Several isoperimetric inequalities of Dirichlet and Neumann eigenvalues of the Witten-Laplacian

Ruifeng Chen, Jing Mao*

Faculty of Mathematics and Statistics,
Key Laboratory of Applied Mathematics of Hubei Province,
Hubei University, Wuhan 430062, China
Emails: gchenruifeng@163.com (R. F. Chen), jiner120@163.com (J. Mao)

Abstract

In this paper, by mainly using the rearrangement technique and suitably constructing trial functions, under the constraint of fixed weighted volume, we can successfully obtain several isoperimetric inequalities for the first and the second Dirichlet eigenvalues, the first nonzero Neumann eigenvalue of the Witten-Laplacian on bounded domains in space forms. These spectral isoperimetric inequalities extend those classical ones (i.e. the Faber-Krahn inequality, the Hong-Krahn-Szegő inequality and the Szegő-Weinberger inequality) of the Laplacian.

1 Introduction

The study of extremum problems of prescribed functionals is of great significance in Mathematics. For instance, a famous isoperimetric problem, which might be known for nearly all the mathematicians, in the n-dimensional ($n \geq 2$) Euclidean space $\mathbb{R}^n$ is to study the following extremum problem:

$$\min\left\{|\partial\Omega|\Big||\Omega| = const.\right\} \tag{1.1}$$

for bounded domains $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$, where, by the abuse of notations, $|\cdot|$ stands for the Hausdorff measure of a given geometric object. The above extremum problem can be asked in another way as follows:

- *Among all bounded domains in $\mathbb{R}^n$ with fixed volume, which one minimizes the area functional of the boundary?*

$*$ Corresponding author
MSC 2020: 35P15, 35J10, 35J15.
Key Words: Witten-Laplacian, Dirichlet eigenvalues, Neumann eigenvalues, Laplacian, isoperimetric inequalities.

This classical problem has been answered completely and one knows that the unique minimizer of the area functional should be a ball with the volume equal to $|\Omega| = const.$ – see, e.g., [34, Chapter 1] for an interesting derivation for classical isoperimetric inequalities in Euclidean space by using the Schwarz symmetrization. In fact, for any bounded domain Ω in $\mathbb{R}^n$ with smooth boundary, one has:

$$\frac{|\partial\Omega|^n}{|\Omega|^{n-1}} \geq \frac{|\mathbb{S}^{n-1}|^n}{|\mathbb{B}^n|^{n-1}}, \tag{1.2}$$

with equality holding if and only if Ω is a Euclidean ball. Obviously, the RHS of (1.2) is independent of the choice of radius for the Euclidean n-ball $\mathbb{B}^n$ and the corresponding Euclidean $(n-1)$-sphere $\mathbb{S}^{n-1}$. That is to say, the quantity $|\mathbb{S}^{n-1}|^n/|\mathbb{B}^n|^{n-1}$ is scale invariant. So, for convenience and simplification, we denote by $\mathbb{B}^n$, $\mathbb{S}^{n-1}$ the unit Euclidean n-ball and the unit Euclidean $(n-1)$-sphere, respectively. By (1.2), one easily knows that:

- Among all bounded domains in $\mathbb{R}^n$ having the same volume, Euclidean balls minimize the boundary area.
- Among all bounded domains in $\mathbb{R}^n$ having the same boundary area, Euclidean balls maximize the volume.

Clearly, (1.2) gives the answer to the problem (1.1) completely – for a ball B_Ω with $|B_\Omega| = |\Omega| = const.$, it follows that[1]

$$|\partial\Omega| \geq |\partial B_\Omega|, \tag{1.3}$$

with equality holding if and only if Ω is a ball in $\mathbb{R}^n$ (which is congruent with B_Ω). Following the convention in [11], we prefer to call (1.2)-(1.3) the *geometric isoperimetric inequalities*.

The purpose of this paper is to investigate isoperimetric inequalities from the viewpoint of spectral quantities of the Witten-Laplacian. However, in order to state our conclusions clearly, we prefer to first recall several classical results of the Laplacian.

Let $(M^n, \langle\cdot,\cdot\rangle)$ be an n-dimensional ($n \geq 2$) complete Riemannian manifold with the metric $g := \langle\cdot,\cdot\rangle$. Let $\Omega \subset M^n$ be a bounded domain in M^n with smooth[2] boundary $\partial\Omega$. Denote by Δ, ∇ the Laplace and the gradient operators on M^n associated with the metric g, respectively.[3] On Ω, one can consider the Dirichlet eigenvalue problem of the Laplacian as follows

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega \subset M^n, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

[1] Clearly, ∂B_Ω stands for the boundary sphere of the ball B_Ω.

[2] The smoothness assumption for the regularity of the boundary $\partial\Omega$ is strong to consider the eigenvalue problems (1.4) and (1.8). For instance, a weaker regularity assumption that $\partial\Omega$ is Lipschitz continuous can also assure the validity about the description of discrete spectrum of the Neumann eigenvalue problem (1.8) of the Laplacian on the 4th page of this paper. However, the Lipschitz continuous assumption might not be enough to consider some other geometric problems involved Neumann eigenvalues of (1.8). Therefore, in order to avoid a little bit boring discussion on the regularity of the boundary $\partial\Omega$ (which is also not important for the topic investigated in our paper here), without specification, we prefer to assume that $\partial\Omega$ is smooth in this paper. This setting leads to the situation that some conclusions of this paper may still hold under a weaker regularity assumption for the boundary $\partial\Omega$, readers who are interested in this situation could try to seek the weakest regularity.

[3] Without specifications, generally, in this paper same symbols have the same meanings.

which is also known as the *fixed membrane problem* of the Laplacian. In fact, for the eigenvalue problem (1.4), when M^n is chosen to be $\mathbb{R}^3$, this system can be used to describe the vibration of a membrane with boundary fixed, and this is the reason why it is called fixed membrane problem. Because of this physical background, eigenvalues of a prescribed eigenvalue problem of some self-adjoint differentiable elliptic operator are called *frequencies*. It is well-known that the operator $-\Delta$ in (1.4) only has the discrete spectrum and all the elements (i.e., eigenvalues) can be listed non-decreasingly as follows

$$0 < \lambda_1(\Omega) < \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \cdots \uparrow \infty. \tag{1.5}$$

For each eigenvalue $\lambda_i(\Omega)$, $i = 1, 2, \cdots$, all the nontrivial functions satisfying (1.4) form a vector space, which has finite dimensional and is called eigenspace of $\lambda_i(\Omega)$. Moreover, all the elements in this eigenspace are called eigenfunctions belonging to $\lambda_i(\Omega)$. The dimension of this eigenspace is called multiplicity of the eigenvalue $\lambda_i(\Omega)$. Each eigenvalue $\lambda_i(\Omega)$ in the sequence (1.5) is repeated according to its multiplicity. By variational principle, the k-th Dirichlet eigenvalue $\lambda_k(\Omega)$ is characterized as follows

$$\lambda_k(\Omega) = \inf \left\{ \frac{\int_\Omega |\nabla f|^2 dv}{\int_\Omega f^2 dv} \middle| f \in W_0^{1,2}(\Omega), f \neq 0, \int_\Omega f f_i dv = 0 \right\},$$

where dv denotes the Riemannian volume element of M^n, and f_i, $i = 1, 2, \cdots, k-1$, denotes an eigenfunction of $\lambda_i(\Omega)$. Here, as usual, $W_0^{1,2}(\Omega)$ is the Sobolev space with compact support, i.e. the completion of the set of smooth functions (with compact support) $C_0^\infty(\Omega)$ under the following Sobolev norm

$$\|f\|_{1,2} := \left(\int_\Omega f^2 dv + \int_\Omega |\nabla f|^2 dv \right)^{1/2}. \tag{1.6}$$

See, e.g., [11] for the above fundamental facts of the eigenvalue problem (1.4). Besides, for convenience and without confusion, in the sequel, except specification we will write $\lambda_i(\Omega)$ as λ_i directly. This convention would be also used when we meet with other possible eigenvalue problems.

Similar to (1.1), for bounded domains $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$, $n \geq 2$, it should be interesting and important to ask the following extremum problem:

$$\min \left\{ \lambda_k(\Omega) \middle| |\Omega| = const. \right\} \tag{1.7}$$

for each $k = 1, 2, 3, \cdots$. In fact, (1.7) is a natural and classical isoperimetric problem in the study of Spectral Geometry. To the best of our knowledge, for $k = 1, 2$, there exist affirmative answers to the problem (1.7) as follows:

- (Faber-Krahn inequality, [16, 22]) $\lambda_1(\Omega) \geq \lambda_1(B_\Omega)$, and the equality holds if and only if Ω is a ball in $\mathbb{R}^n$ (which is congruent with B_Ω, $|B_\Omega| = |\Omega| = const.$). That is to say, among all bounded domains in $\mathbb{R}^n$ having the same volume, Euclidean balls minimize the first Dirichlet eigenvalue of the Laplacian.

- (Hong-Krahn-Szegő inequality, [20, 23]) $\lambda_2(\Omega) \geq \lambda_1(\widetilde{B}_\Omega)$, where $\widetilde{B}_\Omega$ is a ball in $\mathbb{R}^n$ such that $2|\widetilde{B}_\Omega| = const. = |\Omega|$. That is to say, the minimum of the second Dirichlet eigenvalue of the Laplacian on bounded domains Ω (whose volume equals some prescribed positive constant) should be equal to the first Dirichlet eigenvalue of the Laplacian on a ball $\widetilde{B}_\Omega$ with $|\widetilde{B}_\Omega| = |\Omega|/2$.

Hong-Krahn-Szegő inequality implies that under the constraint that the volume of bounded domains is fixed, the second Dirichlet eigenvalue (of the Laplacian) is minimized by two balls of the same volume. However, if one additionally requires that Ω is connected, then under the constraint of volume fixed ($|\Omega| = const.$), this minimizer of $\lambda_2(\Omega)$ cannot be attained but can be approximated by the domain Ω_ϵ, obtained by joining the union of the two congruent balls (whose volumes equal $|\Omega|/2$) by a thin pipe of width ϵ (sufficiently small) – see [19] for the precise description of this interesting example and see, e.g., [7, 9] for the strict proof of this approximation (as $\epsilon \to 0$). In two dimensional case, it has long been conjectured that the ball minimizes $\lambda_3(\Omega)$, but there did not have much progress in this direction. For higher order Dirichlet eigenvalues, not much is known. However, there is an interesting result we prefer to mention, that is, Berger [2] proved that for planar bounded domain $\Omega \subset \mathbb{R}^2$, the i-th ($i > 4$) Dirichlet eigenvalue $\lambda_i(\Omega)$ is not minimized by any union of disks.

For a bounded domain Ω (with smooth boundary) on a given complete Riemannian n-manifold M^n, one can also consider the Neumann eigenvalue problem of the Laplacian as follows

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega \subset M^n, \\ \frac{\partial u}{\partial \vec{v}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

which is also known as the *free membrane problem* of the Laplacian. In fact, for the eigenvalue problem (1.8), when M^n is chosen to be $\mathbb{R}^3$, this system can be used to describe the vibration of a membrane with free boundary, and this is the reason why it is called free membrane problem. It is well-known that the operator $-\Delta$ in (1.8) only has the discrete spectrum and all the eigenvalues can be listed non-decreasingly as follows

$$0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \cdots \uparrow \infty. \tag{1.9}$$

The eigenvalue $\mu_0(\Omega) = 0$ has nonzero constant functions as its eigenvalues. Each eigenvalue $\mu_i(\Omega)$ in the sequence (1.9) is repeated according to its multiplicity (which is finite and actually equals the dimension of $\mu_i(\Omega)$'s eigenspace). By variational principle, the k-th nonzero Neumann eigenvalue $\mu_k(\Omega)$ is characterized as follows

$$\mu_k(\Omega) = \inf \left\{ \frac{\int_\Omega |\nabla f|^2 dv}{\int_\Omega f^2 dv} \middle| f \in W^{1,2}(\Omega), f \neq 0, \int_\Omega f f_i dv = 0 \right\},$$

where f_i, $i = 0, 1, \cdots, k-1$, denotes an eigenfunction of $\mu_i(\Omega)$. Here, as usual, $W^{1,2}(\Omega)$ is the Sobolev space, i.e. the completion of the set of smooth functions $C_0^\infty(\Omega)$ under the Sobolev norm $\|\cdot\|_{1,2}$ defined by (1.6).

