The solenoidal Heisenberg Virasoro algebra and its simple weight modules

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Abstract

Let $A_n = \mathbb{C}[t_i^{\pm 1},\ 1 \le i \le n]$ and $\mathbf{W}(n)_\mu = A_n d_\mu$ the solenoidal Lie algebra introduced by Y.Billig and V.Futorny in [6], where $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{C}^n$ is a generic vector and

$$d_\mu = \sum_{i=1}^{n} \mu_i t_i \frac{\partial}{\partial t_i}.$$

We consider the semi-direct product Lie algebra $\mathbf{WA}(n)_\mu := \mathbf{W}(n)_\mu \ltimes A_n$.

In the first part, We prove that $\mathbf{WA}(n)_\mu$ has a unique three-dimensional universal central extension. In fact we construct a higher rank Heisenberg-Virasoro algebra (see [11, 14]). It will be denoted by $\mathbf{HVir}(n)_\mu$ and it will be called the solenoidal Heisenberg-Virasoro algebra. Then we will study Harish-Chandra modules of $\mathbf{HVir}(n)_\mu$ following [14]. We will obtain two classes of Harich-Chandra modules: generalized highest weight modules(**GHW** modules) and intermediate series modules. Our results are particular cases of [14]. In the end, we will construct $\mathbf{HVir}(n)_\mu$ Verma modules using the lexicographic order on $\mathbb{Z}^n$. In particular we give examples of irreducible weight modules which have infinite dimensional weight spaces.

Key words: Heisenberg-Virasoro algebra, solenoidal algebra, solenoidal Heisenberg Virasoro algebra, central extension, Harish-Chandra modules, cuspidal modules

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1 Introduction

The Heisenberg-Virasoro algebra **HVir** was first introduced in [3], where highest weight modules were studied and a determinant formula for the Shapovalov form on Verma modules was obtained. In [15] (see also [12], [9]), Lu and Zhao classified the irreducible Harish-Chandra modules over **HVir**, which turn out to be modules of intermediate series and highest/lowest weight modules. Whittaker modules for **HVir** were studied by [13]. Recently, a large class of irreducible non-weight modules were constructed in [8]. The generalized Heisenberg Virasoro algebras are generalization of the Heisenberg-Virasoro algebras where the grading by $\mathbb{Z}$ is replaced by an additive subgroup G of $\mathbb{C}$. Their representation theory was considered by several authors, see for example [11, 18].

Recently in [4, 5], Y. Billig and V. Futorny study weight modules of finite weight spaces of the Lie algebra $\mathbf{W}(n)$ of vector fields on the torus. They prove that such modules are highest modules or quotients of modules of tensor fields. In [6], they introduced so called solenoidal Lie algebra $\mathbf{W}(n)_\mu := A_n d_\mu$ as a bridge between the Lie algebra $\mathbf{W}(1)$ and the Lie algebra $\mathbf{W}(n)$ where $\mu = (\mu_1, \dots \mu_n)$ is a generic element in $\mathbb{C}^n$ and $d_\mu = \sum_{i=1}^{n} \mu_i t_i \frac{\partial}{\partial t_i}$. Then they give a classification of the simple cuspidal $\mathbf{W}(n)_\mu$-modules. In a forthcoming paper (see [2]), we compute the second cohomology space $H^2(\mathbf{W}(n)_\mu, \mathbb{C})$. The universal central extension of $\mathbf{W}(n)_\mu$ is a new generalization of the Virasoro algebra, denoted $\mathbf{Vir}(n)_\mu$ and is called the solenoidal-Virasoro algebra. Then we give a complete classification of its Harish-Chandre modules.

In this paper we consider the semi-direct product $\mathbf{WA}(n)_\mu := \mathbf{W}(n)_\mu \ltimes A_n$, the analogue of the Lie algebra $\mathbf{WA}(1) = \mathbf{W}(1) \ltimes A_1$ in the case $n = 1$. The **first section** of this paper contains our main result given by Theorem 2.1. We compute three generating 2-cocycles and then we classify the universal central extension of $\mathbf{WA}(n)_\mu$. The obtained three-dimensional central extension of $\mathbf{WA}(n)_\mu$ is called the solenoidal Heisenberg-Virasoro algebra and is denoted by $\mathbf{HVir}(n)_\mu$. In the **second section**, we study Harish-Chandra modules over $\mathbf{HVir}(n)_\mu$. In [14], G.Liu and X.Guo give the definition of generalized Heisenberg-Virasoro algebras $\mathbf{HVir}[G]$ where G is an additive subgroup of $\mathbb{C}$. When $G \simeq \mathbb{Z}^n, \mathbf{HVir}[G]$ is called rank n Heisenberg-Virasoro algebra. Our algebra $\mathbf{HVir}(n)_\mu$ is an example of rank n Heisenberg-Virasoro algebra.

In the second section, following [14], we classify Harish-Chandra modules of $\mathbf{HVir}(n)_\mu$. We obtain tow kinds of modules, generalized highest weight modules (**GHW** modules) or intermediate series modules. For $n = 1$, we obtain the classification results for the classical Heisenberg-Virasoro algebra given by R. Lü and K. Zhao (see [15]).

In the **third section**, we introduce a triangular decomposition of $\mathbf{HVir}(n)_\mu$ using the lexicographic order on $\mathbb{Z}^n$, then we define Verma modules and anti-Verma modules. As the usual highest weight theory, we obtain irreducible highest weight modules and irreducible lowest weight modules of $\mathbf{HVir}(n)_\mu$ by taking respectively quotients of Verma modules and anti-Verma modules. In the end, we provide that these modules have infinite dimensional weight spaces.

2 The solenoidal Heisenberg-Virasoro algebra $\mathbf{HVir}(n)_\mu$

Let $A_n = \mathbb{C}[t_i^{\pm 1},\ 1 \leq i \leq n]$ be the algebra of Laurent polynomials and let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{C}^n$ generic, that is, for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, $\mu \cdot \alpha := \sum_{i=1}^{n} \mu_i \alpha_i \neq 0$. Let $d_\mu := \sum_{i=1}^{n} \mu_i D_{t_i}$, where $D_{t_i} = t_i \frac{\partial}{\partial t_i}$. Y. Billig and V. Futorny [6], introduced the solenoidal-Witt Lie algebra $\mathbf{W}(n)_\mu := A_n d_\mu$ as the Lie subalgebra of the Lie algebra $\mathbf{W}(n) = Der(A_n)$. Let

$$\Gamma_\mu = \{\mu \cdot \alpha; \alpha \in \mathbb{Z}^n\}.$$

It is the image of $\mathbb{Z}^n$ by the map :

$$\sigma_\mu : \quad \begin{aligned} &\mathbb{Z}^n \longrightarrow \mathbb{C} \\ &\alpha \mapsto \mu \cdot \alpha \end{aligned}$$

Γ_μ is a subgroup of $(\mathbb{C}, +)$. A canonical basis of $\mathbf{W}(n)_\mu$ is given by:

$$\{e_{\mu\cdot\alpha} := t^\alpha d_\mu, \mu \cdot \alpha \in \Gamma_\mu\}.$$

The commutators of the $e_{\mu\cdot\alpha}$ are given by:

$$[e_{\mu\cdot\alpha}, e_{\mu\cdot\beta}] = \mu \cdot (\beta - \alpha) e_{\mu\cdot(\alpha+\beta)},\ \mu \cdot \alpha, \mu \cdot \beta \in \Gamma_\mu. \tag{2.1}$$

In the case of $n = 1$, we take $\mu \in \mathbb{C}^*$ then $\Gamma_\mu = \mu\mathbb{Z}$ and $\mathbf{W}(n)_\mu$ is isomorphic to $\mathbf{W}(1)$ by taking $d_m \to \gamma d_m$ where γ is the square root of μ . In particular if $\mu = 1$ we obtain the classical Witt algebra $\mathbf{W}(1)$.

