SERRE FUNCTOR AND COMPLETE TORSION PAIRS

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Abstract. Given a torsion pair $(\mathcal{T}, \mathcal{F})$ in an abelian category $\mathcal{A}$, there is a t-structure $(\mathcal{U}_{\mathcal{T}}, \mathcal{V}_{\mathcal{T}})$ determined by $\mathcal{T}$ on the derived category $D^b(\mathcal{A})$. The existence of derived equivalence between heart $\mathcal{B}$ of the t-structure and $\mathcal{A}$ which naturally extends the embedding $\mathcal{B} \to D^b(\mathcal{A})$ is determined by the completeness of the torsion pair [6]. When $\mathcal{A}$ is the module category of a finite-dimensional hereditary algebra and $\mathcal{U}_{\mathcal{T}}$ is closed under Serre functor, then there exists a triangle equivalence $D^b(\mathcal{B}) \to D^b(\mathcal{A})$ [21]. In this case, we give a straightforward proof of the fact torsion pair $(\mathcal{T}, \mathcal{F})$ is complete if and only if $\mathcal{U}_{\mathcal{T}}$ is closed under the Serre functor.

1. Introduction

Given a t-structure on a triangulated category $\mathcal{D}$, the Serre functor plays an important role in the relation between the derived category of the heart and $\mathcal{D}$ [21]. On the other hand, given an abelian category $\mathcal{A}$ with a torsion pair $(\mathcal{T}, \mathcal{F})$, the heart $\mathcal{B}$ of the t-structure $(\mathcal{U}_{\mathcal{T}}, \mathcal{V}_{\mathcal{T}})$ on $D^b(\mathcal{A})$ defined by the torsion pair is an abelian category[11, 2]. We call the category $\mathcal{B}$ Happel-Reiten-Smalø tilt (HRS tilt) of $\mathcal{A}$ with respect to $(\mathcal{T}, \mathcal{F})$. By [2], there is a realization functor $F\colon D^b(\mathcal{A}) \to D^b(\mathcal{B})$ which extends the embedding $\mathcal{B} \to D^b(\mathcal{A})$. In general, the realization functor F is not an equivalence. There is a criterion on $(\mathcal{T}, \mathcal{F})$ for the functor F being an equivalence [6]. In this paper, we consider how the aisle $\mathcal{U}_{\mathcal{T}}$ closed under the Serre functor relates to the completeness property of $(\mathcal{T}, \mathcal{F})$.

We call a torsion pair $(\mathcal{T}, \mathcal{F})$ of $\mathcal{A}$ is *complete* if for each object X in $\mathcal{A}$, there is an exact sequence $0 \to F_1 \to F_2 \to X \to T_1 \to T_2 \to 0$ with $T_i \in \mathcal{T}$ and $F_i \in \mathcal{F}$ such that the corresponding class in $\mathrm{Ext}^3_{\mathcal{A}}(T_2, F_1)$ vanishes. According to [6, Theorem A], the realization functor F is an equivalence if and only if $(\mathcal{T}, \mathcal{F})$ is a complete torsion pair. In [21], the authors consider the sufficient and necessary condition for F being equivalence from the point of view of bounded t-structure. Assume that the derived category $D^b(\mathcal{A})$ admits a Serre functor $\mathbb{S}$. The functor F being an equivalence is closely related to the aisle $\mathcal{U}$ satisfying $\mathbb{S}\mathcal{U} \subseteq \mathcal{U}$. However, it is sufficient and necessary for the module category $\mathcal{A}$ of a finite-dimensional hereditary algebra.

From [6] and [21], there does not exist a straightforward approach to prove the equivalence between $\mathbb{S}\mathcal{U}_{\mathcal{T}} \subseteq \mathcal{U}_{\mathcal{T}}$ and completeness of $(\mathcal{T}, \mathcal{F})$. The completeness of $(\mathcal{T}, \mathcal{F})$ is sufficient to imply $\mathbb{S}\mathcal{U}_{\mathcal{T}} \subseteq \mathcal{U}_{\mathcal{T}}$. Conversely, there is no explicit construction of the 5-term exact sequence in (2.1). The reason is that the completeness of $(\mathcal{T}, \mathcal{F})$ depends on the calculation of the preimage of objects in $\mathcal{A}$ under the realization functor F. The main

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result of the paper gives a straightforward proof of the equivalence which also provides a construction of preimages of the realization functor.

Theorem 1.1. *Let Λ be a finite-dimensional hereditary algebra and $(\mathcal{T},\mathcal{F})$ be a torsion pair in $\mathcal{A} = \operatorname{mod}\Lambda$. Then the following statements are equivalent.*

(1) The aisle $\mathcal{U}$ that determined by $\mathcal{T}$ is closed under the Serre functor.
(2) The torsion pair $(\mathcal{T},\mathcal{F})$ is complete.

Note that one could not generalize the result to any hereditary abelian category $\mathcal{A}$, see [21, Example 10.4] and [6, Theorem A]. From the proof of Theorem 1.1, we actually construct the preimage of objects in $\mathcal{A}$ under the realization functor $F\colon D^b(\mathcal{B}) \to D^b(\mathcal{A})$.

In Section 2, we recall basic facts about tilting theory and perpendicular categories defined by exceptional objects. The twist functors associated with exceptional objects play an essential role in proving Theorem 1.1. In Section 3, we prove the completeness of torsion pair $(\mathcal{T},\mathcal{F})$ implies the aisle $\mathcal{U}_{\mathcal{T}}$ closed under the Serre functor. Furthermore, we prove the converse holds either $\mathcal{T}$ is finitely generated or without any Ext-projective objects. In Section 4 and Section 5, we apply the indecomposable exceptional objects to reduce the general case of $\mathcal{T}$ to the special case $\mathcal{T}$ without any Ext-projective objects.

2. Preliminary

For any category $\mathcal{D}$, a subcategory $\mathcal{C}$ of $\mathcal{D}$ is called *reflective* if the embedding $\iota : \mathcal{C} \hookrightarrow \mathcal{D}$ has a *left adjoint* F, i.e., there is a bifunctorial isomorphism

$$\operatorname{Hom}_{\mathcal{D}}(X,\iota Y) \cong \operatorname{Hom}_{\mathcal{C}}(FX,Y), \forall X \in \mathcal{D}, \forall Y \in \mathcal{C}.$$

The *coreflective* subcategory is defined dually.

For any subcategory $\mathcal{C}$ of a triangulated category $\mathcal{D}$, let $\operatorname{add}\mathcal{C}$ be the additive closure of $\mathcal{C}$. For any pair of subcategories $(\mathcal{C}_1,\mathcal{C}_2)$, we write $\mathcal{C}_1 * \mathcal{C}_2$ the full subcategory of $\mathcal{D}$ consisting of those objects Z such that there is a triangle $C_1 \to Z \to C_2 \to C_1[1]$ in $\mathcal{D}$, where $C_1 \in \mathcal{C}_1$ and $C_2 \in \mathcal{C}_2$.

We denote by $\operatorname{mod}\Lambda$ the category of finite-dimensional right Λ-modules over a finite-dimensional algebra Λ.

2.1. **Tilting theory and wide subcategory.**

Definition 2.1. A full subcategory $\mathcal{T}$ of an abelian category $\mathcal{A}$ is called a *torsion class* if it is reflective and closed under quotients and extensions. A torsion class $\mathcal{T}$ is called *tilting* if for all $X \in \mathcal{A}$, there is an monomorphism $X \to T_X$ for some $T_X \in \mathcal{T}$.

A pair $(\mathcal{T},\mathcal{F})$ of full subcategories of $\mathcal{A}$ is called a *torsion pair* if $\mathcal{T}$ is a torsion class and $\operatorname{Hom}_{\mathcal{A}}(T,F) = 0$ for any $T \in \mathcal{T}$ and $F \in \mathcal{F}$. The subcategory $\mathcal{F}$ is called a *torsion free class*.

For an abelian category $\mathcal{A}$ and a subcategory $\mathcal{C}$, an object $M \in \mathcal{C}$ is called an *Ext-projective* in $\mathcal{C}$ if $\operatorname{Ext}^i_{\mathcal{C}}(M,C) = 0$ for each $C \in \mathcal{C}$. A torsion class $\mathcal{T}$ is called *finitely generated* if each object is a quotient of an object in $\operatorname{add}E$ for an Ext-projective object E in $\mathcal{T}$. We call a torsion pair $(\mathcal{T},\mathcal{F})$ of $\mathcal{A}$ is *complete* if for each object X in $\mathcal{A}$, there is an exact sequence

$$0 \to F_1 \to F_2 \to X \to T_1 \to T_2 \to 0 \tag{2.1}$$

with $T_i \in \mathcal{T}$ and $F_i \in \mathcal{F}, i = 1, 2$ such that the corresponding class in $\mathrm{Ext}^3_{\mathcal{A}}(T_2, F_1)$ vanishes.

Remark 2.2. In [15], Krause and Šťovíček defined the complete Ext-orthogonal pair. The definition of complete torsion pair in their setting is not equivalent to the notation here.

Wide subcategories of a hereditary abelian category have been extensively studied in the representation theory of algebras. There are close relations between the wide subcategory and combinatorics of the Coxeter groups [9, 13]. A full additive subcategory $\mathcal{W} \subseteq \mathcal{A}$ is called a *wide* if it is closed under kernels, cokernels, and extensions. For any full subcategory $\mathcal{C}$ of $\mathcal{A}$, the *wide closure* of $\mathcal{C}$ in $\mathcal{A}$ is the intersection of all wide subcategories of $\mathcal{A}$ containing $\mathcal{C}$. We denote $\mathfrak{a}_{\mathcal{A}}(\mathcal{C})$ the wide closure of $\mathcal{C}$ in $\mathcal{A}$.

In general, there is no canonical way to obtain the wide closure $\mathcal{C}$ for a subcategory$\mathcal{C}$. The following result gives a construction of $\mathfrak{a}_{\mathcal{A}}(\mathcal{C})$ in the case $\mathcal{A}$ is hereditary.