Similar to (1.7), for bounded domains $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$, $n \geq 2$, the following extremum problem

$$\max \left\{ \mu_k(\Omega) \middle| |\Omega| = const. \right\} \tag{1.10}$$

can be asked for each $k = 1, 2, 3, \cdots$. To the best of our knowledge, for $k = 1, 2$, there exist affirmative answers to the problem (1.10) as follows:

- (Szegő-Weinberger inequality, [37, 38]) $\mu_1(\Omega) \leq \mu_1(B_\Omega)$, and the equality holds if and only if Ω is a ball in $\mathbb{R}^n$ (which is congruent with B_Ω, $|B_\Omega| = |\Omega| = const.$). That is to say, among all bounded domains in $\mathbb{R}^n$ having the same volume, Euclidean balls maximize the first nonzero Neumann eigenvalue of the Laplacian.

- (Bucur-Henrot [8]) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set such that the Sobolev space $W^{1,2}(\Omega)$ is compactly embedded[4] in $L^2(\Omega)$. Then

$$|\Omega|^{2/n}\mu_2(\Omega) \leq 2^{2/n}|B|^{2/n}\mu_1(B), \tag{1.11}$$

  where B is any ball in $\mathbb{R}^n$. If equality in (1.11) occurs, then Ω coincides a.e. with the union of two disjoint, equal balls. Clearly, the quantity $2^{2/n}|B|^{2/n}\mu_1(B)$ is scale invariant. Using (1.11) directly, one has $\mu_2(\Omega) \leq 2^{2/n}\mu_1(B_\Omega)$, with a ball B_Ω satisfying $|B_\Omega| = |\Omega| = const.$, which gives an affirmative answer to the problem (1.10) for $k = 2$.

For higher order ($k \geq 3$) Neumann eigenvalues, not much is known. However, recent years, some works have shown numerical approaches which propose candidates for the optimizers for Dirichlet/Neumann eigenvalues of the Laplacian and related spectral problems, and which also suggest conjectures about their qualitative properties – see, e.g., [1, 4, 33] for details.

As mentioned above, in some situation, the eigenvalue problems (1.4) and (1.8) have physical backgrounds, and hence eigenvalues in discrete spectrum are called frequencies. So, sometimes, spectral isoperimetric inequalities introduced above are also called *physical isoperimetric inequalities*. There is also one more thing we prefer to say here, that is, spectral isoperimetric inequalities mentioned above hold may not only in Euclidean spaces but also some curved spaces – for instance, at least one also has the Faber-Krahn inequality in hyperbolic spaces and spheres. In fact, a more general version of Faber-Krahn inequality says that (see, e.g., [11, Chapter IV]):

- Let $\mathbb{M}^n(\kappa)$ be the complete, simply connected, n-dimensional ($n \geq 2$) space form of constant sectional curvature κ, and let $\mathbb{D}$ denote a geodesic disk in $\mathbb{M}^n(\kappa)$. For a complete Riemannian n-manifold M^n, $n \geq 2$, and each open set Ω, consisting of a finite disjoint union of regular[5] domains in M^n, and satisfying

$$|\Omega| = |\mathbb{D}|. \tag{1.12}$$

  (If $\kappa > 0$, then only consider those Ω for which $|\Omega| < |\mathbb{M}^n(\kappa)|$.) If, for all such Ω in M^n, equality (1.12) implies the geometric isoperimetric inequality

$$|\partial\Omega| \geq |\partial\mathbb{D}|, \tag{1.13}$$

[4] In fact, the regularity that $\partial\Omega$ is Lipschitz continuous is sufficient such that $W^{1,2}(\Omega)$ is compactly embedded in $L^2(\Omega)$. Therefore, the smoothness assumption for the boundary $\partial\Omega$ is much enough to investigate the maximum of $\mu_2(\Omega)$ under the constraint of fixed volume.

[5] Here, following the convention in [11], "*regular*" means that the domain considered has compact closure and smooth boundary, while the word "*normal*" also in this statement means that the domain considered has compact closure and piecewise smooth boundary.

with equality in (1.13) if and only if Ω is isometric to $\mathbb{D}$, then we also have, for every normal domain Ω in M^n, that equality (1.12) implies the inequality

$$\lambda_1(\Omega) \geq \lambda_1(\mathbb{D}), \tag{1.14}$$

with equality in (1.14) if and only if Ω is isometric to $\mathbb{D}$.

This fact can be simply summarized as "*under the constraint of volume fixed, the geometric isoperimetric inequality (1.13) would imply the physical isoperimetric inequality (1.14)*". It is known that in space forms, (1.13) holds once $|\Omega| = |\mathbb{D}|$. Hence, in space forms, one has the physical isoperimetric inequality (1.14) under the volume constraint (1.12). From this example, one might have a recognition that geometric isoperimetric inequalities have a close relation with physical isoperimetric inequalities (of differential operators). A natural question is "*except space forms, whether one could find other spaces on which the geometric isoperimetric inequality (1.13) holds under the volume constraint (1.12)?*". One might refer to [11, Chapter IV] for some interesting progresses on this question.

In the sequel, we will show that how to extend the Faber-Krahn inequality, the Hong-Krahn-Szegő inequality and the Szegő-Weinberger inequality of the Laplacian to the case of the Witten-Laplacian.

For a given complete Riemannian n-manifold ($n \geq 2$) with the metric g, let $\Omega \subset M^n$ be a bounded domain (with boundary $\partial\Omega$) in M^n, and $\phi \in C^\infty(M^n)$ be a smooth[6] real-valued function defined on Ω. In this setting, one can define the following elliptic operator

$$\Delta_\phi := \Delta - \langle \nabla\phi, \nabla\cdot \rangle$$

on Ω, which is called the *Witten-Laplacian*[7] (also called the *drifting Laplacian* or the *weighted Laplacian*) w.r.t. the metric g. Consider the Dirichlet eigenvalue problem of the Witten-Laplacian as follows

$$\begin{cases} \Delta_\phi u + \lambda u = 0 & \text{in } \Omega \subset M^n, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.15}$$

and it is not hard to check that the operator Δ_ϕ in (1.15) is **self-adjoint** w.r.t. the following inner product

$$\widetilde{(h_1, h_2)} := \int_\Omega h_1 h_2 d\eta = \int_\Omega h_1 h_2 e^{-\phi} dv, \tag{1.16}$$

with $h_1, h_2 \in \widetilde{W}_0^{1,2}(\Omega)$, where $\widetilde{W}_0^{1,2}(\Omega)$ is the Sobolev space with compact support w.r.t. the weighted measure $d\eta := e^{-\phi}dv$, i.e. the completion of the set of smooth functions (with compact support) $C_0^\infty(\Omega)$ under the following Sobolev norm

$$\widetilde{\|f\|}_{1,2} := \left(\int_\Omega f^2 d\eta + \int_\Omega |\nabla f|^2 d\eta \right)^{1/2}. \tag{1.17}$$

[6] In fact, one might see that $\phi \in C^2(\Omega)$ is suitable to derive our main conclusions in this paper. However, in order to avoid a little bit boring discussion on the regularity of ϕ and following the assumption on conformal factor $e^{-\phi}$ for the notion of *smooth metric measure spaces* in many literatures (including of course those cited in this paper), without specification, we prefer to assume that ϕ is smooth on the domain Ω.

[7] It is interesting and important to study geometric problems related to the Witten-Laplacian – we refer to [12, Introduction] for a detailed explanation. Except the paper here, the corresponding author, Prof. J. Mao, also has some other interesting works related to the Witten-Laplacian – see, e.g., [15, 25, 28, 29, 31, 39].

Then using similar arguments to those of the classical fixed membrane problem of the Laplacian (i.e., the discussions about the existence of discrete spectrum, Rayleigh's theorem, Max-min theorem, etc. These discussions are standard, and for details, please see for instance [11]), it is not hard to know:

- The self-adjoint elliptic operator $-\Delta_\phi$ in (1.15) *only* has discrete spectrum, and all the eigenvalues in this discrete spectrum can be listed non-decreasingly as follows

$$0 < \lambda_{1,\phi}(\Omega) < \lambda_{2,\phi}(\Omega) \leq \lambda_{3,\phi}(\Omega) \leq \cdots \uparrow +\infty. \tag{1.18}$$

  Each eigenvalue $\lambda_{i,\phi}$, $i = 1, 2, \cdots$, in the sequence (1.18) according to its multiplicity (which is finite and equals to the dimension of the eigenspace of $\lambda_{i,\phi}$). By applying the standard variational principles, one can obtain that the k-th Dirichlet eigenvalue $\lambda_{k,\phi}(\Omega)$ can be characterized as follows

$$\lambda_{k,\phi}(\Omega) = \inf\left\{\frac{\int_\Omega |\nabla f|^2 e^{-\phi} dv}{\int_\Omega f^2 e^{-\phi} dv}\middle| f \in \widetilde{W}_0^{1,2}(\Omega), f \neq 0, \int_\Omega f f_i e^{-\phi} dv = 0\right\}, \tag{1.19}$$

  where f_i, $i = 1, 2, \cdots, k-1$, denotes an eigenfunction of $\mu_{i,\phi}(\Omega)$. Moreover, the first Dirichlet eigenvalue $\lambda_{1,\phi}(\Omega)$ of the eigenvalue problem (1.15) satisfies

$$\lambda_{1,\phi}(\Omega) = \inf\left\{\frac{\int_\Omega |\nabla f|^2 d\eta}{\int_\Omega f^2 d\eta}\middle| f \in \widetilde{W}^{1,2}(\Omega), f \neq 0\right\}. \tag{1.20}$$

On Ω, one can also define a notion *weighted volume* (or ϕ-volume) as follows:

$$|\Omega|_\phi := \int_\Omega d\eta = \int_\Omega e^{-\phi} dv.$$

Using the constraint of fixed weighted volume, we can obtain several spectral isoperimetric inequalities for the first and the second Dirichlet eigenvalues of the Witten-Laplacian. However, in order to state our conclusions clearly, we need to impose an assumption on the function ϕ as follows:

- (**Property 1**) Furthermore, ϕ is a function of the Riemannian distance parameter $t := d(o, \cdot)$ for some point $o \in M^n$.

Clearly, if a given open Riemannian n-manifold (M^n, g) was endowed with the weighted density $e^{-\phi} dv$ with ϕ satisfying **Property 1**, then ϕ would be a ***radial*** function defined on M^n w.r.t. the radial distance t, $t \in [0, \infty)$. Especially, when the given open n-manifold is chosen to be $\mathbb{R}^n$ or $\mathbb{H}^n$ (i.e., the n-dimensional hyperbolic space of sectional curvature -1), we additionally require that o is the origin of $\mathbb{R}^n$ or $\mathbb{H}^n$.

First, we have the following Faber-Krahn type inequality for the Witten-Laplcian in the Euclidean space.

Theorem 1.1. *Assume that the function ϕ satisfies **Property 1** (with M^n chosen to be $\mathbb{R}^n$) and is concave. Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^n$, and let $B_R(o)$*

be a ball of radius R and centered at the origin o of $\mathbb{R}^n$ such that $|\Omega|_\phi = |B_R(o)|_\phi$, i.e. $\int_\Omega d\eta = \int_{B_R(o)} d\eta$. Then

$$\lambda_{1,\phi}(\Omega) \geq \lambda_{1,\phi}(B_R(o)),$$

and the equality holds if and only if (up to measure zero) Ω is the ball $B_R(o)$, which lies entirely in the region $B_{\mathcal{R}(h)}$ defined by (1.21).

Remark 1.2. (1) Unlike the Neumann case described in Theorems 1.11 and 1.12 below, for the Dirichlet case we do not need to require that the point o locates in the convex hull of the domain Ω in Theorem 1.1. The same situation also happens in Theorem 1.3,
(2) From the introduction on the Faber-Krahn inequality of the Laplacian, one knows that under the volume constraint (1.12), the geometric isoperimetric inequality (1.13) makes an important role in the derivation process. What about the Witten-Laplacian case? Does some weighted geometric isoperimetric inequality play an important role also? The answer is affirmative. We would like to recall a recent breakthrough of Chambers [10] to the Log-Convex Density Conjecture. Given a positive function h in $\mathbb{R}^n$, $n \geq 2$, one can define the weighted perimeter and weighted volume of a set $A \subset \mathbb{R}^n$ of locally finite perimeter as

$$\mathrm{Per}(A) = \int_{\partial A} h d\mathcal{H}^{n-1}, \qquad \mathrm{Vol}(A) = \int_A h d\mathcal{H}^n,$$

where following the usage of notations in [10], $\mathcal{H}^m$ indicates the m-dimensional Hausdorff measure, and ∂A denotes the essential boundary of A. Such positive function h is called a *density* on $\mathbb{R}^n$. If one fixes a positive weighted volume $m > 0$, does there exist a set $A \subset \mathbb{R}^n$ such that $\mathrm{Vol}(A) = m$ and

$$\mathrm{Per}(A) = \inf_{Q \subset \mathbb{R}^n, \mathrm{Vol}(Q)=m} \mathrm{Per}(Q)?$$

Rosales, Cañete, Bayle and Morgan considered this problem and gave a partial answer that in $\mathbb{R}^n$ with the density $e^{c|x|^2}$, $c > 0$, round balls about the origin uniquely minimize perimeter for given volume (see [35, Theorem 5.2]). Moreover, they showed that for any radial, smooth density $h = e^{f(|x|)}$, balls around the origin are stable[8] if and only f is convex ([35, Theorem 3.10]). This fact motivates the following conjecture (3.12 in their article), first stated by Kenneth Brakke:

- (Log-Convex Density Conjecture) *In $\mathbb{R}^n$ with a smooth, radial, log-convex[9] density, balls around the origin provide isoperimetric regions of any given volume.*

Chambers [10, Theorem 1.1] gave an answer to the above conjecture as follows:

- (**Fact A**) *Given a density $h(x) = e^{f(|x|)}$ on $\mathbb{R}^n$ with f smooth, convex and even, balls around the origin are isoperimetric regions with respect to weighted perimeter and volume.*

[8] Here "*stable*" means that $\mathrm{Per}''(0) \geq 0$ under smooth, volume-conserving variations.
[9] Clearly, for a density h here, the log-convex assumption means $(\log h)'' \geq 0$.

