On the Ashbaugh-Benguria type conjecture about lower-order Neumann eigenvalues of the Witten-Laplacian

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Abstract

An isoperimetric inequality for lower order nonzero Neumann eigenvalues of the Witten-Laplacian on bounded domains in a Euclidean space or a hyperbolic space has been proven in this paper. About this conclusion, we would like to point out two things:

- It strengthens the well-known Szegő-Weinberger inequality for nonzero Neumann eigenvalues of the classical free membrane problem given in [J. Rational Mech. Anal. **3** (1954) 343–356] and [J. Rational Mech. Anal. **5** (1956) 633–636];
- Recently, Xia-Wang [Math. Ann. **385** (2023) 863–879] gave a very important progress to the celebrated conjecture of M. S. Ashbaugh and R. D. Benguria proposed in [SIAM J. Math. Anal. **24** (1993) 557–570]. It is easy to see that our conclusion here covers Xia-Wang's this progress as a special case.

In this paper, we have also proposed two open problems which can be seen as a generalization of Ashbaugh-Benguria's conjecture mentioned above.

1 Introduction

Let $(M^n, \langle\cdot,\cdot\rangle)$ be an n-dimensional ($n \geq 2$) complete Riemannian manifold with the metric $g := \langle\cdot,\cdot\rangle$. Let $\Omega \subseteq M^n$ be a domain in M^n, and $\phi \in C^\infty(M^n)$ be a smooth[1] real-valued function defined on M^n. In this setting, on Ω, the following elliptic operator

$$\Delta_\phi := \Delta - \langle\nabla\phi, \nabla\cdot\rangle$$

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MSC 2020: 35P15, 49Jxx, 35J15.

Key Words: Witten-Laplacian, Neumann eigenvalues, Laplacian, the free membrane problem, isoperimetric inequalities.

[1] In fact, one might see that $\phi \in C^2$ is suitable to derive our main conclusions in this paper. However, in order to avoid a little bit boring discussion on the regularity of ϕ and following the assumption on conformal factor $e^{-\phi}$ for the notion of *smooth metric measure spaces* in many literatures (including of course those cited in this paper), without specification, we prefer to assume that ϕ is smooth on the domain Ω.

can be well-defined, where ∇, Δ are the gradient and the Laplace operators on M^n, respectively. The operator Δ_ϕ w.r.t. the metric g is called the *Witten-Laplacian* (also called the *drifting Laplacian* or the *weighted Laplacian*). The K-dimensional Bakry-Émery Ricci curvature Ric_ϕ^K on M^n can be defined as follows

$$\mathrm{Ric}_\phi^K := \mathrm{Ric} + \mathrm{Hess}\phi - \frac{d\phi \otimes d\phi}{K-n-1},$$

where Ric denotes the Ricci curvature tensor on M^n, and Hess is the Hessian operator on M^n associated to the metric g. Here $K > n+1$ or $K = n+1$ if ϕ is a constant function. When $K = \infty$, the so-called ∞-dimensional Bakry-Émery Ricci curvature Ric_ϕ (simply, *Bakry-Émery Ricci curvature* or *weighted Ricci curvature*) can be defined as follows

$$\mathrm{Ric}_\phi := \mathrm{Ric} + \mathrm{Hess}\phi.$$

These notions were introduced by D. Bakry and M. Émery in [2]. Many interesting results (under suitable assumptions on the Bakry-Émery Ricci curvature) have been obtained, and we prefer to mention briefly several ones:

- For Riemannian manifolds[2] (M^n, g) endowed with a weighted measure $d\eta := e^{-\phi}dv$, where dv denotes the Riemannian volume element (or Riemannian density) w.r.t. the metric g, Wei and Wylie [19] proved mean curvature and volume comparison results when the Bakry-Émery Ricci curvature Ric_ϕ is bounded from below and ϕ or $|\nabla\phi|$ is bounded, improving the classical ones (i.e., when ϕ is constant). As described by J. Mao (the corresponding author here) in [14, pp. 31-32], one might have an illusion that smooth metric measure spaces are not necessary to be studied since they are simply obtained from corresponding Riemannian manifolds by adding a conformal factor to the Riemannian measure. However, they do have many differences. For instance, when Ric_ϕ is bounded from below, the Myer's theorem, Bishop-Gromov's volume comparison, Cheeger-Gromoll's splitting theorem and Abresch-Gromoll's excess estimate cannot hold as in the Riemannian case. Moreover, in order to let readers have a deep impression and a nice comprehension on those differences, Mao [14, page 32] has also repeated briefly an interesting example (given in [19, Example 2.1]) to make an explanation therin. More precisely, for the metric measure space $(\mathbb{R}^n, g_{\mathbb{R}^n}, e^{-\phi}dv_{\mathbb{R}^n})$, where $g_{\mathbb{R}^n}$ is the usual Euclidean metric of the Euclidean n-space $\mathbb{R}^n$, and $dv_{\mathbb{R}^n}$ denotes the Euclidean volume density related to $g_{\mathbb{R}^n}$, if $\phi(x) = \frac{\lambda}{2}|x|^2$ for $x \in \mathbb{R}^n$, then $\mathrm{Hess} = \lambda g_{\mathbb{R}^n}$ and $\mathrm{Ric}_\phi = \lambda g_{\mathbb{R}^n}$. Therefore, from this example we know that unlike in the case of Ricci curvature bounded from below uniformly by some positive constant, a metric measure space is not necessarily compact provided $\mathrm{Ric}_\phi \geq \lambda$ and $\lambda > 0$. Hence, it is meaningful to study geometric problems in smooth metric measure spaces.

- Perelman's $\mathcal{W}$-entropy formula for the heat equation associated with the Witten Laplacian on complete Riemannian manifolds via the Bakry-Émery Ricci curvature tensor has been investigated by Li [10]. In fact, under the assumption that the m-dimensional Bakry-Émery Ricci curvature is bounded from below, Li [10, Theorem 2.3] obtained

[2] Without specifications, generally, in this paper same symbols have the same meanings.

an analogue of Perelman's entropy formula for the $\mathcal{W}$-entropy of the heat kernel of the Witten Laplacian on complete Riemannian manifolds with some natural geometric conditions. In particular, by this fact, he proved a monotonicity theorem and a rigidity theorem for the $\mathcal{W}$-entropy on complete Riemannian manifolds with *nonnegative* m-dimensional Bakry-Émery Ricci curvature.

- Mao and his collaborators [8, Theorems 4.1 and 4.4, Corollary 4.2] investigated the buckling problem of the drifting Laplacian, and **firstly** obtained some universal inequalities for eigenvalues of the same problem on bounded connected domains in the Gaussian shrinking solitons

$$\left(\mathbb{R}^n, g_{\mathbb{R}^n}, e^{-\frac{|x|^2}{4}} dv_{\mathbb{R}^n}, \frac{1}{2}\right)$$

and some general product solitons of the type

$$\left(\Sigma \times \mathbb{R}, g, e^{-\frac{\kappa t^2}{2}} dv, \kappa\right),$$

where Σ is an Einstein manifold with constant Ricci curvature κ, and $t \in \mathbb{R}$ is the parameter defined along the line $\{x\} \times \mathbb{R}$, $x \in \Sigma$. Besides, as interpreted in [8, Remark 4.3], for a self-shrinker, if the weighted function ϕ was chosen to be $\phi = \frac{|x|^2}{4}$, then the drifting Laplacian considered in [8] degenerates into the operator $\mathcal{L} := \Delta - \frac{1}{2}\langle x, \nabla(\cdot)\rangle$ which was introduced by Colding-Minicozzi [7] to study self-shrinker hypersurfaces. For the Dirichlet eigenvalue problem of the operator $\mathcal{L}$, Cheng-Peng [6] have obtained some universal inequalities. From this viewpoint, [8, Theorem 4.1 and Corollary 4.2] can be regarded as conclusions for the buckling problem of the operator $\mathcal{L}$.

