

Confinement of N -body systems and non-integer dimensions

E. Garrido^{1*} and A.S. Jensen^{2†}

^{1*}Instituto de Estructura de la Materia, IEM-CSIC, Madrid, E-28006, Spain.

²Department of Physics and Astronomy, Aarhus University, Aarhus C, DK-8000, Denmark.

*Corresponding author(s). E-mail(s): e.garrido@csic.es;

Contributing authors: asj@phys.au.dk;

†These authors contributed equally to this work.

Abstract

The squeezing process of a three-dimensional quantum system by use of an external deformed one-body oscillator potential can also be described by the $\mathbf{d}$ -method, without external field and where the dimension can take non-integer values. In this work we first generalize both methods to $\mathbf{N}$ particles and any transition between dimensions below $\mathbf{3}$. Once this is done, the use of harmonic oscillator interactions between the particles allows complete analytic solutions of both methods, and a direct comparison between them is possible. Assuming that both methods describe the same process, leading to the same ground state energy and wave function, an analytic equivalence between the methods arises.

The equivalence between both methods and the validity of the derived analytic relation between them is first tested for two identical bosons and for squeezing transitions from 3 to 2 and 1 dimensions, as well as from 2 to 1 dimension. We also investigate the symmetric squeezing from 3 to 1 dimensions of a system made of three identical bosons. We have in all the cases found that the derived analytic relations between the two methods work very well. This fact permits to relate both methods also for large squeezing scenarios, where the brute force numerical calculation with the external field is too much demanding from the numerical point of view, especially for systems with more than two particles.

1 Introduction

A system may be constrained to a phase-space smaller than allowed in our ordinary three-dimensional world [1–4]. The corresponding effective dimension does not have to be an integer like 3, 2, 1, or 0 [5]. The magnetic-optical trapping is extremely flexible and able to simulate almost any geometry [2, 6], as for example carving out a deformed space in three or perhaps in smaller dimensions. One practical example is harmonic oscillator squeezing of one coordinate from being fully allowed to completely forbidden. Theoretically this is achieved by inserting an oscillator potential with movable wall in one of the coordinates. The practical importance is especially due to the experimental availability. To reduce active dimensions by either simulating or directly squeezing has been discussed before, f.ex. in Refs. [7–13].

Practical investigations are obviously possible by adding an external oscillator confinement to the otherwise ordinary treatment of relative degrees-of-freedom. This introduces necessarily one extra three-dimensional degree-of-freedom on top of those used in the normal description. This does not sound, and is not, in any way prohibitive. However, considering the complications added in going up one particle number in a given system, as exemplified in the well-known two and three-body systems, it is maybe desirable, or at least an advantage, to minimize complications.

Recently, an equivalent method precisely offering this possibility has been established. The method was initially formulated [5, 14], developed and interpreted [15] in several publications, where the case of d varying between 2 and 3 especially is discussed for two and three short-range interacting particles for both, bound [15, 16] and continuum [17, 18] states. Particular emphasis has been made in the possibility of inducing the appearance of the Efimov effect [19–22].

This d -method is precisely as complicated as solving the corresponding relative two- and three-body problems. The formulation is in terms of non-integer dimensions, which is a mathematical analytic continuation formulating interpolation between integer dimensions. The formulation provides a translation between non-integer dimensions and external field constrained three-dimensional calculations. All necessary non-integer calculations are without explicit use of the external confinement, but the results can be interpreted as obtained with such a potential. The translation or interpretation extends to an approximate, deformed wave function available for calculations of any desired observable, still obtained entirely within this d -method.

In this report, we first formulate the d -method for an arbitrary number of particles. We derive general translations to the brute force method of confining with an external deformed field. Obviously the intrinsic many-body structure cannot be complicated, and ground states are therefore most suited. We specify to two and three particles, where the assumptions can be substantially fewer, and compare in details the specific transition between integer dimensions.

The paper is organized as follows. In Section 2 we describe the main ingredients of the theoretical calculations with explicit inclusion of the squeezing potential in three dimensions as well as for arbitrary values of the dimension. We also provide analytic solutions for both methods assuming harmonic oscillator particle-particle interactions, and use these results to derive an analytic relation between both methods for different initial and final integer dimensions. In Section 3 we particularize to the case of identical

bosons. The results for two- and three-body systems are shown in Section 4. We finish in Section 5 with the summary and the conclusions.

2 Formulation for N particles

In this section we first give a formalism for an interacting system of N particles, which are exposed to external one-body deformed oscillator potentials. The choice of parameters of the varying deformation decides the transition between spatial dimensions. The goal is then to derive an equivalent simpler method, the d -method, based on the use of non-integer dimensions, and compare it with the brute force formulation using an external potential.

Once both methods have been presented, we solve them analytically by means of two-body harmonic oscillator particle-particle interactions. This permits a direct comparison of the solutions from the two formulations, and we arrive at an explicit formal translation between the two methods. Finally, these simple results for different initial and final spatial dimensions are given.

In order to simplify the analysis of the numerical solutions, we are considering only two-body short-range particle-particle potentials. In principle, we could as well introduce three-body potentials into the game, but provided they are of short-range character, they should not produce significant changes in the conclusions. In any case, a careful investigation should be done to confirm this point.

2.1 General theory

Let us consider N particles with masses m_i , coordinates $\mathbf{r}_i = (x_i, y_i, z_i)$, two-body interaction, $V_{ij}(\mathbf{r}_{ij})$, between pair i and j with $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$, and an external harmonic oscillator one-body interaction, $V_{ho}(\mathbf{r}_i)$, acting on each of the particles. The Schrödinger equation for the system is given by:

$$\left[- \sum_{i=1}^N \frac{\hbar^2}{2m_i} \Delta_{\mathbf{r}_i} + \sum_{i=1}^N V_{ho}(\mathbf{r}_i) + \sum_{i < j} V_{ij}(\mathbf{r}_{ij}) - E_N \right] \Psi_{\text{tot}} = 0, \quad (1)$$

where E_N and Ψ_{tot} are the corresponding energy and wave function, and $\Delta_{\mathbf{r}}$ is the Laplace kinetic energy operator related to the vector $\mathbf{r}$.

Introducing the total mass of the system, $M = \sum_{i=1}^N m_i$, and the center-of-mass coordinate, $\mathbf{R}_{\text{cm}} = (X_{\text{cm}}, Y_{\text{cm}}, Z_{\text{cm}}) = \frac{1}{M} \sum_{i=1}^N m_i \mathbf{r}_i$, the external oscillator potential can be separated into relative and center-of-mass parts, $V_{ho} = V_{ho}^{\text{rel}} + V_{ho}^{\text{cm}}$. The explicit division for this, possibly deformed, oscillator potential is:

$$V_{ho} = \sum_{i=1}^N V_{ho}(\mathbf{r}_i) = \frac{1}{2} \sum_{i=1}^N m_i (\omega_x^2 x_i^2 + \omega_y^2 y_i^2 + \omega_z^2 z_i^2) \quad (2)$$

$$V_{ho}^{\text{cm}} = \frac{1}{2} M (\omega_x^2 X_{\text{cm}}^2 + \omega_y^2 Y_{\text{cm}}^2 + \omega_z^2 Z_{\text{cm}}^2), \quad (3)$$

$$\begin{aligned}
V_{\text{ho}}^{\text{rel}} &= \frac{1}{2} \sum_{i=1}^N m_i (\omega_x^2 (x_i - X_{\text{cm}})^2 + \omega_y^2 (y_i - Y_{\text{cm}})^2 + \omega_z^2 (z_i - Z_{\text{cm}})^2) \\
&= \frac{1}{2} m (\omega_x^2 \rho_x^2 + \omega_y^2 \rho_y^2 + \omega_z^2 \rho_z^2),
\end{aligned} \tag{4}$$

which is only correct assuming the same oscillator frequencies, $(\omega_x, \omega_y, \omega_z)$, for each particle.