Moreover, Chambers [10, Theorem 1.2] characterized the uniqueness of isoperimetric regions as follows:

- (**Fact B**) *Up to sets of measure* 0*, the only isoperimetric regions are balls centered at the origin, and balls that lie entirely in*

$$B_{\mathcal{R}(h)} = \{x \big| |x| \leq \mathcal{R}(h)\}, \tag{1.21}$$

where $\mathcal{R}(h) = \sup\{|x| \big| h(x) = h(0)\}$.

Fact A and **Fact B** would make an important role in the proof of Theorem 1.1 – see Subsection 2.1 for details.
(3) Since Chambers' weighted geometric isoperimetric inequality in $\mathbb{R}^n$ (i.e. **Fact A**) makes an important role in the proof of Theorem 1.1, which implies that similar to the potential precondition of [10, Theorem 1.1], we also need to require that the boundary $\partial\Omega$ has finite area (or following the convention in [10], "*perimeter*") here. However, we think this setting is so natural when considering the isoperimetric problems, we prefer not to list it out individually in every statement of our main conclusions in this paper. But, of course, $\partial\Omega$ should have this natural setting throughout the paper, which we do not mention again anymore.

We can prove the following:

Theorem 1.3. *Let* $\mathbb{S}^n_+$ *be an* n*-dimensional hemisphere of radius* 1*, and let* $\Omega \subset \mathbb{S}^n_+$ *be a bounded domain whose boundary* $\partial\Omega$ *has positive constant mean curvature. Assume that the function* ϕ *satisfies* ***Property 1*** *(with* M^n *chosen to be* $\mathbb{S}^n_+$*) and moreover* $\phi = -\log\cos t$*, where the point* o *mentioned in* ***Property 1*** *should additionally be required to be the base point of* $\mathbb{S}^n_+$*. Then*

$$\lambda_{1,\phi}(\Omega) \geq \lambda_{1,\phi}(B_R(o)),$$

where $B_R(o)$ *denotes a geodesic ball of radius* R *and centered at the base point* o *of* $\mathbb{S}^n_+$ *such that* $|\Omega|_\phi = |B_R(o)|_\phi$*. The equality holds if and only if* Ω *is isometric to the geodesic ball* $B_R(o)$*.*

Remark 1.4. (1) When investigating the above Faber-Krahn type isoperimetric inequality, there is no essential difference between $\mathbb{S}^n_+$ and a hemisphere with radius not equal to 1.
(2) In order to let readers who might not know the concept "*the base point*" clearly, we prefer to give an explanation here. It is better to start the explanation with spherically symmetric manifolds, which is also called generalized space forms (suggested in the work of Katz-Kondo [21]). We refer readers to [17, 26, 32] for a detailed description about the accurate definition, the basic properties and some interesting applications of spherically symmetric manifolds. The corresponding author has used spherically symmetric manifold as the model space to derive some interesting comparison theorems (for volume, eigenvalues of different types, heat kernel, and some other geometric quantities) – see, e.g., [17, 27, 30, 39]. In fact, one has:

- ([17, Definition 2.1]) For a given complete n-manifold M^n, a domain $\mathcal{D} = \exp_p([0,l) \times S^{n-1}_p) \subset M^n \setminus Cut(p)$, with $l < inj(p)$, is said to be spherically symmetric with respect to a point $p \in \mathcal{D}$, if and only if the matrix $\mathbb{A}(t,\xi)$ satisfies $\mathbb{A}(t,\xi) = f(t)I$, for a function $f \in C^2([0,l))$ with $f(0) = 0$, $f'(0) = 1$ and $f|_{(0,l)} > 0$.

Here S_p^{n-1} denotes the unit sphere of the tangent space T_pM^n, $Cut(p)$ stands for the cut-locus of the point p, $inj(p)$ denotes the injectivity radius at p, $\xi \in S_p^{n-1}$, and $\mathbb{A}(t,\xi) : \xi^\perp \to \xi^\perp$ is the path of linear transformations well-defined in [17, Section 2]. A standard model for spherically symmetric manifolds is given by the quotient of the warped product $[0,l) \times_f \mathbb{S}^{n-1}$ with the metric

$$ds^2 = dt^2 + f^2(t)|d\xi|^2, \qquad \forall p \in S_p^{n-1},\ 0 < t < l,$$

where usually $|d\xi|^2$ denotes the round metric of the unit $(n-1)$-sphere $\mathbb{S}^{n-1}$. In this model, all pairs $(0,\xi)$ are identified with the single point p, which is called *the base point* of the spherically symmetric domain $\mathcal{D} = [0,l) \times_f \mathbb{S}^{n-1}$. Clearly, as revealed already in (2.12) of [17], a space form with constant sectional curvature κ is also a spherically symmetric manifold and in this particular situation the warping function f satisfies

$$f(t) = \begin{cases} \frac{\sin(\sqrt{\kappa}t)}{\sqrt{\kappa}}, & l = \frac{\pi}{\sqrt{\kappa}} \quad \kappa > 0, \\ t, & l = +\infty \quad \kappa = 0, \\ \frac{\sinh(\sqrt{-\kappa}t)}{\sqrt{-\kappa}}, & l = +\infty \quad \kappa < 0. \end{cases}$$

(3) Since o is required to be the base point of $\mathbb{S}^n_+$, then for the domain $\Omega \subset \mathbb{S}^n_+$ in Theorem 1.3, the range of the Riemannian distance parameter $t = d(o,\cdot)$ should be $(0,\pi/2)$, which implies that the choice of the function $\phi = -\log\cos t$ makes sense. Besides, in fact, $\mathbb{S}^n_+$ can be modeled as $[0,\pi/2] \times_{\sin t} \mathbb{S}^{n-1}$ with the metric $dt^2 + (\sin t)^2|d\xi|^2$, and its base point o should be the vertex of $\mathbb{S}^n_+$.

We can also get the following:

Theorem 1.5. *Assume that the function ϕ satisfies **Property 1** (with M^n chosen to be $\mathbb{H}^n$) and is strictly concave, where the point o mentioned in **Property 1** should additionally be required to be the origin of $\mathbb{H}^n$. Let $\Omega \subset \mathbb{H}^n$ be a bounded domain with boundary. Then*

$$\lambda_{1,\phi}(\Omega) \geq \lambda_{1,\phi}(B_R(o)),$$

where $B_R(o)$ denotes a geodesic ball of radius R and centered at the origin o of $\mathbb{H}^n$ such that $|\Omega|_\phi = |B_R(o)|_\phi$. The equality holds if and only if Ω is isometric to the geodesic ball $B_R(o)$.

Remark 1.6. (1) The hyperbolic space $\mathbb{H}^n$ can be modeled as $[0,\infty) \times_{\sinh t} \mathbb{S}^{n-1}$ with the metric

$$dt^2 + (\sinh t)^2|d\xi|^2.$$

Since hyperbolic spaces are two-point homogenous, the base point of $\mathbb{H}^n$ is not unique and any point of $\mathbb{H}^n$ can be chosen as the base point, which is different with the case of hemisphere $\mathbb{S}^n_+$. However, for $\mathbb{H}^n$ once its globally defined coordinate system was set up, the origin o would be determined uniquely w.r.t. this system. As shown above, in order to get the main conclusion in Theorem 1.5, we need to assume that ϕ is radial w.r.t. some fixed point and is also concave, which leads to the situation that in the statement of Theorem 1.5, it is better to choose the point o to be the origin of $\mathbb{H}^n$ (not the base point), and correspondingly ϕ is

concave w.r.t. the radial Riemannian distance parameter $t = d(o, \cdot)$.
(2) As mentioned before, one knows two facts: (a) under the constraint of fixed volume, the Faber-Krahn inequality for the first Dirichlet eigenvalue of the Laplacian also holds in hyperbolic spaces; (b) under the constraint of fixed weighted volume, **Fact A** (i.e., a weighted geometric isoperimetric inequality in $\mathbb{R}^n$) makes an important role in the proof of the Faber-Krahn type inequality for the Witten-Laplcian in $\mathbb{R}^n$ (i.e. Theorem 1.1). So, it is natural to ask:

- *Could one expect to get a hyperbolic version of **Fact A** which makes a contribution in the proof of Theorem 1.5?*

The answer is affirmative. In fact, Li-Xu [24, Theorem 1.1] obtained a partial result to the hyperbolic version of **Fact A** for specified density through suitably applying Chambers' result [10] by projecting the hyperbolic space onto $\mathbb{R}^n$ and employing a comparison argument. Very recently, L. Silini [36] solved the above question completely. For an arbitrary base point $o \in \mathbb{H}^n$, and a density h given by $h := e^{f(d(o,\cdot))}$, where $h : \mathbb{R} \to \mathbb{R}$ is a smooth, (strictly) convex, even function, and, similar as before, $d(o, \cdot)$ denotes the Riemannian distance to the point o on $\mathbb{H}^n$, one can define the weighted perimeter and volume of a set with finite perimeter $E \subset \mathbb{H}^n$ as follows

$$\mathrm{P}_h(E) = \int_{\partial^* E} h d\mathcal{H}^{n-1}, \qquad \mathrm{V}_h(E) = \int_E h d\mathcal{H}^n,$$

where following the usage of notations in [36], $\partial^* E$ denotes the reduced boundary of E, and $\mathcal{H}^m$ indicates the m-dimensional Hausdorff measure. Silini [36, Theorem 1.1] proved the following:

- (**Fact C**) *For any strictly radially log-convex density h, geodesic balls centered at $o \in \mathbb{H}^n$ uniquely minimize the weighted perimeter for any given weighted volume with respect to P_h and V_h.*

Fact C would make an important role in the proof of Theorem 1.5 – see Section 3 for details. Using a comparison argument between $H^n_{\mathbb{C}} = U(n,1)/U(n)$ (i.e. the n-dimensional complex hyperbolic space of constant curvature -1) and $\mathbb{H}^{2n}$, together with **Fact C**, Silini [36] can get further:

- In $H^n_{\mathbb{C}}$, geodesic balls are uniquely isoperimetric in the class of Hopf-symmetric sets for all volumes.

This conclusion gives a partial answer to an open conjecture proposed by Gromov-Ros in [18] as follows:

- (**Conjecture**) *Geodesic balls are isoperimetric for all volumes in the complex hyperbolic space $H^n_{\mathbb{C}}$.*

Silini's above result on the isoperimetric problem for the class of Hopf-symmetric sets in $H^n_{\mathbb{C}}$ might inspire readers to try to extend the spectral isoperimetric inequality in Theorem 1.5 to a more general space, which we think it is possible. However, due to the structure of this paper, here we just focus on investigating spectral isoperimetric inequalities for the

Witten-Laplacian on bounded domains in space forms.
(3) As explained in [36, Remark 1.7], since technical difficulties arising from the presence of regions with constant weight, for simplicity it was decided to to assume the weight to be strictly log-convex rather than simply log-convex in extending the proof of Brakke's conjecture from the Euclidean space to the hyperbolic space. This is the reason why in Theorem 1.5 we assume that the radial function ϕ is strictly concave (i.e., $-(\log\phi)'' > 0$). Besides, if the domain Ω has a constant weight (i.e., a constant density), then the Witten-Laplacian degenerates into the classical Laplacian, and correspondingly, in $\mathbb{H}^n$ one naturally has the Faber-Krahn inequality for the first Dirichlet eigenvalue. In this situation, it is no need to write down Theorem 1.5 any more. Based on this truth, in Theorem 1.5 it is acceptable to assume that the radial function ϕ is strictly concave.