Except [8, 14], Mao also has some other interesting works related to the Witten-Laplacian – see, e.g., [9, 13, 15, 16, 23].

Using the conformal measure $d\eta = e^{-\phi}dv$, the notion, *smooth metric measure space* $(M^n, g, d\eta)$, can be well-defined, which is actually the given Riemannian manifold (M^n, g) equipped with the weighted measure $d\eta$. Smooth metric measure space $(M^n, g, d\eta)$ sometimes is also called the *weighted measure space*. For the smooth metric measure space $(M^n, g, d\eta)$, one can define a notion, weighted volume (or ϕ-volume), as follows:

$$|M^n|_\phi := \int_{M^n} d\eta = \int_{M^n} e^{-\phi} dv.$$

On a compact smooth metric measure space $(\Omega, \langle\cdot,\cdot\rangle, d\eta)$, one can naturally consider the Neumann eigenvalue problem of the Witten-Laplacian Δ_ϕ as follows

$$\begin{cases} \Delta_\phi u + \mu u = 0 & \text{in } \Omega \subset M^n, \\ \frac{\partial u}{\partial \vec{\nu}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\vec{\nu}$ is the unit normal vector along the smooth[3] boundary $\partial\Omega$. The eigenvalue problem (1.1) can also be called the *free membrane problem* of the operator Δ_ϕ. It is not hard to

[3] The smoothness assumption for the regularity of the boundary $\partial\Omega$ is strong to consider the eigenvalue

check that the operator Δ_ϕ in (1.1) is **self-adjoint** w.r.t. the following inner product

$$\widetilde{(f_1, f_2)} := \int_\Omega f_1 f_2 d\eta = \int_\Omega f_1 f_2 e^{-\phi} dv,$$

with $f_1, f_2 \in \widetilde{W}^{1,2}(\Omega)$, where $\widetilde{W}^{1,2}(\Omega)$ is the Sobolev space w.r.t. the weighted measure $d\eta$, i.e. the completion of the set of smooth functions $C^\infty(\Omega)$ under the following Sobolev norm

$$\widetilde{\|f\|}_{1,2} := \left(\int_\Omega f^2 d\eta + \int_\Omega |\nabla f|^2 d\eta\right)^{1/2}.$$

Then using similar arguments to those of the classical free membrane problem of the Laplacian (i.e., the discussions about the existence of discrete spectrum, Rayleigh's theorem, Max-min theorem, etc. These discussions are standard, and for details, please see for instance [4]), it is not hard to know:

- The self-adjoint elliptic operator $-\Delta_\phi$ in (1.1) *only* has discrete spectrum, and all the elements (i.e., eigenvalues) in this discrete spectrum can be listed non-decreasingly as follows

$$0 = \mu_{0,\phi}(\Omega) < \mu_{1,\phi}(\Omega) \leq \mu_{2,\phi}(\Omega) \leq \cdots \uparrow +\infty. \tag{1.2}$$

  For each eigenvalue $\mu_{i,\phi}(\Omega)$, $i = 0, 1, 2, \cdots$, all the possible nontrivial functions u satisfying (1.1) are called eigenfunctions belonging to $\mu_{i,\phi}(\Omega)$. Since the first equation in (1.1) is linear, the space of $\mu_{i,\phi}(\Omega)$'s eigenfunctions should be a vector space. This vector space of $\mu_{i,\phi}(\Omega)$ is called eigenspace. Each eigenspace has finite dimension, and usually the dimension of each eigenspace is called multiplicity of the eigenvalue. It is easy to know that eigenfunctions of the first eigenvalue $\mu_{0,\phi}(\Omega) = 0$ are nonzero constant functions, and correspondingly, the eigenspace of $\mu_{0,\phi}(\Omega) = 0$ has dimension 1. Eigenvalues in the sequence (1.2) are repeated according to its multiplicity. By applying the standard variational principles, one can obtain that the k-th nonzero Neumann eigenvalue $\mu_{k,\phi}(\Omega)$ can be characterized as follows

$$\mu_{k,\phi}(\Omega) = \inf\left\{ \frac{\int_\Omega |\nabla f|^2 e^{-\phi} dv}{\int_\Omega f^2 e^{-\phi} dv} \middle| f \in \widetilde{W}^{1,2}(\Omega), f \neq 0, \int_\Omega f f_i e^{-\phi} dv = 0 \right\}, \tag{1.3}$$

  where f_i, $i = 0, 1, 2, \cdots, k-1$, denotes an eigenfunction of $\mu_{i,\phi}(\Omega)$. Specially, the first nonzero Neumann eigenvalue $\mu_{1,\phi}(\Omega)$ of the eigenvalue problem (1.1) satisfies

$$\mu_{1,\phi}(\Omega) = \inf\left\{ \frac{\int_\Omega |\nabla f|^2 d\eta}{\int_\Omega f^2 d\eta} \middle| f \in \widetilde{W}^{1,2}(\Omega), f \neq 0, \int_\Omega f d\eta = 0 \right\}. \tag{1.4}$$

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problem (1.1) of the Witten-Laplacian. In fact, a weaker regularity assumption that $\partial\Omega$ is Lipschitz continuous can also assure the validity about the description of the discrete spectrum of the eigenvalue problem (1.1). However, the Lipschitz continuous assumption might not be enough to consider some other geometric problems involved Neumann eigenvalues of (1.1). Therefore, in order to avoid a little bit boring discussion on the regularity of the boundary $\partial\Omega$ (which is also not important for the topic investigated in our paper here), we prefer to assume that $\partial\Omega$ is smooth. This setting leads to the situation that some conclusions of this paper may still hold under a weaker regularity assumption for the boundary $\partial\Omega$, readers who are interested in this situation could try to seek the weakest regularity.

For convenience and without confusion, in the sequel, except specification we will write $\mu_{i,\phi}(\Omega)$ as $\mu_{i,\phi}$ directly. This convention would be also used when we meet with other possible eigenvalue problems.

In this paper, we focus on the Neumann eigenvalue problem (1.1) of the Witten-Laplacian and can prove an isoperimetric inequality for the sums of the reciprocals of the first $(n-1)$ nonzero Neumann eigenvalues of the Witten-Laplacian on bounded domains in $\mathbb{R}^n$ or a hyperbolic space. However, in order to state our conclusions clearly, we need to impose an assumption on the function ϕ as follows:

- (**Property I**) Furthermore, ϕ is a function of the Riemannian distance parameter $t := d(o, \cdot)$ for some point $o \in \mathrm{hull}(\Omega)$, and ϕ is also a non-increasing convex function defined on $[0, \infty)$.

Here $\mathrm{hull}(\Omega)$ stands for the convex hull of the domain Ω. Clearly, if a given open Riemannian n-manifold (M^n, g) was endowed with the weighted density $e^{-\phi}dv$ with ϕ satisfying **Property I**, then ϕ would be a ***radial*** function defined on M^n w.r.t. the radial distance t, $t \in [0, \infty)$. Especially, when the given open n-manifold is chosen to be $\mathbb{R}^n$ or $\mathbb{H}^n$ (i.e., the n-dimensional hyperbolic space of sectional curvature -1), we additionally require that o is the origin of $\mathbb{R}^n$ or $\mathbb{H}^n$.

