arXiv:2403.07054v2 [cond-mat.str-el] 26 Mar 2024

Minimal Fractional Topological Insulator in half-filled conjugate moiré Chern bands

Chao-Ming Jian,[1] Meng Cheng,[2] and Cenke Xu[3]

[1]Department of Physics, Cornell University, Ithaca, New York 14853, USA
[2]Department of Physics, Yale University, New Haven, Connecticut 06511-8499, USA
[3]Department of Physics, University of California, Santa Barbara, CA 93106

(Dated: March 27, 2024)

We propose a "minimal" fractional topological insulator (mFTI), motivated by the recent experimental report on the signatures of FTI at total filling factor $\nu_{\rm tot} = 3$ in a transition metal dichalcogenide moiré system. The observed FTI at $\nu_{\rm tot} = 3$ is likely given by a topological state living in a pair of half-filled conjugate Chern bands with Chern numbers $C = \pm 1$ on top of another pair of fully-filled conjugate Chern bands. We propose the mFTI as a strong candidate topological state in the half-filled conjugate Chern bands. The mFTI is characterized by the following features: (1) It is a fully gapped topological order (TO) with 16 Abelian anyons if the electron is considered trivial (32 including electrons); (2) the minimally-charged anyon carries electric charge $e^* = e/2$, together with the fractional quantum spin-Hall conductivity, implying the robustness of the mFTI's gapless edge state whenever time-reversal symmetry and charge conversation are present; (3) the mFTI is "minimal" in the sense that it has the smallest total quantum dimension (a metric for the TO's complexity) within all the TOs that can potentially be realized at the same electron filling and with the same Hall transports; the mFTI is also the *unique* one that respects time-reversal symmetry. (4) the mFTI is the common descendant of multiple valley-decoupled "product TOs" with larger quantum dimensions. It can also be viewed as the result of gauging multiple symmetry-protected topological states. Similar mFTIs can be constructed for a pair of $1/q$-filled conjugate Chern bands. We classify the mFTIs via the stability of the gapless interfaces between them.

I. INTRODUCTION

The recent discovery of fractional Chern insulator[1,2] and fractional quantum anomalous Hall insulator at zero magnetic field[3–7] have led to a new excitement on the strongly correlated states of matter in the moiré systems. Especially, the fractional topological insulator (FTI) reported most recently in transition metal dichalcogenide (TMD) homobilayer moiré heterostructure[8], twisted bilayer $MoTe_2$ to be more precise, represents a potentially new topological state of matter which could be fundamentally different from the previously known fractional quantum Hall state. In this TMD moiré system with spin-valley locking, the electron bands relevant to the experimental observations include two bands with Chern number $C = +1$ at valley-1 (spin-up), and two bands with opposite Chern number $C = -1$ at valley-2 (spin-down).[9] The observed FTI signal appears at the total filling $\nu_{\rm tot} = 3$. If the system preserves the time-reversal symmetry, and one ignores the fully occupied lower bands in each valley, the most relevant bands are a pair of time-reversal conjugate Chern bands with $C = \pm 1$, each with electron filling factor $\nu = 1/2$. The experiment[8] on twisted bilayer $MoTe_2$ observed a series of appealing evidence that supports the existence of a certain type of time-reversal-invariant FTI with fractionalized helical edge states and spin-Hall conductivity $\sigma^{\rm sh} = 1/2$ (when we only consider the contribution from the half-filled bands). The experimental evidence includes the nonlocal transport signal, the quantization of the edge conductance at fractional filling, and the suppression of the edge conductance when the time-reversal symmetry is explicitly broken by an in-plane magnetic field, analogous to the quantum spin Hall insulator[10]. It is worth noting that the relevant twisted bilayer $MoTe_2$ system exhibits an S_z spin quantum number conservation in addition to the charge conservation and time-reversal symmetry.[8]

The experimental facts have not yet completely pinned down the very nature of the reported FTI. A possiblity is that the FTI is simply a product of a pair of time-reversal conjugate topological orders (TO) in each valley, like some of the FTIs considered theoretically in the past[11,12] (For a review of previous theoretical discussion of FTI, please refer to Ref. 13. We also refer to the seminal works Ref. 14 and 15 for general properties of generic FTIs). In the context of the TMD moiré system, the Chern band at valley-1 (valley-2) with Chern number $C = 1$ ($C = -1$) can be viewed as a Landau level due to the real space pseudo magnetic field[16,17] (as illustrated in the upper panel of Fig. 1). For a half-filled Landau level, the candidate topological order of electrons could be the nonabelian Pfaffian state[18–20], anti-Pfaffian state[21], PH-Pfaffian state[22], the Abelian $U(1)_8$ state[23], or the "331" state[24,25]. Hence, the FTI could be the product of one of the TOs mentioned above in one valley, and its time-reversal conjugate in the other valley. In such product TOs, the TOs from the two valleys essentially decouple.

In this work, we explore what kind of FTI is the "minimal" one, in the sense that it gives the desired experimental signals (especially the spin-Hall conductivity) and also has the minimal total quantum dimension (which is, in simple terms, a metric for the topological order's complexity). We take charge conversation, spin-S_z conservation, and time-reversal symmetry into account in the search for the minimal FTI. It turns out that the minimal FTI (mFTI) is *unique*, and it is *NOT* a simple product of a pair of conjugate TOs of electrons from the two valleys, though it can be viewed as the common descendant of multiple "product TOs" via anyon condensations. Our mFTI also has a spin-Hall conductivity $\sigma^{\text{sh}} = 1/2$, which is consistent with the experimental observation once the fully filled bands are included. This spin-Hall conductivity is also half of the value for the elementary non-interacting quantum spin-Hall insulators[26,27]. The minimally charged anyon of the mFTI carries an electric charge $e^* = e/2$. In contrast, all the product TOs with the same response must have a minimal charge equal to or smaller than $e/4$. Additionally, according to the criterion established in Ref. 14, our mFTI has robust gapless edge states that remain protected by time-reversal symmetry and charge conservation even when S_z is not conserved.

If we only impose charge conservation, S_z conservation, and the Hall transport signals (but not time-reversal symmetry) in the search, we find that the mFTI belongs to a list of minimal TOs following an 8-fold classification. 4 of the 8 minimal TOs, albeit time-reversal broken, are interesting non-Abelian TOs that exhibit quantized thermal Hall effects in addition to the required Hall responses associated with the charge and S_z quantum numbers.

We generalize the construction of mFTI (and the minimal TOs) to a pair of conjugate $1/q$-filled Chern bands (for positive integer q). Under the symmetry and Hall transport constraints, mFTIs are found to be unique for a general q. We classify all mFTIs via the stability of the gapless interfaces between them.

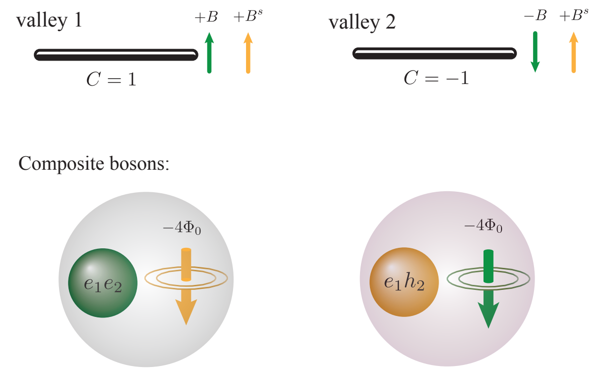

FIG. 1. Mutual flux attachment construction of the mFTI. Upper panel: The pair of conjugate Chern bands with Chern number $C = \pm 1$ are both half-filled. The green and yellow arrows indicate their contribution to the effective magnetic field seen by the inter-valley Cooper pair and the effective spin-magnetic field seen by the inter-valley excitons. In total, the Cooper pair sees no net magnetic field while the exciton sees a spin flux $\Phi^s = 2\Phi_0$ per moire unit cell. Lower panel: The inter-valley Cooper pair (green ball) is attached with flux $-4\Phi_0$ seen by the spin-1 inter-valley exciton (yellow ball), forming a composite boson. Hence, the exciton sees a total zero magnetic field when each valley has a filling factor $\nu = 1/2$. The Cooper pair also sees the exciton as a $-4\Phi_0$ flux. This flux attachment turns the exciton into another composite boson. $e_{1,2}$ and $h_{1,2}$ stand for the electrons and holes in the two valleys.

II. CONSTRUCTION OF THE MFTI

The mFTI may be perceived and constructed in various ways. Let us first explore the most intuitive flux-attachment picture of the mFTI, which is based on the kinematics of both the inter-valley Cooper pair, and also the inter-valley exciton. In the TMD system, the motion of electrons in each valley is governed by a real-space pseudo magnetic field. Hence, the physics of a Chern band with Chern number $C = +1$ can be viewed as a Landau level in a magnetic field[16,17]. More precisely, an electron in valley-1(2) will see magnetic flux $\pm\Phi_0 = 2\pi$ in each moiré unit cell. If one performs the standard flux attachment to electrons (independently for each valley), it is most natural to construct the conjugate composite Fermi liquid state discussed recently[28–31], which is a compressible state in the bulk. However, there is another alternative flux attachment which naturally leads to an incompressible FTI.

