1_G)$. Since $E \subset E^3 \subset F \subset FE$, we deduce from our construction that $p|_E = s|_E$ and $q|_E = t|_E$. To conclude, we only need to check that $\sigma_q \circ \sigma_p = \mathrm{Id}$.

Let $g \in FE^2\setminus FE$. Then $gE \subset FE^3\setminus F$ since $E = E^{-1}$. Consequently, $s|_{gE} = c|_{gE} = p|_{gE}$ and thus $s|_{gM} = c|_{gM} = p|_{gM}$ since $M \subset E$. Hence, the condition $\sigma_t(\sigma_s(x))(g) = x(g)$ for all $x \in A^G$ is equivalent to $t(g) \circ f^{+M}_{M,c|_M} = \pi$ by Lemma 3.1. Similarly, the condition $\sigma_q(\sigma_p(x))(g) = x(g)$ for all $x \in A^G$ amounts to $q(g) \circ f^{+M}_{M,c|_M} = \pi$. Since $q(g) = t(g_0)$ and $\sigma_t \circ \sigma_s = \mathrm{Id}$, we conclude from the above discussion that $\sigma_q(\sigma_p(x))(g) = x(g)$ for all $x \in A^G$.

Let $g \in FE$. Since $s|_{FE^3\setminus F} = c|_{FE^3\setminus F}$, $p|_{FE} = s|_{FE}$, and $p|_{G\setminus FE} = c|_{G\setminus FE}$ by construction, we have $p|_{FE^3} = s|_{FE^3}$. In particular, $p|_{gM} = s|_{gM}$ since $gM \subset (FE)E = FE^2 \subset FE^3$. Therefore, we can infer from the relations $\sigma_t \circ \sigma_s = \mathrm{Id}$ and $q(g) = t(g)$ that $\sigma_q(\sigma_p(x))(g) = x(g)$ for all $x \in A^G$.

Finally, let $g \in G\setminus FE^2$. Since $p|_{G\setminus FE} = c|_{G\setminus FE}$, $q|_{G\setminus FE} = d|_{G\setminus FE}$, and since c, d are constant, we deduce that $q(g) = d(g_0) = t(g_0)$ and $p|_{gM} = c|_{gM}$.

The condition $\sigma_q(\sigma_p(x))(g) = x(g)$ for all $x \in A^G$ is thus equivalent to $t(g_0) \circ f^{+M}_{M,c|_M} = \pi$ by Lemma 3.1. But since $\sigma_t \circ \sigma_s = \mathrm{Id}$ and $s|_{g_0M} = c|_{g_0M}$, another application of Lemma 3.1 shows that $t(g_0) \circ f^{+M}_{M,c|_M} = \pi$. Thus, $\sigma_t(\sigma_s(x))(g) = x(g)$ for all $x \in A^G$. Therefore, we conclude that $\sigma_q \circ \sigma_p = \mathrm{Id}$ and the proof is complete. □

We are now in position to prove Theorem B.

Proof of Theorem B. Since σ_s is stably injective by hypothesis, we deduce from [38, Theorem A] that there exists a finite subset $N \subset G$ and $t \in T^G$, where $T = A^{A^N}$, such that $\sigma_t \circ \sigma_s = \mathrm{Id}$. In particular, it suffices to show that σ_s is surjective for σ_s to be invertible. Up to enlarging M and N, we can suppose that $M = N$ and thus $S = T$. We suppose on the contrary that σ_s is not surjective. Since $\Gamma = \sigma_s(A^G)$ is closed in A^G with respect to the prodiscrete topology by [38, Theorem 4.4], there must exists a finite subset $E \subset G$ such that $\Gamma_E \subsetneq A^E$. Since s has uniformly bounded singularity, we infer from Lemma 6.1 that there exists asymptotically constant configurations $p, q \in S^G$ such that $p|_{EM} = s|_{EM}$, $q|_{EM} = t|_{EM}$, and $\sigma_q \circ \sigma_p = \mathrm{Id}$. Let $\Lambda = \sigma_p(A^G)$ then it follows from $p|_{EM} = s|_{EM}$ that $\Lambda_E = \Gamma_E \subsetneq A^E$. We deduce that σ_p is not surjective. In particular, σ_p is not invertible. On the other hand, the condition $\sigma_q \circ \sigma_p = \mathrm{Id}$ shows that σ_q is stably injective by [38, Theorem A]. We can thus apply Theorem A to deduce that σ_p is invertible. Therefore, we obtain a contradiction and the proof is complete. □

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