Proposition 2.3 ([22, Propostion 3.5])**.** *If $\mathcal{A}$ is a hereditary abelian category and $\mathcal{C}$ is a full subcategory closed under extensions and cokernels. Then each object in $\mathfrak{a}_{\mathcal{A}}(\mathcal{C})$ is the kernel of a map in $\mathcal{C}$.*

2.2. **t-structure and co-t-structure.** Denote by $D^b(\mathcal{A})$ the bounded derived category of $\mathcal{A}$. If $\mathcal{A}$ is hereditary, we have

$$\mathcal{D}^b(A) = \bigcup_{n\in\mathbb{Z}} \mathcal{A}[n].$$

Definition 2.4. A pair $(\mathcal{U}, \mathcal{V})$ of full subcategories of $D^b(\mathcal{A})$ is called a *t-structure* (resp. *co-t-structure*) if

- $\mathcal{U}[1] \subseteq \mathcal{U}$ and $\mathcal{V}[-1] \subseteq \mathcal{V}$ (resp. $\mathcal{U}[-1] \subseteq \mathcal{U}$ and $\mathcal{V}[1] \subseteq \mathcal{V}$);
- $\mathrm{Hom}_{D^b(\mathcal{A})}(\mathcal{U}, \mathcal{V}) = 0$, and
- for any $X \in D^b(\mathcal{A})$ there is a triangle
$$X_{\mathcal{U}} \to X \to X_{\mathcal{V}} \to X_{\mathcal{U}}[1]$$
with $X_{\mathcal{U}} \in \mathcal{U}, X_{\mathcal{V}} \in \mathcal{V}$.

We call subcategories $\mathcal{U}$ and $\mathcal{V}$ an *aisle* and *coaisle*, respectively. A t-structure $(\mathcal{U}, \mathcal{V})$ is said to be *bounded* if $\bigcup_{n\in\mathbb{Z}} \mathcal{U}[n] = D^b(\mathcal{A}) = \bigcup_{n\in\mathbb{Z}} \mathcal{V}[n]$.

An object $E \in D^b(\mathcal{A})$ is called *presilting* if $\mathrm{Hom}_{D^b(\mathcal{A})}(E, E[1]) = 0$. A bounded t-structure $(\mathcal{U}, \mathcal{V})$ is called *finitely generated* $\mathcal{U} = \bigcup_{n\in\mathbb{Z}} \mathrm{add}\, E[n] * \cdots \mathrm{add}\, E[1] * \mathrm{add}\, E$ for some presilting object E.

The *heart* of a t-structure $(\mathcal{U}, \mathcal{V})$ is defined to be the full subcategory $\mathcal{U} \cap \mathcal{V}[1]$ of $D^b(\mathcal{A})$, which is an abelian category by [2, Theorem I.3.6].

For each torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$, let

$$\mathcal{U}_{\mathcal{T}} = \{X \in D^b(\mathcal{A}) \mid H^0(X) \in \mathcal{T}, H^n(X) = 0 \text{ for any } n > 0\}$$

and

$$\mathcal{V}_{\mathcal{T}} = \{Y \in D^b(\mathcal{A}) \mid H^0(Y) \in \mathcal{F}, H^n(Y) = 0 \text{ for any } n < 0\}.$$

It is direct to check that $(\mathcal{U}_{\mathcal{T}}, \mathcal{V}_{\mathcal{T}})$ is a bounded t-structure in $D^b(\mathcal{A})$ [11, Proposition 2.1]. The heart $\mathcal{B} = \mathcal{F}[1] * \mathcal{T}$ of $(\mathcal{U}, \mathcal{V})$ is called the *HRS-tilt* of $\mathcal{A}$.

Conversely, given a bounded t-structure $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ on a triangulate category $\mathcal{D}$ with heart $\mathcal{H}$, each t-structure $(\mathcal{U}, \mathcal{V})$ with $\mathcal{D}^{\leqslant 0}[1] \subset \mathcal{U} \subset \mathcal{D}^{\leqslant 0}$ determines a torsion class $\mathcal{T} := H^0_{\mathcal{H}}(\mathcal{U})$ in $\mathcal{H}$, where $H^0_{\mathcal{H}}$ denotes the cohomological functor corresponding to $\mathcal{H}$. The following lemma shows that such a correspondence is a bijection.

Proposition 2.5. [23, Proposition 2.3] *Let $\mathcal{H}$ be the heart of a bounded t-structure $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ on a triangulate category $\mathcal{D}$. Then there is a bijection between the set of all torsion classes $\mathcal{T}$ of $\mathcal{H}$ and the set of t-structures $(\mathcal{U}, \mathcal{V})$ of $\mathcal{D}$ satisfying*

$$\mathcal{D}^{\leqslant 0}[1] \subseteq \mathcal{U} \subseteq \mathcal{D}^{\leqslant 0},$$

given by $\mathcal{T} \mapsto \mathcal{U}_{\mathcal{T}}$

Moreover, if $\mathcal{T}$ is finitely generated with a generator E, one may check that E is exactly the presilting object that generates $\mathcal{U}$.

2.3. **Serre functor.** Let $\mathcal{C}$ be a Hom-finite $\mathbf{k}$-linear category, where $\mathbf{k}$ is an algebraically closed field. A *Serre functor* is a $\mathbf{k}$-linear autoequivalence $\mathbb{S}$ of $\mathcal{C}$ such that for any objects $A, B \in \mathcal{C}$, there exists an isomorphism

$$\eta_{A,B} \colon \operatorname{Hom}(A, B) \to \operatorname{Hom}(B, \mathbb{S}A)^*,$$

where $(-)^* = \operatorname{Hom}_{\mathbf{k}}(-, \mathbf{k})$ is the standard duality.

Recall that an abelian category $\mathcal{A}$ is called *Ext-finite* if $\dim_{\mathbf{k}} \operatorname{Ext}^i_{\mathcal{A}}(X, Y) < \infty$ for any $X, Y \in \mathcal{A}$ and all $i \in \mathbb{Z}$. An Ext-finite abelian category $\mathcal{A}$ has *Serre duality* if $D^b(\mathcal{A})$ admits a Serre functor $\mathbb{S}$. A t-structure $(\mathcal{U}, \mathcal{V})$ or an aisle $\mathcal{U}$ is called closed under the Serre functor $\mathbb{S}$ if $\mathbb{S}\mathcal{U} \subseteq \mathcal{U}$.

Lemma 2.6 ([21])**.** *Let Λ be a finite-dimensional hereditary algebra and $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\operatorname{mod} \Lambda$. The aisle $\mathcal{U}_{\mathcal{T}}$ determined by $\mathcal{T}$ is closed under the Serre functor if and only if for each projective object $P \in \mathcal{T}$, the corresponding injective object $P \otimes_{\Lambda} \Lambda^*$ is in $\mathcal{T}$.*

2.4. **Twist functors and perpendicular categories.** Let $\mathcal{D}$ be a triangulated category and $\mathcal{S}$ a full triangulated subcategory, the *right orthogonal* $\mathcal{S}^{\perp}$ of $\mathcal{S}$ is the full subcategory consisting of those objects

$$\{X \in \mathcal{D} \mid \operatorname{Hom}(Y, X) = 0, \forall Y \in \mathcal{S}\}.$$

The *left orthogonal* ${}^{\perp}\mathcal{S}$ is defined similarly. A full triangulated subcategory of $\mathcal{D}$ is called *thick* if it is closed under taking direct summands. For any object X in $\mathcal{D}$, we write $\langle X \rangle$ for the thick subcategory generated by X. Furthermore, we write $X^{\perp}$ and ${}^{\perp}X$ for $\langle X \rangle^{\perp}$ and ${}^{\perp}\langle X \rangle$ respectively.

For a $\mathbf{k}$-linear, Hom-finite, algebraic triangulated category $\mathcal{D}$, an object E in $\mathcal{D}$ is called *exceptional* if $\operatorname{Hom}(E, E) \cong \mathbf{k}$ and $\operatorname{Hom}(E, E[n]) = 0$ for all $n \neq 0$. An object E in an abelian category $\mathcal{A}$ is exceptional if it is exceptional in $D^b(\mathcal{A})$.

Let $\mathcal{A}$ be an Ext-finite abelian category of finite global dimension and $S \in D^b(\mathcal{A})$ an exceptional object. Then the thick subcategory $\langle S \rangle$ generated by S in $D^b(\mathcal{A})$ is equivalent to $D^b(\operatorname{mod} \mathbf{k})$ and the embedding $\langle S \rangle \to D^b(\mathcal{A})$ has a left and right adjoint, given on objects by $\lambda_S(X) = R\operatorname{Hom}(S, X) \otimes^L_{\mathbf{k}} S$ and $\mu_S(X) = R\operatorname{Hom}(X, S)^* \otimes^L_{\mathbf{k}} S$, see [3, Theorem 3.2] and [21, Section 2.4].

The embedding ${}^{\perp}S \to D^b(\mathcal{A})$ has a right adjoint $T_S^*\colon D^b(\mathcal{A}) \to {}^{\perp}S$ and left adjoint $T_S\colon D^b(\mathcal{A}) \to {}^{\perp}S$, which are defined by the following triangles

$$T_S^*(X) \to X \xrightarrow{\alpha_X} \mu_S(X) \to T_S^*(X)[1]$$

$$T_S(X)[-1] \to \lambda_S(X) \xrightarrow{\beta_X} X \to T_S(X),$$

see [3, Lemma 3.1]. By the assumption of S, we have that

$$\lambda_S(X) \cong \oplus_{i\in\mathbb{Z}} \operatorname{Hom}(S, X[i]) \otimes_{\mathbf{k}} S[-i], \quad \mu_S(X) = \oplus_{i\in\mathbb{Z}} \operatorname{Hom}(X, S[i])^* \otimes_{\mathbf{k}} S[i].$$

The functors T_S and T_S^* could be viewed as endofunctors of $D^b(\mathcal{A})$ by composing the corresponding embedding. We call the functors T_S^* and T_S the *twist functors* associated with S. The functor T_S is the right adjoint of T_S^* [10, Section 3.2].