Let's start by considering the bosonic objects of the system: the inter-valley Cooper pairs and inter-valley excitons. A spin singlet inter-valley Cooper pair (whose annihilation operator is labeled as b^c) carrying charge-$2e$ sees zero net magnetic fields from the two valleys. An inter-valley exciton (labeled as b^s) carrying zero charges and spin-1 sees "spin" flux $\Phi^s = 2\Phi_0$ in each moiré unit cell. b^s should be viewed as a bosonic rotor. It can have both positive and negative density fluctuation but has zero average density when the system is time-reversal invariant. At half-filling of the conjugate pair of Chern bands, the density of b^c is at $1/2$ particle per moiré unit cell. It is energetically favorable for each b^c to be attached with flux $-4\Phi_0$ seen by b^s, such that b^s sees zero total spin flux. Likewise, b^c will view each b^s as $-4\Phi_0$ flux. Here, we note that since b^s has zero average density, b^c still sees zero total flux. A schematic illustration of the flux attachment is shown in Fig. 1. After the mutual flux attachment, the "composite bosons", formed by b^c and b^s bound with fluxes, can condense. The condensate of the composite bosons is an incompressible state without any spontaneous symmetry breaking due to the attached gauge fluxes.

The state constructed using the mutual flux attachment picture described above is described by the following Chern-Simons theory:

$$\mathcal{L}_{\text{CS}} = \frac{4\mathrm{i}}{2\pi} a^s da^c + \frac{\mathrm{i}}{2\pi} A^s da^s + \frac{2\mathrm{i}}{2\pi} A^e da^c. \tag{1}$$

Here $a^s da^c$ is a short-handed notation for $\epsilon_{\mu\nu\rho} a^s_\mu \partial_\nu a^c_\rho$. a^s_μ and a^c_μ are the "dual gauge field" of the current of b^s and b^c respectively: $J^{c,s}_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu\rho}\partial_\nu a^{c,s}_\rho$. A^e_μ and A^s_μ are the background charge and "spin gauge fields":

$A^e_\mu = (A_{1,\mu} + A_{2,\mu})/2$, $A^s_\mu = A_{1,\mu} - A_{2,\mu}$. Here $A_{1,2}$ are introduced as the background gauge fields of valley-1 (spin-up) and valley-2 (spin-down) electrons. They arise from the conservation of electron number in each valley/spin species. If we integrate out the dynamical gauge fields a^c and a^s from Eq. (1), the response theory for this state is

$$\mathcal{L}_{\rm res} = \frac{1}{2}\frac{\mathrm{i}}{2\pi}A^e dA^s = \frac{1}{2}\frac{\mathrm{i}}{4\pi}A_1 dA_1 - \frac{1}{2}\frac{\mathrm{i}}{4\pi}A_2 dA_2, \quad (2)$$

indicating the desired spin-Hall conductivity $\sigma^{\rm sh} = 1/2$. Also, there are no "diagonal" Hall responses for the electric charge and S_z quantum numbers separately, which is consistent with time-reversal symmetry.

The mutual Chern-Simons theory Eq. (1) implies that the mFTI is actually a $\mathbb{Z}_4$ TO "enriched" by two U(1) symmetries and time-reversal symmetry $\mathcal{T}$, i.e. it is a type of symmetry-enriched topological orders. Throughout this work, we assume that $\mathcal{T}^2 = -1$ when acting on an electron or a hole, i.e. electrons/holes are Kramers doublets. Within the $\mathbb{Z}_4$ TO, the "minimally-charged" anyon $\mathbf{e}$ carries physical electric charge $e/2$, and the "minimal-spin" anyon $\mathbf{m}$ carries a physical spin quantum number $S_z = \hbar/4$. Both of these anyons are self-bosons, but they see each other as a $\Phi_0/4$ flux, i.e. they have mutual statistical angle $\pi/2$. The $\mathbf{em}$ bound state carries an electric charge and a spin quantum number $(e/2, \hbar/4)$, and it is an anyon with self-statistical angle $\pi/2$, i.e. its topological spin is $\theta_{\mathbf{em}} = e^{\mathrm{i}\pi/2} = \mathrm{i}$. This $\mathbb{Z}_4$ TO also preserves the time-reversal symmetry $\mathcal{T}$, which keeps the anyon $\mathbf{e}$ invariant and maps the anyon $\mathbf{m}$ to its anti-particle $\mathbf{m}^3$. Curiously, $\mathbf{m}^2$ is a Kramers doublet under time reversal with $\mathcal{T}^2 = -1$. This result can be seen using both the edge theory in Sec. III and the general arguments of Ref. 14, which we will elaborate more in Sec. IV.

This $\mathbb{Z}_4$ TO have 16 anyons modulo the electrons (and holes), which can be represented as composite states $\mathbf{e}^p\mathbf{m}^q$ with $p, q = 0, \cdots 3$. However, since this $\mathbb{Z}_4$ TO is constructed with bosonic objects b^c and b^s, the background gapped electrons and holes were not yet taken into account. Hence, there exist another 16 anyons, which are bound states of $\mathbf{e}^p\mathbf{m}^q$ and the electron. Note that the bound state of $\mathbf{e}^p\mathbf{m}^q$ and the electron has a different topological spin from $\mathbf{e}^p\mathbf{m}^q$. Here, we've effectively included the electron as a "transparent" Abelian anyon (that braids trivially with all other anyons), which is a convenient technical choice for showing the minimality of this $\mathbb{Z}_4$ TO later. With the electrons and holes included, the $\mathbb{Z}_4$ topological order has in total 32 Abelian anyons, resulting in a squared total quantum dimension $\mathcal{D}^2 = 32$. In Sec. IV, we will prove that $\mathcal{D}^2 = 32$ is the minimal squared total quantum dimension compatible with the given Hall transports associated with the electric charge and spin quantum number S_z. If one views electrons/holes as trivial anyons and identifies the anyons differed only by an electron/hole, then the mFTI effectively has a squared total quantum dimension $\tilde{\mathcal{D}}^2 = 16$. In the Appendix, we further prove that the mFTI proposed here is the *unique* mFTI compatible with charge conservation, S_z conservation, time-reversal symmetry, and the desired Hall responses.

III. EXPERIMENTAL SIGNATURES

Following the standard derivation of the edge states from the bulk CS theory[32,33], we obtain the Lagrangian for the $1d$ edge based on the bulk CS theory Eq. 1

$$\mathcal{L}_{\rm edge} = \frac{4}{4\pi}\mathrm{i}\partial_\tau\phi^c\partial_x\phi^s - \frac{4}{4\pi}\mathrm{i}\partial_\tau\phi^s\partial_x\phi^c + \cdots, \quad (3)$$

where the "$\cdots$" is the kinetic energy that will be written in a different basis below. $\partial_x\phi^s$ and $\partial_x\phi^c$ are respectively the density of the Cooper pair b^c and spin-1 exciton b^s at the boundary. We have the identification $e^{\mathrm{i}4\phi^c} \sim b^c$ and $e^{\mathrm{i}4\phi^s} \sim b^s$. Under the time-reversal symmetry $\mathcal{T}$, the Cooper pair b^c we introduced is invariant, while the spin-1 exciton operator b^s shares the same behavior as the lowering operator of a spin-1/2 object, i.e. $\mathcal{T}b^s\mathcal{T}^{-1} = -b^{s\dagger}$. Therefore, the time-reversal action on the edge theory is given by $\mathcal{T}: \phi^c \to -\phi^c, \phi^s \to \phi^s + \pi/4$.[34] The anyon $\mathbf{m}^2$ is identified with the field $e^{\mathrm{i}2\phi_s}$, which implies that $\mathbf{m}^2$ is a Kramers doublet under time reversal with $\mathcal{T}^2 = -1$.

If the spin S_z conservation is broken but time-reversal symmetry is still preserved, an extra term $\cos(8\phi_s)$ is allowed in the edge theory Eq. 3. When this term is relevant, it will lead to two-fold degenerate ground states at the edge, which are characterized by nonzero expectation values of gauge invariant and time-reversal odd operator $\cos(4\phi_s)$, or $\sin(4\phi_s)$. Hence, the edge theory remains gapless unless there is either spontaneous or explicit time-reversal symmetry breaking (assuming charge is conserved). A different way to understand the stability of the gapless edge using only the bulk data will be given below.

By recombining ϕ^s and ϕ^c into $\phi_{1,2} = \phi^s \pm \phi^c$, we obtain a pair of counter-propagating (or helical) modes

$$\mathcal{L}_{\rm edge} = \frac{2}{4\pi}\mathrm{i}\partial_\tau\phi_1\partial_x\phi_1 - \frac{2}{4\pi}\mathrm{i}\partial_\tau\phi_2\partial_x\phi_2 - V_{IJ}\partial_x\phi_I\partial_x\phi_J. \quad (4)$$

The density $\partial_x\phi_1$ (or $\partial_x\phi_2$) carries charge e (or $-e$) and spin $\hbar/2$. We have added the kinetic energy with the velocity matrix V_{IJ} in Eq. (4) which arises from the non-topological part of the system. If V_{IJ} is proportional to Pauli matrix σ^z, the left and right counter-propagating modes do not interact with each other, and this edge theory would lead to a fractionally quantized $1d$ conductance per edge:

$$G = \frac{1}{2}\frac{e^2}{h}. \quad (5)$$

In a two-terminal measurement, the two-terminal conductance receives contributions from two edges connecting the two leads. At the total filling factor $\nu_{\rm tot} = 3$, there are two completely filled bands with $C = \pm 1$ in addition to the mFTI state in the half-filled conjugate Chern bands. Therefore, the total two-terminal conductance should be $G = 3e^2/h$, which was observed experimentally in the twisted bilayer $MoTe_2$ at $\nu_{\mathrm{tot}} = 3$.[8]

To give a full analysis of the $1d$ edge conductance with counter-propagating fractionalized modes, one needs careful analysis of the physics in the $1d$ channel as well as in the metallic lead and the contact[35–37]. We expect the full analysis to be more involved than the edge state of the quantum spin-Hall insulator without fractionalization[38,39], and we defer the systematic study to the future.