Theorem 1.1. *Assume that the function ϕ satisfies **Property I**. Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^n$, and let $B_R(o)$ be a ball of radius R and centered at the origin o of $\mathbb{R}^n$ such that $|\Omega|_\phi = |B_R(o)|_\phi$, i.e. $\int_\Omega d\eta = \int_{B_R(o)} d\eta$. Then*

$$\frac{1}{\mu_{1,\phi}(\Omega)} + \frac{1}{\mu_{2,\phi}(\Omega)} + \cdots + \frac{1}{\mu_{n-1,\phi}(\Omega)} \geq \frac{n-1}{\mu_{1,\phi}(B_R(o))}. \tag{1.5}$$

The equality case holds if and only if Ω is the ball $B_R(o)$.

By applying the sequence (1.2), i.e. the monotonicity of Neumann eigenvalues of the Witten-Laplacian, from (1.5) one has:

Corollary 1.2. *Under the assumptions of Theorem 1.1, we have*

$$\mu_{1,\phi}(\Omega) \leq \mu_{1,\phi}(B_R(o)), \tag{1.6}$$

*with equality holding if and only if Ω is the ball $B_R(o)$. That is to say, among all bounded domains in $\mathbb{R}^n$ having the same weighted volume, the ball $B_R(o)$ maximizes the first nonzero Neumann eigenvalue of the Witten-Laplacian, provided the function ϕ satisfies **Property I**.*

Remark 1.3. The spectral isoperimetric inequality (1.6) in Corollary 1.2 has already been proven by the authors in [5] by suitably constructing the trail function. However, we still prefer to list it here to show the close relation between (1.5) and (1.6), and show of course the significance of the spectral isoperimetric inequality (1.5) as well.

Remark 1.4. (1) Topologically, the Euclidean n-space $\mathbb{R}^n$ is two-points homogenous, so generally it seems like there is no need to point out the information of the center for the ball $B_R(o)$ when describing the isometry conclusion in Theorem 1.1. However, for the eigenvalue

problem (1.1), by (1.3) one knows that even on Euclidean balls, the Neumann eigenvalues $\mu_{i,\phi}$ also depend on the weighted function ϕ (except the situation that ϕ is a constant function). This implies that for Euclidean balls with the same radius but different centers, they might have different Neumann eigenvalues $\mu_{i,\phi}$ since generally the radial function ϕ here has different distributions on different balls. Therefore, we need to give the information of the center for the ball $B_R(o)$ when we investigate the possible rigidity for the equality case of (1.5).
(2) A slightly sharper version of (1.5) has also been obtained – for details, see Theorem 3.1 in Section 3 below.
(3) As we know, if $\phi = const.$ is a constant function, then the Witten-Laplacian Δ_ϕ degenerates into the Laplacian Δ, and correspondingly the eigenvalue problem (1.1) becomes the classical free membrane problem of the Laplacian Δ as follows

$$\begin{cases} \Delta u + \mu u = 0 & \text{in } \Omega \subset M^n, \\ \frac{\partial u}{\partial \vec{\nu}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

Clearly, the Laplacian $-\Delta$ in (1.7) only has the discrete spectrum, and all the eigenvalues in this discrete spectrum can be listed non-decreasingly as follows

$$0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \cdots \uparrow +\infty.$$

The corresponding Neumann eigenvalues μ_k can be characterized similarly as (1.3)-(1.4) with $\phi = const.$ instead. Now, we would like to recall some results on isoperimetric inequalities of Neumann eigenvalues of the eigenvalue problem (1.7). For simply connected bounded domains $\Omega \subset \mathbb{R}^2$, by using the conformal mapping techniques, Szegő [18] obtained

$$\mu_1(\Omega)A(\Omega) \leq \mu_1(\mathbb{D})A(\mathbb{D}) = \pi p_{1,1}^2, \tag{1.8}$$

where $\mathbb{D}$ stands for a disk in the plane $\mathbb{R}^2$, and $A(\cdot)$ denotes the area of a given geometric object. Later, this result was improved by Weinberger [20] to the higher dimensional case, that is, for bounded domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$, he proved

$$\mu_1(\Omega) \leq \left(\frac{w_n}{|\Omega|}\right)^{2/n} p_{n/2,1}^2, \tag{1.9}$$

where w_n, $|\Omega|$ denote[4] the volume of the unit ball in $\mathbb{R}^n$ and the volume of Ω, respectively. Here $p_{v,k}$ in (1.8)-(1.9) stands for the k-th positive zero of the derivative of $x^{1-v}J_v(x)$, with $J_v(x)$ the Bessel function of the first kind of order v. The equality case in (1.8) (or (1.9)) if and only if Ω is a disk (or a ball in $\mathbb{R}^n$). Clearly, from the Szegő-Weinberger's isoperimetric inequality (1.9), one knows:

- (**Fact A**) Among all bounded domains in $\mathbb{R}^n$ having the same volume, the ball maximizes the first nonzero Neumann eigenvalue of the Laplacian.

[4] Similarly, without confusion, in the sequel $|\cdot|$ would denote the volume of a given geometric object.

It is not hard to see that **Fact A** was covered by Corollary 1.2 as a special case (corresponding to $\phi = const.$). Szegő and Weinberger found that Szegő's proof of (1.8) for simply connected domains in $\mathbb{R}^2$ can be improved to get the estimate

$$\frac{1}{\mu_1(\Omega)} + \frac{1}{\mu_2(\Omega)} \geq \frac{2A(\Omega)}{\pi p_{1,1}^2} \tag{1.10}$$

for such domains. Brasco and Pratelli [3] made a quantitative improvement of (1.8) – for any bounded domain with smooth boundary $\Omega \subset \mathbb{R}^n$, they have proven

$$w_n^{2/n} p_{n/2,1}^2 - \mu_1(\Omega)|\Omega|^{2/n} \geq c(n)\mathcal{A}(\Omega),$$

where $c(n)$ is a positive constant depending only on n, and $\mathcal{A}(n)$ is the so-called *Fraenkel asymmetry* defined by

$$\mathcal{A}(n) := \left\{ \frac{|\Omega \triangle B|}{|\Omega|} \middle| B \text{ is a ball in } \mathbb{R}^n \text{ such that } |\Omega| = |B| \right\},$$

with $\Omega \triangle B$ the symmetric difference of Ω and B. An interesting quantitative improvement of (1.10) obtained by Nadirashvilli [17] states that there exists a constant $C > 0$ such that for every smooth simply connected bounded open set $\Omega \subset \mathbb{R}^2$, it holds

$$\frac{1}{|\Omega|}\left(\frac{1}{\mu_1(\Omega)} + \frac{1}{\mu_2(\Omega)}\right) - \frac{1}{|B|}\left(\frac{1}{\mu_1(B)} + \frac{1}{\mu_2(B)}\right) \geq \frac{\mathcal{A}(\Omega)^2}{C},$$

with B any disk in $\mathbb{R}^2$. Is it possible to improve (1.10) to the higher dimensional case? The answer is affirmative. In fact, for any bounded domain $\Omega \subset \mathbb{R}^n$ (with smooth boundary), Ashbaugh and Benguria [1] obtained the estimate

$$\frac{1}{\mu_1(\Omega)} + \frac{1}{\mu_2(\Omega)} + \cdots + \frac{1}{\mu_n(\Omega)} \geq \frac{n}{n+2}\left(\frac{|\Omega|}{w_n}\right)^{2/n}. \tag{1.11}$$

Some interesting generalizations to (1.11) have been done – see, e.g., [12, 21]. Based on the estimate (1.11), Ashbaugh and Benguria [1] proposed an important open problem as follows:

- **Conjecture I**. ([1]) For any bounded domain Ω with smooth boundary in $\mathbb{R}^n$, we have

$$\frac{1}{\mu_1(\Omega)} + \frac{1}{\mu_2(\Omega)} + \cdots + \frac{1}{\mu_n(\Omega)} \geq \frac{n}{p_{n/2,1}^2}\left(\frac{|\Omega|}{w_n}\right)^{2/n},$$

  with equality holding if and only if Ω is a ball in $\mathbb{R}^n$.