In the recent paper (see [2]) , we study the central extension of the solenoidal Lie algebra $\mathbf{W}(n)_\mu$ introduced by Y. Billig and V.Futorny (see [6]), we obtain an analogue of the Virasoro algebra and we called it the solenoidal Virasoro algebra and we denoted it by $\mathbf{Vir}(n)_\mu$. Then we give a classification of Harish-Chandra modules over $\mathbf{Vir}(n)_\mu$. Also, we construct $\mathbf{Vir}(n)_\mu$-modules with infinite dimensional weight spaces by using the lexicographic order on $\mathbb{Z}^n$.

In this paper we consider the Lie algebra $\mathbf{WA}(n)_\mu := \mathbf{W}(n)_\mu \ltimes A_n$. Its canonical basis is:

$$\{e_{\mu\cdot\alpha} = t^\alpha d_\mu, h_\alpha = t^\alpha, \mu \cdot \alpha \in \Gamma_\mu,\ \alpha \in \mathbb{Z}^n\}$$

Its Lie structure generated by the following brackets:

$$[e_{\mu\cdot\alpha}, e_{\mu\cdot\beta}] = \mu \cdot (\beta - \alpha) e_{\mu\cdot(\alpha+\beta)}.$$

$$[h_\alpha, h_\beta] = 0.$$

$$[e_{\mu\cdot\alpha}, h_\beta] = (\mu \cdot \beta) h_{\alpha+\beta}.$$

The main purpose of this paper is to compute central extensions of the algbera $\mathbf{WA}(n)_\mu$.

The following theorem is a generalization to multidimensional case of Theorem 3 and Proposition 3 in [17] where the extension of the Lie algebra $Vect(S^1)$ of vector fields on the circle by modules of tensor densities $\mathcal{F}_\lambda$ is study.

Theorem 2.1. *The second cohomology space $H^2(\mathbf{WA}(n)_\mu, \mathbb{C})$ is three dimensional and it is generated by the following 2-cocycles $C_{\mu,1},\ C_{\mu,2},\ C_{\mu,3} : \mathbf{WA}(n)_\mu \times \mathbf{WA}(n)_\mu \longrightarrow \mathbb{C}$ defined by:*

$$\begin{cases} C_{\mu,1}(e_{\mu\cdot\alpha}, e_{\mu\cdot\beta}) := \delta_{\alpha,-\beta}\frac{(\mu\cdot\alpha)^3-(\mu\cdot\alpha)}{12}c_{\mu,1} \\ 0 \ otherwise \end{cases}, \tag{2.2}$$

$$\begin{cases} C_{\mu,2}(e_{\mu\cdot\alpha}, h_{\beta}) := \delta_{\alpha,-\beta}((\mu\cdot\alpha)^2-(\mu\cdot\alpha))c_{\mu,2} \\ 0 \ otherwise \end{cases}, \tag{2.3}$$

$$\begin{cases} C_{\mu,3}(h_{\alpha}, h_{\beta}) := \delta_{\alpha,-\beta}\frac{(\mu\cdot\alpha)}{3}c_{\mu,3} \\ 0 \ otherwise \end{cases}, \tag{2.4}$$

Proof. The fact that the 2-cochains

$$C_{\mu,1},\ C_{\mu,2},\ C_{\mu,3} : \mathbf{WA}(n)_\mu \times \mathbf{WA}(n)_\mu \longrightarrow \mathbb{C}$$

are 2-cocycles is a straight forward computations using the 2-cocycle condition:

$$C_{\mu,i}([X,Y],Z) + C_{\mu,i}([Y,Z],X) + C_{\mu,i}([Z,X],Y) = 0, \tag{2.5}$$

For $i = 1,2,3;\ X,Y,Z \in \mathbf{WA}(n)_\mu$.

Let us now prove the unicity of the 2-cocycles $C_{\mu,1},\ C_{\mu,2},\ C_{\mu,3}$.

Denote $X_{\alpha,1} = e_{\mu\cdot\alpha}$ and $X_{\alpha,2} = h_\alpha$. The first step, we prove that for $i \in \{1,2,3\}$ and $j,k \in \{1,2\}$ each cocycle has the following form:

$$C_{\mu,i}(X_{\alpha,j}, X_{\beta,k}) = \delta_{i,j+k-1}\delta_{\alpha,-\beta}\theta_i(\mu\cdot\alpha)c_{\mu,i},\ \text{for all } \alpha,\beta \in \mathbb{Z}^n.$$

The second step, we apply known results on functional equations (see [10], [1]) to give the final expressions.

Take $X = X_{\alpha,j}, Y = X_{\beta,k}$ and $Z = X_{\gamma,l}$. Since condition (2.5) is cyclic in X,Y,Z, it suffices to take $(j,k,l) \in \{(1,1,1),(1,1,2),(1,2,2)\}$ corresponding respectively to $\{C_{\mu,1},\ C_{\mu,2},\ C_{\mu,3}\}$ since the left hand side in condition (2.5) is equal to zero for the other possibilities.

Let us start by proving the unicity of $C_{\mu,1}$. So we take $(j,k,l) = (1,1,1)$, that is $(X_{\alpha,1}, X_{\beta,1}, X_{\gamma,1}) = (e_{\mu\cdot\alpha}, e_{\mu\cdot\beta}, e_{\mu\cdot\gamma})$. Assume that there exists $\Psi_1 : \Gamma_\mu \times \Gamma_\mu \to \mathbb{C}$ such that:

$$[e_{\mu\cdot\alpha}, e_{\mu\cdot\beta}]_{HVir_\mu} = (\mu\cdot\beta - \mu\cdot\alpha)e_{\mu\cdot(\alpha+\beta)} + \Psi_1(\mu\cdot\alpha, \mu\cdot\beta)c_{\mu,1}. \tag{2.6}$$

The function $\Psi_1(\mu\cdot\alpha, \mu\cdot\beta)$ can not be chosen arbitrary because of the anti-commutativity of the bracket and of the Jacobi identity. We observe from (2.6) that if we put:

$$e'_{\mu\cdot 0} = e_{\mu\cdot 0}, e'_{\mu\cdot\alpha} = e_{\mu\cdot\alpha} + \frac{\Psi_1(0,\mu\cdot\alpha)}{\mu\cdot\alpha}c_{\mu,1},\ (\alpha \neq \overrightarrow{0}),$$

then we will have

$$[e'_{\mu\cdot 0}, e'_{\mu\cdot\alpha}]_{HVir_\mu} = (\mu\cdot\alpha)e'_{\mu\cdot\alpha} \text{ for all } \mu\cdot\alpha \in \Gamma_\mu.$$