Lemma 2.7 ([14, Lemma 1])**.** *Keep the notation as above, the subcategory ${}^{\perp}S$ has the Serre functor $\mathbb{S}' = T_S^* \circ \mathbb{S} \circ T_S$.*

Lemma 2.8 ([4, Lemma 1.9])**.** *Keep the notation as above, the twist functors on $D^b(\mathcal{A})$ induces equivalent functors $T_S\colon {}^{\perp}S \to S^{\perp}$ and $T_S^*\colon S^{\perp} \to {}^{\perp}S$ which are quasi-inverse to each other.*

For an abelian category $\mathcal{A}$ and a class $\mathcal{C}$ of objects in $\mathcal{A}$, there is a full subcategory $\mathcal{C}_{\mathcal{A}}^{\perp}$ consisting of objects $\{X \in \mathcal{A} \mid \forall n \in \mathbb{Z}, \operatorname{Ext}^n_{\mathcal{A}}(\mathcal{C}, X) = 0\}$. The subcategory ${}^{\perp}\mathcal{C}_{\mathcal{A}}$ is defined similarly. In general, the categories $\mathcal{C}_{\mathcal{A}}^{\perp}$ and ${}^{\perp}\mathcal{C}_{\mathcal{A}}$ are neither wide subcategories of $\mathcal{A}$ nor abelian categories.

If $\mathcal{A} = \operatorname{mod}\Lambda$ for some finite-dimensional algebra Λ and $\mathcal{C}$ consisting of $M \in \operatorname{mod}\Lambda$ with projective dimension $\operatorname{Proj.dim} M \leqslant 1$, then $\mathcal{C}_{\mathcal{A}}^{\perp}$ is an abelian subcategory of $\mathcal{A}$. Note that the condition $\operatorname{Proj.dim}(M) \leqslant 1$ is necessary for $M^{\perp}$ to be abelian. If $\mathcal{A}$ is hereditary, then both $\mathcal{C}_{\mathcal{A}}^{\perp}$ and ${}^{\perp}\mathcal{C}_{\mathcal{A}}$ are wide subcategories of $\mathcal{A}$ [8, Proposition 1.1].

Lemma 2.9 ([12, Proposition 3])**.** *Let $\mathcal{A} = \operatorname{mod}\Lambda$ for some finite-dimensional hereditary algebra Λ and S be an indecomposable exceptional object in $\mathcal{A}$. Then ${}^{\perp}S \subseteq D^b(\Lambda)$ is equivalent to $D^b(\Lambda')$ for a finite-dimensional hereditary algebra with one fewer distinct simple objects of Λ and ${}^{\perp}S_{\mathcal{A}}$ is equivalent to* $\operatorname{mod}\Lambda'$.

3. The equivalence for special torsion classes

For a finite-dimensional algebra Λ, we denote the category $\operatorname{mod}\Lambda$ by $\mathcal{A}$. In this section, we prove that for a hereditary algebra Λ, the torsion pair $(\mathcal{T}, \mathcal{F})$ in a module category $\mathcal{A}$ of a hereditary algebra Λ is complete implies that the aisle $\mathcal{U}_{\mathcal{T}}$ in $D^b(\Lambda)$ is closed under the Serre functor. We also prove the converse holds for the torsion class $\mathcal{T}$ either is finitely generated or has no non-zero Ext-projective objects.

3.1. **Necessity.** For a finite-dimensional algebra Λ with a finite global dimension, the category $\mathcal{A} = \operatorname{mod}\Lambda$ has Serre duality. The Serre functor $\mathbb{S}$ is given by $\mathbb{S} \cong - \otimes^L_{\mathbf{k}} \Lambda^*\colon D^b(\mathcal{A}) \to D^b(\mathcal{A})$. The category $D^b(\mathcal{A})$ has a Serre functor $\mathbb{S}$ if and only if $D^b(\mathcal{A})$ has Auslander-Reiten triangles. For every indecomposable object $X \in D^b(\mathcal{A})$, the Auslander-Reiten triangle is $\mathbb{S}[-1]X \to Y \to X \to \mathbb{S}(X)$, see [20, Proposition I.2.3] and [5, Proposition 2.8].

If Λ is a hereditary algebra, then the Auslander-Reiten translation functor on $D^b(\mathcal{A})$ and the Auslander-Reiten translation on $\mathcal{A}$ coincide on non-projective indecomposable objects in $\mathcal{A}$. We write the Auslander-Reiten translation on $D^b(\mathcal{A})$ as τ.

Lemma 3.1. *Let Λ be a finite-dimensional algebra of finite global dimension. $(\mathcal{T}, \mathcal{F})$ a torsion pair on $\operatorname{mod}\Lambda$. Assume that $P \in \mathcal{T}$ is projective, and the corresponding injective object I fits in a 5-term exact sequence of the form (2.1). Then I is also in $\mathcal{T}$.*

Proof. Considering the 5-term exact sequence 2.1 with $X = I$, we have that $F_2 \to I$ is zero. Otherwise, there exists a non-trivial map $P \to F_2$ by Serre duality, which contradicts the fact that $P \in \mathcal{T}$. So the sequence 2.1 for I is a short exact sequence of the form

$$0 \to I \to T_1 \to T_2 \to 0. \tag{3.1}$$

Since I is injective, the sequence (3.1) splits. This implies that I is a direct summand of T_1 and $I \in \mathcal{T}$. □

Now assume that Λ is a hereditary algebra. Then $D^b(\mathcal{A})$ has a Serre functor $\mathbb{S} = \tau[1]$. So $\mathbb{S}X \subseteq \mathcal{A}[\geqslant 1]$ for any $X \in \mathcal{A}[\geqslant 0]\backslash \operatorname{Proj}\mathcal{A}$, and $\mathbb{S}P = P \otimes_\Lambda \Lambda^*$ for any $P \in \operatorname{Proj}\mathcal{A}$. Combining Lemma 2.6 and Lemma 3.1, we have that $\mathcal{U}_\mathcal{T}$ is closed under the Serre functor. By Lemma 2.4, we have the following result.

Corollary 3.2. *Let Λ be a finite-dimensional hereditary algebra. If a torsion pair $(\mathcal{T}, \mathcal{F})$ on $\operatorname{mod}\Lambda$ is complete. Then the aisle $\mathcal{U}_\mathcal{T}$ is closed under the Serre functor.*

Remark 3.3. The corollary holds for any finite-dimensional algebra with finite global dimension by [6, Theorem A] and [21, Corollary 4.13]. We do not have a straightforward proof of this argument.

3.2. Aisle determined by special torsion classes.

Lemma 3.4. *Let Λ be a finite-dimensional algebra of finite global dimension. Assume $\mathcal{T}$ is a finitely generated torsion class in $\operatorname{mod}\Lambda$ and the aisle $\mathcal{U}_\mathcal{T}$ is closed under the Serre functor. Then the torsion pair $(\mathcal{T}, \mathcal{F})$ is complete.*

Proof. Recall that for each t-structure $(\mathcal{U}, \mathcal{V})$ with $\mathcal{U}$ finitely generated, we have that $(\mathcal{V}, \mathbb{S}\mathcal{U})$ is a co-t-structure, see [1, Proposition 2.23].

If $\mathcal{T}$ is finitely generated, then so is $\mathcal{U}_\mathcal{T}$. Now we take the the co-t-structure $(\mathcal{V}_\mathcal{T}, \mathbb{S}\mathcal{U}_\mathcal{T})$. For any $X \in \mathcal{A}$, let

$$N \to X \to M \to N[1]$$

be the triangle of X with respect to $(\mathcal{V}_\mathcal{T}, \mathbb{S}\mathcal{U}_\mathcal{T})$. Taking cohomology with respect to canonical t-structure, we have the following long exact sequence

$$H^{-1}(X) = 0 \to H^{-1}(M) \to H^0(N) \to X \to H^0(M) \to H^1(N) \to H^1(X) = 0$$

where the first term $H^{-1}(M)$ lies in $\mathcal{F}$ because $N \in \mathcal{V}_\mathcal{T}$ and $H^0(N) \in \mathcal{F}$. Similarly, $H^1(N) \in \mathcal{T}$ implies that $H^0(M) \in \mathcal{T}$. Thus $(\mathcal{T}, \mathcal{F})$ is complete. □

If a torsion class $\mathcal{T}$ is not finitely generated, then we do not have the co-t-structure $(\mathcal{V}_\mathcal{T}, \mathbb{S}\mathcal{U}_\mathcal{T})$. In this case, there is no direct approach to prove $(\mathcal{T}, \mathcal{F})$ is complete under the assumption $\mathcal{U}_\mathcal{T}$ is closed under the Serre functor. For the category $\mathcal{A}$ with Λ being a hereditary algebra, we could show that $\mathbb{S}\mathcal{U}_\mathcal{T} \subseteq \mathcal{U}_\mathcal{T}$ implies $(\mathcal{T}, \mathcal{F})$ is complete under the assumption the torsion class $\mathcal{T}$ without Ext-projectives.