Our mFTI is essentially a bosonic symmetry-enriched TO in the sense that although the system is made of correlated electrons/holes, a single electron/hole is always gapped in the bulk and the boundary. Hence, there is always a single particle gap in the system, despite the existence of the gapless charge modes at the edge, contrary to most fractional quantum Hall states. This effect is analogous to the bosonic SPT state proposed to be realized in the bilayer graphene[40].

The smallest electric charge carried by the anyons of the mFTI is $e^* = e/2$. The odd value of $\sigma^{\mathrm{sh}}/e^* = 1$ in the mFTI guarantees the gaplessness of the edge whenever time-reversal symmetry and charge conversation are present (even without the S_z conservation).[14] In contrast, using similar arguments as Ref. 14, one can show that, given the spin-Hall conductivity $\sigma^{\mathrm{sh}} = 1/2$ (and the vanishing "diagonal" Hall responses associated with the charge and spin quantum numbers), any TO that is a product of two decoupled TOs from the two valleys must have excitations with fractional electric charges equal to or smaller than $e/4$. Therefore, the mFTI and the product TOs from the two valleys can be experimentally distinguished by probing the electric charge fractionalization. Examples of such probes include the shot noise of quantum point contacts[41], local capacitance probe of localized charge states[42], and Coulomb oscillations in antidots[43].

IV. BOOTSTRAPPING THE MINIMAL FTI

In this section, we bootstrap the minimal topological order (TO) allowed by the Hall responses under the charge and the S_z-spin U(1) symmetries and show that $\mathcal{D}^2 = 32$ is the minimal squared total quantum dimension compatible with a spin-Hall conductivity $\sigma^{\mathrm{sh}} = 1/2$, zero electric Hall responses, and zero Hall response with respect to the spin S_z quantum number. The vanishing of the diagonal Hall responses associated with either the electric charge or S_z is required by the time-reversal symmetry.

For technical convenience, we prefer to re-organize the charge and spin U(1) symmetries into $\mathrm{U}(1)_\uparrow \times$ $\mathrm{U}(1)_\downarrow$, where $\mathrm{U}(1)_{\uparrow,\downarrow}$ are associated with the charge conversation within each of the valleys 1 and 2 (which are locked to electron spins $\uparrow$ and $\downarrow$). The Hall responses we focus on in this paper are equivalent to the combination of (1) $\mathrm{U}(1)_\uparrow$-Hall conductivity $\sigma_1^{\mathrm{h}} = 1/2$, (2) $\mathrm{U}(1)_\downarrow$-Hall conductivity $\sigma_2^{\mathrm{h}} = -1/2$, and (3) zero mixed $\mathrm{U}(1)_\uparrow \times \mathrm{U}(1)_\downarrow$ response. For an anyon x, we denote its fractional charge under $\mathrm{U}(1)_\uparrow \times \mathrm{U}(1)_\downarrow$ as $q_x = (q_{x,1}, q_{x,2})$, which is defined modulo the charge of local bosonic objects, i.e. $q_x \sim q_x + (n, m)$ for $n, m \in \mathbb{Z}$ with $m + n$ even. The fractional electric charge and S_z-spin of the anyon x is then given by $(q_{x,1} + q_{x,2})e$ and $(q_{x,1} - q_{x,2})\hbar/2$. In the TO, the electrons and holes are all identified as the transparent fermionic particle f with $q_f = (\pm 1, 0) \sim (0, \pm 1)$. Two such f particles fuse into a trivial anyon $\mathbf{1}$, i.e. $f \times f = \mathbf{1}$

To bootstrap the minimal TO, we start by considering the anyons $v_{1,2}$ created by the adiabatic insertion of the 2π flux of $\mathrm{U}(1)_{\uparrow,\downarrow}$. They must be Abelian anyons[44]. Their fractional charges are given by the Hall conductivities $\sigma_{1,2}^{\mathrm{h}}$: $q_{v_1} = (1/2, 0)$ and $q_{v_2} = (0, -1/2)$. Their topological spins are $\theta_{v_1} = e^{\mathrm{i}\pi\sigma_1^{\mathrm{h}}} = \mathrm{i}$ and $\theta_{v_2} = e^{\mathrm{i}\pi\sigma_2^{\mathrm{h}}} = -\mathrm{i}$, which are the consequences of the Aharonov-Bohm (AB) phase between charge and flux. Based on the AB effect, the mutual statistics between v_1 and v_2 must be trivial because v_1 (v_2) does not carry any fractional $\mathrm{U}(1)_\downarrow$-charge ($\mathrm{U}(1)_\uparrow$-charge). The anyons $v_{1,2}^2$ are associated with the 4π flux of $\mathrm{U}(1)_{\uparrow,\downarrow}$. They are both self-bosons, i.e. $\theta_{v_1^2} = \theta_{v_1}^4 = 1$ and $\theta_{v_2^2} = \theta_{v_2}^4 = 1$. They have charges $q_{v_1^2} = (1, 0)$ and $q_{v_2^2} = (0, -1)$, which is non-trivial compared to local bosonic objects. Hence, v_1^2 and v_2^2 are non-trivial anyons. Since $\theta_{v_1^2} = \theta_{v_2^2}$ and $q_{v_1^2} \sim q_{v_2^2}$, there are two scenarios: (1) v_1^2 and v_2^2 belongs to the same anyon type, and (2) v_1^2 and v_2^2 are different anyons.

For the second scenario with $v_1^2 \neq v_2^2$, there are at least 32 different anyons of the form $v_1^n v_2^m f^k$ with $n, m = 0, 1, 2, 3$ and $k = 0, 1$. They can be distinguished from each other based on their topological spins, fractional charges, and the assumption that $v_1^2 \neq v_2^2$. Moreover, based on the AB-phase, v_2^2 and v_1^2 braids trivially with any of the 32 anyons listed above. For a TO to be "complete"[45], there must be some extra anyons that braid non-trivially v_1^2 and v_2^2. Consequently, the squared total quantum dimension must be larger than 32 in this scenario. Therefore, to search for the minimal TO, we only need to focus on the first scenario.

For the first scenario with $v_1^2 = v_2^2$, we show below that the minimal squared total quantum dimension is exactly $\mathcal{D}^2 = 32$. Let s denote the minimal (positive) integer s such that $v_1^{2s} = v_2^{2s}$ equals the trivial anyon $\mathbf{1}$. Given that $v_1^2 = v_2^2$ is non-trivial, $s \geq 2$. In the spirit of bootstrapping the minimal TO, we should focus on the case with $s = 2$. Similar arguments as below, when applied to the cases with $s \geq 3$, will lead to a squared total quantum dimension larger than 32.

From now on, we focus on the first scenario with $s = 2$. A list of 8 distinct Abelian anyons purely consists of $2\pi\mathbb{Z}$ fluxes of $\mathrm{U}(1)_{\uparrow,\downarrow}$ is given by

$$V = \{\mathbf{1}, v_1, v_1^2, v_1^3, v_2, v_2^3, v_1 v_2, v_1^3 v_2\}. \tag{6}$$

By including the transparent fermion f, one finds 16 different Abelian anyons $\{\mathbf{1}, f\} \times V$. They can be distinguished by their topological spins and fractional charges. Note that v_1^2 braids trivially with all anyons in $\{\mathbf{1}, f\}\times V$. Therefore, there must be an extra anyon, called it y, that braids non-trivially with $v_1^2 = v_2^2$. However, it must braid trivially with $v_1^4 = v_2^4 = \mathbf{1}$. Using the AB-phase interpretation of the braiding with v_1^n and v_2^m, we conclude that the fractional charge of y must be $q_y = (1/4+\mathbb{Z}, 1/4+\mathbb{Z})$. Without loss of generality, we can pick

$$q_y = (1/4, 1/4). \tag{7}$$

For other choices of q_y, we can find another anyon with the fractional charge $(1/4, 1/4)$ by fusing y with one of the Abelian anyons in $\{\mathbf{1}, f\} \times V$. From the AB effect, we can obtain mutual statistics between y and $v_1^n v_2^m f^k$:

$$M_{y,v_1^n v_2^m f^k} = e^{\mathrm{i}\frac{2\pi}{4}(n+m)}. \tag{8}$$

Therefore, the topological spins of the anyons $yv_1^n v_2^m f^k$ are all related to the topological spin of y via

$$\theta_{yv_1^n v_2^m f^k} = \theta_y \times (\mathrm{i})^{n^2} (-\mathrm{i})^{m^2} e^{\mathrm{i}\frac{2\pi}{4}(n+m)} (-1)^k. \tag{9}$$

Using this result of the topological spins (and fractional charges), we can examine the set of anyons $\{1, y\} \times \{1, f\} \times V$. This set may have redundancy because y and $yv_1^2 f$ share the same topological spin and fractional charge. As a result, there are two types, type-I and type-II, of minimal TOs.

For *type-I* minimal TOs, y and $yv_1^2 f$ are distinct. Consequently, there are 32 different anyons in

$$\mathcal{C}_{\mathrm{I}} = \{1, y\} \times \{1, f\} \times V. \tag{10}$$

For minimality, $\mathcal{C}_{\mathrm{I}}$ should be the entire set of anyons, and each anyon should be Abelian, which yields a squared total quantum dimension $\mathcal{D}^2 = 32$. The $\mathbb{Z}_4$ FTI constructed in Sec. II provides an example of such type-I minimal TOs. v_1, v_2, and y correspond to the anyons $\mathbf{em}$ and $\mathbf{e}^3\mathbf{m}$, $\mathbf{e}$ respectively. Note that the $\mathbb{Z}_4$ FTI not only has the desired Hall responses but also preserves the time-reversal symmetry. Moreover, $\mathbf{m}^2$, which corresponds to $v_1 v_2$, is the anyon generated by the insertion of a 2π flux in each valley. Its time-reversal conjugate should be associated with a -2π flux in each valley, which still belongs to the same anyon type $\mathbf{m}^2$ given that $v_1^2 v_2^2 \sim \mathbf{1}$. Knowing that $\mathbf{m}^2$ is related to its time-reversal partner by a 4π flux in each valley and that the spin Hall conductivity is $\sigma^{\mathrm{sh}} = 1/2$, $\mathbf{m}^2$ must be a Kramers doublet under time reversal[14], which is consistent with the analysis in Sec. III. This statement has used the fact that the system is ultimately built out of the electron and holes that are Kramers doublets with $\mathcal{T}^2 = -1$.