Conjecture I is still open until now. Recently, Xia-Wang [22] gave a very important progress to this celebrated conjecture, and actually they proved that

$$\frac{1}{\mu_1(\Omega)} + \frac{1}{\mu_2(\Omega)} + \cdots + \frac{1}{\mu_{n-1}(\Omega)} \geq \frac{n-1}{p_{n/2,1}^2}\left(\frac{|\Omega|}{w_n}\right)^{2/n} \tag{1.12}$$

holds for any bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, where the equality holds if and only if Ω is a ball in $\mathbb{R}^n$. Clearly, the isoperimetric inequality (1.12) gives a partial answer to **Conjecture I** and also supports its validity. It is not hard to see that our conclusion in Theorem 1.1 here covers Xia-Wang's spectral isoperimetric inequality (1.12) as a special case (corresponding to $\phi = const.$).

Theorem 1.5. *Assume that the function ϕ satisfies **Property I**. Let Ω be a bounded domain in $\mathbb{H}^n$, and let $B_R(o)$ be a geodesic ball of radius R and centered at the origin o of $\mathbb{H}^n$ such that $|\Omega|_\phi = |B_R(o)|_\phi$. Then*

$$\frac{1}{\mu_{1,\phi}(\Omega)} + \frac{1}{\mu_{2,\phi}(\Omega)} + \cdots + \frac{1}{\mu_{n-1,\phi}(\Omega)} \geq \frac{n-1}{\mu_{1,\phi}(B_R(o))}. \tag{1.13}$$

The equality case holds if and only if Ω is isometric to the geodesic ball $B_R(o)$.

Similarly, by applying the sequence (1.2), from (1.13) one has:

Corollary 1.6. *Under the assumptions of Theorem 1.5, we have*

$$\mu_{1,\phi}(\Omega) \leq \mu_{1,\phi}(B_R(o)), \tag{1.14}$$

*with equality holding if and only if Ω is isometric to $B_R(o)$. That is to say, among all bounded domains in $\mathbb{H}^n$ having the same weighted volume, the geodesic ball $B_R(o)$ maximizes the first nonzero Neumann eigenvalue of the Witten-Laplacian, provided the function ϕ satisfies **Property I**.*

Remark 1.7. Similar to the Euclidean case, the spectral isoperimetric inequality (1.14) in Corollary 1.6 has already been proven by the authors in [5] by suitably constructing the trail function. However, we still prefer to list it here to show the close relation between (1.13) and (1.14), and show of course the significance of the spectral isoperimetric inequality (1.13) as well.

Remark 1.8. (1) Similar to (1) of Remark 1.4, except the situation that ϕ is a constant function, one also needs to give the information of the center for the geodesic ball $B_R(o)$ mentioned in Theorem 1.5 and Corollary 1.6.
(2) When investigating spectral isoperimetric inequalities (1.13)-(1.14), there is no essential difference between $\mathbb{H}^n$ and a hyperbolic n-space with constant curvature not equal to -1.
(3) Ashbaugh and Benguria [1] also proposed another important open problem as follows:

- **Conjecture II**. ([1]) Let $\mathbb{M}^n(\kappa)$ be an n-dimensional complete simply connected Riemannian manifold of constant sectional curvature $\kappa \in \{1, -1\}$, and Ω be a bounded domain in $\mathbb{M}^n(\kappa)$ which is contained in a hemisphere in the case that $\kappa = 1$. Let B_Ω be a geodesic ball in $\mathbb{M}^n(\kappa)$ such that $|\Omega| = |B_\Omega|$. Then

$$\frac{1}{\mu_1(\Omega)} + \frac{1}{\mu_2(\Omega)} + \cdots + \frac{1}{\mu_n(\Omega)} \geq \frac{n}{\mu_1(B_\Omega)},$$

  with equality holding if and only if Ω is isometric to B_Ω.

Conjecture II is still open until now. Recently, Xia-Wang [22] also gave a very important progress to this celebrated conjecture, and actually they proved that

$$\frac{1}{\mu_1(\Omega)} + \frac{1}{\mu_2(\Omega)} + \cdots + \frac{1}{\mu_{n-1}(\Omega)} \geq \frac{n-1}{\mu_1(B_\Omega)} \tag{1.15}$$

holds for any bounded domain $\Omega \subset \mathbb{H}^n$ with smooth boundary, and for a geodesic ball $B_\Omega \subset \mathbb{H}^n$ with $|\Omega| = |B_\Omega|$. Moreover, the equality in (1.15) holds if and only if Ω isometric to B_Ω in $\mathbb{H}^n$. Clearly, the isoperimetric inequality (1.15) gives a partial answer to **Conjecture II** and also supports its validity. It is not hard to see that our conclusion in Theorem 1.5 here covers Xia-Wang's spectral isoperimetric inequality (1.15) as a special case (corresponding to $\phi = const.$).

Based on the deriving process of our main conclusions in Theorems 1.1 and 1.5, we would like to propose the following two open problems, which we think it should be suitable to call them ***the Ashbaugh-Benguria type conjecture***.

Question A. Consider the eigenvalue problem (1.1) with choosing M^n to be $M^n = \mathbb{R}^n$, and assume that the function ϕ satisfies **Property I**. Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^n$, and let $B_R(o)$ be a ball of radius R and centered at the origin o of $\mathbb{R}^n$ such that $|\Omega|_\phi = |B_R(o)|_\phi$. Then

$$\frac{1}{\mu_{1,\phi}(\Omega)} + \frac{1}{\mu_{2,\phi}(\Omega)} + \cdots + \frac{1}{\mu_{n-1,\phi}(\Omega)} + \frac{1}{\mu_{n,\phi}(\Omega)} \geq \frac{n}{\mu_{1,\phi}(B_R(o))}.$$

The equality case holds if and only if Ω is the ball $B_R(o)$.

Question B. Consider the eigenvalue problem (1.1) with choosing M^n to be $M^n = \mathbb{H}^n$, and assume that the function ϕ satisfies **Property I**. Let Ω be a bounded domain with smooth boundary in $\mathbb{H}^n$, and let $B_R(o)$ be a geodesic ball of radius R and centered at the origin o of $\mathbb{H}^n$ such that $|\Omega|_\phi = |B_R(o)|_\phi$. Then

$$\frac{1}{\mu_{1,\phi}(\Omega)} + \frac{1}{\mu_{2,\phi}(\Omega)} + \cdots + \frac{1}{\mu_{n-1,\phi}(\Omega)} + \frac{1}{\mu_{n,\phi}(\Omega)} \geq \frac{n}{\mu_{1,\phi}(B_R(o))}.$$

The equality case holds if and only if Ω is isometric to $B_R(o)$.

Remark 1.9. Obviously, Theorems 1.1 and 1.5 give a partial answer to the Ashbaugh-Benguria type conjecture and also support its validity.