Lemma 3.5. *Assume that $\mathcal{A}$ is a hereditary category and $\mathcal{T}$ is a torsion class without Ext-projective objects, then all projective objects of $\mathcal{A}$ are in the torsion-free class.*

Proof. For any projective object $P \in \mathcal{A}$, there is a short exact sequence

$$0 \to P_{\mathcal{T}} \to P \to P_{\mathcal{F}} \to 0$$

with $P_{\mathcal{T}} \in \mathcal{T}$, and $P_{\mathcal{F}} \in \mathcal{F}$. Since $\mathcal{A}$ is hereditary, we have that $P_{\mathcal{T}}$ is either zero or projective. Since $\mathcal{T}$ does not contain any nonzero projective objects, we have that $P_{\mathcal{T}} = 0$ and hence $P \in \mathcal{F}$. □

Lemma 3.6. *Assume that $\mathcal{A} = \operatorname{mod}\Lambda$ for a hereditary algebra Λ. Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{A}$ with $\mathcal{T}$ having no nonzero Ext-projectives. If $\mathcal{U}_{\mathcal{T}}$ is closed under the Serre functor then $(\mathcal{T}, \mathcal{F})$ is complete.*

Proof. If $\mathcal{T}$ is a torsion class of $\mathcal{A}$, then the wide closure $\mathfrak{a}_{\mathcal{A}}(\mathcal{T})$ is coreflective and reflective wide subcategory of $\mathcal{A}$ by [22, Proposition 7.3]. It implies that there is a right approximation $X \to W$ with $W \in \mathfrak{a}_{\mathcal{A}}(\mathcal{T})$ for any object $X \in \mathcal{A}$. On the other hand, $\mathcal{T}$ is a tilting torsion class in $\mathfrak{a}_{\mathcal{A}}(\mathcal{T})$ by [22, Corollary 3.7]. So there exists an object T_1 in $\mathcal{T}$ such that there exists an embedding $W \hookrightarrow T_1$. We denote the composition $X \to W \to T_1$ by $g\colon X \to T_1$.

Denote $T_2 := \operatorname{Coker} g$, then $T_2 \in \mathcal{T}$. Thus we get a sequence from the morphism $g\colon X \to T_1$

$$0 \to \operatorname{Ker} g \to X \xrightarrow{g} T_1 \to T_2 \to 0$$

with $T_1, T_2 \in \mathcal{T}$. By Lemma 3.5, all projective objects are contained in $\mathcal{F}$. We take the projective cover $P_2 \to \operatorname{Ker} g$ of $\operatorname{Ker} g$. Denote $P_1 := \operatorname{Ker}(P_2 \to \operatorname{Ker} g)$, we have the desired exact sequence

$$0 \to P_1 \to P_2 \to X \to T_1 \to T_2 \to 0.$$

Note that $P_1 \in \mathcal{F}$ since $P_2 \in \mathcal{F}$ and $\mathcal{A}$ is hereditary. By definition, $(\mathcal{T}, \mathcal{F})$ is complete. □

4. The Reduction via exceptional objects

Throughout this section, we assume that category $\mathcal{A}$ is an abelian category with Serre duality $\mathbb{S}$. Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair of $\mathcal{A}$ such that the corresponding aisle $\mathcal{U}_{\mathcal{T}}$ is closed under the Serre functor, i.e. $\mathbb{S}\mathcal{U}_{\mathcal{T}} \subset \mathcal{U}_{\mathcal{T}}$. The HRS-tilt of $\mathcal{A}$ with respect to $(\mathcal{T}, \mathcal{F})$ is denoted by $\mathcal{B}$.

We will consider an Ext-projective object E in $\mathcal{U}_{\mathcal{T}}$. It is actually a projective object in $\mathcal{B}$, and we fix its simple top S in $\mathcal{B}$. The object S is an exceptional object in $D^b(\mathcal{A})$. If Λ is hereditary, we could reduce the category $\mathcal{A}$ to a new category $\mathcal{W}$ and the category $D^b(\mathcal{A})$ to the left orthogonal ${}^{\perp}S$.

4.1. The aisle on ${}^{\perp}S$.

Like the Ext-projective objects in an abelian category, we consider the Ext-projective object in a subcategory of a triangulated category. Given a triangulated category $\mathcal{D}$ and a full additive category $\mathcal{C}$, an object E in $\mathcal{C}$ is called Ext-projective if $\operatorname{Hom}_{\mathcal{D}}(E, C[i]) = 0$ for all $i > 0$. The following result shows that the indecomposable Ext-projective object $E \in \mathcal{U}_{\mathcal{T}}$ is projective in $\mathcal{B}$ and has a simple top S.

Proposition 4.1 ([21, Proposition 6.4]). *Let $\mathcal{A}$ be an abelian category with Serre duality and $(\mathcal{U}, \mathcal{V})$ the bounded t-structure in $D^b(\mathcal{A})$ closed under the Serre functor. Then for the Ext-projective E of $\mathcal{U}$, and the Auslander-Reiten triangle $\tau E \to M \to E \to \mathbb{S}E$ given by E, we have that*

(1) E is projective in $\mathcal{B}$ and $\mathbb{S}E$ is injective in $\mathcal{B}$;
(2) The object E has a simple top S in $\mathcal{B}$;
(3) there is a short exact sequence $0 \to S \to \mathbb{S}E \to M_{\mathcal{V}}[1] \to 0$ in $\mathcal{B}$;
(4) $\mathrm{Hom}_{D^b(\mathcal{A})}(S, S[n]) = 0$ for $n \neq 0$, and $\mathrm{End}\, S = \mathbf{k}$.

Recall that a torsion pair $(\mathcal{T}, \mathcal{F})$ of $\mathcal{A}$ is called *splitting* if $\mathrm{Ext}^1_{\mathcal{A}}(F, T) = 0$ for all $F \in F$ and $T \in \mathcal{T}$. In this case, any object in $\mathcal{A}$ is isomorphic to a direct sum $F \oplus T$ for some $F \in \mathcal{F}$ and $T \in \mathcal{T}$. The following result is obvious.

Lemma 4.2. *If $\mathcal{A}$ is a hereditary category with a torsion pair $(\mathcal{T}, \mathcal{F})$, then HRS-tilt $\mathcal{B}$ has a splitting torsion pair $(\mathcal{F}[1], \mathcal{T})$.*

In the following, we assume that $\mathcal{A} = \mathrm{mod}\, \Lambda$ for a finite-dimensional hereditary algebra Λ. By Lemma 4.2, we have that either $S \in \mathcal{F}[1]$ or $S \in \mathcal{T}$. Let R be S if $S \in \mathcal{T}$ and $S[-1]$ if $S \in \mathcal{F}[1]$. Since S is exceptional, so is R. Hence, by [8, Proposition 1.1] and Lemma 2.9, the subcategory

$$\mathcal{W} = \{X \in \mathcal{A} \mid \mathrm{Hom}(X, R) = 0, \mathrm{Ext}^1_{\mathcal{A}}(X, R) = 0\}$$

is a wide subcategory of $\mathcal{A}$ that equivalent to the module category of some hereditary algebra. The following lemma is dual to [19, Lemma 3.6].

Lemma 4.3. *There exists equivalences $D^b(\mathcal{W}) \cong D^b_{\mathcal{W}}(\mathcal{A}) \cong {}^{\perp}S$, where $D^b_{\mathcal{W}}(\mathcal{A})$ the full subcategory of $D^b(\mathcal{A})$ consisting of objects $\{X \in D^b(\mathcal{A}) | H^i(X) \in \mathcal{W}, \forall i \in \mathbb{Z}\}$.*

Proof. Denote by Σ the suspension functor in $D^b(\mathcal{W})$. Since $\mathcal{W}$ is hereditary, each object of $D^b(\mathcal{W})$ is isomorphic to a direct sum of shifts of objects in $\mathcal{W}$. Let $\Phi\colon D^b(\mathcal{W}) \to D^b_{\mathcal{W}}(\mathcal{A})$ be the functor which is induced by $\Sigma^n X \mapsto X[n]$ for $X \in \mathcal{W}$. It is obvious that Φ is dense. For any two objects $X, Y \in \mathcal{W}$, $\mathrm{Hom}_{D^b(\mathcal{W})}(\Sigma^i X, \Sigma^j) \neq 0$ if and only if $i = j$ or $j - i = 1$. In the former case, we have

$$\mathrm{Hom}_{D^b(\mathcal{W})}(\Sigma^i X, \Sigma^j Y) \cong \mathrm{Hom}_{D^b(\mathcal{A})}(X[i], Y[j]).$$

In the case $j = i + 1$, we have

$$\mathrm{Hom}_{D^b(\mathcal{W})}(\Sigma^i X, \Sigma^j Y) \cong \mathrm{Ext}^1_{\mathcal{W}}(X, Y) \cong \mathrm{Ext}^1_{\mathcal{A}}(X, Y) \cong \mathrm{Hom}_{D^b(\mathcal{A})}(X[i], Y[j]),$$

where the second isomorphism follows because $\mathcal{W}$ is closed under extensions. So Φ is fully faithful and hence $D^b(\mathcal{W}) \cong D^b_{\mathcal{W}}(\mathcal{A})$.

Since $\mathcal{A}$ is hereditary, we have $X = \oplus_{i \in \mathbb{Z}} H^i(X)[-i]$ for any $X \in D^b(\mathcal{A})$. Thus, $X \in {}^{\perp}S$ if and only if $\mathrm{Hom}(H^i(X), S[n]) = 0$, $\forall i, n \in \mathbb{Z}$ if and only if $H^i(X) \in \mathcal{W}, \forall i \in \mathbb{Z}$. So ${}^{\perp}S \cong D^b_{\mathcal{W}}(\mathcal{A})$. □

Recall that ${}^{\perp}S$ admits a Serre functor $\mathbb{S}' = T^*_S \circ \mathbb{S} \circ T_S$ by Lemma 2.7. Let $\mathcal{U}' = \mathcal{U} \cap {}^{\perp}S$. By [21, Lemma 9.1], $T^*_S(X) \in \mathcal{U}'$ for any $X \in \mathcal{U}$.

Lemma 4.4 ([21, Proposition 9.2]). *For any bounded aisle $\mathcal{U} \subseteq D^b(\mathcal{A})$ closed under the Serre functor $\mathbb{S}$, then $\mathcal{U}'$ is an aisle in ${}^{\perp}S$ and $\mathbb{S}'\mathcal{U}' \subseteq \mathcal{U}'$.*

Note that the coaisle $\mathcal{V}'$ is not $\mathcal{V} \cap {}^{\perp}S$, see [21, Lemma 9.3]. The following result controls the add R-approximation in the category $\mathcal{A}$.