For *type-II* minimal TOs, y and $yv_1^2 f$ are identical. In other words, the fusion of y and $v_1^2 f$, a self-fermion, yields y. Consequently, y is a non-Abelian anyon whose quantum dimension of y is at least $\sqrt{2}$. Now, there are

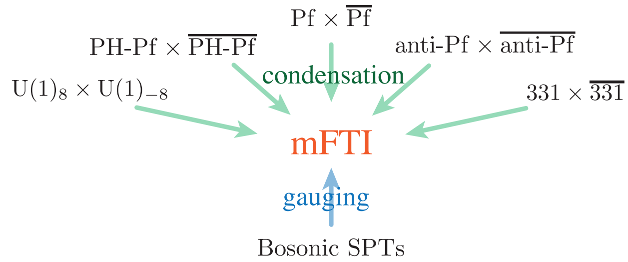

FIG. 2. The mFTI we propose is the common descendant of multiple product TOs with larger quantum dimensions. "Pf" stands for Pfaffian. It can also be viewed as a symmetry-enriched TO promoted from multiple bosonic SPTs by "gauging" the discrete subgroup of the symmetries.

24 distinct anyons (or 12 if one treats the electron/hole f as a trivial anyon):

$$\mathcal{C}_{\mathrm{II}} = (\{1, f\} \times V) \cup (y \times V). \tag{11}$$

For minimality, $\mathcal{C}_{\mathrm{II}}$ should be the entire set of anyons, and the quantum dimension of each anyon in $y \times V$ should be exactly $\sqrt{2}$, which again yields a squared total quantum dimension of $\mathcal{D}^2 = 32$. Note that all the anyons in $y \times V$ share the same quantum dimension because of the Abelian nature of anyons in V. Such type-II minimal TOs must be non-Abelian. They also require non-trivial couplings between the two valleys. More importantly, it turns out that they must break the time-reversal symmetry $\mathcal{T}$, as we showed in the Appendix (where a full 8-fold classification of all minimal TOs is also given). In the next section, we briefly discuss examples of such type-II minimal TO.

We emphasize that the bootstrap of type-I and type-II minimal TOs only uses charge conservation, spin-S_z conservation, and the Hall transport signals. In the Appendix, we further show that once time-reversal symmetry $\mathcal{T}$ is considered, the $\mathbb{Z}_4$ FTI constructed in Sec. II is the *unique* mFTI with the minimal total quantum dimension.

V. ALTERNATIVE CONSTRUCTIONS OF THE MFTI

A. Descending from larger topological orders

In this section, we present other constructions of the same $\mathbb{Z}_4$ mFTI. The mFTI can be constructed through a "top-down" approach by descending from "larger" TOs, such as a product of two separate TOs in each valley. The simplest example of such TO, which can be realized at the $\nu = 1/2$ filling of both valleys, is the $\mathrm{U}(1)_8 \times \mathrm{U}(1)_{-8}$ state. The $\mathrm{U}(1)_8$ TO realized in a half-filled Landau level can be viewed as a $\nu_b = 1/8$ bosonic Laughlin state for Cooper pairs[23], described by a level-8 Chern-Simons theory for a dynamical U(1) gauge field a, whose flux is dual to the density of the *intra-valley* Cooper pair. The anyons of the $\mathrm{U}(1)_8 \times \mathrm{U}(1)_{-8}$ state correspond to the gauge charges (n_1, n_2) of the two gauge fields, whose topological spins are given by $e^{\mathrm{i}\pi(n_1^2-n_2^2)/8}$. This anyon also carries electric charge $(n_1 - n_2)e/4$, and spin $S_z = (n_1 + n_2)\hbar/8$.

The anyon $(4, 4)$ is obviously a self-boson, hence it can condense. The anyon $(4, 4)$ can be created by adiabatically inserting 4π fluxes of $\mathrm{U}(1)_\uparrow$ and $\mathrm{U}(1)_\downarrow$ in both valleys. It carries spin $S_z = \hbar$. To avoid spontaneously breaking the spin S_z symmetry in the condensate, we bind the $(4, 4)$ anyon with the physical inter-valley exciton b^s with $S_z = -\hbar$ to neutralize its S_z spin. The mutual braiding statistical between the neutralized bosonic anyon $(4, 4)$ and a general anyon (n_1, n_2) is $e^{\mathrm{i}\pi(n_1-n_2)}$. Hence, the condensate of $(4, 4)$ would confine all the anyons whose n_1 and n_2 have opposite parity. The condensate also identifies anyons (n_1, n_2) and (n_1+4, n_2+4). Consequently, there are, in total, 16 anyons left (before including the electrons/holes), the same as the mFTI. Using the notation of Sec. IV, the anyon v_1 and v_2 correspond to $(2, 0)$, and $(0, 2)$. One of the most elementary deconfined anyons is $(1, 1)$, which is a self-boson with electric charge 0 and spin $S_z = \hbar/4$. Hence, $(1, 1)$ is identified with the anyon $\mathbf{m}$ of the mFTI. Similarly, the anyon $(1, -1)$ can be identified as anyon $\mathbf{e}$ of the mFTI. The anyons $(1, 1)$ and $(1, -1)$ also have the desired mutual braiding statistics $e^{\mathrm{i}\pi/2}$. At this point, we can confirm that the condensate of the neutralized anyon $(4, 4)$ is the mFTI with $\mathbb{Z}_4$ TO. As an alternative approach, how the $\mathrm{U}(1)_8 \times \mathrm{U}(1)_{-8}$ TO descends, via the anyon condensation of (4,4), to the desired mFTI can also be shown explicitly using the K-matrix formalism.

In fact, one can show that if we start with the other obvious product TOs, such as the Pfaffian $\times$ $\overline{\text{Pfaffian}}$, $331 \times \overline{331}$, anti-Pfaffian $\times$ $\overline{\text{anti-Pfaffian}}$, or PH-Pfaffian $\times$ $\overline{\text{PH-Pfaffian}}$, there is always a self-bosonic anyon created by inserting 4π fluxes in both valleys. Then, condensing this anyon after neutralizing its S_z spin will always lead to the same mFTI state.

As an aside, we can perform a similar "descending" construction starting from a product TO with *different* Abelian TOs in the two valleys, such as $331 \times \mathrm{U}(1)_{-8}$. A similar anyon condensation in $331 \times \mathrm{U}(1)_{-8}$ results in a type-I minimal TO that breaks time-reversal symmetry (manifested by its net chiral central charge $c = 1$). Hence, the so-constructed type-I minimal TO is not a candidate for mFTI even though they share the same Hall responses with respect to the electric charge and the S_z quantum number. All such type-I minimal TOs are classified in the Appendix.

B. "Promoting" SPT states

As we remarked earlier, the mFTI is a symmetry-enriched topological (SET) order. The subject of SET has attracted enormous theoretical interest in the past decade[46–49]. One very general way to construct nontrivial SET states is by "gauging" a part of the discrete symmetries of symmetry-protected topological (SPT) states[50,51], i.e. by "promoting" a part of the discrete global symmetries to local gauge invariance, and coupling the state to the gauge fields accordingly.

In our current case, the mFTI can be constructed through the same procedure. The SPT state we start with could be constructed by the inter-valley exciton operator b^s, and "one quarter" of the inter-valley Cooper pair b^c. In other words, we formally fractionalize b^c into four bosonic partons each carrying charge-$e/2$: $b^c = (\tilde{b}^c)^4$. Then, $\tilde{b}^c$ must coupled to a Z_4 gauge field. Now, we construct a minimal level-1 bosonic SPT state between b^s and $\tilde{b}_c$[52], which can be described in various formulations, including the Chern-Simons theory[53], or the nonlinear sigma model[52,54]. The most convenient formulation for our current purpose is the Chern-Simons theory developed in Ref. 53. The Chern-Simons theory for many $(2 + 1)d$ bosonic SPT states takes a universal form:

$$\mathcal{L}_{\mathrm{spt},1} = \frac{\mathrm{i}}{2\pi}\tilde{a}^c da^s + \frac{\mathrm{i}}{2\pi}A^s da^s + \frac{1}{2}\frac{\mathrm{i}}{2\pi}A^e d\tilde{a}^c, \qquad (12)$$

where $\tilde{a}^c_\mu$ and a^s_μ are the dual of the currents of $\tilde{b}^c$ and b^s. Now, we need to gauge this SPT by coupling $\tilde{b}^c$ to a Z_4 gauge field, which can be captured by adding the following terms

$$\mathcal{L}_{g,1} = \frac{\mathrm{i}}{2\pi}c_1 d\tilde{a}^c + \frac{4\mathrm{i}}{2\pi}c_1 dc_2. \qquad (13)$$

The gauge fields $c_{1,\mu}$ and $c_{2,\mu}$ are introduced as auxiliary dynamical gauge fields that describe the Z_4 gauge field through the last mutual Chern-Simons term. The coupling $\frac{\mathrm{i}}{2\pi}c_1 d\tilde{a}^c$ is the minimal coupling between the Z_4 gauge field and the current of $\tilde{b}^c$. Combining the Lagrangians of Eq. (12) and Eq. (13), the Chern-Simons couplings of all the dynamical gauge fields $a_I = (\tilde{a}^c, a^s, c_1, c_2)$ can be organized into a K-matrix theory

$$\mathcal{L}_{\mathrm{cs},1} = \frac{\mathrm{i}}{4\pi}K_1^{IJ} a_I da_J, \quad K_1 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \end{pmatrix}. \qquad (14)$$

The charge-1 particle of the gauge field c_1 is the minimal $\mathbf{e}$ anyon of the Z_4 gauge field, it carries a 2π flux of $\tilde{a}^c$, which carries physical electric charge $e/2$, as expected. The charge-1 particle of gauge field c_2 carries $\pi/2$ flux of c_1, which is equivalent to $\pi/2$ flux of a^s through the equation of motion of the gauge fields. The $\pi/2$ flux of a^s carries S_z-spin $\hbar/4$, hence the charge-1 of c_2 corresponds to the anyon $\mathbf{m}$ of our mFTI.