This paper is organized as follows. By suitably constructing trial functions, we successfully give a proof to Theorems 1.1 and 1.5 in Section 2. BTW, since originally the proof of Theorem 1.1 is highly similar to that of Theorem 1.5, this leads to the situation that we prefer to unify those two proofs into a single one, which finally appears as its present version shown in Section 2. A refined result of Theorem 1.1 would be given in Section 3 – see Theorem 3.1 for details.

2 A proof of Theorems 1.1 and 1.5

First, we would like to recall a property of the eigenfunction corresponding to the first nonzero Neumann eigenvalue of the Witten-Laplacian on geodesic balls (in space forms) if the function ϕ is radial w.r.t. some chosen point. This property has been carefully proven in [5, Appendix], and readers can check all the details therein.

Lemma 2.1. *([5, Theorem 4.1]) Assume that $B_R(o)$ is a geodesic ball of radius R and centered at some point o in the n-dimensional complete simply connected Riemannian manifold $\mathbb{M}^n(\kappa)$ with constant sectional curvature $\kappa \in \{-1,0,1\}$, and that ϕ is a radial function w.r.t. the distance parameter $t := d(o,\cdot)$, which is also a non-increasing convex function. Then the eigenfunctions of the first nonzero Neumann eigenvalue $\mu_{1,\phi}(B_R(o))$ of the Witten-Laplacian on $B_R(o)$ should have the form $T(t)\frac{x_i}{t}$, $i = 1,2,\cdots,n$, where $T(t)$ satisfies*

$$\begin{cases} T'' + \left(\frac{(n-1)C_\kappa}{S_\kappa} - \phi'\right)T' + \left(\mu_{1,\phi}(B_R(o)) - (n-1)S_\kappa^{-2}\right)T = 0, \\ T(0) = 0, \ T'(R) = 0, \ T'|_{[0,R)} \neq 0. \end{cases} \tag{2.1}$$

Here $C_\kappa(t) = (S_\kappa(t))'$ and

$$S_\kappa(t) = \begin{cases} \sin t, & \text{if } \mathbb{M}^n(\kappa) = \mathbb{S}^n_+, \\ t, & \text{if } \mathbb{M}^n(\kappa) = \mathbb{R}^n, \\ \sinh t, & \text{if } \mathbb{M}^n(\kappa) = \mathbb{H}^n, \end{cases}$$

with $\mathbb{S}^n_+$ the n-dimensional hemisphere of radius 1.

Remark 2.2. From [5, Appendix], it is not hard to know that x_i, $i = 1,2,\cdots,n$, are coordinate functions of the globally defined orthonormal coordinate system set up in $\mathbb{M}^n(\kappa)$.

A proof of Theorems 1.1 and 1.5. Due to the fact that ϕ is radial w.r.t. o, one can define a radial function f as follows

$$f(t) = \begin{cases} T(t), & \text{if } 0 \leq t \leq R, \\ T(R), & \text{if } t > R, \end{cases} \tag{2.2}$$

where R is the radius of the (geodesic) ball $B_R(o)$ satisfying the volume constraint $|\Omega|_\phi = |B_R(o)|_\phi$. The origin o would be chosen as follows: in fact, by the Brouwer fixed point theorem and using a similar argument to that of Weinberger given in [20], one can always choose a suitable origin $o \in \mathrm{hull}(\Omega)$ such that

$$\int_\Omega f(t)\frac{x_i}{t}d\eta = 0, \qquad i = 1,2,\cdots,n. \tag{2.3}$$

Denote by $\{e_1, e_2, \cdots, e_n\}$ the orthonormal basis (of $\mathbb{R}^n$ or $\mathbb{H}^n$) corresponding to the coordinates $x_1, x_2, \cdots, x_n$. Then (2.3) can be rewritten as

$$\int_\Omega \langle x, e_i\rangle \frac{f(t)}{t}d\eta = 0, \qquad i = 1,2,\cdots,n, \tag{2.4}$$

with $\langle\cdot,\cdot\rangle$ denoting the inner product. Denote by u_i the eigenfunction corresponding to the i-th Neumann eigenvalue $\mu_{i,\phi}$ of the eigenvalue problem (1.1). Then (2.4) implies that

$$\langle x, e_i\rangle \frac{f(t)}{t} \perp u_0$$

in the sense of L^2-norm w.r.t. the weighted density $d\eta$. Our purpose now is to construct suitable trail function ψ_i for the eigenvalue $\mu_{i,\phi}$ such that ψ_i is orthogonal to the preceding eigenfunctions $u_0, u_1, \cdots, u_{i-1}$. That is to say, $\psi_i \perp \text{span}\{u_0, u_1, \cdots, u_{i-1}\}$ in the sense of L^2-norm w.r.t. the weighted density $d\eta$. Define an $n \times n$ matrix $Q = (q_{ij})_{n\times n}$ with q_{ij} given by

$$q_{ij} := \int_\Omega \langle x, e_i\rangle \frac{f(t)}{t} u_j d\eta, \qquad i, j = 1, 2, \cdots, n.$$

Using the orthogonalization of Gram and Schmidt (QR-factorization theorem), one knows that there exist an upper triangle matrix $\mathcal{M} = (\mathcal{M}_{ij})_{n\times n}$ and an orthogonal matrix $U = (a_{ij})_{n\times n}$ such that $\mathcal{M} = UQ$, which implies

$$\mathcal{M}_{ij} = \sum_{k=1}^{n} a_{ik} q_{kj} = \int_\Omega a_{ik} \langle x, e_k\rangle \frac{f(t)}{t} u_j d\eta, \qquad 1 \le j < i \le n.$$

Set $e'_i = \sum_{k=1}^n a_{ik} e_k$, $i = 1, 2, \cdots, n$, and then

$$\int_\Omega \langle x, e'_i\rangle \frac{f(t)}{t} u_j d\eta = 0 \tag{2.5}$$

holds for $j = 1, 2, \cdots, i-1$ and $i = 2, 3, \cdots, n$. BTW, it is easy to see that $\{e'_1, e'_2, \cdots, e'_n\}$ is also an orthonormal basis (of $\mathbb{R}^n$ or $\mathbb{H}^n$), which is actually formed by making an orthogonal transformation to the orthonormal basis $\{e_1, e_2, \cdots, e_n\}$. Denote by $y_1, y_2, \cdots, y_n$ the coordinate functions corresponding to the basis $\{e'_1, e'_2, \cdots, e'_n\}$, that is, $y_i = \langle x, e'_i\rangle$. Then from (2.5) one has

$$\int_\Omega y_i \frac{f(t)}{t} u_j d\eta = 0, \qquad j = 1, 2, \cdots, i-1 \text{ and } i = 2, 3, \cdots, n. \tag{2.6}$$

For convention and by the abuse of notations, we prefer to use x_i as coordinate functions – based on this, we still write y_i as x_i, $i = 2, 3, \cdots, n$. Then in this setting, (2.6) can be rewritten as

$$\int_\Omega x_i \frac{f(t)}{t} u_j d\eta = 0, \qquad j = 1, 2, \cdots, i-1 \text{ and } i = 2, 3, \cdots, n. \tag{2.7}$$

Together with (2.3) and (2.7), one has that there exists an orthonormal basis $\{e_1, e'_2, e'_3, \cdots, e'_n\}$ such that the coordinate functions $x_1, x_2, \cdots, x_n$ corresponding to this basis satisfy

$$\int_\Omega x_i \frac{f(t)}{t} u_j d\eta = 0, \qquad j = 0, 1, 2, \cdots, i-1 \text{ and } i = 1, 2, 3, \cdots, n. \tag{2.8}$$

Here the eigenfunction u_0 of the eigenvalue $\mu_{0,\phi}$ can be chosen as $u_0 = 1/\sqrt{|\Omega|_\phi}$. Set in (2.8) that

$$\psi_i := x_i \frac{f(t)}{t}, \qquad i = 1, 2, 3, \cdots, n,$$

and then one has

$$\int_\Omega \psi_i u_j d\eta = 0, \qquad j = 0, 1, 2, \cdots, i-1 \text{ and } i = 1, 2, 3, \cdots, n. \tag{2.9}$$

Hence, our purpose of constructing trail functions ψ_i, $i = 1, 2, 3, \cdots, n$, has been achieved. To prove our main conclusions, we also need the following truth.