Lemma 4.5. *(1) If $S \in \mathcal{T}$ then for any short exact sequence $0 \to M \to S^n \xrightarrow{f} T \to 0, n \in \mathbb{Z}_{>0}$ in $\mathcal{A}$, we have $M = M' \oplus S^l$ with $l \leqslant n$ and $M' \in \mathcal{F}$. In particular, $T \in \operatorname{add} S$ if $M' = 0$.*

(2) If $S \in \mathcal{F}[1]$ then for any $0 \to F \xrightarrow{g} S[-1]^n \to M \to 0, n \in \mathbb{Z}_{>0}$ in $\mathcal{A}$, we have $M = M'' \oplus S[-1]^l$ with $l \leqslant n$ and $M'' \in \mathcal{T}$. In particular, $F \in \operatorname{add} S[-1]$ if $M'' = 0$.

Proof. We only prove the first assertion since the proof of the second statement is similar. Since $S \in \mathcal{T}$ and $\mathcal{T}$ is closed under images, we have $T \in \mathcal{T}$. Write $f = (f_1, \cdots, f_{n-l}, 0, \cdots, 0)$ with $f_i \in \operatorname{Hom}_{\mathcal{A}}(S, T), 1 \leqslant i \leqslant n-l$ nonzero. Since S is simple in $\mathcal{B}$, $f' = (f_1, \cdots, f_{n-1}) : S^{n-l} \to T$ is injective in $\mathcal{B}$. There is a short exact sequence $0 \to S^{n-1} \xrightarrow{f'} T \to \operatorname{Coker} f' \to 0$ in $\mathcal{B}$ which induces a triangle

$$T[-1] \to \operatorname{Coker} f'[-1] \oplus S^l \to S^{n-l} \oplus S^l \xrightarrow{f} T$$

in $D^b(\mathcal{A})$.

The short exact sequence $0 \to M \to S^l \xrightarrow{f} T \to 0$ in $\mathcal{A}$ gives another triangle

$$T[-1] \to M \to S^n \xrightarrow{f} T$$

in $D^b(\mathcal{A})$. It implies that $S^l \oplus \operatorname{Coker} f'[-1] \cong M \in \mathcal{A}$. So $\operatorname{Coker} f'[-1] \in \mathcal{F}$ and the assertion follows by taking $M' = \operatorname{Coker} f'[-1]$. □

4.2. **The torsion pairs on $\mathcal{W}$.** By [21, Proposition 2.2], $\mathcal{W}$ is a reflective and coreflective wide subcategory of $\mathcal{A}$. The torsion class $(\mathcal{T}, \mathcal{F})$ induces a torsion pair $(\mathcal{T}', \mathcal{F}')$ on $\mathcal{W}$ with the torsion class $\mathcal{T}' = \mathcal{T} \cap \mathcal{W}$. By Lemma 4.3 and Lemma 4.4, the aisle $\mathcal{U}_{\mathcal{T}} \subseteq D^b(\mathcal{A})$ induces the aisle $\mathcal{U}' = \bigcup_{n \in \mathbb{Z}} \mathcal{W}[n] * \cdots * \mathcal{W}[1] * \mathcal{T}'$ in ${}^{\perp}S$. Moreover, $\mathcal{T}'$ and $\mathcal{T}$ are related by the twist functor T_S^* as the following.

Lemma 4.6. *Suppose $S \in \mathcal{T}$ as above. Then $T_S^*(T) \in \mathcal{T}'$ for any $T \in \mathcal{T}$.*

Proof. By the definition of the functor T_S^*, we have the triangle $\mu_S(T)[-1] \xrightarrow{g} T_S^*(T) \to T \xrightarrow{\alpha_T} \mu_S(T)$ for any $T \in \mathcal{T}$. Since $\mathcal{A}$ is hereditary, we have $\mu_S(T) = \operatorname{Hom}(T, S)^* \otimes S \oplus \operatorname{Hom}(T, S[1])^* \otimes S[1]$. Taking cohomology, we have $H^i(T_S^*(T)) = 0$ for $i \neq 0, 1$, and a long exact sequence

$$0 \to H^{-1}(\mu_S(T)) \to H^0(T_S^*(T)) \to T \xrightarrow{h} H^0(\mu_S(T)) \to H^1(T_S^*(T)) \to 0.$$

So we get short exact sequences

$$0 \to \operatorname{Ker} h \to T \xrightarrow{p} \operatorname{Im} h \to 0, \tag{4.1}$$

and thus $\operatorname{Im} h \in \mathcal{T}$. By Lemma 4.5 and the short exact sequence

$$0 \to \operatorname{Im} h \xrightarrow{i} H^0(\mu_S(T)) \to H^1(T_S^*(T)) \to 0$$

we have that $\operatorname{Im} h \in \operatorname{add} S$ and hence $H^1(T_S^*(T)) \in \operatorname{add} S$. Since $T_S^*(T) \in {}^{\perp}S$, we have $H^1(T_S^*(T)) = 0$. So $\operatorname{Im} h \cong H^0(\mu_S(T)) \in \operatorname{add} S$. Since S is simple in $\mathcal{B}$, we have that p

in (4.1) is also an epimorphism in $\mathcal{B}$. So (4.1) is an exact sequence in $\mathcal{B}$, which implies $\operatorname{Ker} h \in \mathcal{T}$ by Lemma 4.2. So our claim follows from the short exact sequence

$$0 \to H^{-1}(\mu_S(T)) \to H^0(T_S^*(T)) \to \operatorname{Ker} h \to 0.$$

□

The torsion class $\mathcal{T}'$ in $\mathcal{W}$ is given by $\mathcal{T} \cap \mathcal{W}$ but the corresponding torsion free class $\mathcal{F}' \neq \mathcal{F} \cap \mathcal{W}$. Indeed, $\mathcal{F}'$ is determined by the exceptional object S as the following lemmas.

Lemma 4.7. *Suppose $S \in \mathcal{T}$. Then $X \in \mathcal{F}'$ if and only if $X \cong T_S^*(F)$ for some $F \in \mathcal{F}$ with* $\operatorname{Hom}(F, S) = 0$.

Proof. For $F \in \mathcal{F}$ such that $\operatorname{Hom}(F, S) = 0$, we have a short exact sequence in $\mathcal{A}$

$$\zeta : 0 \to \operatorname{Hom}(F, S[1])^* \otimes S \to T_S^*(F) \to F \to 0.$$

Applying functor $\operatorname{Hom}(T', -)$ for any $T \in \mathcal{T}' = \mathcal{T} \cap \mathcal{W}$ to the short exact sequence, we have $\operatorname{Hom}(T', T_S^*(F)) = 0$. Thus $T_S^*(F) \in \mathcal{F}'$.

Conversely, for any $X \in \mathcal{F}'$, we consider the following diagram, where the second row is given by the canonical triangle of X with respect to $(\mathcal{T}, \mathcal{F})$.

$$\begin{tikzcd}
T_S^*(X_\mathcal{T}) \arrow[r, "\beta"] \arrow[d, "h"'] & X_\mathcal{T} \arrow[r, "\alpha"] \arrow[d, equal] & \mu_S(X_\mathcal{T}) \arrow[r] \arrow[d, "f_1"] \arrow[dl, "\alpha'"'] & T_S^*(X_\mathcal{T})[1] \arrow[d] \\
X_\mathcal{F}[-1] \arrow[r, "\beta'"'] & X_\mathcal{T} \arrow[r, "f"'] & X \arrow[r, "\delta"'] & X_\mathcal{F}
\end{tikzcd}$$

By Lemma 4.6, we have $T_S^*(X_\mathcal{T}) \in \mathcal{T}'$.

Since $X \in \mathcal{F}'$, we have $f\beta = 0$. So there exists a morphism $f_1 \in \operatorname{Hom}(\mu_S(X_\mathcal{T}), X)$ such that $f = f_1\alpha$. Since $\mu_S(T) \in \operatorname{add}(S \oplus S[1])$, $S \in \mathcal{T}$ and $X_\mathcal{F} \in \mathcal{F}$, we have $\delta f_1 = 0$. It implies there exists $\alpha' \in \operatorname{Hom}(\mu_S(X_\mathcal{T}), X_\mathcal{T})$ such that $f_1 = f\alpha'$. Thus $f(\operatorname{Id}_{X_\mathcal{T}} - \alpha'\alpha) = 0$.

Since $\alpha'\alpha$ is a morphism in $\mathcal{A}$ and f is monomorphism in $\mathcal{A}$, we have $\alpha'\alpha = \operatorname{Id}_{X_\mathcal{T}}$ and $X_\mathcal{T} \in \operatorname{add} S$. Since $X \in {}^\perp S$, applying $\operatorname{Hom}(-, S)$ to the second row of the above commutative diagram, we have that $\operatorname{Hom}(X_\mathcal{F}, S) = 0$ and $\operatorname{Hom}(X_\mathcal{T}, S) \cong \operatorname{Hom}(F, S[1])$. This implies $X_\mathcal{T} \cong \operatorname{Hom}(F, S[1])^* \otimes S$ and $X \cong T_S^*(X_\mathcal{F})$. □

If the simple top S belongs to $\mathcal{F}[1]$, then for some $F \in \mathcal{F}$, the object $T_S^*(F)$ may be not in $\mathcal{A}$. The torsion free class $\mathcal{F}'$ has one more constraint as follows.

Lemma 4.8. *Suppose $S \in \mathcal{F}[1]$ as above. Then $X \in \mathcal{F}'$ if and only if there exists some $F \in \mathcal{F}$ with* $\operatorname{Hom}(F, S) = 0$ *such that X is a direct summand of $T_S^*(F)$ in $\mathcal{A}$.*

Proof. Let $F \in \mathcal{F}$ such that $\operatorname{Hom}(F, S) = 0$. By the definition of μ_S, we have $\mu_S(F) = \operatorname{Hom}(F, S[-1])^* \otimes S[-1]$. Since S is simple in $\mathcal{B}$, the canonical morphism $\alpha_F[1] \colon F[1] \to \mu_S(F)[1]$ is an epimorphism in $\mathcal{B}$. Hence it gives a short exact sequence

$$0 \to T_S^*(F)[1] \to F[1] \to \mu_S(F)[1] \to 0$$

in $\mathcal{B}$. Thus we could write $T_S^*(F) = K_1 \oplus K_2$ with $K_1 \in \mathcal{F}$ and $K_2 \in \mathcal{T}[-1]$ by Lemma 4.2. It follows that $K_1 \in {}^\perp S \cap \mathcal{F} = \mathcal{F}'$.