To make a direct connection to the original CS theory Eq. (1), we note that the 2×2 block of the K-matrix associated with $(\tilde{a}^c, a^s)$ has determinant -1. Hence, $\tilde{a}^c$ and a^s can be integrated out safely without compromising the nature of the TO. After integrating out $(\tilde{a}^c, a^s)$, and shifting $c_1 + \frac{1}{2}A^e \to c_1$, the theory Eq. 14 returns to the form of Eq. (1), except now (c_1, c_2) plays the role of (a^s, a^c).

Another way of constructing the same mFTI state is by introducing the "half-partons" for both b^c and b^s: $b^c \sim (\tilde{b}^c)^2$, and $b^s \sim (\tilde{b}^s)^2$. Now, $\tilde{b}^c$ and $\tilde{b}^s$ carry electric charge e and quantum number $S_z = \hbar/2$, respectively. And, they are both coupled to their own Z_2 gauge fields. We can make $\tilde{b}^c$ and $\tilde{b}^s$ form a level-1 bosonic SPT state. The entire theory now reads

$$\begin{aligned}\mathcal{L}_{\text{cs},2} &= \frac{\mathrm{i}}{2\pi}\tilde{a}^c d\tilde{a}^s + \frac{\mathrm{i}}{2\pi}(A^e + c_1)d\tilde{a}^c + \frac{2\mathrm{i}}{2\pi}c_1 dc_2 \\ &+ \frac{\mathrm{i}}{2\pi}(\frac{1}{2}A^s + c_1')d\tilde{a}^s + \frac{2\mathrm{i}}{2\pi}c_1' dc_2'. \end{aligned} \tag{15}$$

Again, $\tilde{a}^c$ and $\tilde{a}^s$ are dual to the currents of $\tilde{b}^c$ and $\tilde{b}^s$ respectively, and they can be integrated out safely. After integrating out $\tilde{a}^c$ and $\tilde{a}^s$, the $\mathcal{L}_{\text{cs},2}$ reduces to a CS theory for (c_1, c_2, c_1', c_2') with K-matrix

$$K_2 = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix} \simeq K_2' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \end{pmatrix}. \tag{16}$$

One can show that the K_2 matrix is equivalent to the K_2'-matrix up to an $\mathrm{SL}(4,\mathbb{Z})$ transformation, and K_2' again describes a Z_4 gauge field. The last two components of gauge fields of the K_2'-matrix couple to the external gauge field A^e and A^s in the same way as a^c and a^s in Eq. (1).

C. Example of type-II minimal TO

The non-Abelian "type-II minimal TO" discussed in Sec. IV can be obtained as descendants of the product TOs (with larger total quantum dimensions). As an example, We start from the product TO Pfaffian $\times\, \mathrm{U}(1)_{-8}$. The anyons of the Pfaffian TO is a subset of the topological order Ising $\times\, \mathrm{U}(1)_8$. Here, "Ising" stands for the Ising TO, which has three anyons $\{\mathbb{1}, \sigma, \psi\}$ with $\mathbb{1}$ the trivial anyon, σ the Ising anyon with quantum dimension $\sqrt{2}$, and ψ the self fermion. Their topological spins are $\theta_{\mathbb{1}} = 1$, $\theta_\sigma = e^{\mathrm{i}2\pi/16}$, and $\theta_\psi = -1$ respectively. For the $\mathrm{U}(1)_8 \times \mathrm{U}(1)_{-8}$ sector of Pfaffian $\times\, \mathrm{U}(1)_{-8}$, we can still use the charge (n_1, n_2) of the gauge fields of the $\mathrm{U}(1)_8 \times \mathrm{U}(1)_{-8}$ Chern-Simons theory as a part of the anyon label. The contribution to the topological spins from the gauge charge (n_1, n_2) is $\theta_{(n_1,n_2)} = e^{\mathrm{i}2\pi(n_1^2 - n_2^2)/16}$. The full set of anyons of Pfaffian $\times\, \mathrm{U}(1)_{-8}$ is given by

$$\begin{aligned} &(\mathbb{1}, n_1, n_2) \quad \text{with} \quad n_1 = 0,2,4,6 \text{ and } n_2 = 0,1,...,7, \\ &(\psi, n_1, n_2) \quad \text{with} \quad n_1 = 0,2,4,6 \text{ and } n_2 = 0,1,...,7, \\ &(\sigma, n_1, n_2) \quad \text{with} \quad n_1 = 1,3,5,7 \text{ and } n_2 = 0,1,...,7, \end{aligned} \tag{17}$$

where $(\psi, 4, 0)$ is identified as the transparent fermion f, i.e. the electron or the hole. The electric charge and S_z quantum numbers of these anyons are given by $(n_1 - n_2)e/4$ and $(n_1 + n_2)\hbar/8$. Note that $(\mathbb{1}, 4, 4)$ is self-boson without electric charge. Its S_z quantum number can be neutralized by adding a local boson, i.e. an intervalley exciton.

By condensing the neutralized version of $(\mathbb{1}, 4, 4)$ in Pfaffian $\times\, \mathrm{U}(1)_{-8}$, we obtain a type-II minimal TO with 24 deconfined anyons. Using the notation of Sec. IV, the 16 Abelian anyons in $\{1, f\} \times V$ are given by

$$(\mathbb{1}, n_1, n_2) \text{ and } (\psi, n_1, n_2) \quad \text{with} \quad n_{1,2} = 0,2,4,6 \tag{18}$$

with the identification $(\mathbb{1}/\psi, n_1, n_2) \sim (\mathbb{1}/\psi, n_1 + 4, n_2 + 4)$.[55] In particular, we can identify $v_1 = (\mathbb{1}, 2, 0)$ and $v_2 = (\mathbb{1}, 0, 2)$. The 8 non-Abelian quantum-dimension-$\sqrt{2}$ anyons are given by

$$(\sigma, n_1, n_2) \quad \text{with} \quad n_{1,2} = 1,3,5,7 \tag{19}$$

with the identification $(\sigma, n_1, n_2) \sim (\sigma, n_1 + 4, n_2 + 4)$. In particular, we have $y = (\sigma, 1, 7)$.

This non-Abelian TO, described by the condensate of $(\mathbb{1}, 4, 4)$, is one of the type-II minimal TO that can be realized in a pair of half-filled conjugate Chern bands. It has the same Hall responses with respect to the electric charge and the S_z quantum number as the mFTI. However, this state breaks the time-reversal symmetry because it has the same chiral central charge $c = 1/2$ as the original Pfaffian $\times\, \mathrm{U}(1)_{-8}$ TO. Consequently, this type-II minimal TO also has a quantized thermal-Hall response.

More examples of type-II minimal TOs can be constructed as the descendants of other combinations of a non-Abelian TO (which could be the Pfaffian TO, the anti-Pfaffian TO, the PH-Pfaffian TO, and their conjugates) in one valley and an Abelian TO (which could be $\mathrm{U}(1)_8$, the 331 state, and their conjugates) in the other valley. A systematic construction of all the type-II minimal TOs is given in the Appendix. All four of them break time-reversal symmetry.

D. mFTIs with $\sigma^{\text{sh}} = \frac{1}{q}$: bootstrap and classification

Now, we generalize the study of mFTI to a pair of conjugate Chern bands with $1/q$ electron filling in each band. Here, q is a positive integer. It is natural to expect such a mFTI to have a fractional spin-Hall conductivity $\sigma^{\text{sh}} = 1/q$ and vanishing diagonal Hall responses for the electric charge and the spin-S_z quantum number. The symmetry of such a mFTI includes the charge conservation, S_z conservation, and time-reversal symmetry $\mathcal{T}$. Per our definition, the mFTI is minimal in the sense that it has the smallest total quantum dimension among all the TOs with the required symmetries and Hall transport signals with the given q. We will separately discuss the mFTIs with even and odd q's. And we will provide a classification of them.

For even q, by generalizing Sec. IV, we can show that there is a list of, in total, eight different minimal TOs compatible with charge conversation, S_z conservation, and the Hall transports described above. The details are given in the Appendix. After further imposing time-reversal symmetry $\mathcal{T}$, the list narrows down to a *unique* mFTI, which has the $\mathbb{Z}_{2q}$ TO, the same TO as the $\mathbb{Z}_{2q}$ toric code. The squared total quantum dimension (with the electron included) is $\mathcal{D}^2 = 8q^2$. The minimally-charged anyon has an electric charge $e^* = e/q$, and the minimal-S_z anyon has the S_z quantum number $\hbar/2q$. Such an mFTI is not a valley-decoupled product TO. Since σ^{sh}/e^* is odd, based on the result of Ref. 14, the gapless edge state of the mFTI is protected by charge conservation and time-reversal symmetry (even when S_z is not conserved). The edge theory is a straightforward generalization of Eq. (3)

$$\mathcal{L}_{\text{edge}} = \frac{2q}{4\pi}\mathrm{i}\partial_\tau\phi^c\partial_x\phi^s - \frac{2q}{4\pi}\mathrm{i}\partial_\tau\phi^s\partial_x\phi^c + \cdots, \tag{20}$$

Again, $e^{\mathrm{i}2q\phi^c}$ is the annihilation operator of the charge-$2e$ Cooper pair, while $e^{\mathrm{i}2q\phi^s}$ is that of an intervalley exciton with the S_z quantum number $\hbar$. Under time-reversal symmetry, the edge modes transform as

$$\mathcal{T}\phi_c\mathcal{T}^{-1} = -\phi_c, \mathcal{T}\phi_s\mathcal{T}^{-1} = \phi_s + \frac{\pi}{2q}. \tag{21}$$

For odd q, we find that the unique mFTI is given by the product of the $1/q$ Laughlin state and its time-reversal conjugate. The minimally-charged anyon has an electric charge $e^* = e/q$. The squared total quantum dimension is $\mathcal{D}^2 = 2q^2$.