Inspired by the technique used in [3], under other assumptions on ϕ and the constraint of weighted volume fixed, we can also get the following Faber-Krahn type inequality for the Witten-Laplcian in the Euclidean space, which can be seen as a complement to Theorem 1.1.

Theorem 1.7. *Assume that the function ϕ satisfies **Property 1** (with M^n chosen to be $\mathbb{R}^n$), ϕ is monotone non-increasing, and for $z \geq 0$, the function*

$$\left(e^{-\phi(z^{\frac{1}{n}})} - e^{-\phi(0)}\right) z^{1-\frac{1}{n}}$$

is convex. Let Ω be a bounded domain with Lipschitz boundary in $\mathbb{R}^n$, and let $B_R(o)$ be a ball of radius R and centered at the origin o of $\mathbb{R}^n$ such that $|\Omega|_\phi = |B_R(o)|_\phi$. Then

$$\lambda_{1,\phi}(\Omega) \geq \lambda_{1,\phi}(B_R(o)).$$

Remark 1.8. Since ϕ satisfies **Property 1** and moreover when M^n is chosen to be $\mathbb{R}^n$, we additionally require that o is the origin of $\mathbb{R}^n$, so o corresponds to $z = 0$, and then $\phi(0)$ is actually the value of the function ϕ at the origin o.

For the second Dirichlet eigenvalue of the Witten-Laplacian, we can obtain the following Hong-Krahn-Szegő type inequalities.

Theorem 1.9. *Assume that the function ϕ satisfies **Property 1** (with M^n chosen to be $\mathbb{R}^n$) and is concave. Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^n$, and let $B_{\widetilde{R}}(o)$ be a ball of radius $\widetilde{R}$ and centered at the origin o of $\mathbb{R}^n$ such that $|\Omega|_\phi/2 = |B_{\widetilde{R}}(o)|_\phi$, i.e. $\frac{1}{2}\int_\Omega d\eta = \int_{B_{\widetilde{R}}(o)} d\eta$. Then*

$$\lambda_{2,\phi}(\Omega) \geq \lambda_{1,\phi}(B_{\widetilde{R}}(o)).$$

That is to say, under the assumptions for ϕ described above, the minimum of the second Dirichlet eigenvalue of the Witten-Laplacian on bounded domains Ω in $\mathbb{R}^n$ (whose weighted volume equals some prescribed positive constant) should be equal to the first Dirichlet eigenvalue of the Witten-Laplacian on a ball $B_{\widetilde{R}}(o)$ (of radius $\widetilde{R}$ and centered at the origin $o \in \mathbb{R}^n$) such that $|\Omega|_\phi/2 = |B_{\widetilde{R}}(o)|_\phi$.

Theorem 1.10. *Assume that the function ϕ satisfies **Property 1** (with M^n chosen to be $\mathbb{H}^n$) and is strictly concave, where the point o mentioned in **Property 1** should additionally be required to the origin of $\mathbb{H}^n$. Let $\Omega \subset \mathbb{H}^n$ be a bounded domain with boundary. Then*

$$\lambda_{2,\phi}(\Omega) \geq \lambda_{1,\phi}(B_{\widetilde{R}}(o)),$$

where $B_{\widetilde{R}}(o)$ denotes a geodesic ball of radius $\widetilde{R}$ and centered at the origin o of $\mathbb{H}^n$ such that $|\Omega|_\phi/2 = |B_{\widetilde{R}}(o)|_\phi$. That is to say, under the assumptions for ϕ described above, the minimum of the second Dirichlet eigenvalue of the Witten-Laplacian on bounded domains Ω in $\mathbb{H}^n$ (whose weighted volume equals some prescribed positive constant) should be equal to the first Dirichlet eigenvalue of the Witten-Laplacian on a geodesic ball $B_{\widetilde{R}}(o)$ (of radius $\widetilde{R}$ and centered at the origin $o \in \mathbb{H}^n$) such that $|\Omega|_\phi/2 = |B_{\widetilde{R}}(o)|_\phi$.

For a bounded domain Ω (with boundary $\partial\Omega$) on a given n-dimensional ($n \geq 2$) complete Riemannian manifold M^n, we can also consider the following Neumann eigenvalue problem of the Witten-Laplacian

$$\begin{cases} \Delta_\phi u + \mu u = 0 & \text{in } \Omega \subset M^n, \\ \frac{\partial u}{\partial \vec{\nu}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.22}$$

and it is easy to check that the operator Δ_ϕ in (1.22) is **self-adjoint** w.r.t. the inner product (1.16) with $h_1, h_2 \in \widetilde{W}^{1,2}(\Omega)$, where $\widetilde{W}^{1,2}(\Omega)$ is the Sobolev space w.r.t. the weighted measure $d\eta$, i.e. the completion of the set of smooth functions $C^\infty(\Omega)$ under the Sobolev norm $\widetilde{\|\cdot\|}$ defined by (1.17). Then using similar arguments to those of the classical free membrane problem of the Laplacian (see, e.g., [11]), it is not hard to know:

- The operator $-\Delta_\phi$ in (1.22) *only* has discrete spectrum, and all the eigenvalues in this discrete spectrum can be listed non-decreasingly as follows

$$0 = \mu_{0,\phi}(\Omega) < \mu_{1,\phi}(\Omega) \leq \mu_{2,\phi}(\Omega) \leq \mu_{3,\phi}(\Omega) \leq \cdots \uparrow +\infty. \tag{1.23}$$

  Each eigenvalue $\mu_{i,\phi}$, $i = 0, 1, 2, \cdots$, in the sequence (1.23) according to its multiplicity (i.e., the dimension of the eigenspace of $\mu_{i,\phi}$). Specially, the zero eigenvalue $\mu_{0,\phi}$ has multiplicity 1 and has nonzero constant function as its eigenfunction. By applying the standard variational principles, one can obtain that the k-th Dirichlet eigenvalue $\mu_{k,\phi}(\Omega)$ can be characterized as follows

$$\mu_{k,\phi}(\Omega) = \inf\left\{ \frac{\int_\Omega |\nabla f|^2 e^{-\phi} dv}{\int_\Omega f^2 e^{-\phi} dv} \middle| f \in \widetilde{W}^{1,2}(\Omega), f \neq 0, \int_\Omega f f_i e^{-\phi} dv = 0 \right\}, \tag{1.24}$$

  where f_i, $i = 1, 2, \cdots, k-1$, denotes an eigenfunction of $\mu_{i,\phi}(\Omega)$. Moreover, the first nonzero Neumann eigenvalue $\mu_{1,\phi}(\Omega)$ of the eigenvalue problem (1.22) satisfies

$$\mu_{1,\phi}(\Omega) = \inf\left\{ \frac{\int_\Omega |\nabla f|^2 d\eta}{\int_\Omega f^2 d\eta} \middle| f \in \widetilde{W}^{1,2}(\Omega), f \neq 0, \int_\Omega f d\eta = 0 \right\}. \tag{1.25}$$

In fact, the above facts have been explained more clearly in [12, Section 1]. Here we prefer to keep writing down the above content for two reasons: the one is for the completion of the brief introduction to the eigenvalue problem (1.22) here; the other one is that the characterization (1.25) would be used to derive spectral isoperimetric inequalities for the first nonzero Neumann eigenvalue $\mu_{1,\phi}(\cdot)$ below.

We can prove the following Szegő-Weinberger type inequalities for the Witten-Laplacian.

Theorem 1.11. *Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^n$. Assume that the function ϕ satisfies **Property I** (with M^n chosen to be $\mathbb{R}^n$ and additionally the point o required to be in the convex hull of Ω, i.e. $o \in \mathrm{hull}(\Omega)$), and ϕ is also a non-increasing convex function defined on $[0,\infty)$. Let $B_R(o)$ be a ball of radius R and centered at the origin o of $\mathbb{R}^n$ such that $|\Omega|_\phi = |B_R(o)|_\phi$, i.e. $\int_\Omega d\eta = \int_{B_R(o)} d\eta$. Then*

$$\mu_{1,\phi}(\Omega) \leq \mu_{1,\phi}(B_R(o)),$$

with equality holding if and only if Ω is the ball $B_R(o)$.

Theorem 1.12. *Let Ω be a bounded domain with smooth boundary in $\mathbb{H}^n$. Assume that the function ϕ satisfies **Property I** (with M^n chosen to be $\mathbb{H}^n$ and additionally $o \in \mathrm{hull}(\Omega)$), and ϕ is also a non-increasing convex function defined on $[0,\infty)$. Let $B_R(o)$ be a geodesic ball of radius R and centered at the origin o of $\mathbb{H}^n$ such that $|\Omega|_\phi = |B_R(o)|_\phi$. Then*

$$\mu_{1,\phi}(\Omega) \leq \mu_{1,\phi}(B_R(o)),$$

with equality holding if and only if Ω is isometric to the geodesic ball $B_R(o)$.

Remark 1.13. (1) In fact, in our very recent work [12, Theorems 1.1 and 1.5], we can prove an isoperimetric inequality for the sums of the reciprocals of the first $(n-1)$ nonzero Neumann eigenvalues of the Witten-Laplacian on bounded domains in $\mathbb{R}^n$ or $\mathbb{H}^n$, which together with the monotonicity of Neumann eigenvalues (i.e. the sequence (1.23)) yields directly our Theorem 1.11 and Theorem 1.12 here. This fact has been already pointed out in [12, Corollaries 1.2 and 1.6], and readers can check there for details.
(2) Based on two reasons, we keep writing down Theorem 1.11 and Theorem 1.12 here. The one is for the completion of the whole structure of this paper, and the other one is that our approach here for proving Theorem 1.11 and Theorem 1.12 is somehow different from the one used in [12].
(3) Different with the Dirichlet case, we need to require that $o \in \mathrm{hull}(\Omega)$ in Theorem 1.11 and Theorem 1.12. This is because we have to use the Brouwer fixed point theorem to make sure the existence of an orthonormal frame field such that the origin of the coordinate system (corresponding to the orthonormal frame field) locates in the convex hull of Ω, and then all the computations involved trail functions constructed are valid. See the proofs of Theorem 1.11 and Theorem 1.12 in Section 3 for details.

The paper is organized as follows. The proofs of the Faber-Krahn type inequalities, the Hong-Krahn-Szegő type inequalities and the Szegő-Weinberger type inequalities for the Witten-Laplcian will be given in Sections 2, 3 and 4 respectively. In Section 5, we will give the detailed information about the first nonzero Neumann eigenvalue and its eigenfunctions of the Witten-Laplacian on prescribed (geodesic) balls in space forms.

2 The Faber-Krahn type inequalities for the Witten-Laplcian

2.1 The Euclidean case

Assume that f is an eigenfunction corresponding to the first Dirichlet eigenvalue $\lambda_{1,\phi}(\Omega)$. Since f does not change sign on Ω, without loss of generality, we can assume $f > 0$ on Ω (see Lemma 3.1 below for the explanation). Consider the sets $\Omega_s := \{x \in \Omega | f(x) > s\}$, and let Ω_s^* be balls in $\mathbb{R}^n$ with center at the origin o and satisfying $|\Omega_s|_\phi = |\Omega_s^*|_\phi$. Let $B_R(o)$ be a ball of radius R and centered at o of $\mathbb{R}^n$ such that $|\Omega|_\phi = |B_R(o)|_\phi$, i.e. $\int_\Omega d\eta = \int_{B_R(o)} d\eta$. Define a function f^* on $B_R(o)$ having the following properties:

- f^* is a radial decreasing function;
- f^* takes the value s on the boundary sphere $\partial\Omega_s^*$ of the ball Ω_s^* (for a fixed s).

It is not hard to see that $\Omega_0 = \Omega$ and correspondingly $\Omega_0^* = B_R(o)$. The existence of the balls Ω_s^* can be assured by using the Schwarz symmetrization. Readers can check e.g. [3, 19] for details on how to use symmetrization to get balls Ω_s^* under the constraint of having the same weighted volume.

Now, we make an agreement on the notations used right below. Denote by $\widehat{dv}$ the $(n-1)$-dimensional Hausdorff measure of the boundary associated to the Riemannian volume element[10] dv, and this convention will be used throughout the paper. Similarly, $\widehat{d\eta} = e^{-\phi}\widehat{dv}$ would be the weighted volume element of the boundary. Besides, for convenience, set $G(s) := \partial\Omega_s$, $S_{t(s)} := (G(s))^* = G^*(s) = \partial\Omega_s^*$ which denotes the sphere with center at the origin and radius $t(s)$. The following formula is known as the co-area formula (see, e.g., [6, 11]):

- For any continuous function h defined on Ω, one has

$$\int_\Omega h dv = \int_0^{\sup f} \int_{G(s)} h|\nabla f|^{-1}\widehat{dv_s} ds, \tag{2.1}$$

  where following the above agreement $\widehat{dv_s}$ denotes the volume element of the hypersurface $G(s) = f^{-1}(s)$.