Conversely, for any $X \in \mathcal{F}'$, we have the following commutative diagram

$$\begin{tikzcd}
 & X \arrow[r, equal] \arrow[d, "f'"'] & X \arrow[d, "f"] & \\
\mu_S(F)[-1] \arrow[r] \arrow[d, equal] & T_S^*(F) \arrow[r, "g"] \arrow[d] & F \arrow[r, "\alpha"] \arrow[d] & \mu_S(F) \arrow[d, equal] \\
\mu_S(F)[-1] \arrow[r] & M \arrow[r] \arrow[d] & X_{\mathcal{T}}[1] \arrow[r, "h"'] \arrow[d] & \mu_S(F) \\
 & X[1] \arrow[r, equal] & X[1] &
\end{tikzcd}$$

where the third column is given by the canonical sequence of X with respect to $(\mathcal{T}, \mathcal{F})$ on $\mathcal{A}$. Since $X \in \mathcal{F}'$, we have $\alpha f = 0$ and a morphism $f' \in \mathrm{Hom}(X, T_S^*(F))$ such that $gf' = f$. The commutative diagram is followed by the octahedral axiom.

Since $\mu_S(F) \in \mathrm{add}(S[-1] \oplus S) \subseteq \mathcal{F} \cup \mathcal{F}[1]$ and $X_{\mathcal{T}} \in \mathcal{T}$, we have that $h = 0$ and thus $\alpha = 0$. It implies that $\mu_S(F) = 0$. Applying $\mathrm{Hom}(-, S)$ to the third column of the commutative diagram, we have that $\mathrm{Hom}(X_{\mathcal{T}}, S) = 0$ and $X_{\mathcal{T}} \in {}^{\perp}S$. It follows that $X_{\mathcal{T}} \in \mathcal{T}'$ and hence $X_{\mathcal{T}} = 0$. Hence $X \cong F$, which is exactly $T_S^*(F)$. $\square$

Corollary 4.9. *If $S \in \mathcal{F}[1]$ then the torsion class $\mathcal{F}' \subseteq \mathcal{F}$.*

Proposition 4.10. *Let X be an object in $\mathcal{A}$ such that $T_S^*(X) \in \mathcal{F}' * \mathcal{T}'[-1]$ for a simple object S of $\mathcal{B}$ which lies in $\mathcal{T}$, then $X \in \mathrm{add}\, S * \mathcal{F}$.*

Proof. Assume $S \in \mathcal{T}$. Let $0 \to X_{\mathcal{T}} \to X \to X_{\mathcal{F}} \to 0$ be the canonical sequence of X with respect to $(\mathcal{T}, \mathcal{F})$. Applying the twist functor T_S^* to X and $X_{\mathcal{T}}$ respectively, we have the following diagram

$$\begin{tikzcd}
T_S^*(X_{\mathcal{T}}) \arrow[r, "\beta"] \arrow[d, "h"'] & X_{\mathcal{T}} \arrow[r, "\alpha"] \arrow[d, "f"'] & \mu_S(X_{\mathcal{T}}) \arrow[r] \arrow[d, "p"] & T_S^*(X_{\mathcal{T}})[1] \arrow[d] \\
T_S^*(X) \arrow[r] \arrow[d] & X \arrow[r, "g"'] \arrow[d] & \mu_S(X) \arrow[r] \arrow[d] & T_S^*(X)[1] \arrow[d] \\
M \arrow[r, "{(\delta_1, \delta_2)}"'] \arrow[d] & X_{\mathcal{F}} \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & M[1] \arrow[d] \\
T_S^*(X_{\mathcal{T}})[1] \arrow[r] & X_{\mathcal{T}} \arrow[r] & \mu_S(X_{\mathcal{T}}) \arrow[r] & T_S^*(X_{\mathcal{T}})[1]
\end{tikzcd}$$

By Lemma 4.6, $T_S^*(X_{\mathcal{T}}) \in \mathcal{T}'$. Since $\mu_S(X) \in \langle S \rangle$, we have $gf\beta = 0$, and hence there exists a morphism $h \colon T_S^*(X_{\mathcal{T}}) \to T_S^*(X)$ such that the top-left square commutes. So we get the whole commutative diagram by 3×3 lemma.

The assumption $T_S^*(X) \in \mathcal{F}' * \mathcal{T}'[-1]$ implies $h = 0$. Thus $M = T_S^*(X) \oplus T_S^*(X_{\mathcal{T}})[1]$. Since $\delta_2 \in \mathrm{Hom}(T_S^*(X_{\mathcal{T}})[1], X_{\mathcal{F}}) = 0$, we get that $T_S^*(X_{\mathcal{T}})[1]$ is a direct summand of $N \in \langle S \rangle$. Since $T_S^*(X_{\mathcal{T}}) \in {}^{\perp}S$, we have $T_S^*(X_{\mathcal{T}}) = 0$. So the first row shows that $X_{\mathcal{T}} \cong \mu_S(X_{\mathcal{T}}) \in \mathrm{add}\, S$ and the assertion follows.

□

5. Proof of Theorem 1.1

Keep the notation as in the previous subsection. Throughout this section, we assume that Λ is a finite-dimensional hereditary algebra. Let $\mathcal{B}' = \mathcal{F}'[1] * \mathcal{T}'$ the HRS tilt of $\mathcal{W}$ with respect to $(\mathcal{T}', \mathcal{F}')$. In this section, we will prove the following proposition and Theorem 1.1. We actually give a construction of the objects B and C in the following results. This construction closely relates to the image of the realization functor, see Section 5.3.

Proposition 5.1. *If $(\mathcal{T}', \mathcal{F}')$ is complete in $\mathcal{W}$, then $(\mathcal{T}, \mathcal{F})$ is complete in $\mathcal{A}$*

Remark 5.2. For each $A \in \mathcal{A}$, the torsion pair $(\mathcal{T}, \mathcal{F})$ is complete in $\mathcal{A}$ if and only if there is a triangle $A \to B \to C \to A[1]$ in $D^b(\mathcal{A})$ such that $B, C \in \mathcal{B}$. Such triangles and the 5-term exact sequences determined by $(\mathcal{T}, \mathcal{F})$ are the same thing in different forms, see [6, Proposition 3.2] for more details. We will use this equivalence in this section without mentioning that.

5.1. Proof of Proposition 5.1.

5.1.1. *The case $S \in \mathcal{F}[1]$.*

Lemma 5.3. *Assume $S \in \mathcal{F}[1]$. If $(\mathcal{T}', \mathcal{F}')$ is complete in $\mathcal{W}$ then for each $X \in \mathcal{A}$ with* $\mathrm{Hom}(X, S[-1]) = 0$*, there is a triangle $X \to M \to N \to X[1]$ in $D^b(\mathcal{A})$ such that $M, N \in \mathcal{B}$.*

Proof. Let $X \in \mathcal{A}$ such that $\mathrm{Hom}(X, S[-1]) = 0$. Then $\mu_S(X) = \mathrm{Hom}(X, S)^* \otimes S$. By the triangle $\mu_S(X)[-1] \to T_S^*(X) \to X \to \mu_S(X)$, we have $T_S^*(X) \in {}^{\perp}S \cap \mathcal{A} = \mathcal{W}$. By the octahedral axiom, we have the following diagram, where the second column is from the assumption, i.e. $B', C' \in \mathcal{B}'$.

$$\begin{tikzcd}
 & C'[-1] \arrow[r, equal] \arrow[d] & C'[-1] \arrow[d] & \\
\mu_S(X)[-1] \arrow[r, "g"] \arrow[d, equal] & T_S^*(X) \arrow[r] \arrow[d, "h"] & X \arrow[r, "\alpha"] \arrow[d] & \mu_S(X) \arrow[d, equal] \\
\mu_S(X)[-1] \arrow[r, "hg"'] & B' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & \mu_S(X) \\
 & C' \arrow[r, equal] & C' &
\end{tikzcd}$$

Note that $\mathcal{B}' = \mathcal{F}'[1] * \mathcal{T}' \subseteq \mathcal{F}[1] * \mathcal{T} = \mathcal{B}$ by Corollary 4.9. Taking the cohomology of the third row, we have a long exact sequence in $\mathcal{A}$

$$0 \to B_1' \to M_1 \to \mu_S(X)[-1] \xrightarrow{f} B_0' \to M_0 \to 0.$$

Since $B_0' \in \mathcal{T}' \subseteq \mathcal{T}$, we have $M_0 \in \mathcal{T}$. Taking the cohomology of the third column, we have a long exact sequence

$$0 \to M_1 \xrightarrow{g} C_1' \to X \xrightarrow{f} M_0 \to C_0' \to 0.$$

By Corollary 4.9, $C_1' \in \mathcal{F}' \subseteq \mathcal{F}$. Let $N = C'$ then the triangle in the third column is the desired triangle. □

Lemma 5.4. *Assume that $S \in \mathcal{F}[1]$. If $(\mathcal{T}', \mathcal{F}')$ is complete in $\mathcal{W}$ then for each $A \in \mathcal{A}$, there is a triangle $A \to B \to C \to A[1]$ in $D^b(\mathcal{A})$ such that $B, C \in \mathcal{B}$.*

Proof. For any indecomposable $A \in \mathcal{A}$, consider the triangle

$$X \to A \xrightarrow{\alpha} G \xrightarrow{g} X[1]$$

where $G = \mathrm{Hom}(A, S[-1])^* \otimes S[-1]$ and α is the canonical map. Applying $\mathrm{Hom}(-, S[-1])$ to the triangle, we get $\mathrm{Hom}(X, S[-1]) = 0$. Taking cohomology, we have a long exact sequence

$$0 \to H^0(X) \to A \to G \xrightarrow{f} H^1(X) \to 0$$

in $\mathcal{A}$. By Lemma 4.5, we have $H^1(X) = S[-1]^l \oplus T$ with $l \in \mathbb{Z}_{>0}$ and $T \in \mathcal{T}$. Since α is minimal, g belongs to radical. So does the component of g between G and $S[-1]^l$. This implies $l = 0$ because S is exceptional. Thus we have $H^1(X) \in \mathcal{T}$.