In Eq.(4) we expressed the external relative oscillator potential in terms of the separated components of the hyperradius coordinate, ρ , where its length is defined by

$$m\rho^2 = \sum_{i=1}^N m_i (\mathbf{r}_i - \mathbf{R}_{\text{cm}})^2 = \sum_{i<j} \frac{m_i m_j}{M} (\mathbf{r}_i - \mathbf{r}_j)^2. \tag{5}$$

The squared hyperradius can in turn be divided into the spatial components used in Eq.(4), that is

$$\begin{aligned}
\rho^2 &= \rho_x^2 + \rho_y^2 + \rho_z^2 = \\
&\sum_{i=1}^N \frac{m_i}{m} (x_i - X_{\text{cm}})^2 + \sum_{i=1}^N \frac{m_i}{m} (y_i - Y_{\text{cm}})^2 + \sum_{i=1}^N \frac{m_i}{m} (z_i - Z_{\text{cm}})^2.
\end{aligned} \tag{6}$$

We emphasize that the hyperradius is not a vector, but expressed as a length of different vectors. The normalization m can be arbitrarily chosen and disappears in all observable quantities.

Eq.(1) can now be separated into intrinsic and center-of-mass coordinates, which amounts to the following equations

$$\left(-\frac{\hbar^2}{2M} \Delta_{\mathbf{R}_{\text{cm}}} + V_{\text{ho}}^{\text{cm}} - E_{\text{cm}} \right) \Psi_{\text{cm}} = 0, \tag{7}$$

$$\left[-\sum_{i=1}^N \frac{\hbar^2}{2m_i} \Delta_{\mathbf{r}_i} + \frac{\hbar^2}{2M} \Delta_{\mathbf{R}_{\text{cm}}} + V_{\text{ho}}^{\text{rel}} + \sum_{i<j} V_{ij}(\mathbf{r}_{ij}) - E_{\text{rel}} \right] \Psi_{\text{rel}} = 0, \tag{8}$$

where the energies and wave functions are related by $\Psi_{\text{tot}} = \Psi_{\text{rel}} \times \Psi_{\text{cm}}$ and $E_N = E_{\text{rel}} + E_{\text{cm}}$. It is important to note that the kinetic energy operator combination in Eq.(8) can be rewritten entirely in terms of relative coordinates. The rigorous proof would be rewriting in terms of Jacobi coordinates (any of the possible sets of Jacobi coordinates is valid), and simultaneously expressing Eq.(8) in those coordinates. For details see Ref. [23], where the precise definition of the N -body Jacobi coordinates can also be found.

The external squeezing oscillator potentials in Eq.(4) are clearly each zero for vanishing frequencies, whereas the other limit of infinite frequency removes motion in the spatial directions. Correspondingly, each infinite frequency leaves one less spatial

dimension. Practical examples are $\omega_x = \omega_y = 0$ and ω_z varying from 0 to ∞ corresponding to a transition from 3 to 2 dimensions (3D to 2D), $\omega_z = 0$ with $\omega_x = \omega_y$ varying from 0 to ∞ corresponding to a transition from 3D to 1D, and $\omega_z = \infty$ with $\omega_y = 0$ and ω_x varying from 0 to ∞ corresponding to a transition from 2D to 1D. The final dimension can even be zero, which would be reached when all frequencies end up being infinitely large.

The squeezing variation from initial to final dimension proceeds by increasing the energy in the squeezed direction from zero to infinity. To reach the proper energy of the lower dimensional system, we have to remove the corresponding zero-point energy, which is related to the removed dimensions. This relative zero-point energy is

$$E_0 = \hbar \sum_{i=1}^{N-1} \left(\omega_x(n_x(i) + 1/2) + \omega_y(n_y(i) + 1/2) + \omega_z(n_z(i) + 1/2) \right), \quad (9)$$

where the quantum numbers now refer to the relative degrees-of-freedom. The meaningful relative lower-dimensional energy, $E_{\text{ext}} = E_{\text{rel}} - E_0$, therefore has to be the relative energy, E_{rel} , minus E_0 , where the latter changes from zero to infinity during the squeezing process.

For bosons the lowest quantum state is occupied by all N particles in the $N - 1$ available relative degrees of freedom, that is $n_x(i) = n_y(i) = n_z(i) = 0$. Then E_0 becomes

$$E_0 = \hbar(\omega_x + \omega_y + \omega_z)(N - 1)/2. \quad (10)$$

2.2 The d -method

The basis of the method is to consider the dimension d as a parameter that varies continuously between some initial and final integer dimension. The external field is omitted as we are looking for another formulation with a different phase-space that replaces the brute force use of a deformed field producing the squeezing.

In order to obtain the N -body wave function of the system for a given value of the dimension d we use the hyperspherical coordinates, which in the present context only requires specification of the length, ρ , defined in Eq.(5). All other coordinates, the hyperangles (Ω), are dimensionless variables. In this way it is possible to separate the complicated hyperangular dependence of the kinetic energy operator into the hyperangular operator, D_Ω^2 , and the corresponding Schrödinger equation then reads:

$$\left(-\frac{\partial^2}{\partial \rho^2} + \frac{\ell_{d,N}(\ell_{d,N} + 1)}{\rho^2} + \frac{D_\Omega^2}{\rho^2} + \frac{2m}{\hbar^2} \left(\sum_{i < j} V_{ij}(\mathbf{r}_{ij}) - E_d \right) \right) (\rho^{\ell_{d,N}+1} \Psi_d) = 0, \quad (11)$$

where Ψ_d is the total d -dimensional wave function. The pair interaction terms, explicitly written in the equation, are understood as expressed in hyperspherical coordinates, and the centrifugal barrier is given in terms of the generalized angular momentum $\ell_{d,N} = [(N - 1)d - 3]/2$.

A large amount of possible structures exist for N particles. This is a tremendous problem and not addressed in general here, where we make simplifying assumptions. In particular we assume relative s -waves between the particles, in such a way that the dependence of Ψ_d on the angles describing the direction of the relative coordinates disappears. The wave function Ψ_d can then be expanded as $\rho^{\ell_{d,N}+1}\Psi_d = \sum_K F_d^{(K)}(\rho)\mathcal{Y}_K^{(d)}(\alpha)$, where $\mathcal{Y}_K^{(d)}(\alpha)$ are the s -wave hyperspherical harmonics in d dimensions, which depend only on the hyperangle α and the hypermomentum K , and whose explicit form can be found in the appendix of Ref. [16]. Introducing this wave function expansion into Eq.(11), one easily gets that the radial functions, $F_d^{(K)}$, can be obtained as the solution of the coupled set of equations:

$$\left[-\frac{\partial^2}{\partial \rho^2} + \frac{\ell_{d,N}(\ell_{d,N}+1) + K(K+d(N-1)-2)}{\rho^2} - \frac{2mE_d}{\hbar^2} \right] F_d^{(K)}(\rho) + \sum_{K'} \langle \mathcal{Y}_K(\alpha) | \sum_{i<j} V_{ij}(\mathbf{r}_{ij}) | \mathcal{Y}_{K'}(\alpha) \rangle_{\Omega} F_d^{(K')}(\rho) = 0, \quad (12)$$

where $\langle \rangle_{\Omega}$ indicates integration over the hyperangles, and where we have used that $D_{\Omega}^2 \mathcal{Y}_K = K(K+d(N-1)-2)\mathcal{Y}_K$ [23].

Although these expressions are in principle derived for integer values of the dimension, d , which simply is the number of spatial degrees-of-freedom in our applications, we however shall analytically continue all occurring functions to any desired non-integer value.

2.3 Harmonic oscillator solutions

Proceeding to obtain results from short-range potentials almost necessarily requires numerical treatment no matter which method is chosen. This is true for even the simplest Gaussian two-body interactions. We postpone this application and pursue instead the analytical insight obtained from harmonic oscillator potentials between the particles. The obvious problem is here that oscillators are far from short-range potentials, although a number of applications suggest the effectivity. The reason is that the short-distance behavior is very similar for short-range potentials and properly adjusted oscillators.

First we define the two-body interactions, V_{ij} , where the sum is the combination entering in the Schrödinger equation, Eq.(8), that is

$$\begin{aligned} \sum_{i<j} V_{ij}(\mathbf{r}_{ij}) &= \frac{1}{2}\omega_{\text{pp}}^2 \sum_{i<j} \frac{m_i m_j}{M} (\mathbf{r}_i - \mathbf{r}_j)^2 = \frac{1}{2}m\omega_{\text{pp}}^2 \rho^2 = \\ &= \frac{1}{2}\omega_{\text{pp}}^2 \sum_{i=1}^N m_i (\mathbf{r}_i - \mathbf{R}_{cm})^2 = \frac{1}{2}\omega_{\text{pp}}^2 \left(-M\mathbf{R}_{cm}^2 + \sum_{i=1}^N m_i \mathbf{r}_i^2 \right), \end{aligned} \quad (13)$$

where Eq.(5) has been used, and where ω_{pp} describes the overall strength of the given particle-particle interaction. The mass dependence of the individual two-body potential is not exactly the reduced mass, since the sum of the two masses in the denominator

is missing. The chosen form in Eq.(13) allows expressing the combined two-body interaction in terms of the hyperradius, and it also allows the final expression in Eq.(13) as a one-body interaction, separating center-of-mass and relative coordinates. For this it is essential to use the same frequency, ω_{pp} , for all particle-particle interactions. As we shall see, these subtle choices in Eq.(13) are essential to obtain the simple analytical solutions, which on the other hand maintain the crucial properties from short-range solutions.