It is also interesting to consider the stability of the gapless states at the interface between different mFTIs, based on which one can deduce a full classification of all the mFTIs with $\sigma^{\text{sh}} = \frac{1}{q}$. For convenience, we denote the mFTI with $\sigma^{\text{sh}} = \frac{1}{q}$ as mFTI$_q$. For our classification, if mFTI$_q$ and mFTI$_{q'}$ can admit a gapped interface that respects charge conservation and time-reversal symmetry, we consider them as equivalent, i.e. mFTI$_q \sim$ mFTI$_{q'}$. Note that we allow the breaking of the S_z conversation on the interface as far as this equivalence relation is concerned. Asking whether mFTI$_q$ and mFTI$_{q'}$ have a stable gapless interface is equivalent to asking whether a system obtained from stacking mFTI$_q$ and mFTI$_{q'}$ has a stable gapless edge state. It is helpful to write $q = 2^k p$ with $k = 0, 1, 2, 3, ...$ and p odd. It turns out that each k corresponds to a distinct class of mFTIs, which we show in the following.

Consider mFTI$_{q_1}$ and mFTI$_{q_2}$ with $q_1 = 2^{k_1}p_1$ and $q_2 = 2^{k_2}p_2$, where p_1 and p_2 are odd integers. Without loss of generality, let us assume $k_1 \geq k_2$. When we stack mFTI$_{q_1}$ and mFTI$_{q_2}$, the total spin-Hall conductance is

$$\sigma^{\text{sh}}_{\text{tot}} = \frac{1}{q_2} + \frac{1}{q_2} = \frac{2^{k_1-k_2}p_1 + p_2}{2^k_1 p_1 p_2}. \tag{22}$$

mFTI$_{q_1}$ (mFTI$_{q_2}$) by itself has anyon with the minimal electric charge $\frac{e}{2^{k_1}p_1}$ ($\frac{e}{2^{k_2}p_2}$). Hence, the stacked system admits a minimal charge of

$$e^* = \frac{\gcd(2^{k_1-k_2}p_1, p_2)}{2^{k_1}p_1p_2}e = \frac{\gcd(p_1, p_2)}{2^{k_1}p_1p_2}e. \tag{23}$$

Knowing that $p_{1,2}$ are odd, we can conclude that $\sigma^{\text{sh}}_{\text{tot}}/e^*$ is odd when $k_1 \neq k_2$ and even when $k_1 = k_2$. By the criteria obtained in Ref. 14, when $\sigma^{\text{sh}}_{\text{tot}}/e^*$ is odd, i.e. $k_1 \neq k_2$, the stacked system has a stable gapless edge protected by charge conservation and time-reversal symmetry. When $\sigma^{\text{sh}}_{\text{tot}}/e^*$ is even, i.e. $k_1 = k_2$, the stacked system admits a charge-conserving, time-reversal-symmetric gapped edge[15]. Therefore, the mFTI$_q$ with $q = 2^k p$ is classified by the non-negative integer k according to the equivalence relation defined above. For each class, we can choose the state mFTI$_{q=2^k}$ as the representative.

VI. SUMMARY

Motivated by a recent experiment[8], in this work, we proposed a minimal fractional topological insulator that can potentially be realized in a pair of conjugate Chern bands, both with electron filling $\nu = 1/2$. The mFTI we proposed has the minimal total quantum dimension amongst all the topological orders that can be potentially realized in the same setting (especially with the same symmetry and the same experimentally observed fractionally quantized transport signal). We caution that a more thorough analysis, similar to that for the $\nu = 2/3$ fractional quantum state[35–37], is required to understand the transport quantization. We found that the minimally-charged anyon in mFTI has a fractional charge $e^* = e/2$, which is different from all product TOs. The gapless edge state of this mFTI is protected by charge conservation and time-reversal symmetry even when S_z is not conserved. We also showed that the mFTI is the common descendant of various product TOs of the two valleys, which have larger quantum dimensions. The mFTI is also the common symmetry-enriched TO that can be promoted from different bosonic SPT states by gauging the discrete subgroup of their symmetries.

If the constraint on time-reversal symmetry is released (while maintaining the Hall transports requirements associated with the electric charge and S_z), we find mFTI belonging to an 8-fold classified list of minimal TOs. Four of them, albeit time-reversal broken, are non-Abelian TOs that exhibit quantized thermal Hall effects in addition to the required Hall responses associated with the charge and S_z quantum numbers.

We generalize the construction of mFTI (and the minimal TOs) to a pair of conjugate $1/q$-filled Chern bands (for positive integer q). We classify all the mFTIs via the stability of the gapless interfaces between them.

The authors thank Leon Balents and Matthew Fisher for very helpful discussions. C.X. is supported by the Simons foundation through the Simons investigator program. M.C. acknowledges support from NSF under award number DMR-1846109. C.-M.J. is supported by a faculty startup grant at Cornell University. While finishing this paper, we became aware of another independent work[56] studying the edge state of FTIs that are a product of conjugate TOs from two valleys, which, as we discussed, should have a larger quantum dimension than the mFTI constructed in this paper.

Appendix A: Classifying minimal TOs with or without time-reversal symmetry

In this appendix, we perform a systematic classification of minimal TOs for gapped topological states of electrons with fractional spin-Hall conductivity $\sigma^{\mathrm{sh}} = \frac{1}{q}$ (with positive integer q), but no diagonal charge or spin quantum Hall effect (i.e. no transverse spin current responding to spin gauge field). This classification of minimal TOs only assumes the charge and the spin-S_z conservations. Then, we identify the mFTIs by further imposing the time-reversal symmetry $\mathcal{T}$. It turns out that there is a unique mFTI for each q. We separately discuss the cases with even and odd q's. For even q, we find that there are exactly eight minimal TOs, all with squared total quantum dimension $\mathcal{D}^2 = 8q^2$. One of them is the $\mathbb{Z}_{2q}$ TO, the same TO as the $\mathbb{Z}_{2q}$ toric code. Further imposing time-reversal symmetry, we find that this $\mathbb{Z}_{2q}$ TO is singled out as the unique mFTI that satisfies all the symmetry and transport requirements. For odd q, there is a unique minimal TO given by a product TO $\mathrm{U}(1)_q \boxtimes \mathrm{U}(1)_{-q}$, which can also be compatible with time-reversal symmetry.

1. Minimal TOs without assuming time-reversal symmetry

Let's first discuss the case with even q. Following the notations in the main text, define $v_{1,2}$ as the anyons generated by the 2π fluxes of $\mathrm{U}(1)_{\uparrow,\downarrow}$. We denote the $\mathrm{U}(1)_{\uparrow,\downarrow}$-charges of an anyon x by $q_x = (q_{x,1}, q_{x,2})$. The charge and spin Hall responses translate into the following conditions on the self and mutual statistics of $v_{1,2}$:

$$\theta_{v_1} = e^{\frac{\mathrm{i}\pi}{q}}, \theta_{v_2} = e^{-\frac{\mathrm{i}\pi}{q}}, M_{v_1,v_2} = 1. \tag{A1}$$

Note that v_1 has charge $q_{v_1} = (\frac{1}{q}, 0)$ and v_2 has $q_{v_2} = (0, -\frac{1}{q})$.

We see that $v_1^q v_2^q$ is a boson carrying charge $(1, -1)$. Similarly, the boson v_1^{2q} (v_2^{2q}) has charge $(2, 0)$ ($(0, 2)$). Their charges can be neutralized by attaching physical bosons (Cooper pairs or excitons). Thus, we can always condense $v_1^q v_2^q, v_1^{2q}$ and v_2^{2q} to reduce the TO, without breaking the $\mathrm{U}(1)_\uparrow \times \mathrm{U}(1)_\downarrow$ symmetry. After the condensation, we find that the TO contains the subcategory $\mathcal{C}_0$ generated by v_1, v_2 subject to the relations $v_1^q = v_2^q = b,\ b^2 = 1$. Here, we've introduced the notation b, which labels a self-boson with a $\mathbb{Z}_2$ fusion rule. In addition, we have

$$M_{b,v_1} = M_{v_1^q,v_1} = \theta_{v_1}^{2q} = 1, M_{b,v_2} = M_{v_2^q,v_2} = \theta_{v_2}^{2q} = 1.$$

If we further condense v_1^q and v_2^q, the resulting TO is the product of the $\mathrm{U}(1)_q$ TO generated by v_1 and the $\mathrm{U}(1)_{-q}$ TO generated by v_2. Therefore, we conclude that $\mathcal{C}_0$ describes the "symmetrization" (or "equivariantization") of the $\mathrm{U}(1)_q \boxtimes \mathrm{U}(1)_{-q}$ TO enriched by a $\mathbb{Z}_2$ symmetry. The $b = v_1^q = v_2^q$ anyon is identified as the $\mathbb{Z}_2$ charge. We infer from the fusion rule that both v_1 and v_2 carry "$1/q$ charge" under this $\mathbb{Z}_2$ symmetry.