Lemma 2.3. *The function* $\frac{f(t)}{S_\kappa(t)}$ *is monotone decreasing in the bounded domain* Ω *with smooth boundary in* $\mathbb{R}^n$ *(or* $\mathbb{H}^n$*).*

Proof. By (2.1) and the definition of the function f, we observe first that

$$\lim_{t\to 0}\frac{f(t)}{S_\kappa(t)} = f'(0).$$

Without loss of generality, we may assume $f>0$. Since

$$\frac{d}{dt}\left(\frac{f(t)}{S_\kappa(t)}\right) = \frac{f'(t) - \frac{C_\kappa(t)}{S_\kappa(t)}f(t)}{S_\kappa(t)},$$

similarly, one has

$$\lim_{t\to 0}\left(f'(t) - \frac{C_\kappa(t)}{S_\kappa(t)}f(t)\right) = 0, \qquad f'(R) - \frac{C_\kappa(R)}{S_\kappa(R)}f(R) < 0.$$

If there exists a point t_0 such that $f'(t_0) - \frac{C_\kappa(t_0)}{S_\kappa(t_0)}f(t_0) > 0$, then there exists t_1 such that

$$f'(t_1) - \frac{C_\kappa(t_1)}{S_\kappa(t_1)}f(t_1) > 0,$$

$$\frac{d}{dt}\left(f'(t) - \frac{C_\kappa(t)}{S_\kappa(t)}f(t)\right)(t_1) = 0. \tag{2.10}$$

Combining the first equation in (2.1) and (2.10) yields at point t_1 that

$$-\frac{nC_\kappa}{S_\kappa}f' - \mu_{1,\phi}f + \phi' f' + \frac{nf}{S_\kappa^2} = 0. \tag{2.11}$$

Therefore, due to $\phi' \le 0$, it follows from (2.11) that

$$\left(f' - \frac{f}{C_\kappa S_\kappa}\right)(t_1) \le 0.$$

So, we have

$$\begin{aligned}\left(f' - \frac{C_\kappa}{S_\kappa}f\right)(t_1) &\le \left(\frac{f}{S_\kappa C_\kappa} - \frac{fC_\kappa}{S_\kappa}\right)(t_1)\\ &= \left(\frac{f(1-C_\kappa^2)}{S_\kappa C_\kappa}\right)(t_1)\\ &\le 0.\end{aligned}$$

This is contradict with $\left(f' - \frac{C_\kappa}{S_\kappa}f\right)(t_1) > 0$. Hence, we have $\frac{d}{dt}(\frac{f(t)}{S_\kappa(t)}) < 0$, and then $\frac{f(t)}{S_\kappa(t)}$ is monotone decreasing. $\square$

By the characterization (1.3) and (2.9), one can obtain

$$\mu_{i,\phi}(\Omega)\int_{\Omega} f^2\frac{x_i^2}{t^2}d\eta \leq \int_{\Omega}\left(f'^2\frac{x_i^2}{t^2}+f^2\left|\overline{\nabla}\left(\frac{x_i}{t}\right)\right|^2 S_\kappa^{-2}(t)\right)d\eta, \tag{2.12}$$

where $\overline{\nabla}$ is the gradient operator defined on the unit $(n-1)$-sphere $\mathbb{S}^{n-1}$. By a direct calculation to (2.12), one has

$$\begin{aligned}
\int_{\Omega} f^2\frac{x_i^2}{t^2}d\eta &\leq \frac{1}{\mu_{i,\phi}(\Omega)}\int_{\Omega}(f')^2\frac{x_i^2}{t^2}d\eta+\frac{1}{\mu_{i,\phi}(\Omega)}\int_{\Omega} f^2\left|\overline{\nabla}\left(\frac{x_i}{t}\right)\right|^2 S_\kappa^{-2}(t)d\eta\\
&= \frac{1}{\mu_{i,\phi}(\Omega)}\int_{\Omega\cap B_R(o)}(f')^2\frac{x_i^2}{t^2}d\eta+\frac{1}{\mu_{i,\phi}(\Omega)}\int_{\Omega} f^2\left|\overline{\nabla}\left(\frac{x_i}{t}\right)\right|^2 S_\kappa^{-2}(t)d\eta\\
&\leq \frac{1}{\mu_{i,\phi}(\Omega)}\int_{B_R(o)}(f')^2\frac{x_i^2}{t^2}d\eta+\frac{1}{\mu_{i,\phi}(\Omega)}\int_{\Omega} f^2\left|\overline{\nabla}\frac{x_i}{t}\right|^2 S_\kappa^{-2}(t)d\eta\\
&= \frac{1}{\mu_{i,\phi}(\Omega)}\frac{1}{n}\int_0^R\int_{\mathbb{S}^{n-1}(1)}(f')^2 S_\kappa^{n-1}(t)\ e^{-\phi}dSdt\\
&\qquad +\frac{1}{\mu_{i,\phi}(\Omega)}\int_{\Omega} f^2\left|\overline{\nabla}\left(\frac{x_i}{t}\right)\right|^2 S_\kappa^{-2}(t)d\eta\\
&= \frac{1}{\mu_{i,\phi}(\Omega)}\frac{1}{n}\int_{B_R(o)}(f')^2 d\eta+\frac{1}{\mu_{i,\phi}(\Omega)}\int_{\Omega} f^2\left|\overline{\nabla}\left(\frac{x_i}{t}\right)\right|^2 S_\kappa^{-2}(t)d\eta,
\end{aligned} \tag{2.13}$$

where dS stands for the volume element on the $(n-1)$-sphere $\mathbb{S}^{n-1}(1)$ of radius 1. By [22], one knows

$$\sum_{i=1}^{n}\frac{1}{\mu_{i,\phi}(\Omega)}\left|\overline{\nabla}\left(\frac{x_i}{r}\right)\right|^2 \leq \sum_{i=1}^{n-1}\frac{1}{\mu_{i,\phi}(\Omega)}. \tag{2.14}$$

Therefore, combining (2.13) with (2.14), and then doing summation over the index i from 1 to n, we can obtain

$$\int_{\Omega} f^2 d\eta \leq \sum_{i=1}^{n}\frac{1}{n\mu_{i,\phi}(\Omega)}\int_{B_R(o)}(f')^2 d\eta+\sum_{i=1}^{n-1}\frac{1}{\mu_{i,\phi}(\Omega)}\int_{\Omega} f^2 S_\kappa^{-2}(t)d\eta. \tag{2.15}$$