Clearly, taking $h = |\nabla f|^2$ and then applying the co-area formula, one has

$$\int_\Omega |\nabla f|^2 dv = \int_0^{\sup f} \int_{G(s)} |\nabla f|\widehat{dv_s} ds.$$

Denote by the Schwarz symmetric rearrangement mapping $t : [0, \sup f] \to [0, R]$, with R the radius of the $B_R(o)$, and ψ the inverse transformation of t, where t additionally satisfies $t(0) = R$, $t(\sup f) = 0$.

[10] In fact, for domains Ω_s and $\Omega = \Omega_0$, they should have the same volume element dv. However, in order to emphasize that the domain Ω_s depends on s, we prefer to additionally write the volume element of Ω_s as dv_s (except $s = 0$).

Lemma 2.1. *If Ω is a bounded region in $\mathbb{R}^n$, and ϕ satisfies* ***Property 1*** *(with M^n chosen to be $\mathbb{R}^n$), then*

$$\int_\Omega f^2 d\eta = \int_{B_R(o)} (f^*)^2 d\eta, \tag{2.2}$$

where $B_R(o) \subset \mathbb{R}^n$ is the ball defined as in Theorem 1.1.

Proof. By a direct calculation, one can obtain

$$\begin{aligned}
\int_{B_R(o)} (f^*)^2 d\eta &= \int_0^R \int_{\partial B_t(o)} (f^*)^2 e^{-\phi(t)} \widehat{dv_t} dt \\
&= \int_0^R \psi^2(t) \int_{\partial B_t(o)} e^{-\phi(t)} \widehat{dv_t} dt \\
&= -\int_0^{\sup f} \psi^2(t(s)) t'(s) \left(\int_{\partial B_{t(s)}(o)} e^{-\phi(t(s))} \widehat{dv_t} \right) ds \\
&= -\int_0^{\sup f} s^2 \left(-\int_{G(s)} |\nabla f|^{-1} e^{-\phi|_{G(s)}} \widehat{dv_s} \right) ds \\
&= \int_\Omega f^2 d\eta,
\end{aligned}$$

which implies (2.2) directly. □

Now, together with **Fact A** and **Fact B**, we can get:

Proof of Theorem 1.1. Applying the co-area formula, we have

$$\int_\Omega |\nabla f|^2 e^{-\phi} dv = \int_0^{\sup f} \int_{G(s)} |\nabla f| e^{-\phi} \widehat{dv_s} ds. \tag{2.3}$$

We can obtain by using the Cauchy-Schwarz inequality that

$$\int_{G(s)} |\nabla f| e^{-\phi|_{G(s)}} \widehat{dv_s} \geq \frac{\left(\int_{G(s)} e^{-\phi|_{G(s)}} \widehat{dv_s} \right)^2}{\int_{G(s)} |\nabla f|^{-1} e^{-\phi|_{G(s)}} \widehat{dv_s}}. \tag{2.4}$$

By **Fact A** and **Fact B**, we have $\int_{G(s)} e^{-\phi|_{G(s)}} \widehat{dv_s} \geq \int_{G^*(s)} e^{-\phi(t(s))} \widehat{dv_s}$, with equality holding if and only if $G(s) \setminus E(s) = G^*(s)$, where the set $E(s)$ denotes a set of measure zero. Substituting this fact into (2.4) yields

$$\int_{G(s)} |\nabla f| e^{-\phi|_{G(s)}} \widehat{dv_s} \geq \frac{\left(\int_{G^*(s)} e^{-\phi(t(s))} \widehat{dv_s} \right)^2}{\int_{G(s)} |\nabla f|^{-1} e^{-\phi|_{G(s)}} \widehat{dv_s}}. \tag{2.5}$$

On the other hand, one has

$$\int_{G^*(s)} |\nabla f^*| e^{-\phi(t(s))} \widehat{dv_s} = \frac{\left(\int_{G^*(s)} e^{-\phi(t(s))} \widehat{dv_s} \right)^2}{\int_{G^*(s)} |\nabla f^*|^{-1} e^{-\phi(t(s))} \widehat{dv_s}},$$

since $|\nabla f^*|$ and $e^{-\phi(s)}$ are constant on the sphere $G^*(s)$. We notice that

$$|\Omega_r|_\phi = \int_{\Omega_r} e^{-\phi} dv = \int_r^{\sup f} \int_{G(s)} |\nabla f|^{-1} e^{-\phi|_{G(s)}} \widehat{dv_s} ds,$$

and so it follows that

$$(|\Omega_r|_\phi)'(s) = -\int_{G(s)} |\nabla f|^{-1} e^{-\phi|_{G(s)}} \widehat{dv_s},$$

which implies

$$-\int_{G(s)} |\nabla f|^{-1} e^{-\phi|_{G(s)}} \widehat{dv_s} = \frac{d}{ds}|\Omega_s|_\phi = \frac{d}{ds}|\Omega_s^*|_\phi. \tag{2.6}$$

Since

$$|\Omega_s^*|_\phi = \int_0^{t(s)} \int_{\partial B_z(o)} e^{-\phi(z)} \widehat{dv_z} dz,$$

one has

$$\frac{d}{ds}|\Omega_s^*|_\phi = t'(s) \int_{S_{t(s)}} e^{-\phi(t(s))} \widehat{dv_s}. \tag{2.7}$$

Putting (2.6)-(2.7) into (2.5) results in

$$\begin{aligned} \int_{G(s)} |\nabla f| e^{-\phi|_{G(s)}} \widehat{dv_s} &\geq \frac{\left(\int_{G^*(s)} e^{-\phi(t(s))} \widehat{dv_s}\right)^2}{\int_{G(s)} |\nabla f|^{-1} e^{-\phi|_{G(s)}} \widehat{dv_s}} \\ &= \frac{\int_{S_{t(s)}} e^{-\phi(t(s))} \widehat{dv_s}}{-t'(s)}. \end{aligned}$$

Therefore, by substituting the above inequality into (2.3), one can obtain

$$\begin{aligned} \int_\Omega |\nabla f|^2 d\eta &= \int_0^{\sup f} \int_{G(s)} |\nabla f| e^{-\phi|_{G(s)}} \widehat{dv_s} ds \\ &\geq -\int_0^{\sup f} \frac{\int_{S_{t(s)}} e^{-\phi(t(s))} \widehat{dv_s}}{t'(s)} ds \\ &= -\int_0^{\sup f} (\psi'(t(s)))^2 t'(s) \int_{S_{t(s)}} e^{-\phi(t(s))} \widehat{dv_s} ds \\ &= \int_0^R (\psi'(t))^2 \int_{S_{t(s)}} e^{-\phi(t(s))} \widehat{dv_s} dt \\ &= \int_{B_R(o)} |\nabla f^*|^2 d\eta. \end{aligned} \tag{2.8}$$

The equality case in (2.8) implies that $\int_{G(0)} e^{-\phi|_{G(0)}}\widehat{dv} = \int_{G^*(0)} e^{-\phi(t)}\widehat{dv}$ holds. So, one has $G(0) \setminus E(0) = G^*(0)$, that is, $\Omega \setminus E(0) = B_R(o)$. Moreover, this domain should lie entirely in the region $B_{\mathcal{R}(h)}$ defined by (1.21). Furthermore, by Lemma 2.1, we have

$$\lambda_{1,\phi}(\Omega) = \frac{\int_\Omega |\nabla f|^2 d\eta}{\int_\Omega f^2 d\eta} \geq \frac{\int_{B_R(o)} |\nabla f^*|^2 d\eta}{\int_{B_R(o)} (f^*)^2 d\eta} \geq \lambda_{1,\phi}(B_R(o)),$$

which completes the proof of Theorem 1.1. □

Proof of Theorem 1.7. Use an almost the same argument as that in the above proof of Theorem 1.1 except replacing the usage of **Fact A** and **Fact B** by the following fact:

- ([3]) Assume that the function $a : [0, +\infty) \to [0, +\infty)$ satisfies $a(t)$ is non-decreasing for $t \geq 0$, $(a(z^{\frac{1}{n}}) - a(0))z^{1-\frac{1}{n}}$ is convex, $z \geq 0$, and assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary $\partial\Omega$. Then

  $$\int_{\partial\Omega} a(|x|)dx \geq \int_{\partial\Omega^*} a(|x|)dx,$$

  where $\partial\Omega^*$ is a sphere with center at the origin and enclosing the weight volume equal to that of Ω.

Then the conclusion in Theorem 1.7 would follow naturally by choosing $a(t) = e^{-\phi(t)}$. □

2.2 The hemisphere case

As we know, the Schwarz symmetrization can also be carried out on hemispheres and hyperbolic spaces. For convenience, we will continue to use notions and also the notations introduced at the beginning of Subsection 2.1 to investigate the Faber-Krahn type inequalities for the Witten-Laplcian in the hemisphere case and the hyperbolic case.

Lemma 2.2. *Assume that the function ϕ satisfies **Property 1** (with M^n chosen to be $\mathbb{S}^n_+$), where the point o mentioned in **Property 1** should additionally be required to be the base point of $\mathbb{S}^n_+$. Then we have*

$$\int_\Omega f^2 d\eta = \int_{B_R(o)} (f^*)^2 d\eta,$$

where $B_R(o) \subset \mathbb{S}^n_+$ is the geodesic ball defined as in Theorem 1.3.

Proof. Formally, the computation for the assertion in Lemma 2.2 is almost the same as that for (2.2), and so we omit the details here. □

We also need the following fact:

Lemma 2.3 ([5]). *Let $\Omega \subset \mathbb{S}^n_+$ be a compact n-dimensional domain with smooth boundary $\partial\Omega$. Let H be the normalized mean curvature of $\partial\Omega$. Let $V(x) = \cos dist_{\mathbb{S}^n}(x, o)$. If H is positive everywhere, then*[11]

$$\int_{\partial\Omega} \frac{V}{H} dA \geq n \int_{\Omega} V d\Omega. \tag{2.9}$$

The equality in (2.9) holds if and only if Ω is isometric to a geodesic ball.

Now, we have:

Proof of Theorem 1.3. Applying the co-area formula, we have

$$\int_{\Omega} |\nabla f|^2 \cos t dv = \int_0^{\sup f} \int_{G(s)} |\nabla f| \cos(t|_{G(s)}) \widehat{dv_s} ds. \tag{2.10}$$

We can obtain by using the Cauchy-Schwarz inequality that

$$\int_{G(s)} |\nabla f| \cos(t|_{G(s)}) \widehat{dv_s} \geq \frac{\left(\int_{G(s)} \cos(t|_{G(s)}) \widehat{dv_s}\right)^2}{\int_{G(s)} |\nabla f|^{-1} \cos(t|_{G(s)}) \widehat{dv_s}}. \tag{2.11}$$

By Lemma 2.3 and the assumption that H is a positive constant, one has $\int_{G(s)} \cos(t|_{G(s)}) \widehat{dv_s} \geq \int_{G^*(s)} \cos t(s) \widehat{dv_s}$, and then (2.11) becomes

$$\int_{G(s)} |\nabla f| \cos(t|_{G(s)}) \widehat{dv_s} \geq \frac{\left(\int_{G^*(s)} \cos t(s) \widehat{dv_s}\right)^2}{\int_{G(s)} |\nabla f|^{-1} \cos(t|_{G(s)}) \widehat{dv_s}}. \tag{2.12}$$

On the other hand, one has

$$\int_{G^*(s)} |\nabla f^*| \cos t(s) \widehat{dv_s} = \frac{\left(\int_{G^*(s)} \cos t(s) \widehat{dv_s}\right)^2}{\int_{G^*(s)} |\nabla f^*|^{-1} \cos t(s) \widehat{dv_s}},$$

since $|\nabla f^*|$ and $\cos t(s)$ are constant on the sphere $G^*(s)$. Notice that

$$|\Omega_r|_\phi = \int_{\Omega_r} \cos t(r) dv = \int_r^{\sup f} \int_{G(s)} |\nabla f|^{-1} \cos(t|_{G(s)}) \widehat{dv_s} ds,$$

and so it follows that

$$(|\Omega_r|_\phi)'(s) = -\int_{G(s)} |\nabla f|^{-1} \cos(t|_{G(s)}) \widehat{dv_s},$$

[11] In (2.9), the Hausdorff measures of the domain Ω and its boundary $\partial\Omega$ are given by $d\Omega$, dA respectively. This usage of notations does not match the agreement made at the beginning of Subsection 2.1, and the reason is that we prefer to list here the original statement of the conclusion in Lemma 2.3 proven firstly in the reference [5].

which implies

$$-\int_{G(s)} |\nabla f|^{-1} \cos(t|_{G(s)}) \widehat{dv_s} = \frac{d}{ds}|\Omega_s|_\phi = \frac{d}{ds}|\Omega_s^*|_\phi. \tag{2.13}$$