To find the minimal TO, we use the fact that the TO is a (fermionic) modular extension of $\mathcal{C}_0$. Using a theorem in Ref. 57 (Proposition 5.1), the total quantum dimension of any modular extension satisfies $\mathcal{D}^2 \geq 8q^2$ (counting the electron as a nontrivial anyon type). In this case, because $\mathcal{C}_0$ has a $\mathbb{Z}_2$ transparent center (generated by b), all minimal modular extensions are obtained from gauging the $\mathbb{Z}_2$ symmetry. One of the modular extensions is just the $\mathbb{Z}_{2q}$ toric code $\mathrm{D}(\mathbb{Z}_{2q})$. However, when gauging the $\mathbb{Z}_2$ symmetry, one also has additional freedom in choosing a topological term for the $\mathbb{Z}_2$ gauge field, or in other words, stacking a $\mathbb{Z}_2$ SPT state. Notice that we are considering a fermionic system. So, more precisely, the stacked SPT state should be a $\mathbb{Z}_2 \times \mathbb{Z}_2^f$ fermionic SPT state, which is classified by $\mathbb{Z}_8$. Hence, all possible fermionic modular extensions of $\mathcal{C}_0$ can be represented as follows:

$$\mathcal{M}_n = \frac{\mathrm{D}(\mathbb{Z}_{2q}) \boxtimes \mathrm{Spin}(n)_1}{(\mathbf{e}^q\mathbf{m}^q, \psi, f)}, n = 0, 1, \ldots, 7. \tag{A2}$$

Here, $\mathrm{Spin}(n)_1$ denotes the TO of the level-1 $\mathrm{Spin}(n)$ Chern-Simons theory, ψ is the neutral fermion in $\mathrm{Spin}(n)_1$, and f represents the physical electron/hole. $\mathbf{e}$ and $\mathbf{m}$ are the two elementary anyons in the $\mathbb{Z}_{2q}$ toric code $\mathrm{D}(\mathbb{Z}_{2q})$. The quotient means condensing the bound state of $\mathbf{e}^q\mathbf{m}^q$, ψ, and f.

One can check that all the eight theories above are distinct from each other. Without further conditions, we have found all possible minimal TOs. Notice that even (odd) n's corresponds to the type-I (type-II) minimal TOs. For example, the TO discussed in Sec. V C corresponds to $\mathcal{M}_1$.

For q odd, we note that v_1^q is actually a fermion carrying charges $(1, 0)$. We can thus identify it with the spin-up electron. Similarly, we can identify v_2^q with the spin-down hole. With this identification, we obtain the minimal TO $\mathrm{U}(1)_q \boxtimes \mathrm{U}(1)_{-q}$, which has same TO as the

$\mathbb{Z}_q$ toric code (in a fermionic system). This state is nothing but the simple fractional QSH state discussed in Ref. 14 whose K-matrix and charge vectors are given by

$$K = \begin{pmatrix} q & 0 \\ 0 & -q \end{pmatrix},\ t_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ t_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{A3}$$

This TO has the minimal squared total quantum dimension $\mathcal{D}^2 = 2q^2$. When $q = 1$, this K-matrix theory above reduces to the theory of the non-interacting topological insulator.

2. Imposing time-reversal symmetry

For odd q, the unique minimal TO is compatible with time-reversal symmetry. Hence, the unique minimal TO $\mathrm{U}(1)_q \boxtimes \mathrm{U}(1)_{-q}$ is the unique mFTI.

For the rest of the discussion, we focus on even q. Among $\mathcal{M}_{n=0,1,\ldots,7}$ in Eq. (A2), the odd n theories all have non-zero chiral central charge and exhibit thermal Hall effect, breaking the time-reversal symmetry. So let us consider the even n theories. The theory $\mathcal{M}_0$ is the mFTI discussed in Sec. II and Sec. V D. As we will see in the following, any other theories $\mathcal{M}_{n\neq 0}$ cannot be made compatible with time-reversal symmetry.

For even n, $\mathcal{M}_n$ is an Abelian TO and can be described by a K-matrix theory. First, recall that the $\mathrm{U}(1)_q \boxtimes \mathrm{U}(1)_{-q}$ theory is described by the following Chern-Simons theory:

$$\mathcal{L} = \frac{\mathrm{i}q}{4\pi} a_1 da_1 - \frac{\mathrm{i}q}{4\pi} a_2 da_2 + \frac{\mathrm{i}}{2\pi} A d(a_1 + a_2). \tag{A4}$$

v_1 (v_2) carries a unit gauge charge under $a_1 (a_2)$. To describe the $\mathbb{Z}_2$ symmetry enrichment, it is convenient to first enlarge the $\mathbb{Z}_2$ symmetry to U(1), under which v_1 and v_2 both carry charge $1/q$. A in Eq. (A4) denotes the background gauge field for the enlarged U(1) symmetry. Then, we Higgs the U(1) to $\mathbb{Z}_2$, which can be implemented by the BF term $\frac{1}{\pi} BdA$ with a U(1) gauge field B. We can then promote both A and B to dynamical gauge fields to gauge the symmetry. The SPT layer can be accounted for by adding a Chern-Simons term $-\frac{n}{8\pi} AdA$ for A. All together, we have found the following K-matrix (gauge fields ordered as a_1, a_2, A, B):

$$K_{\mathcal{M}_n} = \begin{pmatrix} q & 0 & 1 & 0 \\ 0 & -q & 1 & 0 \\ 1 & 1 & -n/2 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}. \tag{A5}$$

Evidently, we have the following identification: $v_1 = (1, 0, 0, 0), v_2 = (0, 1, 0, 0)$. In addition, $\mathbf{m} = (0, 0, 0, -1)$ represents a $\mathbb{Z}_2$ gauge flux anyon. The fusion rule of the Abelian anyons in this theory is given by the anyon group $\mathbb{Z}_{2q} \times \mathbb{Z}_{2q}$, the same as the $\mathbb{Z}_{2q}$ toric code. The two generators of the anyon group can be chosen as $v_1 = (1, 0, 0, 0)$ and $\mathbf{m} = (0, 0, 0, -1)$. And we have $v_2 = v_1^{-1+nq/2}\mathbf{m}^2$. The self and mutual statistics of these generators are given by

$$\theta_{v_1} = e^{\frac{\mathrm{i}\pi}{q}},\ \theta_{\mathbf{m}} = e^{\frac{\mathrm{i}\pi n}{8}},\ M_{v_1,\mathbf{m}} = e^{\frac{\mathrm{i}\pi}{q}}. \tag{A6}$$

We now determine the time-reversal transformations of anyons. First, since v_1 and v_2 are generated by 2π fluxes of $\mathrm{U}(1)_{\uparrow,\downarrow}$, they should have the following transformations:

$$\mathcal{T}(v_1) = v_2^{-1} = v_1^{1-nq/2}\mathbf{m}^{-2}, \mathcal{T}(v_2) = v_1^{-1}. \tag{A7}$$

We can then use the exchange and braiding statistics to fix the transformation of $\mathbf{m}$. In general, we may write $\mathcal{T}(\mathbf{m}) = \mathbf{m}^a v_1^b f^c$ with $a, b, c \in \mathbb{Z}$. $M_{\mathcal{T}(\mathbf{m}),\mathcal{T}(v_1)} = e^{-\frac{\mathrm{i}\pi}{q}}$ yields $a = -1$. Then, the requirement $\theta_{\mathcal{T}(\mathbf{m})} = \theta^*_{\mathbf{m}}$ yields

$$\theta_{\mathcal{T}(\mathbf{m})} = e^{\mathrm{i}\pi\left(\frac{n}{8} + \frac{b(b-1)}{q}\right)}(-1)^c = e^{-\frac{\mathrm{i}\pi n}{8}} = \theta^*_{\mathbf{m}}. \tag{A8}$$

There are extra constraints coming from the compatibility between time-reversal symmetry and the $\mathrm{U}(1)_\uparrow \times \mathrm{U}(1)_\downarrow$ symmetry. More specifically, the $\mathrm{U}(1)_{\uparrow,\downarrow}$ charges should be inter-changed under $\mathcal{T}$. That means the time-reversal partner $\mathcal{T}(x)$ of the anyon x should carry charges:

$$q_{\mathcal{T}(x)} = (q_{x,2}\ ,\ q_{x,1}) + (n_1, n_2) \tag{A9}$$

Here, (n_1, n_2) represents the charge of local bosons, i.e. $n_{1,2} \in \mathbb{Z}$, and $n_1 + n_2 \equiv 0 \bmod 2$.

Combining $\mathcal{T}(\mathbf{m}) = \mathbf{m}^{-1}v_1^b f^c$ with the constrain Eq. (A9), we have

$$\begin{aligned} -q_{\mathbf{m},1} + \frac{b}{q} + c &= q_{\mathbf{m},2} + n_1, \\ -q_{\mathbf{m},2} &= q_{\mathbf{m},1} + n_2, \end{aligned} \tag{A10}$$

which leads to

$$-q_{\mathbf{m},1} - q_{\mathbf{m},2} = n_1 - \frac{b}{q} - c = n_2. \tag{A11}$$

Therefore, b must be an integer multiple of q. Since b is defined modulo $2q$, we only have the options $b = 0$ or $b = q$.

For $b = 0$, we find $n = 0$ or 4 are the only possibilities compatible with Eq. (A8). For $n = 4$, it follows that $c = 1$ and $\mathcal{T}(\mathbf{m}) = \mathbf{m}^{-1}f$.

For $b = q$, we again find $n = 0$ or 4 to be the only possibilities compatible with Eq. (A8). For $n = 4$, it follows that $c = 0$, and we have $\mathcal{T}(\mathbf{m}) = \mathbf{m}^{-1}v_1^q$.