On the other hand, still from (2.14), one has

$$\sum_{i=1}^{n}\frac{1}{\mu_{n,\phi}(\Omega)}\left|\overline{\nabla}\left(\frac{x_i}{r}\right)\right|^2 \leq \sum_{i=1}^{n}\frac{1}{\mu_{i,\phi}(\Omega)}\left|\overline{\nabla}\left(\frac{x_i}{r}\right)\right|^2 \leq \sum_{i=1}^{n-1}\frac{1}{\mu_{i,\phi}(\Omega)},$$

which implies

$$\frac{1}{n\mu_{n,\phi}(\Omega)} \leq \frac{1}{n(n-1)}\sum_{i=1}^{n-1}\frac{1}{\mu_{i,\phi}(\Omega)}.$$

Substituting the above inequality into (2.15) results in

$$\int_{\Omega} f^2(t)d\eta \leq \sum_{i=1}^{n-1}\frac{1}{(n-1)\mu_{i,\phi}(\Omega)}\left[\int_{B_R(o)}(f')^2(t)d\eta+\int_{\Omega}(n-1)f^2(t)S_\kappa^2(t)d\eta\right]. \tag{2.16}$$

Applying Lemma 2.3 and [5, Lemma 4.4 and Appendix], one has

$$\int_{\Omega} f^2(t)d\eta \geq \int_{B_R(o)} f^2(t)d\eta, \qquad \int_{\Omega} \frac{f^2(t)}{S_\kappa^2(t)}d\eta \leq \int_{B_R(o)} \frac{f^2(t)}{S_\kappa^2(t)}d\eta.$$

Putting the above fact into (2.16), we have

$$\frac{1}{n-1}\sum_{i=1}^{n-1}\frac{1}{\mu_{i,\phi}(\Omega)} \geq \frac{\int_{B_R(o)} f^2(t)d\eta}{\int_{B_R(o)}\left[(f')^2+(n-1)\frac{f^2(t)}{S_\kappa^2(t)}\right]d\eta} = \frac{1}{\mu_{1,\phi}(B_R(o))},$$

which implies (1.5) or (1.13) directly. This completes the proof of Theorems 1.1 and 1.5.

3 A sharper estimate

In the last section, we would like to give a more shaper estimate (for the sums of the reciprocals of the first $(n-1)$ nonzero Neumann eigenvalues of the Witten-Laplacian on bounded domains in $\mathbb{R}^n$) than (1.5) shown in Theorem 1.1. In fact, we can prove:

Theorem 3.1. *Assume that Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, and that the function ϕ satisfies* ***Property I****. Then*

$$\begin{aligned}\mu_{1,\phi}(B_R(o)) - &\frac{n-1}{\frac{1}{\mu_{1,\phi}(\Omega)}+\frac{1}{\mu_{2,\phi}(\Omega)}+\cdots+\frac{1}{\mu_{n-1,\phi}(\Omega)}}\\ &\geq \frac{\int_{B_R(o)\setminus B_1}\left[(f')^2+(n-1)\frac{f^2}{t^2}\right]d\eta - f^2(R)\int_{B_2\setminus B_R(o)}\left[(n-1)\frac{1}{t^2}\right]d\eta}{\int_{B_R(o)} f^2 d\eta},\end{aligned} \tag{3.1}$$

where (as in Theorem 1.1) $B_R(o)$ is a ball of radius R and centered at the origin o of $\mathbb{R}^n$ such that $|\Omega|_\phi = |B_R(o)|_\phi$, f is the function defined by (2.2), and B_1, B_2 are two balls centered at the origin o and satisfying $|B_1|_\phi = |\Omega\cap B_R(o)|_\phi$, $|B_2\setminus B_R(o)|_\phi = |\Omega\setminus B_R(o)|_\phi$, respectively. The equality in (3.1) holds if and only if Ω is the ball $B_R(o)$.

Proof. By Lemma 2.3, we have $f'-\frac{1}{t}f\leq 0$ and $f'\geq 0$ in $[0,R]$, which implies

$$(f')^2-\frac{f^2}{t^2}\leq 0.$$

Since $|\overline{\nabla}\frac{x_i}{t}|^2 = 1-\frac{x_i^2}{t^2}$ and $(f')^2-\frac{f^2}{t^2}\leq 0$, with the help of trail functions ψ_i, $i=1,2,\cdots,n$, constructed in Section 2 and by using a similar argument of deriving the inequality (2.31) in [22], it is not hard to get

$$\frac{n-1}{\sum\limits_{i=1}^{n-1}\frac{1}{\mu_{i,\phi}}}\int_{\Omega} f^2 d\eta \leq \int_{\Omega}\left[(f')^2+(n-1)\frac{f^2}{t^2}\right]d\eta. \tag{3.2}$$

Since f is increasing, we can deduce from [5, Lemma 4.4 and Appendix] by the rearrangement technique the following:

$$\int_{\Omega} f^2 d\eta \geq \int_{B_R(o)} f^2 d\eta. \tag{3.3}$$

Putting the above expression into (3.2) yields

$$\frac{n-1}{\sum_{i=1}^{n-1} \frac{1}{\mu_{i,\phi}}} \int_{B_R(o)} f^2 d\eta \leq \int_{\Omega} \left[(f')^2 + (n-1)\frac{f^2}{t^2} \right] d\eta. \tag{3.4}$$

Since $f(t)\frac{x_i}{t}$, $i = 1, 2, \cdots, n$, are the eigenfunctions corresponding to the eigenvalue $\mu_{1,\phi}(B_R(o))$, one can obtain from the characterization (1.4) that

$$\mu_{1,\phi}(B_R(o)) \int_{B_R(o)} f^2 d\eta = \int_{B_R(o)} \left[(f')^2 + (n-1)\frac{f^2}{t^2} \right] d\eta. \tag{3.5}$$

Combining (3.4) and (3.5) results in

$$\begin{aligned} \left(\mu_{1,\phi}(B_R(o)) - \frac{n-1}{\sum_{i=1}^{n-1} \frac{1}{\mu_{i,\phi}}} \right) \int_{B_R(o)} f^2 d\eta \geq & \\ \int_{B_R(o)} \left[(f')^2 + (n-1)\frac{f^2}{t^2} \right] d\eta - \int_{\Omega} \left[(f')^2 + (n-1)\frac{f^2}{t^2} \right] d\eta. & \end{aligned} \tag{3.6}$$

On one hand,

$$\begin{aligned} \int_{\Omega} \left[(f')^2 + (n-1)\frac{f^2}{t^2} \right] d\eta &= \int_{(\Omega \setminus B_R(o)) \cup (\Omega \cap B_R(o))} \left[(f')^2 + (n-1)\frac{f^2}{t^2} \right] d\eta \\ &= \int_{\Omega \setminus B_R(o)} \left[(f')^2 + (n-1)\frac{f^2}{t^2} \right] d\eta + \\ &\qquad \int_{\Omega \cap B_R(o)} \left[(f')^2 + (n-1)\frac{f^2}{t^2} \right] d\eta. \end{aligned} \tag{3.7}$$

By [5], it is not hard to show that $(f')^2 + (n-1)\frac{f^2}{t^2}$ is monotone decreasing along the radial direction of $(\Omega \cap B_R(o)) \setminus B_1$, which implies

$$\begin{aligned} \int_{\Omega \cap B_R(o)} \left[(f')^2 + (n-1)\frac{f^2}{t^2} \right] d\eta &= \int_{\Omega \cap B_R(o) \cap B_1} \left[(f')^2 + (n-1)\frac{f^2}{t^2} \right] d\eta + \\ &\qquad \int_{(\Omega \cap B_R(o)) \setminus B_1} \left[(f')^2 + (n-1)\frac{f^2}{t^2} \right] d\eta \\ &\leq \int_{\Omega \cap B_R(o) \cap B_1} \left[(f')^2 + (n-1)\frac{f^2}{t^2} \right] d\eta + \\ &\qquad \left[(f')^2(R_1) + (n-1)\frac{f^2(R_1)}{R_1^2} \right] \int_{(\Omega \cap B_R(o)) \setminus B_1} d\eta. \end{aligned} \tag{3.8}$$