This transformation is merely a change of basis and we can drop the prime and say that:

$$[e_{\mu\cdot 0}, e_{\mu\cdot\alpha}]_{HVir_\mu} = (\mu\cdot\alpha)e_{\mu\cdot\alpha} \text{ for all } \mu\cdot\alpha \in \Gamma_\mu. \tag{2.7}$$

From the Jacobi identity for $e_{\mu\cdot 0}, e_{\mu\cdot\alpha}, e_{\mu\cdot\beta}$ we get

$$[e_{\mu\cdot 0}, [e_{\mu\cdot\beta}, e_{\mu\cdot\alpha}]_{HVir_\mu}]_{HVir_\mu} = \mu\cdot(\beta+\alpha)[e_{\mu\cdot\beta}, e_{\mu\cdot\alpha}]_{HVir_\mu} \tag{2.8}$$

Substituting (2.6) in (2.8) and using (2.7) we get:

$$\mu \cdot (\alpha+\beta)\Psi_1(\mu\cdot\alpha, \mu\cdot\beta)c_{\mu,1} = 0.$$

But this is equivalent to $\alpha+\beta = \overrightarrow{0}$ or $\Psi_1(\mu\cdot\alpha,\mu\cdot\beta) = 0$. Then Ψ_1 has the following form:

$$\Psi_1(\mu\cdot\alpha,\mu\cdot\beta) = \delta_{\alpha,-\beta}\theta_1(\mu\cdot\alpha) \tag{2.9}$$

where θ_1 is a function from Γ_μ to $\mathbb{C}$.

The Lie bracket (2.6) becomes:

$$[e_{\mu\cdot\alpha}, e_{\mu\cdot\beta}]_{HVir_\mu} = (\mu\cdot\beta - \mu\cdot\alpha)e_{\mu\cdot(\alpha+\beta)} + \delta_{\alpha,-\beta}\theta_1(\mu\cdot\alpha)c_{\mu,1},\ \mu\cdot\alpha,\mu\cdot\beta \in\ \Gamma_\mu. \tag{2.10}$$

By antisymmetry of the bracket, we deduce that θ_1 is an odd function ($\theta_1(\mu\cdot\alpha) = -\theta_1(-\mu\cdot\alpha)$) and by bilinearity of the bracket, we deduce that θ_1 is additive. So, θ_1 is a group morphism from $(\Gamma_\mu, +)$ to $(\mathbb{C}, +)$.

We now work out the 2-cocycle condition (2.5) on $C_{\mu,1}$ for $e_{\mu\cdot\gamma}, e_{\mu\cdot\alpha}, e_{\mu\cdot\beta}$. If $\gamma+\beta+\alpha \neq \overrightarrow{0}$ then (2.5) is satisfied. If $\gamma+\beta+\alpha = \overrightarrow{0}$, using (2.10) and the the fact that θ_1 is odd, we get from (2.5) the following equation:

$$\mu\cdot(\alpha-\beta)\theta_1(\mu\cdot(\alpha+\beta)) - \mu\cdot(2\beta+\alpha)\theta_1(\mu\cdot\alpha) + \mu\cdot(\beta+2\alpha)\theta_1(\mu\cdot\beta) = 0 \tag{2.11}$$

where θ_1 is a continuous function. Substituting β by $-\beta$ in (2.11) we obtain the following equation:

$$\mu\cdot(\alpha+\beta)\theta_1(\mu\cdot(\alpha-\beta)) - \mu\cdot(\alpha-2\beta)\theta_1(\mu\cdot\alpha) - \mu\cdot(2\alpha-\beta)\theta_1(\mu\cdot\beta) = 0 \tag{2.12}$$

by adding (2.11) and (2.12) we get:

$$(\mu\cdot\alpha)[\theta_1(\mu\cdot(\alpha+\beta))+\theta_1(\mu\cdot(\alpha-\beta))-2\theta_1(\mu\cdot\alpha)] = (\mu\cdot\beta)[\theta_1(\mu\cdot(\alpha+\beta))+\theta_1(\mu\cdot(\beta-\alpha))-2\theta_1(\mu\cdot\beta)] \tag{2.13}$$

Let us denoted $x := \mu\cdot\alpha$ and $y := \mu\cdot\beta$ and replace them in (2.13) we will obtain:

$$x[\theta_1(x+y)+\theta_1(x-y)-2\theta_1(x)] = y[\theta_1(x+y)-\theta_1(x-y)-2\theta_1(y)] \tag{2.14}$$

But (2.14) is equivalent to the following equation:

$$2x\theta_1(x) - 2y\theta_1(y) = (x-y)\theta_1(x+y) + (x+y)\theta_1(x-y). \tag{2.15}$$

Using results on functional equations by PL.Kannappan,T.Riedel and P.K.Sahoo (see [10]), the equation (2.15) has the following general solution:

$$\theta_1(x) = ax^3 + A(x)$$

where $A : \mathbb{C} \mapsto \mathbb{C}$ is an additive function. Since we work with continuous function θ_1, then A will be continuous and additive function, and so it is a linear function $A(x) = bx, b \in \mathbb{C}$.

Finally, $\theta_1(x) = ax^3 + bx$ where $a, b\ \in\ \mathbb{C}$ and for $x = \mu\cdot\alpha$ we have:

$$\theta_1(\mu\cdot\alpha) = a(\mu\cdot\alpha)^3 + b(\mu\cdot\alpha).$$

The 2-cocycle θ_1 is non trivial if and only if $a \neq 0$ while b can be chosen arbitrary. By the convention taken in Virasoro 2-cocycle ($n = 1$), the choice $a = -b = \frac{1}{12}$ and the generating 2-cocycle becomes:

$$C_{\mu,1}(e_{\mu\cdot\alpha}, e_{\mu\cdot\beta}) = \delta_{\alpha,-\beta}\theta_1(\mu\cdot\alpha)c_{\mu,1} = \frac{(\mu\cdot\alpha)^3 - \mu\cdot\alpha}{12}\delta_{\alpha,-\beta}c_{\mu,1}. \tag{2.16}$$

For the unicity of the 2-cocycle $C_{\mu,2}$, we take $(j,k,l) = (1,1,2)$. Assume that there exists $\Psi_2 : \Gamma_\mu \times \Gamma_\mu \to \mathbb{C}$ such that:

$$[e_{\mu\cdot\alpha}, h_\beta]_{HVir_\mu} = (\mu\cdot\beta)h_{\alpha+\beta} + \Psi_2(\mu\cdot\alpha, \mu\cdot\beta)c_{\mu,2}. \tag{2.17}$$

The function $\Psi_2(\mu\cdot\alpha, \mu\cdot\beta)$ can not be chosen arbitrary because of the anti-commutativity of the bracket and of the Jacobi identity. We observe from (2.17) that if we put:

$$e'_{\mu\cdot 0} = e_{\mu\cdot 0},\ \ h'_\alpha = h_\alpha + \frac{\Psi_2(0,\mu\cdot\alpha)}{\mu\cdot\alpha}c_{\mu,2},\ \ (\alpha \neq \overrightarrow{0}),$$

then we will have

$$[e'_{\mu\cdot 0}, h'_\alpha]_{HVir_\mu} = (\mu\cdot\alpha)h'_\alpha \text{ for all } \alpha \in \mathbb{Z}^n.$$

This transformation is merely a change of basis and we can drop the prime and say that:

$$[e_{\mu\cdot 0}, h_\alpha]_{HVir_\mu} = (\mu\cdot\alpha)h_\alpha \text{ for all } \alpha \in \mathbb{Z}^n \tag{2.18}$$