At this point, we only need to examine $\mathcal{M}_4$. With $n = 4$, for both $b = 0$ and $b = q$, we find $n_1 - n_2 = \frac{b}{q} + c = 1$, contradicting the requirement that $n_1 + n_2$ is even. We conclude that it is impossible for the theory $\mathcal{M}_4$ to have a charge assignment under $\mathrm{U}(1)_\uparrow \times \mathrm{U}(1)_\downarrow$ that is compatible with the time-reversal symmetry. Hence, $\mathcal{M}_0$ is the unique minimal TO that is consistent with time-reversal symmetry. Hence, it is the unique mFTI.

[1] E. M. Spanton, A. A. Zibrov, H. Zhou, T. Taniguchi, K. Watanabe, M. P. Zaletel, and A. F. Young, Science **360**, 62 (2018).

[2] Y. Xie, A. T. Pierce, J. M. Park, D. E. Parker, E. Khalaf, P. Ledwith, Y. Cao, S. H. Lee, S. Chen, P. R. Forrester, K. Watanabe, T. Taniguchi, A. Vishwanath, P. Jarillo-Herrero, and A. Yacoby, Nature (London) **600**, 439 (2021), arXiv:2107.10854 [cond-mat.mes-hall].

[3] J. Cai, E. Anderson, C. Wang, X. Zhang, X. Liu, W. Holtzmann, Y. Zhang, F. Fan, T. Taniguchi, K. Watanabe, Y. Ran, T. Cao, L. Fu, D. Xiao, W. Yao, and X. Xu, Nature **622**, 63 (2023).

[4] H. Park, J. Cai, E. Anderson, Y. Zhang, J. Zhu, X. Liu, C. Wang, W. Holtzmann, C. Hu, Z. Liu, T. Taniguchi, K. Watanabe, J.-H. Chu, T. Cao, L. Fu, W. Yao, C.-Z. Chang, D. Cobden, D. Xiao, and X. Xu, Nature **622**, 74 (2023).

[5] Y. Zeng, Z. Xia, K. Kang, J. Zhu, P. Kn¨**1**ppel, C. Vaswani, K. Watanabe, T. Taniguchi, K. F. Mak, and J. Shan, "Integer and fractional chern insulators in twisted bilayer mote2," (2023), arXiv:2305.00973 [cond-mat.mes-hall].

[6] F. Xu, Z. Sun, T. Jia, C. Liu, C. Xu, C. Li, Y. Gu, K. Watanabe, T. Taniguchi, B. Tong, *et al.*, arXiv preprint arXiv:2308.06177 (2023).

[7] Z. Lu, T. Han, Y. Yao, A. P. Reddy, J. Yang, J. Seo, K. Watanabe, T. Taniguchi, L. Fu, and L. Ju, "Fractional quantum anomalous hall effect in a graphene moire superlattice," (2023), arXiv:2309.17436 [cond-mat.mes-hall].

[8] K. Kang, B. Shen, Y. Qiu, K. Watanabe, T. Taniguchi, J. Shan, and K. F. Mak, "Observation of the fractional quantum spin hall effect in moiré mote2," (2024), arXiv:2402.03294 [cond-mat.mes-hall].

[9] F. Wu, T. Lovorn, E. Tutuc, I. Martin, and A. H. MacDonald, Phys. Rev. Lett. **122**, 086402 (2019).

[10] M. König, S. Wiedmann, C. Brüne, A. Roth, H. Buhmann, L. W. Molenkamp, X.-L. Qi, and S.-C. Zhang, Science **318**, 766 (2007), https://www.science.org/doi/pdf/10.1126/science.1148047.

[11] N. H. Lindner, E. Berg, G. Refael, and A. Stern, Phys. Rev. X **2**, 041002 (2012).

[12] M. Cheng, Phys. Rev. B **86**, 195126 (2012).

[13] A. Stern, Annual Review of Condensed Matter Physics **7**, 349–368 (2016).

[14] M. Levin and A. Stern, Phys. Rev. Lett. **103**, 196803 (2009).

[15] M. Levin and A. Stern, Phys. Rev. B **86**, 115131 (2012).

[16] N. Morales-Dur¨¢n, N. Wei, and A. H. MacDonald, "Magic angles and fractional chern insulators in twisted homobilayer tmds," (2023), arXiv:2308.03143 [cond-mat.str-el].

[17] N. Paul, Y. Zhang, and L. Fu, Science Advances **9** (2023), 10.1126/sciadv.abn1401.

[18] G. Moore and N. Read, Nuclear Physics B **360**, 362 (1991).

[19] M. Greiter, X.-G. Wen, and F. Wilczek, Phys. Rev. Lett. **66**, 3205 (1991).

[20] X.-G. Wen, Phys. Rev. Lett. **70**, 355 (1993).

[21] M. Levin, B. I. Halperin, and B. Rosenow, Physical Review Letters **99** (2007), 10.1103/physrevlett.99.236806.

[22] D. T. Son, Phys. Rev. X **5**, 031027 (2015).

[23] M. Greiter, X. Wen, and F. Wilczek, Nuclear Physics B **374**, 567 (1992).

[24] X. G. Wen and A. Zee, Phys. Rev. B **46**, 2290 (1992).

[25] B. J. Overbosch and X.-G. Wen, arXiv e-prints , arXiv:0804.2087 (2008), arXiv:0804.2087 [cond-mat.mes-hall].

[26] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 226801 (2005).

[27] B. A. Bernevig and S.-C. Zhang, Phys. Rev. Lett. **96**, 106802 (2006).

[28] Y.-H. Zhang, "Composite fermion insulator in opposite-fields quantum hall bilayers," (2018), arXiv:1810.03600 [cond-mat.str-el].

[29] N. Myerson-Jain, C.-M. Jian, and C. Xu, "The conjugate composite fermi liquid," (2023), arXiv:2311.16250 [cond-mat.str-el].

[30] Y.-H. Zhang, "Vortex spin liquid with neutral fermi surface and fractional quantum spin hall effect at odd integer filling of moiré chern band," (2024), arXiv:2402.05112 [cond-mat.str-el].

[31] Z. D. Shi, H. Goldman, Z. Dong, and T. Senthil, "Excitonic quantum criticality: from bilayer graphene to narrow chern bands," (2024), arXiv:2402.12436 [cond-mat.str-el].

[32] X. G. Wen, Phys. Rev. Lett. **64**, 2206 (1990).

[33] X. G. Wen, Phys. Rev. B **43**, 11025 (1991).

[34] In principle, one can consider a more general time-reversal action with $\phi^c \to -\phi^c + \alpha_0$ for some finite constant α_0. However, this constant α_0 can be absorbed by redefining the time-reversal symmetry as the original one followed by an extra charge U(1) rotation. The redefined time-reversal action remains to be a symmetry action of order 2. We caution that one cannot redefine the time-reversal symmetry by combining it with arbitrary spin S_z rotations because the resulting action is generically not an order-2 action anymore.

[35] C. L. Kane, M. P. A. Fisher, and J. Polchinski, Phys. Rev. Lett. **72**, 4129 (1994).

[36] C. L. Kane and M. P. A. Fisher, Phys. Rev. B **51**, 13449 (1995).

[37] C. L. Kane and M. P. A. Fisher, Phys. Rev. B **52**, 17393 (1995).

[38] C. Xu and J. E. Moore, Phys. Rev. B **73**, 045322 (2006).

[39] C. Wu, B. A. Bernevig, and S.-C. Zhang, Phys. Rev. Lett. **96**, 106401 (2006).

[40] Z. Bi, R. Zhang, Y.-Z. You, A. Young, L. Balents, C.-X. Liu, and C. Xu, Phys. Rev. Lett. **118**, 126801 (2017).

[41] L. Saminadayar, D. C. Glattli, Y. Jin, and B. Etienne, Phys. Rev. Lett. **79**, 2526 (1997).

[42] J. Martin, S. Ilani, B. Verdene, J. Smet, V. Umansky, D. Mahalu, D. Schuh, G. Abstreiter, and A. Yacoby, Science **305**, 980 (2004), https://www.science.org/doi/pdf/10.1126/science.1099950.

[43] A. Kou, C. M. Marcus, L. N. Pfeiffer, and K. W. West, Phys. Rev. Lett. **108**, 256803 (2012).

[44] M. Cheng, M. Zaletel, M. Barkeshli, A. Vishwanath, and P. Bonderson, Phys. Rev. X **6**, 041068 (2016).

[45] A. Kitaev, Annals of Physics **321**, 2 (2006), arXiv:cond-mat/0506438 [cond-mat.mes-hall].

[46] A. M. Essin and M. Hermele, Phys. Rev. B **87**, 104406 (2013).

[47] A. Mesaros and Y. Ran, Phys. Rev. B **87**, 155115 (2013).

[48] L.-Y. Hung and X.-G. Wen, Phys. Rev. B **87**, 165107 (2013).

[49] Y.-M. Lu and A. Vishwanath, Phys. Rev. B **93**, 155121 (2016).
[50] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B **87**, 155114 (2013).
[51] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Science **338**, 1604 (2012).
[52] T. Senthil and M. Levin, Phys. Rev. Lett. **110**, 046801 (2013).
[53] Y.-M. Lu and A. Vishwanath, Phys. Rev. B **86**, 125119 (2012).
[54] C. Xu and T. Senthil, Phys. Rev. B **87**, 174412 (2013).
[55] $n_{1,2}$ should be viewed as integers mod 8.
[56] J. May-Mann, A. Stern, and T. Devakul, "Theory of half-integer fractional quantum spin hall insulator edges," (2024), arXiv:2403.03964 [cond-mat.mes-hall].
[57] M. Müger, Proceedings of the London Mathematical Society **87**, 291–308 (2003).