Since

$$|\Omega_s^*|_\phi = \int_0^{t(s)} \int_{\partial B_z(o)} \cos z \widehat{dv_z} dz,$$

one has

$$\frac{d}{ds}|\Omega_s^*|_\phi = t'(s) \int_{S_{t(s)}} \cos t(s) \widehat{dv_s}. \tag{2.14}$$

Putting (2.13)-(2.14) into (2.12) results in

$$\begin{aligned} \int_{G(s)} |\nabla f| \cos(t|_{G(s)}) \widehat{dv_s} &\geq \frac{\left(\int_{G^*(s)} \cos t(s) \widehat{dv_s}\right)^2}{\int_{G(s)} |\nabla f|^{-1} \cos(t|_{G(s)}) \widehat{dv_s}} \\ &= \frac{\int_{S_{t(s)}} \cos t(s) \widehat{dv_s}}{-t'(s)}. \end{aligned}$$

Therefore, by substituting the above inequality into (2.10), one has

$$\begin{aligned} \int_\Omega |\nabla f|^2 d\eta &= \int_0^{\sup f} \int_{G(s)} |\nabla f| \cos(t|_{G(s)}) \widehat{dv_s} ds \\ &\geq -\int_0^{\sup f} \frac{\int_{S_{t(s)}} \cos t(s) \widehat{dv_s}}{t'(s)} ds \\ &= -\int_0^{\sup f} (\psi'(t(s)))^2 t'(s) \int_{S_{t(s)}} \cos t(s) \widehat{dv_s} ds \\ &= \int_0^R (\psi'(t))^2 \int_{S_{t(s)}} \cos t(s) \widehat{dv_s} dt \\ &= \int_{B_R(o)} |\nabla f^*|^2 d\eta. \end{aligned}$$

Together with Lemma 2.2, it follows that

$$\lambda_{1,\phi}(\Omega) = \frac{\int_\Omega |\nabla f|^2 d\eta}{\int_\Omega f^2 d\eta} \geq \frac{\int_{B_R(o)} |\nabla f^*|^2 d\eta}{\int_{B_R(o)} (f^*)^2 d\eta} \geq \lambda_{1,\phi}(B_R(o)). \tag{2.15}$$

Especially, if the equality in (2.15) was achieved, then the equality in (2.11) and (2.12) can be attained simultaneously, and the rigidity assertion in Theorem 1.3 follows by using Lemma 2.3 directly. This completes the proof of Theorem 1.3. □

2.3 The hyperbolic case

Proof of Theorem 1.5. It is not hard to see that similar to Lemma 2.1, in the hyperbolic case one also has the L^2 integral (w.r.t. the weighted density $d\eta$) unchanged after the Schwarz symmetrization under the constraint of fixed weighted volume. Then using an almost the same argument as that in the proof of Theorem 1.1, together with the help of **Fact C** (i.e., the geometric isoperimetric inequality in $\mathbb{H}^n$ under the constraint of fixed weighted volume), we can get the spectral isoperimetric inequality and the rigidity in Theorem 1.5. □

3 The Hong-Krahn-Szegő type inequalities for the Witten-Laplcian

For the Dirichlet eigenvalue problem (1.15), we know from Section 1 that its admissible space is the Sobolev space $\widetilde{W}_0^{1,2}(\Omega)$. Using the inner product (1.16), one can define the L^2 space $\widetilde{L}^2(\Omega)$ w.r.t. the weighted density as follows: we say that $u \in \widetilde{L}^2(\Omega)$ if

$$\int_\Omega u^2 e^{-\phi} dv < \infty.$$

Before giving the proof of the Hong-Krahn-Szegő type inequalities for the second Dirichlet eigenvalue of the Witten-Laplcian, we need the following facts.

Lemma 3.1. *(Nodal domain theorem for the Witten-Laplacian, [13]) For the Dirichlet eigenvalue problem (1.15), its eigenvalues consist of a non-decreasing sequence (1.18). Denote by f_i be an eigenfunction of the i-th eigenvalue $\lambda_{1,\phi}$, $i = 1, 2, 3, \cdots$, and $\{f_1, f_2, f_3, \cdots\}$ forms a complete orthogonal basis of $\widetilde{L}^2(\Omega)$. Then each $k = 1, 2, 3, \cdots$, the number of connected components of the nodal domain of f_k is less than or equal to k.*

Remark 3.2. By Lemma 3.1, one easily knows that the eigenfunction f_1 does not change sign on Ω, and $\lambda_{1,\phi}$ has multiplicity 1. Without loss of generality, we can assume $f_1 > 0$ on Ω. Besides, the nodal domain of eigenfunction f_2 of the second Dirichlet eigenvalue $\lambda_{2,\phi}$ has precisely two components.

Lemma 3.3. *([14]) Domain monotonicity of eigenvalues with vanishing Dirichlet data also holds for the Dirichlet eigenvalues of the weighted Laplacian.*

Now, we have:

A proof of Theorem 1.9 or 1.10. By Lemma 3.1, one knows that the eigenfunction f_2 has two nodal domains and its nodal set lies inside Ω. Denote by Γ the nodal set of f_2. Γ divides the domain Ω into two parts D_1 and D_2. Without loss of generality, assume that $f_2|_{D_1} > 0$ and $f_2|_{D_2} < 0$. Then it is easy to see that

$$\begin{cases} \Delta_\phi f_2 + \lambda_{2,\phi}(\Omega) f_2 = 0 & \text{in } D_1, \\ f_2 = 0 & \text{on } \partial D_1, \end{cases} \tag{3.1}$$

and

$$\begin{cases} \Delta_\phi f_2 + \lambda_{2,\phi}(\Omega) f_2 = 0 & \text{in } D_2, \\ f_2 = 0 & \text{on } \partial D_2, \end{cases} \tag{3.2}$$

In fact, the nodal set Γ also divides the boundary $\partial\Omega$ into two parts, and let us call them $\mathcal{C}_1$ and $\mathcal{C}_2$. It is not hard to see that $\mathcal{C}_1$ and Γ surround one of D_1 and D_2, and without loss of generality, let us say D_1. This implies that the boundary ∂D_1 of D_1 satisfies $\partial D_1 = \mathcal{C}_1 \cup \Gamma$. Correspondingly, one has $\partial D_2 = \mathcal{C}_2 \cup \Gamma$. From (3.1) and (3.2), one knows that f_2 satisfies the eigenvalue problem (1.15) with $\Omega = D_1$ or $\Omega = D_2$, and moreover, f_2 does not change sign on D_i, $i = 1, 2$. Hence, we have $\lambda_{1,\phi}(D_1) = \lambda_{2,\phi}(\Omega) = \lambda_{1,\phi}(D_2)$, and f_2 can be treated as an eigenfunction of $\lambda_{1,\phi}(D_i)$, $i = 1, 2$. Denote by $B_{R_i}(o)$ the (geodesic) ball in $\mathbb{R}^n$ (or $\mathbb{H}^n$) centered at the origin o and radius R_i such that its weighted volume equals that of D_i, $i = 1, 2$, that is, $|B_{R_i}(o)|_\phi = |D_i|_\phi$. Then by Theorem 1.1 (or Theorem 1.5), we know that

$$\lambda_{2,\phi}(\Omega) \geq \lambda_{1,\phi}(B_{R_1}(o)), \qquad \lambda_{2,\phi}(\Omega) \geq \lambda_{1,\phi}(B_{R_2}(o))$$

hold simultaneously. Hence, one has

$$\lambda_{2,\phi}(\Omega) \geq \max\{\lambda_{1,\phi}(B_{R_1}(o)), \lambda_{1,\phi}(B_{R_2}(o))\}.$$

We may suppose that $|D_1|_\phi \leq |D_2|_\phi$. So, $R_1 \leq R_2$, and by Lemma 3.3 we have $\lambda_{1,\phi}(B_{R_1}(o)) \geq \lambda_{1,\phi}(B_{R_2}(o))$. Therefore, in this setting, finding the greatest lower bound for the second eigenvalue $\lambda_2(\Omega)$ among domains with the fixed weighted volume $|\Omega|_\phi = const.$, it is sufficient to minimize $\lambda_{1,\phi}(B_{R_1}(o))$. Since $|D_1|_\phi \leq |D_2|_\phi$ and $|D_1|_\phi + |D_2|_\phi = |\Omega|_\phi$, the maximal possibility for the weighted volume of D_1 is that $|D_1|_\phi = |\Omega|_\phi/2$. Hence, there exists $\widetilde{R} > 0$ such that $|B_{\widetilde{R}}(o)|_\phi = |\Omega|_\phi/2$, and by Lemma 3.3, in this situation, the eigenvalue $\lambda_{1,\phi}(B_{\widetilde{R}}(o))$ minimizes the eigenvalue functional $\lambda_{1,\phi}(B_{R_1}(o))$ as R_1 changes. Hence, one has $\lambda_{2,\phi}(\Omega) \geq \lambda_{1,\phi}(B_{\widetilde{R}}(o))$, and the eigenvalue $\lambda_{1,\phi}(B_{\widetilde{R}}(o))$ equals the minimum value of the eigenvalue functional $\lambda_{2,\phi}(\Omega)$ under the constraint of weighted volume $|\Omega|_\phi = const.$ fixed. This completes the proof. □

4 The Szegő-Weinberger type inequalities for the Witten-Laplcian

This section devotes to giving isoperimetric inequalities for the first nonzero Neumann eigenvalue of the Witten-Laplacian under the constraint of weighted volume fixed. Before that, we need the following fact.

Theorem 4.1. *Assume that $B_R(o)$ is a geodesic ball of radius R and centered at some point o in the n-dimensional complete simply connected Riemannian manifold $\mathbb{M}^n(\kappa)$ with constant sectional curvature $\kappa \in \{-1, 0, 1\}$, and that ϕ is a radial function w.r.t. the distance parameter $t := d(o, \cdot)$, which is also a non-increasing convex function. Then the eigenfunctions of the first nonzero Neumann eigenvalue $\mu_{1,\phi}(B_R(o))$ of the Witten-Laplacian on $B_R(o)$ should have the form $T(t)\frac{x_i}{t}$, $i = 1, 2, \cdots, n$, where $T(t)$ satisfies*

$$\begin{cases} T'' + \left(\frac{(n-1)C_\kappa}{S_\kappa} - \phi'\right) T' + (\mu_{1,\phi}(B_R(o)) - (n-1)S_\kappa^{-2})\, T = 0, \\ T(0) = 0,\ T'(R) = 0,\ T'|_{[0,R)} \neq 0. \end{cases} \tag{4.1}$$

Here $C_\kappa(t) = (S_\kappa(t))'$ *and*

$$S_\kappa(t) = \begin{cases} \sin t, & \text{if } \mathbb{M}^n(\kappa) = \mathbb{S}^n_+, \\ t, & \text{if } \mathbb{M}^n(\kappa) = \mathbb{R}^n, \\ \sinh t, & \text{if } \mathbb{M}^n(\kappa) = \mathbb{H}^n, \end{cases}$$

with $\mathbb{S}^n_+$ *the* n*-dimensional hemisphere of radius* 1.

The proof of the above fact is a little bit long, and looks like it does not have close relation with the main content of this section. Hence, we prefer to leave the proof in Appendix – Section 5.

Remark 4.2. it is not hard to see in Section 5 that x_i, $i = 1, 2, \cdots, n$, are coordinate functions of the globally defined orthonormal coordinate system set up in $\mathbb{M}^n(\kappa)$.