From the Jacobi identity for $e_{\mu\cdot 0}, e_{\mu\cdot\alpha}, h_\beta$, we get:

$$[e_{\mu\cdot 0}, [e_{\mu\cdot\alpha}, h_\beta]_{HVir_\mu}]_{HVir_\mu} = \mu\cdot(\beta+\alpha)[e_{\mu\cdot\alpha}, h_\beta]_{HVir_\mu} \tag{2.19}$$

Substituting (2.17) in (2.19) and using (2.18) we get:

$$\mu\cdot(\alpha+\beta)\Psi_2(\mu\cdot\alpha, \mu\cdot\beta)c_{\mu,2} = 0.$$

But this is equivalent to $\alpha + \beta = \overrightarrow{0}$ or $\Psi_2(\mu\cdot\alpha, \mu\cdot\beta) = 0$. Then Ψ_2 has the following form:

$$\Psi_2(\mu\cdot\alpha, \mu\cdot\beta) = \delta_{\alpha,-\beta}\theta_2(\mu\cdot\alpha) \tag{2.20}$$

where θ_2 is a function from Γ_μ to $\mathbb{C}$.

We now work out the 2-cocycle condition on $C_{\mu,2}$ for $(X_{\alpha,1}, X_{\beta,1}, X_{\gamma,2}) = (e_{\mu\cdot\alpha}, e_{\mu\cdot\beta}, h_\gamma)$. If $\gamma + \beta + \alpha \neq \overrightarrow{0}$ then (2.5) is satisfied. If $\gamma + \beta + \alpha = \overrightarrow{0}$, using (2.9) and the the fact that θ_2 is odd, we get from (2.5) the following equation:

$$(\mu\cdot\beta - \mu\cdot\alpha)\theta_2(\mu\cdot(\alpha+\beta)) - (\mu\cdot(\alpha+\beta))\theta_2(\mu\cdot\beta) + (\mu\cdot\beta + \mu\cdot\alpha)\theta_2(\mu\cdot\alpha) = 0$$

Put $x = \mu\cdot\alpha$ and $y = \mu\cdot\beta$, then we will obtain:

$$(y-x)\theta_2(x+y) = (y+x)(\theta_2(y) - \theta_2(x)) \tag{2.21}$$

If $x = y$ or $x = -y$ the equation (2.21) is satisfied. If $x \neq y$ and $x \neq -y$, then (2.21) is equivalent to the following equation:

$$\frac{\theta_2(x+y)}{x+y} = \frac{\theta_2(x) - \theta_2(y)}{x-y} \tag{2.22}$$

If $x \neq 0$, put $h(x) = \frac{\theta_2(2x)}{2x}$, so we have:

$$\frac{\theta_2(x) - \theta_2(y)}{x - y} = h(\frac{x+y}{2}) \tag{2.23}$$

This is the well known Aczél functional equation (see [1]). Its general solution is given by:

$$\theta_2(x) = ax^2 + bx + c, \text{ for } a, b, c \in \mathbb{R}$$

and h is C^1-function such that $h(x) = \theta_2'(x)$. But in our case $\theta_2(0) = 0$ then $c = 0$ and θ_2 becomes:

$$\theta_2(x) = ax^2 + bx \ \forall \ a, b \in \mathbb{R}.$$

Following the choice of the 2-cocycle in the twisted Heisenberg-Virasoro algebra corresponding to one variable ($n = 1$), we take $a = 1$ and $b = -1$ then we obtain:

$$\theta_2(\mu \cdot \alpha) = (\mu \cdot \alpha)^2 - \mu \cdot \alpha.$$

For the unicity of the 2-cocycle $C_{\mu,3}$, we take $(j, k, l) = (1, 2, 2)$. Assume that there exists $\Psi_3 : \Gamma_\mu \times \Gamma_\mu \to \in \mathbb{C}$ such that:

$$[h_\alpha, h_\beta]_{HVir_\mu} = \Psi_3(\mu \cdot \alpha, \mu \cdot \beta) c_{\mu,3}. \tag{2.24}$$

From the Jacobi identity for $e_{\mu \cdot 0}, h_\alpha, h_\beta$ we get:

$$\mu \cdot (\alpha + \beta) \Psi_3(\mu \cdot \alpha, \mu \cdot \beta) c_{\mu,3} = 0.$$

But this is equivalent to $\alpha + \beta = \overrightarrow{0}$ or $\Psi_3(\mu \cdot \alpha, \mu \cdot \beta) = 0$. Then Ψ_3 has the following form:

$$\Psi_3(\mu \cdot \alpha, \mu \cdot \beta) = \delta_{\alpha,-\beta} \theta_3(\mu \cdot \alpha) \tag{2.25}$$

where θ_3 is a function from Γ_μ to $\mathbb{C}$. We will have:

$$[h_\alpha, h_\beta]_{HVir} = \delta_{\alpha,-\beta} \theta_3(\mu \cdot \alpha) c_{\mu,3},$$

with $\theta_3(0) = 0$, $\theta_3(-\mu \cdot \alpha) = -\theta_3(\mu \cdot \alpha)$.

Let $\alpha, \beta, \gamma \in \mathbb{Z}^n$ and $\alpha + \beta + \gamma = \overrightarrow{0}$. We apply the 2-cocycle condition for $(X_{\alpha,1}, X_{\beta,2}, X_{\gamma,2}) = (e_{\mu \cdot \alpha}, h_\beta, h_\gamma)$ we obtain the equation:

$$(\mu \cdot \beta) \theta_3(\mu \cdot (\alpha + \beta)) - (\mu \cdot \alpha + \mu \cdot \beta) \theta_3(\mu \cdot \beta) = 0. \tag{2.26}$$

If we put $x = \mu \cdot \alpha$ and $y = \mu \cdot \beta$, then (2.26) becomes:

$$y \theta_3(x + y) = (y + x) \theta_3(y) \tag{2.27}$$

If $x = 0$ or $y = 0$ the equation (2.27) is satisfies.
If $x \neq 0$ and $y \neq 0$, then (2.27) is equivalent to:

$$\frac{\theta_3(x + y) - \theta_3(y)}{x} = \frac{\theta_3(y)}{y}. \tag{2.28}$$

Let $X = x + y, Y = y$, then (2.28) becomes:

$$\frac{\theta_3(Y) - \theta_3(X)}{Y - X} = \frac{\theta_3(Y)}{Y}. \tag{2.29}$$

If Y approaches X($Y \to X$) in the first member of (2.29), we obtain the following differential equation:

$$\theta_3'(X) = \frac{\theta_3(X)}{X}$$

which has solution $\theta_3(X) = aX, a \in \mathbb{C}$.