We can now solve the external field relative equation in Eq.(8) with the oscillator interactions from Eq.(13). The implicitly used Jacobi coordinates then separates individually and in addition the three spatial coordinates separate as well. The center-of-mass motion is now completely decoupled, and the effective oscillator frequencies obtained from the two-body interactions, Eq.(13), and the external field, Eq.(4), in the three directions are $\sqrt{\omega_{\text{pp}}^2 + \omega_x^2}$, $\sqrt{\omega_{\text{pp}}^2 + \omega_y^2}$, and $\sqrt{\omega_{\text{pp}}^2 + \omega_z^2}$, respectively.

Making use of Eq.(10), we have that the energy $E_{\text{ext}} = E_{\text{rel}} - E_0$, relative to $\hbar\omega_{\text{pp}}$ is then given by

$$\begin{aligned} \frac{E_{\text{ext}}}{\hbar\omega_{\text{pp}}} = & \left(\sqrt{1 + \frac{\omega_x^2}{\omega_{\text{pp}}^2}} - \frac{\omega_x}{\omega_{\text{pp}}} \right) \sum_{i=1}^{N-1} (n_x(i) + 1/2) \\ & + \left(\sqrt{1 + \frac{\omega_y^2}{\omega_{\text{pp}}^2}} - \frac{\omega_y}{\omega_{\text{pp}}} \right) \sum_{i=1}^{N-1} (n_y(i) + 1/2) \\ & + \left(\sqrt{1 + \frac{\omega_z^2}{\omega_{\text{pp}}^2}} - \frac{\omega_z}{\omega_{\text{pp}}} \right) \sum_{i=1}^{N-1} (n_z(i) + 1/2) , \end{aligned} \quad (14)$$

where the oscillator quantum numbers are related to the motion relative to the center-of-mass divided into the three different coordinate directions.

The corresponding N -body relative wave function, solution of Eq.(8) with the two-body interactions in Eq.(13), is again the oscillator solutions for the individual particles, decoupled in the x , y and z -directions, that is

$$\Psi_{\text{rel}} = \prod_{i=1}^N N_x H_{n_x(i)}\left(\frac{x_i}{b_{x,i}}\right) e^{-\frac{x_i^2}{2b_{x,i}^2}} N_y H_{n_y(i)}\left(\frac{y_i}{b_{y,i}}\right) e^{-\frac{y_i^2}{2b_{y,i}^2}} N_z H_{n_z(i)}\left(\frac{z_i}{b_{z,i}}\right) e^{-\frac{z_i^2}{2b_{z,i}^2}} \quad (15)$$

where N_x , N_y , and N_z are the normalization factors and the coordinates are the Jacobi coordinates in the Schrödinger equation. The center-of-mass motion is not included, but it could be derived by solving Eq.(7). The oscillator quantum numbers are related to the relative motion, and the lengths in the different cases are defined as usual in terms of the harmonic oscillator frequencies, that is

$$b_{x,i}^2 = \frac{\hbar}{m_i \sqrt{\omega_x^2 + \omega_{\text{pp}}^2}}, b_{y,i}^2 = \frac{\hbar}{m_i \sqrt{\omega_y^2 + \omega_{\text{pp}}^2}}, b_{z,i}^2 = \frac{\hbar}{m_i \sqrt{\omega_z^2 + \omega_{\text{pp}}^2}} . \quad (16)$$

Turning now to the d -method, the key equation of motion is now Eq.(11). The external field is substituted by the hyperspherical structure and the d -dependence in the centrifugal barrier strength. The angular dependence would in general also contribute to the centrifugal barrier, but our choice of interaction in Eq.(13) leaves only the harmonic oscillators second order dependence in hyperradius. This implies that all hyperspherical partial angular momenta are zero except s -wave types, and only the spherical solution is left. The energy is the energy of a three-dimensional oscillator with angular momentum, $\ell_{d,N}$, and frequency, ω_{pp} , corresponding to

$$E_d = \hbar\omega_{\text{pp}}(2n_r^{(d)} + \ell_{d,N} + \frac{3}{2}) = \hbar\omega_{\text{pp}}(2n_r^{(d)} + \frac{1}{2}(N-1)d), \quad (17)$$

where the number of radial nodes, $n_r^{(d)}$, (multiplied by two) is added to the generalized angular momentum quantum number introduced in Eq.(11), and ω_{pp} describes the oscillator pair interaction. The total radial wave function, $F_d(\rho)$, contained in Ψ_d , and solution of Eq.(12), is

$$F_d(\rho) = N_d e^{-\frac{\rho^2}{2b_{\text{pp}}^2}} L_{n_r^{(d)}}^{\ell_{d,N}+1/2} \left(\frac{\rho^2}{b_{\text{pp}}^2} \right), \quad (18)$$

where $b_{\text{pp}}^2 = \hbar/(m\omega_{\text{pp}})$ is the squared oscillator length, N_d is the normalization factor, that depends on $n_r^{(d)}$ and $\ell_{d,N}$, and L_k^p are the generalized Laguerre polynomials of order k .

2.4 Translation between methods

The above two methods are attempted constructed to be equivalent. This means that computations in one of them can be translated and interpreted in the parameters of the other method. The calculations are simplest with the d -method, whereas the external field method is directly laboratory applicable. Thus, it is the most efficient procedure to calculate with d -parameters and translate the results to laboratory variables.

For equal energies, $E_{\text{ext}} = E_d$, we have from Eqs.(14) and (17):

$$\begin{aligned} 2n_r^{(d)} + \frac{1}{2}(N-1)d &= \left(\sqrt{1 + \frac{\omega_x^2}{\omega_{\text{pp}}^2}} - \frac{\omega_x}{\omega_{\text{pp}}} \right) \sum_{i=1}^{N-1} (n_x(i) + 1/2) \\ &+ \left(\sqrt{1 + \frac{\omega_y^2}{\omega_{\text{pp}}^2}} - \frac{\omega_y}{\omega_{\text{pp}}} \right) \sum_{i=1}^{N-1} (n_y(i) + 1/2) \\ &+ \left(\sqrt{1 + \frac{\omega_z^2}{\omega_{\text{pp}}^2}} - \frac{\omega_z}{\omega_{\text{pp}}} \right) \sum_{i=1}^{N-1} (n_z(i) + 1/2). \end{aligned} \quad (19)$$

The dimension, d , can now be found from the combination of frequency ratios and the quantum numbers of the state. However, it is limited how useful this relation really is, because the quantum numbers of the two methods in general do not describe

precisely the same state. The d -method is formulated as spherical and the kinetic energy accounted for omitted hyperangles, that is all relative angular momentum terms and all relative distance-related terms. Therefore, at best we can compare average properties of angular momentum zero states. This is better indicated by rewriting Eq.(19) as

$$\begin{aligned}
d = & -\frac{4n_r^{(d)}}{N-1} + (2n_x^{(av)} + 1) \left(\sqrt{1 + \frac{\omega_x^2}{\omega_{pp}^2}} - \frac{\omega_x}{\omega_{pp}} \right) \\
& + (2n_y^{(av)} + 1) \left(\sqrt{1 + \frac{\omega_y^2}{\omega_{pp}^2}} - \frac{\omega_y}{\omega_{pp}} \right) \\
& + (2n_z^{(av)} + 1) \left(\sqrt{1 + \frac{\omega_z^2}{\omega_{pp}^2}} - \frac{\omega_z}{\omega_{pp}} \right),
\end{aligned} \tag{20}$$

where the average occupation numbers are given by $n_x^{(av)} = \frac{1}{N-1} \sum_{i=1}^{N-1} n_x(i)$, $n_y^{(av)} = \frac{1}{N-1} \sum_{i=1}^{N-1} n_y(i)$, and $n_z^{(av)} = \frac{1}{N-1} \sum_{i=1}^{N-1} n_z(i)$, respectively.