We construct an auxiliary function $h(t)$ such that

$$h(t) = \begin{cases} T(t), & 0 \le t \le R, \\ T(R), & T > R. \end{cases} \tag{4.2}$$

Lemma 4.3. *Assume that the function* ϕ *satisfies* ***Property I*** *(with* M^n *chosen to be* $\mathbb{R}^n$ *and additionally the point* o *required to be in the convex hull of* Ω*, i.e.* $o \in \mathrm{hull}(\Omega)$*). Assume that* $T(t)$ *is monotonically non-decreasing determined by the system (4.1). Then* $h(t)$ *is monotonically non-decreasing, and* $(h')^2 + (n-1)h^2/t^2$ *is monotonically non-increasing.*

Proof. First, it is easy to check that $h(t)$ defined by (4.2) is non-decreasing. Besides, by a direct calculation, one has

$$\frac{d}{dt}\left[(h')^2 + \frac{(n-1)h^2}{t^2}\right] = 2h'h'' + 2(n-1)\frac{thh' - h^2}{t^3}.$$

Together with (4.1), we have

$$\frac{d}{dt}\left[(h')^2 + \frac{(n-1)h^2}{t^2}\right] = -2\mu_{1,\phi}(B_R(o))hh' - (n-1)\frac{(th'-h)^2}{t^3} + 2(h')^2\phi' \le 0,$$

which implies the second assertion of the lemma directly. □

Lemma 4.4. *Assuming* Ω *is a bounded domain in* $\mathbb{R}^n$ *(or* $\mathbb{H}^n$*) with smooth boundary. If* $|\Omega|_\phi = |B_R(o)|_\phi$*, with* $B_R(o)$ *be the (geodesic) ball defined as in Theorem 1.11 (or Theorem 1.12), and the non-constant functions* $u(t)$ *and* $v(t)$ *defined on* $[0, +\infty)$ *are monotonically non-increasing and non-decreasing, respectively, then*

$$\int_\Omega v(|x|)d\eta \ge \int_{B_R(o)} v(|x|)d\eta,$$
$$\int_\Omega u(|x|)d\eta \le \int_{B_R(o)} u(|x|)d\eta.$$

The equality holds if and only if $\Omega = B_R(o)$ *(or* Ω *is isometric to* $B_R(o)$*).*

Proof. Assume that $Q = \Omega \cap B_R(o)$, and then we have

$$\begin{aligned}\int_\Omega v(|x|)d\eta &= \int_Q v(|x|)d\eta + \int_{\Omega\setminus Q} v(|x|)d\eta \\ &\geq \int_Q v(|x|)d\eta + v(R)\int_{\Omega\setminus Q} d\eta.\end{aligned}$$

Similarly, one has

$$\begin{aligned}\int_{B_R(o)} v(|x|)d\eta &= \int_Q v(|x|)d\eta + \int_{B_R(o)\setminus Q} v(|x|)d\eta \\ &\leq \int_Q v(|x|)d\eta + v(R)\int_{B_R(o)\setminus Q} d\eta.\end{aligned}$$

Since $|\Omega|_\phi = |B_R(o)|_\phi$, then $\int_{\Omega\setminus Q} d\eta = \int_{B_R(o)\setminus Q} d\eta$, and substituting this fact into the above two inequalities yields

$$\int_\Omega v(|x|)d\eta \geq \int_{B_R(o)} v(|x|)d\eta.$$

Specially, when the equality holds, one has

$$\int_{\Omega\setminus Q} v(|x|)d\eta = v(R)\int_{\Omega\setminus Q} d\eta, \qquad \int_{B_R(o)\setminus Q} v(|x|)d\eta = v(R)\int_{B_R(o)\setminus Q} d\eta$$

simultaneously. Since the non-constant function v is non-increasing, Ω is the ball $B_R(o)$ (or Ω is isometric to $B_R(o)$). The situation for the non-constant function u can be dealt with similarly. □

Now, we have:

Proof of Theorem 1.11. Define $f(t) := h(t)\frac{x_i}{t}$, where i is chosen to be an integer of the set $\{1, 2, \cdots, n\}$. Then applying the Brouwer's fixed point theorem and choosing a suitable coordinate origin $o \in \text{hull}(\Omega)$, we can assure $\int_\Omega f d\eta = 0$. This can be done by using a very similar argument to that on [38, pp. 634-635]. In fact, one can also check our another work [12] where we have given a detailed explanation on how to get the suitable coordinate system such that $\int_\Omega f d\eta = 0$. By the characterization (1.25), and by using a similar calculation to (2.9)-(2.10) on [38, page 635], one has

$$\mu_{1,\phi}(\Omega) \leq \frac{\int_\Omega \left[(h')^2 + \frac{(n-1)h^2}{t^2}\right] d\eta}{\int_\Omega h^2 d\eta}.$$

On the other hand, by Lemma 4.3 and Lemma 4.4, we have

$$\int_\Omega \left[(h')^2 + \frac{(n-1)h^2}{t^2}\right] d\eta \leq \int_{B_R(o)} \left[(h')^2 + \frac{(n-1)h^2}{t^2}\right] d\eta$$

and

$$\int_\Omega h^2 d\eta \geq \int_{B_R(o)} h^2 d\eta.$$

Therefore, we have

$$\mu_{1,\phi}(\Omega) \leq \frac{\int_\Omega \left[(h')^2 + \frac{(n-1)h^2}{t^2}\right] d\eta}{\int_\Omega h^2 d\eta} \leq \frac{\int_{B_R(o)} \left[(h')^2 + \frac{(n-1)h^2}{t^2}\right] d\eta}{\int_{B_R(o)} h^2 d\eta} = \mu_{1,\phi}(B_R(o)),$$

which together with the description of the equality case in Lemma 4.4 implies the assertion of Theorem 1.11 directly. □

Proof of Theorem 1.12. We still use $f(t)$ as the trail function, but now the distance should be the Riemannian distance in the hyperbolic space $\mathbb{H}^n$. In the hyperbolic case, using a similar argument to that in the proof of Theorem 1.11, we have

$$\mu_{1,\phi}(\Omega) \leq \frac{\int_\Omega \left[(h')^2 + \frac{(n-1)h^2}{(\sinh t)^2}\right] d\eta}{\int_\Omega h^2 d\eta}. \tag{4.3}$$

On the other hand,

$$\frac{d}{dt}\left[(h')^2 + \frac{(n-1)h^2}{(\sinh t)^2}\right] = 2h'h'' + 2(n-1)\frac{hh'\sinh t - h^2\cosh t}{(\sinh t)^3.}$$

Putting (4.1) and using the facts $\sinh t \geq 0$, $\cosh t \geq 1$ for $t \geq 0$, one has

$$\begin{aligned}
&\frac{d}{dt}\left[(h')^2 + \frac{(n-1)h^2}{(\sinh t)^2}\right] \\
&\quad = -2\mu_{1,\phi}(B_R(o))hh' + 2(h')^2\phi' - \frac{2(n-1)\cosh t}{\sinh t}(h')^2 \\
&\qquad -\frac{2(n-1)\cosh t}{\sinh^3 t}h^2 + \frac{4(n-1)}{\sinh^2 t}hh' \\
&\quad \leq -2\mu_{1,\phi}(B_R(o))hh' + 2(h')^2\phi' - \frac{2(n-1)}{\sinh t}(h')^2 \\
&\qquad -\frac{2(n-1)}{\sinh^3 t}h^2 + \frac{4(n-1)}{\sinh^2 t}hh' \\
&\quad = -2\mu_{1,\phi}(B_R(o))hh' + 2(h')^2\phi' - 2(n-1)\frac{(h')^2\sinh^2 t + h^2 - 2hh'\sinh t}{\sinh^3 t} \\
&\quad = -2\mu_{1,\phi}(B_R(o))hh' + 2(h')^2\phi' - 2(n-1)\frac{(h'\sinh t - h)^2}{\sinh^3 t} \\
&\quad \leq 0.
\end{aligned}$$

Then, by applying Lemma 4.4, we have

$$\int_\Omega \left[(h')^2 + \frac{(n-1)h^2}{\sinh^2 t}\right] d\eta \leq \int_{B_R(o)} \left[(h')^2 + \frac{(n-1)h^2}{\sinh^2 t}\right] d\eta$$

and

$$\int_{\Omega} h^2 d\eta \geq \int_{B_R(o)} h^2 d\eta.$$

Therefore, from (4.3) we can obtain

$$\mu_{1,\phi}(\Omega) \leq \frac{\int_{\Omega}\left[(h')^2 + \frac{(n-1)h^2}{\sinh^2 t}\right] d\eta}{\int_{\Omega} h^2 d\eta} \leq \frac{\int_{B_R(o)}\left[(h')^2 + \frac{(n-1)h^2}{\sinh^2 t}\right] d\eta}{\int_{B_R(o)} h^2 d\eta} = \mu_{1,\phi}(B_R(o)),$$

which together with the description of the equality case in Lemma 4.4 implies the assertion of Theorem 1.12 directly. □

5 Appendix

Now, in this section we give a proof of Theorem 4.1 in details. Assume that f is an eigenfunction of the Witten-Laplace operator Δ_ϕ, and f can be decomposed into $T(t)G(\xi)$, where $t := d(o,\cdot)$ stands for the Riemannian distance to the point o, and $\xi \in S_o^{n-1} \subset T_o\mathbb{M}^n(\kappa)$. A simple calculation gives us that

$$0 = \Delta_\phi f + \mu f = S_\kappa^{1-n}(S_\kappa^{n-1}T')'G - S_\kappa^2 T v_l G - \phi' T' G + \mu T G,$$

where v_l denotes the closed eigenvalue of the Laplacian on the unit $(n-1)$-sphere $\mathbb{S}^{n-1}$, i.e., $v_l = l(l+n-2)$, $l = 0, 1, 2, \cdots$. Simplifying the above equation gives us a second-order ODE as follows

$$T'' + \left[\frac{(n-1)C_\kappa}{S_\kappa} - \phi'\right] T' + \left(\mu - \frac{v_l}{S_\kappa^2}\right) T = 0, \tag{5.1}$$

where $C_\kappa(t) = S_\kappa'(t)$. For the Neumann eigenvalue problem of the Witten-Laplacian Δ_ϕ, in order to ensure the smoothness of the function T, we have:

- When $l = 0$, $T'(0) = 0$;
- $T(t) \sim t^l$, $l = 1, 2, \cdots$;
- T satisfies the Neumann boundary condition $T'(R) = 0$.

Choosing a relatively small positive number ϵ and letting $p(t) = e^{\int_\epsilon^t (\frac{(n-1)C_\kappa}{S_\kappa} - \phi')ds}$, we can simplify (5.1) into a Sturm-Liouville equation

$$(pT')' + (\mu - v_l S_\kappa^{-2})pT = 0. \tag{5.2}$$

Assume that for a fixed v_l, $\mu_{l,j,\phi}$, $j = 1, 2, \cdots$, is the j-th eigenvalue related to v_l, and $T_{l,j,\phi}$ denotes an eigenfunction belonging to $\mu_{l,j,\phi}$. Here the purpose that we put the symbol ϕ in the subscript of $\mu_{l,j,\phi}$ is to emphasize that theoretically $\mu_{l,j,\phi}$, $T_{l,j,\phi}$ have close relation with

the function ϕ since the function $p(t)$ in the equation (5.2) depends on $\phi'(t)$. In this setting, the equation (5.2) can be rewritten as

$$(pT'_{l,j,\phi})' + \left(\mu_{l,j,\phi} - v_l S_\kappa^{-2}\right) pT_{l,j,\phi} = 0, \tag{5.3}$$

which implies

$$\int_0^R T_{l,j,\phi}\, T_{l,k,\phi} p dt = 0, \qquad \text{when} \;\; \mu_{l,j,\phi} \neq \mu_{l,k,\phi}. \tag{5.4}$$

Moreover, one can normalize T such that

$$\int_0^R T_{l,j,\phi} T_{l,j,\phi} p dt = 1.$$

For an equation of the form similar to (5.3), we have the following fact.

Lemma 5.1. *Assume that functions f and g satisfy separately the equations*

$$(pf')' + (\alpha - \sigma(t))pf = 0, \tag{5.5}$$
$$(pg')' + (\beta - \tau(t))pg = 0, \tag{5.6}$$

and also the boundary conditions given as in the system (4.1). Then we have

$$p(fg' - f'g)(t) = \int_0^t \left[\alpha - \beta + (\tau - \sigma)\right] pfg dt.$$

Proof. Multiplying both sides of the equation (5.5) by g, multiplying both sides of the equation (5.6) by f, and then making difference yields

$$(pf')'g - (pg')'f + \left[\alpha - \beta + (\tau(t) - \sigma(t))\right] pfg = 0.$$

Integrating both sides of the above equality from 0 to t, and using the boundary conditions given as in the system (4.1), one can get the assertion of Lemma 5.1 directly. □

By the standard Sturm-Liouville theory for second-order ODEs, we know that $T_{l,j,\phi}$ has exactly $j-1$ zeros on the interval $(0, R)$. So, $T_{l,1,\phi}$ keeps its sign unchanged on $(0, R)$. Without loss of generality, we may assume that $T_{l,1,\phi}$ and $T_{k,1,\phi}$ are both greater than 0, where $l < k$. Then, by Lemma 5.1, when $t = R$, we have $\mu_{l,1,\phi}(R) < \mu_{k,1,\phi}(R)$, $l < k$. Since for the eigenvalue problem (1.22), we know from its sequence (1.23) that $\mu_{1,\phi} = \mu_{0,1,\phi} = 0$. Hence, if one wants to get the first non-zero Neumann eigenvalue $\mu_{1,\phi}$ of the Witten-Laplacian on $B_R(o)$, one only needs to know exactly which one is smaller between $\mu_{0,2,\phi}$ and $\mu_{1,1,\phi}$.

The following lemma is important and fundamental.