Following the choice of the 2-cocycle in the twisted Heisenberg-Virasoro algebra corresponding to one variable ($n = 1$), we take $a = 1/3$ then we obtain

$$\theta_3(\mu \cdot \alpha) = \frac{\mu \cdot \alpha}{3}.$$

■

Definition 2.2. *The central extension of* $\mathbf{WA}(n)_\mu$ *given by the three 2-cocycles* $C_{\mu,1}$, $C_{\mu,2}$ *and* $C_{\mu,3}$ *in Theorem 2.1 is called the solenoidal Heisenberg-Virasoro algebra* $(\mathbf{HVir}(n)_\mu, [.,.]_{HVir_\mu})$ *where*

$$\mathbf{HVir(n)}_\mu := \mathbf{WA}(n)_\mu \oplus \mathbb{C}c_{\mu,1} \oplus \mathbb{C}c_{\mu,2} \oplus \mathbb{C}c_{\mu,3}.$$

and where its Lie bracket $[.,.]_{HVir_\mu}$ *is generated by the following brackets:*

$$[e_{\mu\cdot\alpha}, e_{\mu\cdot\beta}]_{HVir_\mu} = \mu \cdot (\beta - \alpha)e_{\mu\cdot(\alpha+\beta)} + \delta_{\alpha,-\beta}\frac{(\mu \cdot \alpha)^3 - (\mu \cdot \alpha)}{12}c_{\mu,1} \tag{2.30}$$

$$[e_{\mu\cdot\alpha}, h_\beta]_{HVir_\mu} = (\mu \cdot \beta)h_{\alpha+\beta} + \delta_{\alpha,-\beta}((\mu \cdot \alpha)^2 - (\mu \cdot \alpha))c_{\mu,2} \tag{2.31}$$

$$[h_\alpha, h_\beta]_{HVir_\mu} = \delta_{\alpha,-\beta}\frac{(\mu \cdot \alpha)}{3}c_{\mu,3} \tag{2.32}$$

$$[c_{\mu,i}, \mathbf{HVir}(n)_\mu]_{HVir_\mu} = 0 \text{ for all } i = 1, 2, 3. \tag{2.33}$$

Remark 2.3. *1) The name solenoidal Heisenberg-Virasoro algebra comes from the facts that* $\mathbf{HVir}(n)_\mu$ *contains a subalgebra isomorphic to* $\mathbf{Vir}(n)_\mu$ *generated by* $\{e_{\mu\cdot\alpha}, c_{\mu,1} \mid \alpha \in \mathbb{Z}^n\}$ *and a subalgebra*

$$\mathbf{H}(n)_\mu := (\oplus_{\alpha\in\mathbb{Z}^n} \mathbb{C}h_\alpha) \oplus \mathbb{C}c_{\mu,2}$$

which is isomorphic to an infinite dimensional Heisenberg algebra graded by $\mathbb{Z}^n$.

2) For a given 2-cocycle $C_\mu : \mathbf{WA}(n)_\mu \times \mathbf{WA}(n)_\mu \to \mathbb{C}$, *there exists* $(a_1, a_2, a_3) \in \mathbb{C}^3$ *such that* $C_\mu = a_1C_{\mu,1} + a_2C_{\mu,2} + a_3C_{\mu,3}$. *By bilinearity its expression is given as following:*

$$\begin{aligned} C_\mu((e_{\mu\cdot\alpha}, h_\beta), (e_{\mu\cdot\gamma}, h_\eta)) = \quad & a_1C_{\mu,1}(e_{\mu\cdot\alpha}, e_{\mu\cdot\gamma}) + \\ & a_2(C_{\mu,2}(e_{\mu\cdot\alpha}, h_\eta) - C_{\mu,2}(e_{\mu\cdot\gamma}, h_\beta)) + \\ & a_3C_{\mu,3}(h_\beta, h_\eta) \end{aligned}$$

for all $\alpha, \beta, \gamma, \eta \in \mathbb{Z}^n$.

Moreover, The Lie bracket of $\mathbf{HVir}(n)_\mu$ *is given by:*

$$[X, Y]_{HVir_\mu} = [X, Y] + C_\mu(X, Y), \text{ for all } X, Y \in \mathbf{HVir}(n)_\mu.$$

3 Harish Chandra modules for $\mathbf{HVir}(n)_\mu$

3.1 Generalities on Harish-Chandra modules

Let V be a nonzero $\mathbf{HVir}(n)_\mu$-module. Suppose that the central elements $c_{\mu,1}, c_{\mu,2}, c_{\mu,3}$ and h_0 act as scalars c_1, c_2, c_3, F respectively, on V. Set

$$V_\lambda = \{v \in V | d_\mu v = \lambda v\},$$

which is called a weight space of weight λ. Then V is called a weight module if $V = \oplus_{\lambda\in\mathbb{C}} V_\lambda$. Denote $supp(V) = \{\lambda | V_\lambda \neq 0\}$, which is called the support of V.

Definition 3.1. *A weight $\mathbf{HVir}(n)_\mu$-module V is called Harish-Chandra if dim $V_\lambda < \infty$ for all $\lambda \in supp(V)$ and is called uniformly bounded or cuspidal if there is some $N \in \mathbb{N}$ such that $dim V_\lambda < N$ for all $\lambda \in supp(V)$.*

Definition 3.2. *A weight $\mathbf{HVir}(n)_\mu$-module V is called a module of the intermediate series if it is indecomposable and all its weight spaces are at most one dimensional.*

3.2 Intermediate series of $\mathbf{HVir}(n)_\mu$

Proposition 3.3. *Let $T_\mu(a,b,F)$ the Γ_μ-graded vector space:*

$$T_\mu(a,b,F) = \oplus_{\mu\cdot\kappa\in\Gamma_\mu} v_{\mu\cdot\kappa+a}$$

where $a, b, F \in \mathbb{C}$. We define an action of $\mathbf{HVir}(n)_\mu$ on $T_\mu(a,b,F)$ by:

$$\begin{gathered} e_{\mu\cdot\alpha}.v_{\mu\cdot\kappa+a} = (a + \mu\cdot\kappa + b(\mu\cdot\alpha))v_{\mu\cdot(\kappa+\alpha)+a}, \\ h_\alpha.v_{\mu\cdot\kappa+a} = Fv_{\mu\cdot(\kappa+\alpha)+a}, \\ c_{\mu,1}v_{\mu\cdot\kappa+a} = 0, c_{\mu,2}v_{\mu\cdot\kappa+a} = 0, c_{\mu,3}v_{\mu\cdot\kappa+a} = 0 \end{gathered} \tag{3.1}$$

for all $\kappa, \alpha \in \mathbb{Z}^n$. Then $T_\mu(a,b,F)$ is a $\mathbf{HVir}(n)_\mu$-module for this action.

Remark 3.4. *The weight spaces of $T_\mu(a,b,F)$ are one dimensional. Then $T_\mu(a,b,F)$ are called cuspidal or intermediate series modules.*

It is easy to check that the $\mathbf{HVir}(n)_\mu$-module $T_\mu(a,b,F)$ is reducible if and only if $F = 0, a \in \Gamma_\mu$ and $b = 0,1$. The module $T_\mu(0,0,0)$ contains $\mathbb{C}v_0$ as a submodule and the quotient $T_\mu(0,0,0)/\mathbb{C}v_0$ is irreducible. The module $T_\mu(0,1,0)$ contains $\oplus_{\alpha\in\mathbb{Z}^n\setminus\{\vec{0}\}}\mathbb{C}v_{\mu\cdot\alpha}$ as irreducible submodule of codimension one. By duality, it will be isomorphic to $T_\mu(0,0,0)/\mathbb{C}v_0$. We will denote it $\overline{T}_\mu(0,0,0)$.

Let V be a nontrivial irreducible weight $\mathbf{HVir}(n)_\mu$-module with weight multiplicity one. We may assume that $h_0, c_{\mu,1}, c_{\mu,2}, c_{\mu,3}$ act as scalars F, c_1, c_2, c_3 respectively.