Since $H^0(X) \in \mathcal{A}$ and $\mathrm{Hom}(H^0(X), S[-1]) = 0$, by Lemma 5.3, there is a triangle $H^0(X) \to M \to N \to H^0(X)[1]$ with $M, N \in \mathcal{B}$. Now we replace the second column and second row of the commutative diagram in Lemma 5.3 by $N[-1] \oplus H^1(X)[-1] \to X \to M \to N \oplus H^1(X)$ and $G[-1] \to X \to A \to G$. Then we obtained a triangle $N[-1] \oplus H^1(X)[-1] \to A \to B \to N \oplus H^1(X)$ in the third column by the octahedral axiom. By calculation of the cohomology, we have $B \in \mathcal{B}$ and $C = N \oplus H^1(X) \in \mathcal{B}$. □

5.1.2. *The case $S \in \mathcal{T}$.*

Lemma 5.5. *For any nonzero $X \in \mathcal{F}'$, we have $T_S(X) \in \mathcal{A}$. In particular, $T_S(B') \in \mathcal{A} * \mathcal{A}[1]$ for any $B' \in \mathcal{B}'$.*

Proof. Without loss of generality, suppose that X is indecomposable. Taking cohomology of the triangle $\lambda_S(X) \to X \to T_S(X) \to \lambda_S(X)[1]$, we have a long exact sequence,

$$0 \to H^{-1}(T_S(X)) \to H^0(\lambda_S(X)) \to X \to H^0(T_S(X)) \to H^1(\lambda_S(X)) \to 0.$$

By Lemma 2.8, we have $T_S(X)$ is indecomposable. So either $H^0(T_S(X)) = 0$ or $H^{-1}(T_S(X)) = 0$. In the former case, since $H^0(\lambda_S(X)) \in \mathrm{add}\, S \subseteq \mathcal{T}$, we have $X \in \mathcal{T} \cap \mathcal{W} = \mathcal{T}'$, a contradiction. So $H^{-1}(T_S(X)) = 0$ and the first assertion follows.

For any $B' \in \mathcal{B}'$, note that $B' = B_1'[1] \oplus B_0'$ with $B_0' \in \mathcal{T}'$ and $B_1' \in \mathcal{F}'$, and $T_S(B_0') \in \mathcal{A} * \mathcal{A}[1]$ by definition. So the last assertion follows from the first assertion. □

Now we could prove the Proposition 5.1 for $S \in \mathcal{T}$ in the following steps.

Lemma 5.6. *If $(\mathcal{T}', \mathcal{F}')$ is complete in $\mathcal{W}$ then for each $A \in \mathcal{A}$ such that $\mathrm{Hom}(A, S) = 0$ and $\mathrm{Hom}(S, A) = 0$, there is a triangle $A \to B \to C \to X[1]$ in $D^b(\mathcal{A})$ such that $B, C \in \mathcal{B}$.*

Proof. Let $A \in \mathcal{A}$ such that $\mathrm{Hom}(A, S) = 0$. Then $\mu_S(A) = \mathrm{Hom}(A, S[1])^* \otimes S[1]$, which implies $T_S^*(A) \in \mathcal{W}$. By assumption, there is a triangle $T_S^*(A) \to B' \to C' \to T_S^*(A)[1]$ in $D^b(\mathcal{W})$ such that $B', C' \in \mathcal{B}'$.

Consider the following diagram. Since $\beta' h\alpha' \in \mathrm{Hom}(\mu_S(A)[-1], T_S(B')) = 0$, we have a morphism $f \in \mathrm{Hom}(\mu_S(A), \lambda_S(B'))$ such that the top-left square commutes. By 3×3 Lemma, we have the following commutative diagram

$$\begin{tikzcd}
\mu_S(A)[-1] \arrow[r, "\alpha'"] \arrow[d, "{f[-1]}"'] & T_S^*(A) \arrow[r] \arrow[d, "h"] & A \arrow[r, "\alpha"] \arrow[d, "g"] & \mu_S(A) \arrow[d, "f"] \\
\lambda_S(B')[-1] \arrow[r] \arrow[d] & B' \arrow[r, "\beta'"] \arrow[d] & T_S(B') \arrow[r, "\beta"] \arrow[d] & \lambda_S(B') \arrow[d] \\
V[-1] \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & N \arrow[r, "\delta"] \arrow[d] & V \arrow[d] \\
\mu_S(A) \arrow[r] & T_S^*(A)[1] \arrow[r] & A[1] \arrow[r] & \mu_S(A)[1]
\end{tikzcd}$$

Note that V is in the category $\langle S\rangle$. We could assume that $V \cong \oplus_{k\in\mathbb{Z}} V_k \otimes S[k]$.

We will fit A into a 5-term exact sequence which is equivalent to the triangle $A \to B \to C \to A[1]$. We construct the exact sequence by computing the cohomologies of $T_S(B')$ and N.

Firstly, we claim that $C' \cong T_S^*(N)$. Since $T_S(B') \in \mathcal{A} * \mathcal{A}[1]$ by Lemma 5.5, we have that $N \in \mathcal{A} * \mathcal{A}[1]$. With $\mathrm{Hom}(-, S)^*$ acting on the last column, we get a long exact sequence of vector spaces, where $f_i = \mathrm{Hom}(f, S[i])^*, i = 0, 1$.

$$\begin{aligned}
0 \quad &\to \mathrm{Hom}(\lambda_S(B'), S[2])^* \to V_2 \to \mathrm{Hom}(\mu_S(A), S[1])^* \xrightarrow{f_1} \mathrm{Hom}(\lambda_S(B'), S[1])^* \\
&\to V_1 \to \mathrm{Hom}(\mu_S(A), S)^* \xrightarrow{f_0} \mathrm{Hom}(\lambda_S(B'), S)^* \to V_0 \to 0
\end{aligned}$$

On the other hand, with $\mathrm{Hom}(-, S)^*$ acts on the third column, we have a long exact sequence, where $g_i = \mathrm{Hom}(g, S[i])^*$

$$\begin{aligned}
0 \quad &\to \mathrm{Hom}(T_S(B'), S[2])^* \to \mathrm{Hom}(N, S[2])^* \to \mathrm{Hom}(A, S[1])^* \\
&\xrightarrow{g_1} \mathrm{Hom}(T_S(B'), S[1])^* \to \mathrm{Hom}(N, S[1])^* \to \mathrm{Hom}(A, S)^* \\
&\xrightarrow{g_0} \mathrm{Hom}(T_S(B'), S)^* \to \mathrm{Hom}(N, S)^* \to 0
\end{aligned}$$

Since $\alpha_i = \mathrm{Hom}(\alpha, S[i])^*$ and $\beta_i = \mathrm{Hom}(\beta, S[i])^*$ are isomorphisms, and $\beta g = f\alpha$, we have $g_i = \beta_i^{-1} f_i \alpha_i$. Thus, comparing these two sequences, we have $V_i \cong \mathrm{Hom}(N, S[i])^*$. Note that components of δ are non-zero, otherwise $C' \in {}^{\perp}S$ has common summands with $V[-1] \in \langle S\rangle$, a contradiction. So $\delta\colon N \to V$ is the canonical map and the claim follows.

Secondly, we construct B and C via N. Note that $N = H^{-1}(N)[1] \oplus H^0(N)$. Taking the cohomology of the second row and noticing $S \in \mathcal{T}$, we have $H^0(T_S(B')) \in \mathcal{T}$. Taking the cohomology of the third column, we have a long exact sequence

$$0 \to H^{-1}(T_S(B')) \to H^{-1}(N) \xrightarrow{f} A \to H^0(T_S(B')) \to H^0(N) \to 0$$

in $\mathcal{A}$. Since $H^0(T_S(B')) \in \mathcal{T}$, we have $H^0(N) \in \mathcal{T}$.

On the other hand, since $T_S^*(H^{-1}(N))[1]$ is a direct summand of $C' \in \mathcal{F}'[1] * \mathcal{T}'$, by Proposition 4.10, we have a short exact sequence $0 \to S_N \xrightarrow{i} H^{-1}(N) \xrightarrow{p} F_N \to 0$ with

$S_N \in \operatorname{add} S$ and $F_N \in \mathcal{F}$. Since $\operatorname{Hom}(S, A) = 0$ by assumption, we have $fi = 0$ and there exists $f' \in \operatorname{Hom}(F_N, A)$ such that $f = f'p$. So we get the 5-term exact sequence

$$0 \to \operatorname{Ker} f' \to F_N \xrightarrow{f'} A \to H^0(T_S(B')) \to H^0(T_S(C')) \to 0.$$

Thus $B, C \in \mathcal{B}$ are determined by this exact sequence. □

Now we could remove the constraint that $\operatorname{Hom}(S, A) = 0$ in Lemma 5.6.

Lemma 5.7. *If $(\mathcal{T}', \mathcal{F}')$ is complete in $\mathcal{W}$ then for each $A \in \mathcal{A}$ such that $\operatorname{Hom}(A, S) = 0$, there is a triangle $A \to N \to D \to X[1]$ in $D^b(\mathcal{A})$ such that $D, N \in \mathcal{B}$.*

Proof. Suppose without loss of generality that A is indecomposable. Let $S_A \xrightarrow{\alpha} A \to A' \to S_A[1]$ be the triangle with $S_A = \operatorname{Hom}(S, A) \otimes S$ and α the canonical map. Then $\operatorname{Hom}(S, A') = 0$ and $A' \in \mathcal{A}[1]*\mathcal{A}$. In particular, $H^0(A') \in \mathcal{A}$, and $\operatorname{Hom}_{\mathcal{A}}(S, H^0(A')) = 0$. Applying $\operatorname{Hom}(-, S)$ to the triangle, we have also $\operatorname{Hom}_{\mathcal{A}}(H^0(A'), S) = 0$. Thus, by Lemma 5.6, there is a triangle $H^0(A') \to B \to C \to H^0(A')[1]$ with $B, C \in \mathcal{B}$.