The frequencies ω_x , ω_y , and ω_z , are externally controlled, whereas ω_{pp} is an artificial oscillator frequency introduced to allow an analytical solution to be compared with the numerical calculations. However, whereas the realistic two-body interactions used in the calculations are of short-range character, the harmonic oscillator potential is not. Therefore, there is in principle no reason to expect a reasonable description of the system when the harmonic oscillator two-body potential is used. As we shall show in the coming sections, this problem can in practice be solved by choosing ω_{pp} such that it gives the same size for the system as in the calculations for one of the dimensions, in particular for $d = 2$, where an infinitesimal attraction always produces a bound state.

3 The case of identical bosons

Given an arbitrary N -body system, clearly very different structures may be chosen, and the d -method introduced above cannot describe them in the present simplified version. The connection in Eq.(20) may at most approximately describe an average translation for squeezing of a homogeneous state. This may contain some information, but we shall not pursue this possibility even though average structures handled in this way could provide some insight.

Instead, we elaborate on the more comparable case, where all the particles are in the same ground states. This therefore necessarily implies that we are dealing with identical bosons, and all the quantum numbers are zero in the ground state. Also, all the masses, m_i , are identical, and the normalization mass, m , introduced in the definition of the hyperradius, Eq.(5), is also taken equal to the boson mass. Having

all this in mind, we then have

$$d = \sqrt{1 + \frac{\omega_x^2}{\omega_{pp}^2}} - \frac{\omega_x}{\omega_{pp}} + \sqrt{1 + \frac{\omega_y^2}{\omega_{pp}^2}} - \frac{\omega_y}{\omega_{pp}} + \sqrt{1 + \frac{\omega_z^2}{\omega_{pp}^2}} - \frac{\omega_z}{\omega_{pp}}, \quad (21)$$

This expression is simple and, remarkably enough, independent of N .

The total ground state wave functions with the two methods are

$$\begin{aligned} \Psi_{\text{ext}} &= N_x N_y N_z \times \\ &\exp \left[-\frac{1}{2} \sum_{i=1}^N \left(\frac{(x_i - X_{cm})^2}{b_x^2} + \frac{(y_i - Y_{cm})^2}{b_y^2} + \frac{(z_i - Z_{cm})^2}{b_z^2} \right) \right], \end{aligned} \quad (22)$$

where the lengths in Eq.(16) are now equal for all the particles. From the d -method we get

$$\begin{aligned} F_d(\rho) &= N_d e^{-\frac{\rho^2}{2b_{pp}^2}} = \\ &N_d \exp \left[-\frac{1}{2} \sum_{i=1}^N \left(\frac{(x_i - X_{cm})^2}{b_{pp}^2} + \frac{(y_i - Y_{cm})^2}{b_{pp}^2} + \frac{(z_i - Z_{cm})^2}{b_{pp}^2} \right) \right] \end{aligned} \quad (23)$$

where Eq.(5) has been used for identical particles and $m = m_i$.

The wave functions (22) and (23) can be made identical by proper scaling of the coordinates according to their squeezing frequencies. In particular, we can formulate this by interpreting the hyperradius, Eq.(6), in the d -space as a hyperradius in the ordinary three-dimensional space, but deformed along the x , y , and z directions by means of some scaling factor s_x , s_y , s_z . In other words, the hyperradius is redefined in the three dimensional space as

$$\langle \rho^2 \rangle \longrightarrow \langle \rho_x^2 \rangle s_x^2 + \langle \rho_y^2 \rangle s_y^2 + \langle \rho_z^2 \rangle s_z^2, \quad (24)$$

and, after making equal Eqs.(22) and (23), we get that the scaling factors are

$$s_x^2 = \frac{b_{pp}^2}{b_x^2}, \quad s_y^2 = \frac{b_{pp}^2}{b_y^2}, \quad s_z^2 = \frac{b_{pp}^2}{b_z^2}, \quad (25)$$

where b_x^2 , b_y^2 , and b_z^2 are given in Eq.(16) with $m_i = m$, which leads to:

$$\frac{1}{s_x^2} = \sqrt{1 + \frac{\omega_x^2}{\omega_{pp}^2}}, \quad \frac{1}{s_y^2} = \sqrt{1 + \frac{\omega_y^2}{\omega_{pp}^2}}, \quad \frac{1}{s_z^2} = \sqrt{1 + \frac{\omega_z^2}{\omega_{pp}^2}}. \quad (26)$$

Therefore, if the system is not squeezed along some direction, for instance direction x , we then have that $\omega_x = 0$ and therefore $s_x = 1$, which implies that the space is not deformed along that direction. On the contrary, if the system is fully confined along

one direction, say $\omega_x = \infty$, we get that $s_x = 0$. In other words, the scale parameters range between zero and one, $0 \leq s_{x,y,z} \leq 1$, where $s_{x,y,z} = 0$ and $s_{x,y,z} = 1$ mean, respectively, full confinement or no confinement along the given direction.

For these Gaussian wave functions related to the oscillator ground state the translation is exact for the applied harmonic oscillator interactions. Using genuine short-range interactions vanishing at large distances still allows this translation, where the pair interaction frequency, ω_{pp} , must be appropriately adjusted. Such an approximation is similar to the numerical approximation obtained in a variational calculation using only one Gauss function.

The sizes of the oscillator solutions are controlled by $b_{pp}^2 = \hbar/(m\omega_{pp})$, which has to be adjusted to reproduce sizes obtained from realistic short-range two-body interactions. It is perhaps remarkable that this interpretation and the derived scalings are independent of both N and d .

Summarizing, the spherical wave function obtained with the d -method depends on ρ , which subsequently has to be deformed by scaling the different directions. The ordinary three-dimensional wave function found in this way is not restricted to the simple Gaussian form in Eq.(22). The results from this translation provides the correct long-range exponential fall-off for bound states. It is certainly much better than the Gaussian long-range behavior, but how well it resembles the brute force calculated wave function has to be separately evaluated.

3.1 Varying initial and final dimension

The connection between d and the oscillator squeezing parameters is given by the general expression in Eq.(21), which involves three independent squeezing frequencies corresponding to each dimension. It may be worth specifying each transition from initial to final dimension. First we note the two limits

$$\sqrt{1 + \frac{\omega_{ho}^2}{\omega_{pp}^2}} - \frac{\omega_{ho}}{\omega_{pp}} \rightarrow 1 \text{ for } \omega_{ho} = 0, \quad (27)$$

$$\sqrt{1 + \frac{\omega_{ho}^2}{\omega_{pp}^2}} - \frac{\omega_{ho}}{\omega_{pp}} \rightarrow 0 \text{ for } \omega_{ho} = \infty, \quad (28)$$

where $\omega_{ho} = \infty$ or 0 imply that the corresponding dimensions are excluded or untouched throughout the process. If two or three of these frequencies are equal the squeezing has cylindrical or spherical symmetry. However, the squeezing from one dimension to another does not have to be symmetric, and the parameter d would also vary in accordance with the frequency variation in Eq.(21).