Following Lemma 3.1 and Lemma 3.2 in [15], we will prove the following proposition:

Proposition 3.5. *Let $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{C}^n$ a generic element that is:*

$$\mu\cdot\alpha \neq 0, \ \forall\ \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n \setminus \{\vec{0}\}.$$

Let $V := \oplus_{\mu\cdot\kappa\in\Gamma_\mu}\mathbb{C}v_{\mu\cdot\kappa}$ be a $\mathbf{HVir}(n)_\mu$-module the action given by:

$$e_{\mu\cdot\alpha}.v_{\mu\cdot\kappa+a} = (a+\mu\cdot\kappa+b(\mu\cdot\alpha))v_{\mu\cdot(\kappa+\alpha)+a}.$$

$$h_\alpha.v_{\mu\cdot\kappa+a} = F_{\mu\cdot\alpha,\mu\cdot\kappa}v_{\mu\cdot(\kappa+\alpha)+a} \text{ and } c_{\mu,i}v_{\mu\cdot\kappa+a} = c_i v_{\mu\cdot\kappa+a} \text{ for } i\in\{1,2,3\}.$$

Then all $F_{\mu\cdot\alpha,\mu\cdot\kappa}$ are equal to a constant F and $c_i = 0$ for $i\in\{1,2,3\}$ and such module V is isomorphic to $T_\mu(a,b,F)$.

Proof. It is strait forward to prove that $c_1 = 0$ by restriction to $\mathbf{Vir}(n)_\mu$ and using results by [7,16].

It is clear that $supp(V)\subset a+\Gamma_\mu$ for some $a\in\mathbb{C}$. We give a proof by induction on n to prove that $F_{\mu\cdot\alpha,\mu\cdot\kappa} = F$ for all $\alpha,\kappa\in\mathbb{Z}^n$.

For $n=1$, Proposition 3.5 is Lemma 3.1 in the paper [15].
Let us prove the case of $n=2$. Let $h_{(l,m)} = t_1^l t_2^m$ and let $\mathcal{H}(2) = \oplus_{(l,m)\in\mathbb{Z}^2}\mathbb{C}h_{(l,m)}\oplus\mathbb{C}c_{\mu,3}$ be the Heisenberg subalgebra of $\mathbf{HVir}(2)_\mu$ and let $V = \oplus_{(p,q)\in\mathbb{Z}^2}\mathbb{C}v_{\mu_1 p+\mu_2 q}$. Let us fix l and p and consider $\mathcal{H}_l(1) = \oplus_{m\in\mathbb{Z}}\mathbb{C}h_{(l,m)}\oplus\mathbb{C}c_{\mu,3}$ and $V_p = \oplus_{q\in\mathbb{Z}}\mathbb{C}v_{\mu_1 p+\mu_2 q}$. Then $\mathcal{H}_l(1)$ is a subalgebra of $\mathcal{H}(2)$ isomorphic to the Heisenberg algebra $\mathcal{H}(1)$ and V_p is an intermediate module for $\mathcal{H}_l(1)$. By Lemma 3.1 in the paper [15], $h_{(l,m)}$ acts by a constant $F_{l,p}$ which depends on $l,p\in\mathbb{Z}$ but independent of m and q and $c_{\mu,3}$ act by zero on V_p for all p. If we interchange n by m and p by q, then $F_{l,p}$ will be independent of l and p and then it will be a constant F for all $(l,m)\in\mathbb{Z}^2$ and $c_{\mu,3}$ act by zero on all V.

Now, assume that the proposition is true on $\mathbb{Z}^{n-1}$ where $n\in\mathbb{N}$ and $n\geq 2$. Let $\mathcal{H}_m(n-1) = \oplus_{\alpha\in\mathbb{Z}^{n-1}}\mathbb{C}h_{(\alpha,m)}\oplus\mathbb{C}c_{\mu,3}$ and $V_q = \oplus_{\beta\in\mathbb{Z}^{n-1}}\mathbb{C}v_{\mu'\cdot\beta+\mu_n q}$ where $\mu' = (\mu_1,\dots,\mu_{n-1})$. By the induction hypothesis $\mathcal{H}_m(n-1)$ acts by a constant $F_{m,q}$ on V_q which depends only on m and q for the moment and $c_{\mu,3}$ act by zero on V_q. Now if we fix $\alpha,\beta\in\mathbb{Z}^{n-1}$ and consider $\mathcal{H}_\alpha(1) = \oplus_{m\in\mathbb{Z}}\mathbb{C}h_{(\alpha,m)}\oplus\mathbb{C}c_{\mu,3}$ and $V_\beta := \oplus_{q\in\mathbb{Z}}\mathbb{C}v_{\mu'\cdot\beta+\mu_n q}$, then $F_{m,q}$ will be independent of m and q and then it will be a constant F for all $(\alpha,m)\in\mathbb{Z}^n$. ■

3.3 Generalized highest weight modules

For $\mu = (\mu_1,\mu_2,\dots,\mu_n)\in\mathbb{C}^n$, let $\mu' = (\mu_2,\dots,\mu_n)\in\mathbb{C}^{n-1}$. For any $\alpha = (\alpha_1,\dots,\alpha_n)\in\mathbb{Z}^n$ we have $\mu\cdot\alpha = \mu_1\alpha_1+\mu'\cdot\alpha'$ where $\alpha' = (\alpha_2,\dots,\alpha_n)$. This induces a natural embedding of $\Gamma_{\mu'}$ in Γ_μ given by $\mu'\cdot\alpha'\mapsto\mu\cdot(0,\alpha')$. The embedding $\Gamma_{\mu'}\hookrightarrow\Gamma_\mu$ as defined below, induces an embedding of the Lie algebra $\mathbf{HVir}(n-1)_{\mu'}$ into the Lie algebra $\mathbf{HVir}(n)_\mu$ given by:

$$e_{\mu'\cdot\alpha'}\mapsto e_{\mu\cdot(0,\alpha')} \text{ and } h_{\alpha'}\mapsto h_{(0,\alpha')}.$$

Let $A_{n-1} = \mathbb{C}[t_2^{\pm1},\dots,t_n^{\pm1}]$, then we have the following $\mathbb{Z}$-grading of $\mathbf{HVir}(n)_\mu$:

$$\mathbf{HVir}(n)_\mu = \oplus_{i\in\mathbb{Z}}\mathbf{HVir}(n)_\mu^i$$

where $\mathbf{HVir}(n)_\mu^0 = A_{n-1}d_\mu\oplus A_{n-1}\oplus\sum_{i=1}^3\mathbb{C}c_{\mu,i}$ and $\mathbf{HVir}(n)_\mu^i = t_1^i A_{n-1}d_\mu\oplus t_1^i A_{n-1}\oplus\sum_{i=1}^3\mathbb{C}c_{\mu,i}$ if $i\neq 0$. The Lie subalgebra $\mathbf{HVir}(n)_\mu^0$ of $\mathbf{HVir}(n)_\mu$ is isomorphic to $\mathbf{HVir}(n-1)_{\mu'}$. The algebra $\mathbf{HVir}(n)_\mu$ has a triangular decomposition

$$\mathbf{HVir}(n)_\mu^+\oplus\mathbf{HVir}(n)_\mu^0\oplus\mathbf{HVir}(n)_\mu^-$$

where $\mathbf{HVir}(n)_\mu^\pm := \oplus_{i\in\pm\mathbb{N}}\mathbf{HVir}(n)_\mu^i$.