By Proposition 4.1, there is a short exact sequence $0 \to S \to \mathbb{S}E \to M[1] \to 0$ in $\mathcal{B}$ with E the projective cover of S in $\mathcal{B}$. Consider the following diagram, where $M_A = \operatorname{Hom}(S, A) \otimes M$ and $\mathbb{S}E_A = \operatorname{Hom}(S, A) \otimes \mathbb{S}E$.

$$\begin{tikzcd}
M_A \arrow[r] \arrow[d] & D[-1] \arrow[r] \arrow[d] & C[-1] \arrow[r, "u"] \arrow[d, "f"'] & M_A[1] \arrow[d, "i"] \\
S_A \arrow[r, "\alpha"] \arrow[d] & A \arrow[r] \arrow[d] & H^0(A') \oplus H^{-1}(A')[1] \arrow[r, "g"] \arrow[d] & S_A[1] \arrow[d, "h"] \\
\mathbb{S}E_A \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & B \oplus H^{-1}(A')[1] \arrow[r] \arrow[d] & \mathbb{S}E_A[1] \arrow[d] \\
M_A[1] \arrow[r] & D \arrow[r] & C \arrow[r] & M_A[2]
\end{tikzcd}$$

Since $\operatorname{Hom}(C[-1], \mathbb{S}E[1]) = 0$, we have $hgf = 0$ and hence there exists a morphism $u \in \operatorname{Hom}(C[-1], M_A[1])$ such that the top-right square commutes. So we get the whole commutative diagram by 3×3 Lemma.

Since $M[1] \in \mathcal{B}$, we have $H^{-1}(M[1]) \in \mathcal{F}$. Taking the cohomology of the first row, we get $H^{-1}(D) \in \mathcal{F}$ by $H^{-1}(C) \in \mathcal{F}$. Similarly, since $\mathbb{S}E \in \mathcal{B}$, we have $H^0(\mathbb{S}E_A) \in \mathcal{T}$. Taking the cohomology of the third row, we have $H^0(N) \in \mathcal{T}$ by $H^0(B) \in \mathcal{T}$. Thus we have that $D, N \in \mathcal{B}$. □

Lemma 5.8. *If $(\mathcal{T}', \mathcal{F}')$ is complete in $\mathcal{W}$ then for each $A \in \mathcal{A}$, there is a triangle $A \to B \to C \to X[1]$ in $D^b(\mathcal{A})$ such that $B, C \in \mathcal{B}$.*

Proof. Consider the triangle $S^A[-1] \to X \to A \xrightarrow{\alpha} S^A$ with $S^A = \operatorname{Hom}(A, S)^* \otimes S$ and α the canonical map. Taking the cohomology of this triangle, we have $H^1(X) \in \mathcal{T}$. Note that $H^0(X) \in \mathcal{A}$ and $\operatorname{Hom}(H^0(X), S) = 0$, there is a triangle $H^0(X) \to N \to D \to H_0(X)[1]$ with $N, D \in \mathcal{B}$ by Lemma 5.7.

Combining the above triangles, we have the following commutative diagram by the octahedral axiom, where $C = D \oplus H^1(X)$

$$\begin{tikzcd}
 & C[-1] \arrow[r, equal] \arrow[d] & C[-1] \arrow[d] & \\
S^A[-1] \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & A \arrow[r, "\alpha"] \arrow[d] & S^A \arrow[d, equal] \\
S^A[-1] \arrow[r] & N \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & S^A \\
 & C \arrow[r, equal] & C &
\end{tikzcd}$$

Taking the cohomology of the third row and noticing that $H^0(N) \in \mathcal{T}$, we have $H^0(B) \in \mathcal{T}$. Taking cohomology of the third column and noticing that $H^{-1}(C) = H^{-1}(D) \in \mathcal{F}$, we have $\mathcal{H}^{-1}(B) \in \mathcal{F}$. So the third column is the desired triangle. □

Proposition 5.1 follows from Lemma 5.4 and Lemma 5.8.

5.2. **Proof of Theorem 1.1.** If a torsion pair $(\mathcal{T}, \mathcal{F})$ is complete in $\mathcal{A}$, then Corollary 3.2 shows that the aisle $\mathcal{U}_{\mathcal{T}}$ is closed under the Serre functor.

Conversely, we assume that $\mathcal{U}_{\mathcal{T}}$ is closed under the Serre functor. If $\mathcal{T}$ either is finitely generated or has no Ext-projective objects, then $(\mathcal{T}, \mathcal{F})$ is complete by Lemma 3.4 and Lemma 3.6.

In general, we suppose $\mathcal{T}$ is not finitely generated and has non-zero Ext-projective object. Then $\mathcal{U}_{\mathcal{T}}$ has Ext-projective object E with its simple top S in $\mathcal{B}$. Then for $D^b(\mathcal{W})$, $\mathcal{U}_{\mathcal{T}}$ induces a bounded t-structure $(\mathcal{U}', \mathcal{V}')$ (and hence a torsion pair $(\mathcal{T}', \mathcal{F}')$) such that $\mathcal{U}'$ has one fewer Ext-projective generators than $\mathcal{U}_{\mathcal{T}}$.

Repeating this procedure finitely many times, we could get a aisle $\mathcal{U}_0$ without non-zero Ext-projective and thus its corresponding torsion class $\mathcal{T}_0$. Applying finitely many times Proposition 5.1 on $\mathcal{T}_0$, we have that the torsion pair $(\mathcal{T}, \mathcal{F})$ is complete in $\mathcal{A}$.

5.3. **Relation with realization functor.** Given a triangulated category $\mathcal{D}$ with a bounded t-structure $(\mathcal{U}, \mathcal{V})$ and its heart $\mathcal{B}$, the *realization functor* $F\colon D^b(\mathcal{B}) \to \mathcal{D}$ is a triangle functor which naturally extends the embedding $\mathcal{B} \to \mathcal{D}$ [2, Section 3.1]. In [2], the existence of the realization functor depends on filtered structures over $\mathcal{D}$. Note that a realization functor exists if $\mathcal{D}$ is algebraic [7, 16]. It follows that the realization functor $F\colon D^b(\mathcal{B}) \to D^b(\mathcal{A})$ exists for the t-structure $(\mathcal{U}_{\mathcal{T}}, \mathcal{V}_{\mathcal{T}})$ given by a torsion pair $(\mathcal{T}, \mathcal{F})$ on $\mathcal{A}$. For more details about tilting theory and realization functor, we refer to [18]. We remark that the uniqueness of the functor is not known [6, 21].

We fix a realization functor $F\colon D^b(\mathcal{B}) \to D^b(\mathcal{A})$ with respect to the heart $\mathcal{B}$. Let A' be the HRS titling of $\mathcal{B}$ with respect to $(\mathcal{F}[1], \mathcal{T})$. The functor F is t-exact, where $D^b(\mathcal{B})$ is endowed with the t-structure given by the heart $\mathcal{A}'$ and $D^b(\mathcal{A})$ has the canonical t-structure. Moreover, the restriction $F|_{\mathcal{A}'}\colon \mathcal{A}' \to \mathcal{A}$ is exact [6, Section 3].

We have the following description of the essential image of the realization functor F, i.e. the full subcategory of $D^b(\mathcal{A})$ consisting of those objects X, which are isomorphic to $F(Y)$ for some object Y in $D^b(\mathcal{B})$.

Proposition 5.9 ([6, Proposition 3.2])**.** *Keep the notation as above. Assume that $A \in \mathcal{A}$. Then the following statements are equivalent:*

(1) The object A belongs to $\operatorname{Im} F$*.*
(2) There is an exact triangle $A \to B^0 \to B^1 \to \Sigma(A)$ in $D^b(\mathcal{A})$ with $B^i \in \mathcal{B}$;

In general, it is difficult to construct the essential image or preimage of a realization functor. In the simplest case of realization functor, i.e., the t-structure $(\mathcal{U}, \mathcal{V})$ is the standard t-structure, we have that the realization functor is the identity functor. Except for this trivial case, we do not know how to construct the preimage of the realization functor for the t-structure $(\mathcal{U}_{\mathcal{T}}, \mathcal{V}_{\mathcal{T}})$ given by a torsion pair $(\mathcal{T}, \mathcal{F})$.

If the aisle $\mathcal{U}_{\mathcal{T}}$ is closed under the Serre functor $\mathbb{S}$ or $(\mathcal{T}, \mathcal{F})$ is complete, then realization functor F is an equivalence and each object X in $\mathcal{A}$ has preimage in $D^b(\mathcal{B})$. By Proposition 5.1 and Proposition 5.9, we could construct the preimage of the realization functor for any object X in $\mathcal{A}$. Unfortunately, we do not know how to construct the preimage of realization functor in general.

Remark 5.10. In [17], Neeman considered the problem when a long exact sequence is obtained from the cohomology of a triangle in a derived category. Given a 5-term exact sequence

$$0 \to X_0 \to X_1 \to X_2 \to X_3 \to X_4 \to 0$$

in derived category $D^b(\mathcal{A})$, this sequence defines a class in $\operatorname{Ext}^3(X_4, X_0)$. Neeman showed that the long exact sequence is obtained from taking the cohomology of a triangle if and only if the above class in $\operatorname{Ext}^3(X_4, X_0)$ under realization functor vanishes. Note that Neeman's argument is in a more general setting. He also claimed the class in $\operatorname{Ext}^3(X_4, X_0)$ vanishing is a necessary condition but not known whether it suffices. It looks interesting to find the relationship between the Serre functor and the completeness of the torsion class in a more general case.

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