When more than one direction is simultaneously squeezed, the frequencies can all be either equal (symmetric) or unequal (asymmetric). For symmetric squeezing Eq.(21) can be rewritten as

$$d = d_{fin} + (d_{ini} - d_{fin}) \left(\sqrt{1 + \frac{\omega_{ho}^2}{\omega_{pp}^2}} - \frac{\omega_{ho}}{\omega_{pp}} \right), \quad (29)$$

where $\omega_{\text{ho}} = \hbar/(mb_{\text{ho}}^2)$ is the squeezing frequency along all the squeezed directions, and d_{fin} and d_{ini} are, respectively, the final and initial integer dimensions. Solving to express the squeezing frequency in terms of dimensions we get

$$\frac{\omega_{\text{ho}}}{\omega_{\text{pp}}} = \frac{b_{\text{pp}}^2}{b_{\text{ho}}^2} = \frac{(d_{\text{ini}} - d)(d_{\text{ini}} + d - 2d_{\text{fin}})}{2(d - d_{\text{fin}})(d_{\text{ini}} - d_{\text{fin}})}, \quad (30)$$

which, when explicitly written for the possible symmetric transitions, becomes:

$$\frac{\omega_{\text{ho}}}{\omega_{\text{pp}}} = \frac{b_{\text{pp}}^2}{b_{\text{ho}}^2} = \frac{(3-d)(d-1)}{2(d-2)} \quad \text{for } 3\text{D} \rightarrow 2\text{D} \quad (31)$$

$$\frac{\omega_{\text{ho}}}{\omega_{\text{pp}}} = \frac{b_{\text{pp}}^2}{b_{\text{ho}}^2} = \frac{(3-d)(1+d)}{4(d-1)} \quad \text{for } 3\text{D} \rightarrow 1\text{D} \quad (32)$$

$$\frac{\omega_{\text{ho}}}{\omega_{\text{pp}}} = \frac{b_{\text{pp}}^2}{b_{\text{ho}}^2} = \frac{(3-d)(3+d)}{6d} \quad \text{for } 3\text{D} \rightarrow 0\text{D} \quad (33)$$

$$\frac{\omega_{\text{ho}}}{\omega_{\text{pp}}} = \frac{b_{\text{pp}}^2}{b_{\text{ho}}^2} = \frac{(2-d)d}{2(d-1)} \quad \text{for } 2\text{D} \rightarrow 1\text{D} \quad (34)$$

$$\frac{\omega_{\text{ho}}}{\omega_{\text{pp}}} = \frac{b_{\text{pp}}^2}{b_{\text{ho}}^2} = \frac{(2-d)(d+2)}{4d} \quad \text{for } 2\text{D} \rightarrow 0\text{D} \quad (35)$$

$$\frac{\omega_{\text{ho}}}{\omega_{\text{pp}}} = \frac{b_{\text{pp}}^2}{b_{\text{ho}}^2} = \frac{(1-d)(d+1)}{2d} \quad \text{for } 1\text{D} \rightarrow 0\text{D} . \quad (36)$$

We emphasize that these (inverse) relations only are possible for symmetric squeezing, since d otherwise depends on more than one frequency ratio. Clearly this is the case when the transition covers more than one dimension.

The symmetric squeezing implies that the involved scalings, s_x , s_y and s_z correspondingly must be equal along the squeezed directions, whereas asymmetric squeezing using Eq.(21), also must lead to asymmetric scaling. The related wave functions in the transitions for these different cases therefore necessarily must differ when reconstructed in three dimensions.

4 Numerical investigations

For identical bosons in the ground state, the derived relations between the non-integer dimension, d , and the frequencies in both symmetric and asymmetric squeezing, Eq.(21), are independent of the particle number, N . The present spherical d -method calculation is directly most applicable for ground states of a small number of particles, where the dominant large-distance behavior is universal. The angular momentum barrier increases with N , and tends to push the wave function to either smaller or larger distances. Distances comparable to the constituent particle sizes is problematic. On the other hand, large distances are not allowed for spherical bound states of large N , where the angular dependence of the kinetic energy operator becomes important and effectively would tend to split the system into clusters each with fewer particles.

In this report, we therefore concentrate on $N = 2, 3$, and analyze different aspects of the transition between integer dimensions not investigated in previous works.

4.1 Two-body systems

For two-body systems the 3D→2D, 3D→1D, and 2D→1D transitions were investigated in detail in Refs.[5, 15]. In those works three different Gaussian-shape and another three Morse-shape potentials were considered to describe the interaction between two spinless particles. Following those works, we shall refer to them as Potentials I, II, and III for both, Gaussian and Morse, shapes. The details about these six potentials can be found in Table I of Ref. [15]. It is interesting to note that their scattering lengths in 3D range from modest values comparable to the range of the interaction (potentials I) to a large scattering length of about 30 times the potential range (potentials III).

In Ref. [15] the connection between the dimension d and the harmonic oscillator parameter of the squeezing potential, b_{ho} , was obtained by making equal the ground state energy of the system in both calculations. The goal was then to establish universal relations between the squeezing harmonic oscillator length and the dimension d for each of the transitions investigated, as well as to provide a universal interpretation of the results. This was made basically through an analytic fit of the results, which in principle allows predictions from non-integer dimension calculations of observables in trap experiments with external potentials.

In this work we have however derived analytic expressions connecting the squeezing harmonic oscillator frequency, ω_{ho} (or length b_{ho}), and the dimension, d , by using a particle-particle harmonic oscillator potential with frequency ω_{pp} (or length b_{pp}). The analytic relations for each possible transition are given in Eqs.(31) to (36).

The harmonic oscillator length of the particle-particle potential, b_{pp} , can be easily related to the root-mean-square (rms) radius of the system in 1D ($r_{1\text{D}}$) or 2D ($r_{2\text{D}}$), since for a harmonic oscillator potential it is well known that $r_{1\text{D}} = b_{\text{pp}}/\sqrt{2}$ and $r_{2\text{D}} = b_{\text{pp}}$. In this way, by means of Eqs.(31) to (36), it is simple to express the dimension d as a function of $b_{\text{ho}}/r_{1\text{D}}$ or $b_{\text{ho}}/r_{2\text{D}}$.

In particular, here we focus on 3D→2D, 3D→1D, and 2D→1D transitions, and by use of Eqs.(31), (32), and (34) we get:

$$\frac{b_{\text{ho}}}{r_{2\text{D}}} = \sqrt{\frac{2(d-2)}{(3-d)(d-1)}} \text{ for } 3\text{D} \rightarrow 2\text{D}, \quad (37)$$

$$\frac{b_{\text{ho}}}{r_{2\text{D}}} = \sqrt{\frac{4(d-1)}{(3-d)(d+1)}} \text{ and } \frac{b_{\text{ho}}}{r_{1\text{D}}} = \sqrt{\frac{8(d-1)}{(3-d)(d+1)}} \text{ for } 3\text{D} \rightarrow 1\text{D}, \quad (38)$$

$$\frac{b_{\text{ho}}}{r_{2\text{D}}} = \sqrt{\frac{2(d-1)}{d(2-d)}} \text{ and } \frac{b_{\text{ho}}}{r_{1\text{D}}} = \sqrt{\frac{4(d-1)}{d(2-d)}} \text{ for } 2\text{D} \rightarrow 1\text{D}. \quad (39)$$

In Fig.1 we compare the relation between d and $b_{\text{ho}}/r_{2\text{D}}$, or $b_{\text{ho}}/r_{1\text{D}}$, obtained after the analytical estimates given in Eqs.(37), (38), and (39) (dashed light-blue curves) and the numerical calculations for the Gaussian (thick curves) and Morse potentials (thin curves) mentioned above. The upper row in the figure shows the relation between

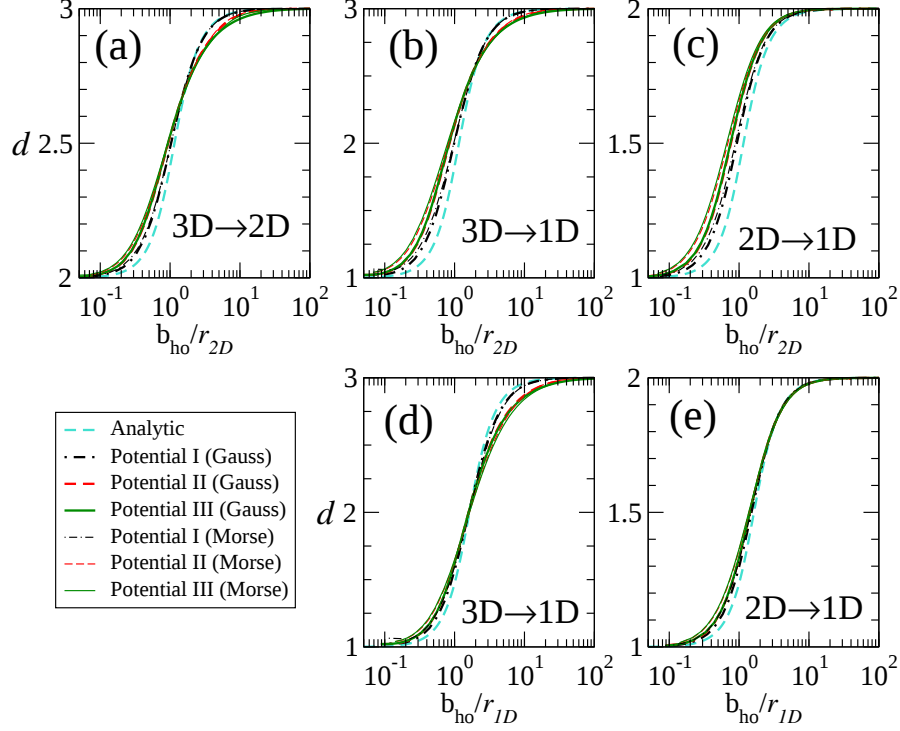


Fig. 1 Relation between the dimension d and b_{ho}/r_{2D} (upper row) or b_{ho}/r_{1D} (lower row) for the different confinement transitions and potentials considered in this work. The dashed light-blue curves correspond to the analytical estimates given in Eqs.(37), (38), and (39).

d and b_{ho}/r_{2D} for the 3D→2D, 3D→1D, and 2D→1D transitions (panels (a), (b), and (c)), and the lower row shows the relation between d and b_{ho}/r_{1D} for the 3D→1D and 2D→1D transitions (panels (d) and (e)).