For $a, b \in \mathbb{C}$, we denote $T_{\mu'}(a, b, F)$ the $\mathbf{HVir}(n)^0_\mu$ module of tensor fields

$$T_{\mu'}(a, b, F) = \oplus_{\mu'\cdot\kappa' \in \Gamma_{\mu'}} \mathbb{C} v_{\mu'\cdot\kappa'}$$

subject to the action:

$$\begin{aligned} & e_{\mu'\cdot\alpha'}.v_{\mu'\cdot\kappa'} = (a + \mu'\cdot\kappa' + b(\mu'\cdot\alpha'))v_{\mu'.(\alpha'+\kappa')}, \\ & h_{\alpha'}.v_{\mu'.\kappa'} = F v_{\mu'\cdot(\kappa'+\alpha')}, \\ & c_{\mu,i}.v_{\mu'\cdot\kappa'} = 0 \text{ for } i = 1, 2, 3 \text{ and } \mu'\cdot\kappa', \mu'\cdot\alpha' \in \Gamma'_\mu. \end{aligned} \tag{3.2}$$

We extend the $\mathbf{HVir}(n)^0_\mu$ module structure on $T_{\mu'}(a, b, F)$ given by (3.2) to $\mathbf{HVir}(n)^+_\mu \oplus \mathbf{HVir}(n)^0_\mu$ where the elements of $\mathbf{HVir}(n)^+_\mu$ act by zero on $T_{\mu'}(a, b, F)$. Let

$$\widetilde{M}(a, b, \Gamma_{\mu'}) = Ind^{\mathbf{HVir}(n)_\mu}_{\mathbf{HVir}(n)^+_\mu \oplus \mathbf{HVir}(n)^0_\mu} T_{\mu'}(a, b, F)$$

be the generalized Verma module. As vector spaces we have $\widetilde{M}(a, b, \Gamma_{\mu'}) \cong U(\mathbf{HVir}(n)^-_\mu) \otimes T_{\mu'}(a, b, F)$. The module $\widetilde{M}(a, b, \Gamma_{\mu'})$ has a unique maximal proper submodule $\overline{M}(a, b, \Gamma_{\mu'})$ trivially intersecting $T_{\mu'}(a, b, F)$. The quotient module

$$L(a, b, \Gamma_{\mu'}) := \widetilde{M}(a, b, \Gamma_{\mu'})/\overline{M}(a, b, \Gamma_{\mu'})$$

is uniquely determined by the constants a, b and

$$L(a, b, \Gamma_{\mu'}) = \oplus_{i>0} L_{a-i\mu_1+\Gamma_{\mu'}}$$

where $L_{a-i\mu_1+\Gamma_{\mu'}} = \oplus_{\mu'\cdot\kappa\in\Gamma_{\mu'}} L_{a-i\mu_1+\mu'\cdot\kappa}$ and

$$L_{a-i\mu_1+\mu'\cdot\kappa} = \{v \in L/d_\mu v = (a - i\mu_1 + \mu'\cdot\kappa)v\}$$

We can similarly define $\widetilde{M}_{a+i\mu_1+\Gamma_{\mu'}}$ and $\widetilde{M}_{a-i\mu_1+\Gamma_{\mu'}}$.

Definition 3.6. *Let $(u_1, \ldots, u_n)$ be a $\mathbb{Z}$-basis of Γ_μ and let $\Gamma^{>0}_\mu := \mathbb{Z}^+ u_1 \oplus \ldots \oplus \mathbb{Z}^+ u_n$ and $\mathbf{HVir}(n)^{>0}_\mu := \oplus_{u\in\Gamma^{>0}_\mu}(\mathbf{HVir}(n)_\mu)_u$. Let V be a weight module such that there exists $\lambda_0 \in Supp(V)$ and a nonzero vector $v_{\lambda_0} \in V_{\lambda_0}$ such that : $\mathbf{HVir}(n)^{>0}_\mu v_{\lambda_0} = 0$. Then V is said to be a generalized highest weight module with generalized highest weight λ_0 and generalized highest weight vector v_{λ_0}. Such module V is denoted by $V(\lambda_0)$.*

In G.Liu and X.Guo (see [14] Theorem.16) , it is proved that for a generalized Heisenberg-Virasoro algebra an irreducible weight module with finite dimensional weight spaces is either a cuspidal or a generalized highest weight module. In our particular case, any irreducible $\mathbf{HVir}(n)_\mu$-module is either cuspidal or isomorphic to $L(a, b, \Gamma_{\mu'})$.

Definition 3.7. *A $\mathbf{HVir}(n)_\mu$-module V is called a **dense** module if $supp(V) = a + \Gamma_\mu,\ a \in \mathbb{C}$ and is called a **cut** module if $supp(V) \subset \lambda + \gamma + \Gamma^{(\alpha)}_{\le 0}$ where $\Gamma^{(\alpha)}_{\le 0} := \{\mu\cdot\beta | \beta \in \mathbb{Z}^n$ and $\beta.\alpha \le 0\}$ and $\gamma \in \Gamma_\mu$.*

The modules $T_\mu(a, b, F)$ are irreducible **dense** modules and $L(a, b, \Gamma_{\mu'})$ are irreducible **cut** modules.

The following theorem is a consequence of Theorem 15 and Theorem 16 in [14]. It classifies Harish-chandra modules of $\mathbf{HVir}(n)_\mu$.

Theorem 3.8. *Let V be a nontrivial irreducible weight module with finite dimensional weight spaces over the Heisenberg solenoidal-Virasoro algebra* $\mathbf{HVir}(n)_\mu$.

1) *If $n = 1$ then $\Gamma_\mu = \mu\mathbb{Z} \simeq \mathbb{Z}$, then V is of intermediate series or highest or lowest module (see [12, 15]).*

2) *If $n \geq 2$, then V is isomorphic to one of the following modules:*

   a) *$V \cong T_\mu(a, b, F)$ for $(a, b) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ or $V \cong \overline{T}_\mu(0, 0, 0)$.*

   b) *$V \cong L(a, b, \Gamma_{\mu'})$ for some $a, b \in \mathbb{C}$.*

4 Simple Weight $\mathbf{HVir}(n)_\mu$-modules having infinite dimensional weight spaces

Let $\mathbb{Z}^n$ be the free abelian group of rank n whose elements are sequences of n integers, and operation is the addition. A group order on $\mathbb{Z}^n$ is a total order, which is compatible with addition, that is

$$a < b \quad \text{if and only if} \quad a + c < b + c.$$

The lexicographical order $<_{lex}$ is a group order on $\mathbb{Z}^n$.