Lemma 5.2. *When $l \geq 1$, $T'_{l,j,\phi}$ has only $j-1$ zeros in the interval $(0, R)$.*

Proof. From (5.3), one has

$$pT''_{l,j,\phi} + p'T'_{l,j,\phi} + \mu_{l,j,\phi} pT_{l,j,\phi} - v_l S_\kappa^{-2} pT_{l,j,\phi} = 0. \tag{5.7}$$

Since $T_{l,1,\phi}$ has no zero points on the interval $(0,R)$, we can assume that $T_{l,1,\phi}$ is greater than 0. According to the boundary conditions, if $T'_{l,1,\phi}$ is not constantly greater than 0 on the interval $(0,R)$, then there exists a $t_0 < t_1$ such that $T''_{l,1,\phi}(t_0) \leq 0$, $T'_{l,1,\phi}(t_0) = 0$ and $T''_{l,1,\phi}(t_1) \geq 0$, $T'_{l,1,\phi}(t_1) = 0$ hold true. Together with (5.7), we can obtain

$$S_\kappa^2(t_0) \geq \frac{v_l}{\mu_{l,1,\phi}} \geq S_\kappa^2(t_1).$$

Due to the increasing property of $S_\kappa(t)$, this contradicts with $t_0 < t_1$. So, $T'_{l,1,\phi}$ has no zero points in the interval $(0,R)$. For the case $T'_{l,j,\phi}$, $j > 1$, one only needs to repeat the above argument in each nodal domain. □

It is not hard to know that the function $T_{0,2}$ satisfies

$$\begin{cases} (pT'_{0,2,\phi})' + \mu_{0,2,\phi} p T_{0,2,\phi} = 0, \\ T'_{0,2,\phi}(0) = T'_{0,2,\phi}(R) = 0. \end{cases} \tag{5.8}$$

Since $T_{0,1,\phi}$ is a non-zero constant function, and $T_{0,2,\phi}$ is orthogonal to $T_{0,1,\phi}$ in the sense of (5.4), we know that $T_{0,2,\phi}$ changes sign on the interval $(0,R)$. Therefore, we may assume that $T_{0,2,\phi}$ is positive on some interval $(0,r_0)$ and $T_{0,2,\phi}(r_0) = 0$, $0 < r_0 < R$. If there exists $r^* \in [0,r_0)$ such that $T''_{0,2,\phi}(r^*) \geq 0$ and $T'_{0,2,\phi}(r^*) = 0$, then substituting this fact into (5.8) yields $((pT'_{0,2,\phi})' + \mu_{0,2,\phi} p T_{0,2,\phi})(r^*) > 0$, which contradicts with the first equation in the system (5.8). Hence, we conclude that $T'_{0,2,\phi}$ is negative on the interval $(0,r_0)$. Since ϕ is non-increasing, $p' \geq 0$ can be obtained, and then from (5.8) again, we have $T''_{0,2,\phi}(r_0) \geq 0$ at r_0.

We notice that the function $T_{1,1,j}$ satisfies the following equation

$$(pT'_{1,1,\phi})' + (\mu_{1,1,\phi} - (n-1)S_\kappa^{-2}) p T_{1,1,\phi} = 0. \tag{5.9}$$

Differentiating both sides of the first equation in the system (5.8) results in

$$(pT''_{0,2,\phi})' + \left(\mu_{0,2,\phi} + (\frac{p'}{p})'\right) pT'_{0,2,\phi} = 0. \tag{5.10}$$

Combining (5.9)-(5.10), and applying Lemma 5.1, we can obtain at r_0 that

$$\begin{aligned} &p(T_{1,1,\phi}T''_{0,2,\phi} - T'_{1,1,\phi}T'_{0,2,\phi})(r_0) = \\ &\qquad \int_0^{r_0} \left[\mu_{1,1,\phi} - \mu_{0,2,\phi} + \left(\left(-\frac{p'}{p}\right)' - (n-1)S_\kappa^{-2}\right)\right] pT_{1,1,\phi}T'_{0,2,\phi} dt. \end{aligned} \tag{5.11}$$

Since ϕ is a convex function, $\phi'' \geq 0$, and so we have

$$-\left(\frac{(n-1)C_\kappa}{S_\kappa} - \phi'\right)' - (n-1)S_\kappa^{-2} \geq 0.$$

Substituting the fact

$$p(t) = e^{\int_\epsilon^t \left(\frac{(n-1)C_\kappa}{S_\kappa} - \phi'\right) dt}$$

into the above inequality, one has

$$\left(-\frac{p'}{p}\right)' - (n-1)S_\kappa^{-2} \geq 0.$$

Together with the fact that at r_0, $T_{1,1,\phi} > 0$, $T'_{1,1,\phi} \geq 0$, $T_{0,2,\phi} > 0$, $T'_{0,2,\phi} \leq 0$ and $T''_{0,2,\phi} \geq 0$, it follows from (5.11) that $\mu_{1,1,\phi} < \mu_{0,2,\phi}$. That is to say, the first non-zero Neumann eigenvalue $\mu_{1,\phi}(B_R(o))$ of the Witten-Laplacian on $B_R(o)$ should be $\mu_{1,\phi} = \mu_{1,1,\phi}$. Substituting this fact in (5.1) results in

$$T'' + \left[\frac{(n-1)C_\kappa}{S_\kappa} - \phi'\right] T' + \left(\mu_1(B_R(o)) - v_1 S_\kappa^{-2}\right) T = 0,$$

which is exactly the first equation in the system (4.1). This completes the proof of Theorem 4.1.

Acknowledgments

This research was supported in part by the NSF of China (Grant Nos. 11801496 and 11926352), the Fok Ying-Tung Education Foundation (China), and Hubei Key Laboratory of Applied Mathematics (Hubei University).

References

[1] P. R. S. Antunes, P. Freitas, *Numerical optimization of low eigenvalues of the Dirichlet and Neumann Laplacians*, J. Optim. Theory Appl. **154** (2012) 235–257.

[2] A. Berger, *The eigenvalues of the Laplacian with Dirichlet boundary condition in $\mathbb{R}^2$ are almost never minimized by disks*, Ann. Glob. Anal. Geom. **47** (2015) 285–304.

[3] M. F. Betta, F. Brock, A. Mercaldo, M. R. Posteraro, *A weighted isoperimetric inequality and applications to symmetrization*, J. Inequal. Appl. **4** (1999) 215–240.

[4] B. Bogosel, *The method of fundamental solutions applied to boundary eigenvalue problems*, J. Comput. Appl. Math. **306** (2016) 265–285.

[5] S. Brendle, P. K. Hung, M. T. Wang, *A Minkowski inequality for hypersurfaces in the Anti-de Sitter-Schwarzschild manifold*, Commun. Pure Appl. Math. **69** (2016) 124–144.

[6] F. E. Browder, *On the spectral theory of elliptic differential operators. I*, Math. Ann. **142** (1961) 22–130.

[7] D. Bucur, J. P. Zolesio, *N-dimensional shape optimization under capacitary constraints*, J. Differ. Equat. **123** (1995) 504–522.

[8] D. Bucur, A. Henrot, *Maximization of the second non-trivial Neumann eigenvalue*, Acta Math. **222** (2019) 337–361.

[9] G. Buttazzo, G. Dal Maso, *An existence result for a class of shape optimization problems*, Arch. Rational Mech. Anal. **122** (1993) 183–195.

[10] Gregory R. Chambers, *Proof of the log-convex density conjecture*, J. Eur. Math. Soc. **21** (2019) 2301–2332.

[11] I. Chavel, *Eigenvalues in Riemannian Geometry*, Academic Press, New York (1984).

[12] R. F. Chen, J. Mao, *On the Ashbaugh-Benguria type conjecture about lower-order Neumann eigenvalues of the Witten-Laplacian*, submitted and available online at arXiv:2403.08070v2

[13] R. F. Chen, J. Mao, C. X. Wu, *On eigenfunctions and nodal sets of the Witten-Laplacian*, submitted.

[14] F. Du, J. Mao, *Estimates for the first eigenvalue of the drifting Laplace and the p-Laplace operators on submanifolds with bounded mean curvature in the hyperbolic space*, J. Math. Anal. Appl. **456** (2017) 787–795.

[15] F. Du, J. Mao, Q. L. Wang, C. X. Wu, *Eigenvalue inequalities for the buckling problem of the drifting Laplacian on Ricci solitons*, J. Differ. Equat. **260** (2016) 5533–5564.

[16] G. Faber, *Beweis, dass unter allen homogenen membranen von gleicher fläche und gleicher spannung die kreisförmige den tiefsten Grundton gibt*, Sitz. bayer. Akad. Wiss. (1923), 169–172.

[17] P. Freitas, J. Mao, I. Salavessa, *Spherical symmetrization and the first eigenvalue of geodesic disks on manifolds*, Calc. Var. Partial Differential Equations **51** (2014) 701–724.

[18] M. Gromov, M. Katz, P. Pansu, S. Semmes, J. La Fontaine, *Metric Structures for Riemannian and Non-Riemannian Spaces*, Birkhäuser, Boston (2014).

[19] A. Henrot, *Minimization problems for eigenvalues of the Laplacian*, J. Evol. Equat. **3** (2003) 443–461.

[20] I. Hong, *On an inequality concerning the eigenvalue problem of membrane*, Kodai Math. Sem. Rep. **6** (1954) 113–114.

[21] N. N. Katz, K. Kondo, *Generalized space forms*, Trans. Am. Math. Soc. **354** (2002) 2279–2284.

[22] E. Krahn, *Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises*, Math. Ann. **94** (1925) 97–100.

[23] E. Krahn, *Uber minimaleigenschaften der krugel in drei un mehr dimensionen*, Acta Comm. Univ. Dorpat. **A9** (1926) 1–44.

[24] H. Z. Li, B. T. Xu, *A class of weighted isoperimetric inequalities in hyperbolic space*, Proc. Amer. Math. Soc. **151** (2023) 2155–2168.

[25] W. Lu, J. Mao, C. X. Wu, L. Z. Zeng, *Eigenvalue estimates for the drifting Laplacian and the p-Laplacian on submanifolds of warped products*, Appl. Anal. **100** (2021) 2275–2300.

[26] J. Mao, *Eigenvalue estimation and some results on finite topological type*, Ph.D. thesis, IST-UTL, 2013.

[27] J. Mao, *Eigenvalue inequalities for the p-Laplacian on a Riemannian manifold and estimates for the heat kernel*, J. Math. Pures Appl. **101** (2014) 372–393.

[28] J. Mao, *The Gagliardo-Nirenberg inequalities and manifolds with non-negative weighted Ricci curvature*, Kyushu J. Math. **70** (2016) 29–46.

[29] J. Mao, *Functional inequalities and manifolds with nonnegative weighted Ricci curvature*, Czech. Math. J. **70** (2020) 213–233.

[30] J. Mao, *Geometry and topology of manifolds with integral radial curvature bounds*, Differ. Geom. Appl. **91** (2023), Article No. 102064. https://doi.org/10.1016/j.difgeo.2023.102064

[31] J. Mao, R. Q. Tu, K. Zeng, *Eigenvalue estimates for submanifolds in Hadamard manifolds and product manifolds $N \times \mathbb{R}$*, Hiroshima Math. J. **50** (2020) 17–42.

[32] J. Mao, F. Du, C. X. Wu, *Eigenvalue Problems on Manifolds*, Science Press, Beijing, 2017.

[33] B. Osting, *Optimization of spectral functions of Dirichlet-Laplacian eigenvalues*, J. Comput. Phys. **229** (2010) 8578–8590.

[34] M. Ritoré, C. Sinestrari, *Geometric flows, isoperimetric inequalities and hyperbolic geometry*, Mean Curvature Flow and Isoperimetric Inequalities, Adv. Courses Math. CRM Barcelona, pp. 45–113. Birkhäuser, Basel (2010).

[35] C. Rosales, A. Cañete, V. Bayle, F. Morgan, *On the isoperimetric problem in Euclidean space with density*, Calc. Var. Partial Differential Equations **31** (2008) 27–46.

[36] L. Silini, *Approaching the isoperimetric problem in $H^m_{\mathbb{C}}$ via the hyperbolic log-convex density conjecture*, Calc. Var. Partial Differential Equations **63** (2024), Article No. 11. https://doi.org/10.1007/s00526-023-02617-0

[37] G. Szegő, *Inequalities for certain eigenvalues of a membrane of given area*, J. Rational Mech. Anal. **3** (1954) 343–356.

[38] H. F. Weinberger, *An isoperimetric inequality for the N-dimensional free membrane problem*, J. Rational Mech. Anal. **5** (1956) 633–636.

[39] Y. Zhao, C. X. Wu, J. Mao, F. Du, *Eigenvalue comparisons in Steklov eigenvalue problem and some other eigenvalue estimates*, Revista Matemática Complutense **33** (2020) 389–414.