Even though some differences between the numerical calculations can be seen, they appear mainly when comparing the results obtained with both, Gaussian and Morse, potentials I and the rest of the calculations. This is a consequence of the fact that potentials I have a relatively small scattering length, whereas the universal behavior of systems is related to the presence of large scattering lengths. In any case, when normalizing the squeezing parameter b_{ho} with r_{2D} or r_{1D} , as we can see in the figure, the dependence on the value of the scattering length is very small in all the cases, especially in Fig.1e.

In Ref.[15] a clearly more universal connection between d and b_{ho} was obtained (see Fig.11 in that reference). However, this was made after correcting b_{ho} with some scattering length dependent factor and using some very particular length unit. Furthermore, the universal analytical curve was obtained by fitting the parameters in some very precise d -dependent function. All in all, although the method shown in Ref.[15] permits to relate d and b_{ho} , the procedure is not very direct. For this reason, it is particularly interesting that the very simple method shown here provides a pretty much universal connection between d and b_{ho} . Also, the numerical estimate given in

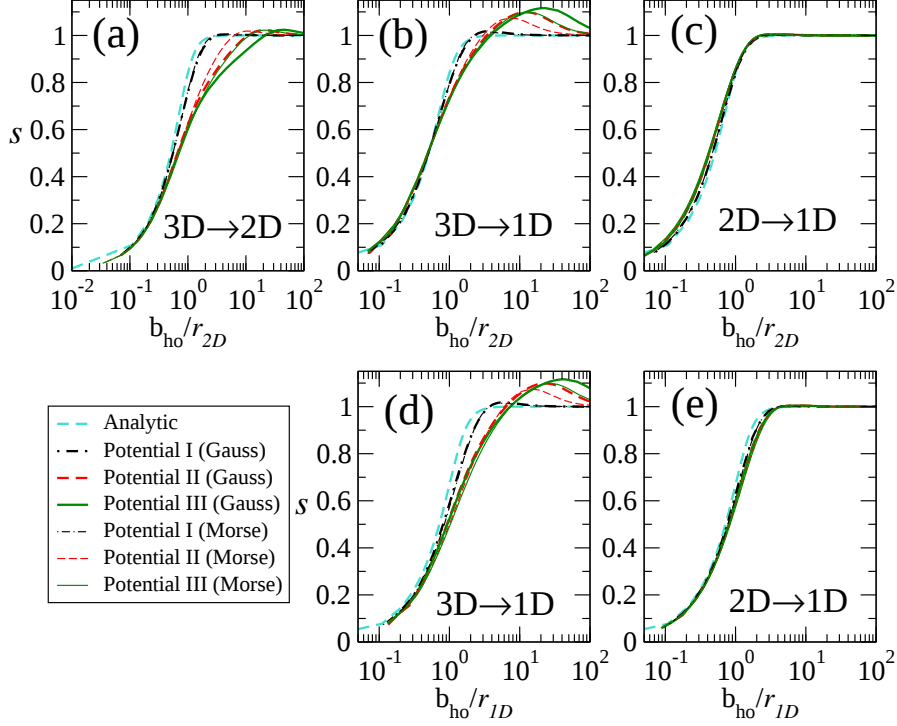


Fig. 2 Relation between the scale parameter s and $b_{\text{ho}}/r_{2\text{D}}$ (upper row) or $b_{\text{ho}}/r_{1\text{D}}$ (lower row) for the different confinement transitions and potentials considered in this work. The dashed light-blue curves correspond to the analytical estimates given in Eq.(40).

Eqs.(37), (38), and (39) follow reasonably well the numerical curves, without need of any numerical fit of the calculations.

At this stage it is interesting as well to investigate how the estimate given in Eq.(26) works for the connection between the scale parameter deforming the three-dimensional wave function and the external squeezing potential. Numerically this is done by choosing s such that the overlap between the deformed three-dimensional d -wave function and the one obtained with the external potential is maximum (and very close to 1, see Ref.[15]).

Of course, in Eq.(26), ω_x , ω_y , and ω_z are non-zero only along the squeezed directions. If we assume a symmetric squeezing for the 3D→1D case ($\omega_x = \omega_y = \omega_{\text{ho}}$), it is then not difficult to see that

$$\frac{1}{s^2} = \sqrt{1 + \left(\frac{r_{2\text{D}}}{b_{\text{ho}}}\right)^4} = \sqrt{1 + 4 \left(\frac{r_{1\text{D}}}{b_{\text{ho}}}\right)^4} \quad (40)$$

for all the three 3D→2D, 3D→1D, and 2D→1D confinement processes.

In Fig. 2 we show the connection between s and $b_{\text{ho}}/r_{2\text{D}}$ (or $b_{\text{ho}}/r_{1\text{D}}$) for all the cases considered in this work. The meaning of the curves is as in Fig. 1. As discussed in Ref. [15], the computed scale parameter for small squeezing is sometimes bigger than

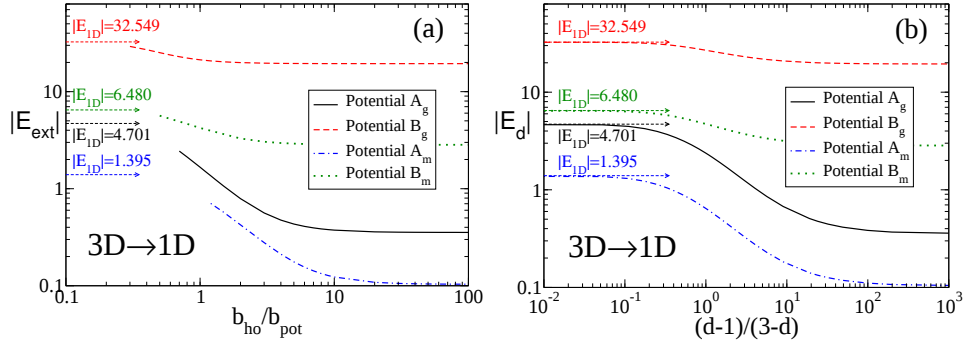


Fig. 3 For each of the Gaussian and Morse potentials used in this work, computed ground-state energies for 3D→1D confinement with an external squeezing harmonic oscillator potential, panel (a), as a function of the harmonic oscillator length, b_{ho} (in units of the range of the interaction, b_{pot}), and after a calculation with the d -method, panel (b), as a function of $(d-1)/(3-d)$. The arrows indicate the computed ground state energy in 1D for each of the potentials.

1. This mainly happens for the 3D→1D process. This is very likely the consequence of using a constant scale parameter.

As we can see in the figure the analytic expressions given in Eq.(40) work extremely well for the 2D→1D confinement process, as observed as well in Fig. 1e. This is the case where the use of a constant scale parameter seems to be particularly appropriate. For the other cases the analytical results work clearly better for small values of b_{ho} (large confinement), which are the most demanding cases from the numerical point of view.

4.2 Three-body systems

In Ref. [16] the case of three identical bosons was investigated for 3D→2D confinement. Two Gaussian and two Morse boson-boson potentials with small (potentials A_g and A_m) and large (potentials B_g and B_m) scattering lengths in three dimensions were considered. The details of these potentials can be found in Table I of Ref. [16].