We transport the lexicographic order $<_{lex}$ on $\mathbb{Z}^n$ to Γ_μ that is

$$\mu \cdot \alpha \prec \mu \cdot \beta \text{ if and only if } \alpha <_{lex} \beta.$$

Let us introduce

$$\Delta^+ := \{\alpha \in \mathbb{Z}^n | \overrightarrow{0} <_{lex} \alpha\}\ ,\ \Delta^- := \{\alpha \in \mathbb{Z}^n | \alpha <_{lex} \overrightarrow{0}\}$$

$$\Gamma_\mu^+ := \sigma_\mu(\Delta^+) := \{\mu \cdot \alpha | \overrightarrow{0} <_{lex} \alpha\}\ ,\ \Gamma_\mu^- := \sigma_\mu(\Delta^-) := \{\mu \cdot \alpha | \alpha <_{lex} \overrightarrow{0}\}$$

Let $(\mathbf{Vir}(n)_\mu)_+, (\mathbf{Vir}(n)_\mu)_-, (\mathbf{Vir}(n)_\mu)_0, (\mathbf{H}(n)_\mu)_+, (\mathbf{H}(n)_\mu)_-$ and $(\mathbf{H}(n)_\mu)_0$ be the subalgebras defined by:

$$\begin{array}{c} (\mathbf{Vir}(n)_\mu)_+ = \bigoplus_{\alpha \in \Delta^+} \mathbb{C}e_{\mu\cdot\alpha},\ (\mathbf{Vir}(n)_\mu)_- = \bigoplus_{\alpha \in \Delta^-} \mathbb{C}e_{\mu\cdot\alpha}, (\mathbf{Vir}(n)_\mu)_0 = \mathbb{C}d_\mu \oplus \mathbb{C}c_{\mu,1}, \\ (\mathbf{H}(n)_\mu)_+ = \bigoplus_{\alpha \in \Delta^+} \mathbb{C}h_\alpha,\ (\mathbf{H}(n)_\mu)_- = \bigoplus_{\alpha \in \Delta^-} \mathbb{C}h_\alpha, (\mathbf{H}(n)_\mu)_0 = \mathbb{C}h_0 \oplus \mathbb{C}c_{\mu,2} \end{array}.$$

The algebra $\mathbf{HVir}(n)_\mu$ has the following triangular decomposition:

$$\mathbf{HVir}(n)_\mu = (\mathbf{HVir}(n)_\mu)_+ \oplus (\mathbf{HVir}(n)_\mu)_0 \oplus (\mathbf{HVir}(n)_\mu)_-$$

where,

$$(\mathbf{HVir}(n)_\mu)_\pm = (\mathbf{H}(n)_\mu)_\pm \oplus (\mathbf{Vir}(n)_\mu)_\pm,$$

$$(\mathbf{HVir}(n)_\mu)_0 = \mathbb{C}e_{\mu\cdot 0} \oplus \mathbb{C}h_0 \oplus \mathbb{C}c_{\mu,1} \oplus \mathbb{C}c_{\mu,2} \oplus \mathbb{C}c_{\mu,3}.$$

Let $\lambda = (\lambda_\mu,\ c_0\ , c_1, c_2, c_3) \in \mathbb{C}^5$ and denote $\mathbf{HB}(n)_+ := (\mathbf{HVir}(n)_\mu)_0 \oplus (\mathbf{HVir}(n)_\mu)_+$. Let the one dimentional $\mathbf{HB}(n)_+$-module $\mathbb{C}_\lambda$ where the action is given by:

$$e_{\mu\cdot 0}.1_\lambda = \lambda_\mu 1_\lambda, h_0.1_\lambda = c_0 1_\lambda, c_{\mu,1}.1_\lambda = c_1 1_\lambda, c_{\mu,2}.1_\lambda = c_2 1_\lambda, c_{\mu,3}.1_\lambda = c_3 1_\lambda.$$

The Verma module of $\mathbf{HVir}(n)_\mu$ is the induced weight module:

$$M(\lambda) = Ind_{\mathbf{HB}(n)_+}^{\mathbf{HVir}(n)_\mu}\mathbb{C}_\lambda := U(\mathbf{HVir}(n)_\mu) \otimes_{U(\mathbf{HB}(n)_+)} \mathbb{C}_\lambda$$

The Verma module $M(\lambda)$ has a maximal proper submodule $\widetilde{M(\lambda)}$ and the quotient $V(\lambda) := M(\lambda)/\widetilde{M(\lambda)}$ will be irreducible and called the irreducible highest module with highest weight λ. Moreover, every irreducible highest module will be constructed with this manner.

The irreducible lowest weight modules $V(\lambda)^\vee$ of lowest weight λ are constructed in the same manner of the ones in the case of the $\mathbf{HVir}(n)_\mu$ algebra.

We can also consider the Verma module of $\mathbf{Vir}(n)_\mu$:

$$K(\nu) := Ind_{(\mathbf{Vir}(n)_\mu)_0 \oplus (\mathbf{Vir}(n)_\mu)_+}^{\mathbf{Vir}(n)_\mu}\mathbb{C}_\nu$$

where $\nu = (\lambda_\mu, c_1), e_{\mu \cdot 0}.1_\nu = \lambda_\mu 1_\nu, c_{\mu,1}.1_\nu = c_1 1_\nu$ and $(\mathbf{Vir}(n)_\mu)_+$ acts by 0.

The module $K(\nu)$ has a maximal proper submodule $\widetilde{K(\nu)}$ and the quotient $L(\nu) := K(\nu)/\widetilde{K(\nu)}$ is an irreducible highest $\mathbf{Vir}(n)_\mu$-module.

The algebra $\mathbf{HVir}(n)_\mu$ has also the following generalized triangular decomposition:

$$\mathbf{HVir}(n)_\mu = (\mathbf{H}(n)_\mu)_- \oplus \mathbf{Vir}(n)_\mu \oplus (\mathbf{H}(n)_\mu)_0 \oplus \mathbb{C}c_{\mu,3} \oplus (\mathbf{H}(n)_\mu)_+.$$

Let $(\mathbf{P}(n)_\mu)_+ := \mathbf{Vir}(n)_\mu \oplus (\mathbf{H}(n)_\mu)_0 \oplus (\mathbf{H}(n)_\mu)_+ \oplus \mathbb{C}c_{\mu,3}$. Let $L(\nu)$ be an irreducible $\mathbf{Vir}(n)_\mu$-module. Extend it to $(\mathbf{P}(n)_\mu)_+$-module by letting h_0 acts by c_0, $c_{\mu,2}$ acts by c_2, $c_{\mu,3}$ acts by c_3 and $(\mathbf{H}(n)_\mu)_+$ acts by 0. Let the generalized Verma module of $\mathbf{HVir}(n)_\mu$:

$$G(\lambda) = Ind_{(\mathbf{P}(n)_\mu)_+}^{\mathbf{HVir}(n)_\mu} L(\nu)$$

where $\lambda = (\nu, c_0, c_2, c_3)$. The module $G(\lambda)$ has a maximal submodule $\widetilde{G(\lambda)}$ and the quotient is irreducible module $V(\lambda)$. As a module of $\mathbf{Vir}(n)_\mu$ it contains $L(\nu)$ as a submodule.

Theorem 4.1. *Let $V(\lambda)$ be the irreducible highest weight module of $\mathbf{HVir}(n)_\mu$, then there exists $\alpha \in supp(V(\lambda))$ such that $V(\lambda)_\alpha$ is an infinite dimensional weight subspace of $V(\lambda)$.*

We have the same assertion for the lowest weight module $V(\lambda)^\vee$.

Proof. As a module of $\mathbf{Vir}(n)_\mu$, $V(\lambda)$ contains $L(\nu)$ as submodule. Using results in [2], $L(\nu)$ has infinite dimensional weight subspaces. We deduce that $V(\lambda)$ has submodules of infinite dimensional weight spaces. ■

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