Similarly, one can obtain

$$\begin{aligned}\int_{B_1}\left[(f')^2+(n-1)\frac{f^2}{t^2}\right]d\eta &= \int_{B_1\cap\Omega\cap B_R(o)}\left[(f')^2+(n-1)\frac{f^2}{t^2}\right]d\eta+\\ &\qquad\int_{B_1\backslash(\Omega\cap B_R(o))}\left[(f')^2+(n-1)\frac{f^2}{t^2}\right]d\eta\\ &\geq \int_{B_1\cap\Omega\cap B_R(o)}\left[(f')^2+(n-1)\frac{f^2}{t^2}\right]d\eta+\\ &\qquad\left[(f'(R_1))^2+(n-1)\frac{f^2(R_1)}{R_1^2}\right]\int_{B_1\backslash(\Omega\cap B_R(o))}d\eta, \qquad (3.9)\end{aligned}$$

where R_1 is the radius of the ball B_1. One has from the assumption $|\Omega\cap B_R(o)|_\phi=|B_1|_\phi$ that

$$\int_{\Omega\cap B_R(o)}\left[(f')^2+(n-1)\frac{f^2}{t^2}\right]d\eta\leq\int_{B_1}\left[(f')^2+(n-1)\frac{f^2}{t^2}\right]d\eta. \qquad (3.10)$$

Since $f(t)=T(R)$ is constant when $t>R$, by a direct calculation one has

$$\begin{aligned}&\int_{\Omega\backslash B_R(o)}\left[(f')^2+(n-1)\frac{f^2}{t^2}\right]d\eta\\ =&\int_{(\Omega\backslash B_R(o))\cap(B_2\backslash B_R(o))}\left[(f')^2+(n-1)\frac{f^2}{t^2}\right]d\eta+\\ &\quad\int_{(\Omega\backslash B_R(o))\backslash(B_2\backslash B_R(o))}\left[(f')^2+(n-1)\frac{f^2}{t^2}\right]d\eta\\ =&\int_{(\Omega\backslash B_R(o))\cap(B_2\backslash B_R(o))}\left[(f')^2+(n-1)\frac{f^2}{t^2}\right]d\eta+\\ &\quad\left[(f'(R))^2+(n-1)\frac{f^2(R)}{R^2}\right]\int_{(\Omega\backslash B_R(o))\backslash(B_2\backslash B_R(o))}d\eta. \qquad (3.11)\end{aligned}$$

Similarly, one can get

$$\begin{aligned}&\int_{B_2\backslash B_R(o)}\left[(f')^2+(n-1)\frac{f^2}{t^2}\right]d\eta\\ =&\int_{(B_2\backslash B_R(o))\cap(\Omega\cap B_R(o))}\left[(f')^2+(n-1)\frac{f^2}{t^2}\right]d\eta+\\ &\quad\int_{(B_2\backslash B_R(o))\backslash(\Omega\backslash B_R(o))}\left[(f')^2+(n-1)\frac{f^2}{t^2}\right]d\eta\\ =&\int_{(B_2\backslash B_R(o))\cap(\Omega\cap B_R(o))}\left[(f')^2+(n-1)\frac{f^2}{t^2}\right]d\eta+\\ &\quad\left[(f'(R))^2+(n-1)\frac{f^2(R)}{R^2}\right]\int_{(B_2\backslash B_R(o))\backslash(\Omega\backslash B_R(o))}d\eta. \qquad (3.12)\end{aligned}$$

One has from the assumption $|\Omega \setminus B_R(o)|_\phi = |B_2 \setminus B_R(o)|_\phi$ that

$$\int_{\Omega\setminus B_R(o)} \left[(f')^2 + (n-1)\frac{f^2}{t^2}\right] d\eta = \int_{B_2\setminus B_R(o)} \left[(f')^2 + (n-1)\frac{f^2}{t^2}\right] d\eta. \tag{3.13}$$

Putting (3.7)-(3.13) into (3.6) yields

$$\left(\mu_{1,\phi}(B_R(o)) - \frac{n-1}{\sum_{i=1}^{n-1}\frac{1}{\mu_{i,\phi}}}\right)\int_{B_R(o)} f^2 d\eta \geq \int_{B_R(o)\setminus B_1} \left[(f')^2 + (n-1)\frac{f^2}{t^2}\right] d\eta - \int_{B_2\setminus B_R(o)} \left[(f')^2 + (n-1)\frac{f^2}{t^2}\right] d\eta,$$

which implies (3.1) directly by using (2.2) first and then multiplying both sides of the above inequality by $\left(\int_{B_R(o)} f^2 d\eta\right)^{-1}$. The equality case of the estimate (3.1) would follow by (3.3) where the equality can be attained if and only if Ω is the ball $B_R(o)$. $\square$

Remark 3.2. The estimate (3.1) is sharper than (1.5) in Theorem 1.1, since the quantity in the RHS of (3.1) is nonnegative. This is because

$$\begin{aligned}
&\int_{B_R(o)\setminus B_1} \left[(f')^2 + (n-1)\frac{f^2}{t^2}\right] d\eta - f^2(R)\int_{B_2\setminus B_R(o)} (n-1)\frac{1}{t^2} d\eta \\
&\geq \left[(f'(R))^2 + (n-1)\frac{f^2(R)}{R^2}\right]\int_{B_R(o)\setminus B_1} d\eta - \frac{(n-1)f^2(R)}{R^2}\int_{B_2\setminus B_R(o)} d\eta \\
&= (n-1)\frac{f^2(R)}{R^2}\left(\int_{B_R(o)\setminus B_1} d\eta - \int_{B_2\setminus B_R(o)} d\eta\right) \\
&= (n-1)\frac{f^2(R)}{R^2}(|B_R(o)\setminus B_1|_\phi - |B_2\setminus B_R(o)|_\phi) \\
&= (n-1)\frac{f^2(R)}{R^2}(|B_R(o)|_\phi - |B_1|_\phi - (|B_2|_\phi - |B_R(o)|_\phi)) \\
&= (n-1)\frac{f^2(R)}{R^2}(2|B_R(o)|_\phi - (|B_1|_\phi + |B_2|_\phi)) \\
&= (n-1)\frac{f^2(R)}{R^2}(2|\Omega|_\phi - (|\Omega\cap B_R(o)|_\phi + |\Omega\setminus B_R(o)|_\phi + |B_R(o)|_\phi)) \\
&= 0.
\end{aligned}$$

Acknowledgments

This research was supported in part by the NSF of China (Grant Nos. 11801496 and 11926352), the Fok Ying-Tung Education Foundation (China), and Hubei Key Laboratory of Applied Mathematics (Hubei University). After we put the first version of our manuscript on arXiv 12$^{\text{th}}$ March 2024, Prof. Kui Wang at Soochow University sent us a digital copy of the reference [11] and told us that for the Euclidean n-space $\mathbb{R}^n$ with Gaussian density (i.e. $e^{-\phi} = (2\pi)^{-n/2}e^{-t^2/2}$), under an extra upper bound constraint for the Gaussian density they could obtain the spectral isoperimetric inequality (1.5) in Theorem 1.1 here. We would like to thank Prof. Wang for sharing us their preprint and do having interest in our manuscript.

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