In this work we have generalized the procedure already described in Ref. [16] for 3D→2D confinement to transitions between any integer dimension. In fact, Eq.(31) is Eq.(23) in Ref. [16]. For this reason here we shall concentrate on investigating to what extent the results shown in Ref. [16] for 3D→2D are valid as well for other confinement processes. In particular we are going to focus on the (symmetric) 3D→1D case using the same three-boson system as the one employed in Ref. [16]. We shall therefore maintain the same notation for the four boson-boson potentials used, potentials A_g and B_g (small and large scattering length with Gaussian shape) and potentials A_m and B_m (small and large scattering length with Morse shape).

As already observed for 3D→2D confinement, the three-body calculations with an external harmonic oscillator squeezing potential become more and more cumbersome when the system is progressively confined. In fact, for small values of b_{ho} (large confinement) the three-body calculations become practically not doable. This is shown in Fig. 3a, where we show, for 3D→1D confinement and for the four potentials used, the ground state energy of the system as a function of the squeezing harmonic oscillator

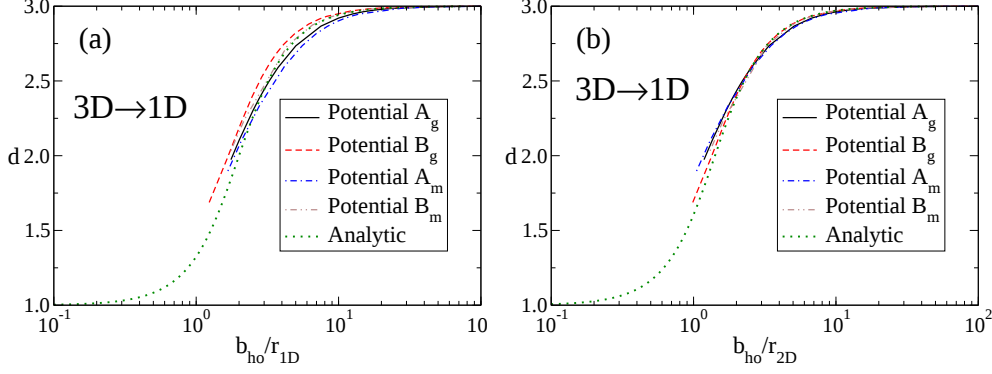


Fig. 4 For each of the Gaussian and Morse potentials used in this work, relation between the dimension d and $b_{\text{ho}}/r_{1\text{D}}$, panel (a), and $b_{\text{ho}}/r_{2\text{D}}$, panel (b), obtained from the values of b_{ho} and d corresponding to the same ground state energy in Fig. 3. The dotted-green curve are the analytic expressions given in Eq. (42).

length (in units of the range of the interaction, b_{pot}). The arrows in the left part of the figure indicate the ground state energies obtained for each potential after a pure 1D calculation. As we can see, when b_{ho} decreases, the computed energies approach the one corresponding to 1D, but it is not easy to go to smaller values of b_{ho} than the ones shown in the figure.

However, when the d -method is used, it is simple to perform the calculations using d as a continuum parameter. There is therefore no problem in reaching the 1D limit in this case. This is shown in Fig. 3b, where the coordinate in the abscissa axis is chosen as $(d-1)/(3-d)$, such that, as in Fig. 3a, the right and left parts of the figure correspond, respectively, to small and large confinement. In this way, it is possible to relate b_{ho} and d as those values providing the same three-body ground state energy. At least this is true for the b_{ho} values for which the calculation with the external potential are feasible.

When a harmonic oscillator with frequency ω_{pp} (or length b_{pp}) is used as boson-boson potential, it is known that the rms radius of the three-body system in 2D and 1D are given, respectively, by $r_{2\text{D}} = b_{\text{pp}}\sqrt{2/3}$ and $r_{1\text{D}} = b_{\text{pp}}\sqrt{1/3}$. We then have that Eq.(31) can be rewritten as

$$\frac{b_{\text{ho}}}{r_{2\text{D}}} = \sqrt{\frac{3(d-2)}{(3-d)(d-1)}}, \quad (41)$$

which is Eq.(38) of Ref. [16], valid for 3D→2D processes, and, similarly, Eq.(32) can be rewritten as

$$\frac{b_{\text{ho}}}{r_{2\text{D}}} = \sqrt{\frac{6(d-1)}{(3-d)(d+1)}} \text{ and } \frac{b_{\text{ho}}}{r_{1\text{D}}} = \sqrt{\frac{12(d-1)}{(3-d)(d+1)}}, \quad (42)$$

which provides an analytic form for the relation between d and b_{ho} valid for 3D→1D confinement.

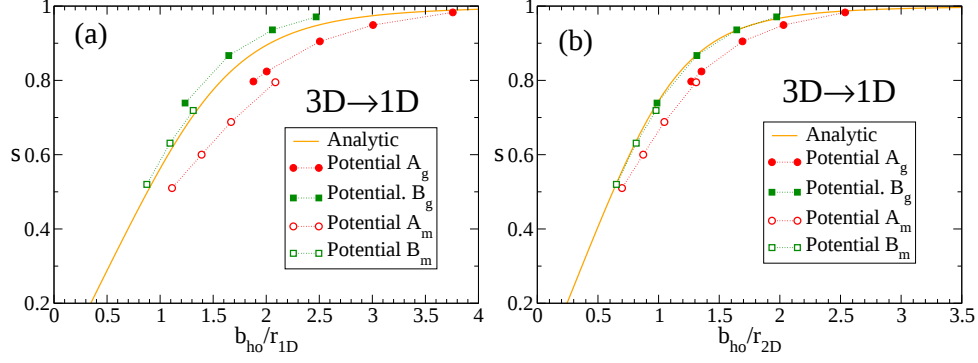


Fig. 5 For each of the Gaussian and Morse potentials used in this work, relation between the scale parameter s and $b_{\text{ho}}/r_{1\text{D}}$, panel (a), and $b_{\text{ho}}/r_{2\text{D}}$, panel (b). The solid-orange curve are the analytic expressions given in Eq.(43).

In Fig. 4 we show, for each of the Gaussian and Morse potentials used in this work, the relation between d and b_{ho} obtained from the values that make equal the ground state energies in Fig.3. In the panels (a) and (b) we show d versus $b_{\text{ho}}/r_{1\text{D}}$ and versus $b_{\text{ho}}/r_{2\text{D}}$, respectively. We can see that in both cases the computed curves tend to merge into a single curve, although this is clearly more evident in Fig. 4b. It seems therefore that $r_{2\text{D}}$ is a more convenient length unit in order to obtain a universal behavior. This same length unit is also giving rise to a pretty much universal curve in the 3D→2D case, as shown in Fig. 5 of Ref. [16].

In Fig. 4 we show as well the analytic relation between d and $b_{\text{ho}}/r_{2\text{D}}$ and $b_{\text{ho}}/r_{1\text{D}}$ given in Eq.(42), obtained assuming a harmonic oscillator boson-boson interaction. The result is shown by the dotted-green curves. We can see that in both panels the analytic curves follow very closely the ones obtained numerically, although, again, using $b_{\text{ho}}/r_{2\text{D}}$ as coordinate, Fig. 4b, seems to be clearly better. We can therefore consider the first equality in Eq.(42) as a very good estimate of the relation between d and b_{ho} pretty much reliable for large confinement scenarios, where the numerical calculations with the external field are extremely complicated.

Let us finally focus on the scale parameter s introduced in Eq.(25), which permits to interpret the d -wave function as an ordinary three-dimensional wave function deformed along the squeezing directions. As discussed in Ref. [16], the scale parameter can be obtained as the value of s that maximizes the overlap between both wave functions, the one obtained after the calculation with the external field and the one obtained after the d -calculation (and interpreted as deformed along the squeezing directions). Also, with Eq.(26) and symmetric confinement for the 3D→1D case, we can see that the scale parameter, s , and b_{ho} are related for oscillator interactions by the analytic expressions

$$\frac{1}{s^2} = \sqrt{1 + \frac{9}{4} \left(\frac{r_{2\text{D}}}{b_{\text{ho}}} \right)^4} = \sqrt{1 + 9 \left(\frac{r_{1\text{D}}}{b_{\text{ho}}} \right)^4}, \quad (43)$$

where we have again used that $r_{2\text{D}} = b_{\text{pp}}\sqrt{2/3}$ and $r_{1\text{D}} = b_{\text{pp}}\sqrt{1/3}$, and where the first equality is the one already given in Eq.(39) of Ref. [16].

In Fig. 5 the closed and open circles and the closed and open squares show, for the potentials A_g , A_m , B_g , and B_m , respectively, the relation between the scale parameter s and $b_{\text{ho}}/r_{1\text{D}}$, panel (a), and $b_{\text{ho}}/r_{2\text{D}}$, panel (b), obtained after the numerical calculation for 3D $\rightarrow$ 1D confinement. In both panels the solid-orange curve is the analytic estimate given in Eq.(43). We again see that the $b_{\text{ho}}/r_{2\text{D}}$ coordinate, panel (b), is clearly more appropriate than $b_{\text{ho}}/r_{1\text{D}}$, panel (a). In fact, when the $r_{2\text{D}}$ length unit is chosen the B -potentials follow extremely well the analytic form given in Eq.(43). For the A -potentials the disagreement with the analytic expression is clearly visible. The result in Fig. 5b is very similar to the one obtained in Ref. [16] for the 3D $\rightarrow$ 2D case, and, as discussed in this reference, the disagreement between the analytic expression and the results for the A -potentials can be a consequence of the use of constant scale parameter combined with the fact that the A -potentials are the ones corresponding to very large scattering length, which could make them much more sensitive to the confinement.

Therefore, the first analytic relation in Eq.(43) can be safely used to extract the connection between the scale parameter, s , and b_{ho} for small values of b_{ho} , i.e. for large confinements. This is precisely the situation where the numerical calculations with the external squeezing field is most complicated. Furthermore, this conclusion is, independently of the potential, also reached in Ref. [16], and similar to the agreement found for the first analytic expression in Eq.(42).

5 Summary and conclusions

We formulate two methods to continuously change the spatial dimension for a system of N particles. The first method, brute force method, employs an external varying one-body deformed oscillator potential. Allowing the oscillator length to take values from zero to infinity, we reduce one or more dimensions from fully present to completely removed from the available space. The two-body particle-particle interaction is of short range. In principle we can study the behavior as the spatial dimension is reduced. The solution involves both relative and center-of-mass degrees-of-freedom and depends clearly on the number of particles and the interactions.

The second method, the d -method, has no external potential, only the hyperradial equation is considered for s -waves, where a centrifugal barrier depends on N and the dimension parameter, d . Only relative motion is then appearing for the short-range interacting particles. The process of reducing the dimension is achieved by varying the parameter d .

To gain insight, we assume the particle-particle interaction is a possibly deformed harmonic oscillator potential with coordinate dependent frequencies, which however must be the same for all particles. These oscillator solutions for both methods are now analytic, and found for arbitrary potentials of this oscillator form and any value of N . Insisting on identical energies for the two methods, we arrive at a simple analytic connection between the dimension parameter, d , and the frequencies of the external one-body potential. We give explicitly the analytic formula for all possible dimension transitions. Furthermore, a similar relation can also be obtained between the squeezing

harmonic oscillator frequency and the scale parameters that describe the deformation of the d -wave function when translated back into the ordinary three-dimensional space. We scale the three coordinates from the spherical d -method, and obtain a wave function very similar to that of the brute force method. The scaling sizes are therefore also given as function of the confining oscillator frequencies.

The generalization to use of genuine short-range potentials is achieved by insisting on identical energies for $d = 2$ from this realistic interaction and the two-body harmonic oscillator. The analytical relation between d and the confinement potential is then modified by substituting the realistic mean-square-radius for the same quantity in the particle-particle oscillator calculation. These results are valid for any d and any N .

When the system under consideration is made of identical bosons they can all occupy the lowest quantum state. The consequence is that the expressions derived present the nice feature of being independent of the number of particles.

In this work, the equivalence between the two methods and the validity of the analytic expressions derived have been tested for the cases of two and three identical bosons. The numerical calculations and two-body potentials employed are as shown in previous works for two and three-body systems. At the two-body level we have investigated the transitions with final dimension 1 or 2, i.e., $3D \rightarrow 2D$, $3D \rightarrow 1D$, and $2D \rightarrow 1D$. For three identical bosons we have focused on the $3D \rightarrow 1D$ process. For both, two- and three-body systems, we have assumed a symmetric confinement in $3D \rightarrow 1D$ case.

From the calculations and the comparison with the derived analytic relations we have observed in all the cases a nice agreement in both, the two- and three-boson cases. We have found that the function describing the relation between the oscillator length of the squeezing potential and the dimension d shows a pretty much universal behaviour provided that the harmonic oscillator length is normalized with the root-mean-square radius of the system in two dimensions, r_{2D} . This is like this for two-body and three-body systems, and agrees with the results shown previously for three identical bosons and $3D \rightarrow 2D$ squeezing. For the relation between the squeezing oscillator length and the scale parameter, s , the use of r_{2D} as unit length gives rise as well to a universal curve, especially in the large squeezing region. The overall agreement is remarkably good, especially for not too small scattering lengths, where the short-range properties are more important.

In conclusion, we have provided a simple method, the d -method, that permits to perform simple numerical calculations, no matter the d -value, equivalent to confinement processes with an external field, which are much more complicated and close to impossible for large confinement scenarios. We have also derived analytic expressions that relate both descriptions of the same process, in such a way that it is most effective to perform the simple d -method calculations for non-integer dimensions, and afterwards translate to find the observables in ordinary three dimensional space.

Acknowledgements. This work has been partially supported by Grant PID2022-136992NB-I00 funded by MCIN/AEI/10.13039/501100011033 and by “ERDF A way of making Europe”.

References

- [1] D.S. Petrov, M. Holzmann, and G.V. Shlyapnikov, Phys. Rev. Lett. 84, 2551 (2000).
- [2] M. Greiner, I. Bloch, O. Mandel, T. W. Hänsch, and T. Esslinger, Appl. Phys. B 73, 769 (2001).
- [3] I. Bloch, Nature Phys. 1, 23 (2005).
- [4] Y. Nishida and S. Tan, Phys. Rev. Lett. 101, 170401 (2008).
- [5] E. Garrido, A.S. Jensen, and R. Álvarez-Rodríguez, Phys. Lett. A 383, 2021 (2019).
- [6] D.S. Petrov and G.V. Shlyapnikov, Phys. Rev. A 64, 012706 (2001).
- [7] J. Levinsen, P. Massignan, and M.M. Parish, Phys. Rev. X 4, 031020 (2014).
- [8] M.T. Yamashita, F.F. Bellotti, T. Frederico, D.V. Fedorov, A.S. Jensen, N. T. Zinner, J. Phys. B: At. Mol. Opt. Phys. 48, 025302 (2015).
- [9] D.S. Rosa, T. Frederico, G. Krein, and M.T. Yamashita, Phys. Rev. A 97, 050701(R) (2018).
- [10] E. R. Christensen, A.S. Jensen, and E. Garrido, Few-Body Syst 59, 136 (2018).
- [11] D.S. Rosa, T. Frederico, G. Krein, and M.T. Yamashita, J. Phys. B: At. Mol. Opt. Phys. 52, 025101 (2019).
- [12] D.S. Rosa, T. Frederico, G. Krein, and M.T. Yamashita, Phys. Rev. A 106, 023311 (2022).
- [13] D.S. Rosa, T. Frederico, G. Krein, and M.T. Yamashita, Phys. Rev. A 108, 033307 (2023).
- [14] E. Nielsen, D.V. Fedorov, A.S. Jensen, and E. Garrido, Phys. Rep. 347, 373 (2001).
- [15] E. Garrido and A.S. Jensen, Phys. Rev. Research 1, 023009 (2019).
- [16] E. Garrido and A.S. Jensen, Phys. Rev. Research 2, 033261 (2020).
- [17] E.R. Christensen, E. Garrido, and A. S. Jensen, Phys. Rev. A 105, 033308 (2022).
- [18] E. Garrido, E.R. Christensen, and A.S. Jensen, Phys. Rev. A 106, 013307 (2022).
- [19] E. Garrido and A.S. Jensen Phys. Lett. A 385, 126982 (2021).
- [20] E. Garrido and A.S. Jensen, Few-body Syst. 62, 25 (2021).

- [21] E. Garrido and A.S. Jensen, Eur. Phys. J. D 77, 46 (2023).
- [22] J.H. Sandoval, F.F. Bellotti, M.T. Yamashita, T. Frederico, D.V. Fedorov, A.S. Jensen, and N.T. Zinner, J. Phys. B: At. Mol. Opt. Phys. 51, 065004 (2018).
- [23] M. Fabre de la Ripelle, Ann. Phys. (N.Y.) 147, 281 (1983).