QUASINORMABILITY AND PROPERTY (Ω) FOR SPACES OF SMOOTH AND ULTRADIFFERENTIABLE VECTORS ASSOCIATED WITH LIE GROUP REPRESENTATIONS

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Abstract. We prove that the spaces of smooth and ultradifferentiable vectors associated with a representation of a real Lie group on a Fréchet space E are quasinormable if E is so. A similar result is shown to hold for the linear topological invariant (Ω). In the ultradifferentiable case, our results particularly apply to spaces of Gevrey vectors of Beurling type. As an application, we study the quasinormability and the property (Ω) for a broad class of Fréchet spaces of smooth and ultradifferentiable functions on Lie groups globally defined via families of weight functions.

1. Introduction

In this article we study the quasinormability and the property (Ω) for spaces of smooth and ultradifferentiable vectors associated with representations of real Lie groups. In particular, we will provide criteria to determine when a Lie group invariant locally convex space of smooth or ultradifferentiable functions possesses one of these linear topological properties. Our considerations shall cover the important instance of spaces of Gevrey vectors of Beurling type, which, together with their Roumieu variants, were introduced and throughly investigated by Goodman and Wallach [20, 21, 22].

The notion of quasinormability for locally convex spaces is due to Grothendieck [23]. The related property (Ω) for Fréchet spaces goes back to Vogt and Wagner [29, 38] and may be seen as a quantified version of quasinormability within the class of Fréchet spaces. Every Fréchet space that satisfies (Ω) is hence quasinormable. Both concepts express approximation properties with respect to families of continuous seminorms. In this regard, our work here is closely connected with classical results of Gårding [17, 18], Nelson [30], and Goodman [20, Section 3] about approximations by smooth, analytic, or Gevrey vectors, respectively.

The interest in these two linear topological properties stems, among other things, from the fact they are often crucial hypotheses for the application of various abstract functional analytic tools. For example, given a surjective continuous linear map $f : X \to Y$ between two Fréchet spaces, the map f lifts bounded sets if $\ker f$

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is quasinormable [5, 29], while f admits a continuous linear right inverse if $\ker f$ satisfies (Ω) (assuming that X is nuclear and Y satisfies the so-called property (DN)) [29, 37]. Furthermore, strong duals of quasinormable Fréchet spaces are ultrabornological and thus barrelled, whence the Banach-Steinhaus theorem and the open mapping and closed graph theorems of De Wilde may be applied to them. We also mention that (Ω) plays an important role in the isomorphic classification theory for Fréchet spaces; in fact, (Ω) is one of the key assumptions to verify if one wants to obtain sequence space representations of function spaces via the structure theory of Fréchet spaces (cf. [9, 27]).

Quasinormability and (Ω) have been studied for a variety of concrete Fréchet function spaces; see [1, 3] for spaces of continuous functions, [8, 28, 40] for spaces of analytic functions, and [10, 36] for smooth kernels of partial differential operators. One of the goals of this work is to provide a systematic method for establishing both properties for function spaces that are invariant under a Lie group action, which we shall achieve here viewing such spaces as spaces of smooth and ultradifferentiable vectors of Lie group representations. We remark that the quasinormability of spaces of smooth vectors associated with representations of $(\mathbb{R}^n, +)$ was studied by the first author in [8].

We now discuss the content of this article in some more detail. For the sake of simplicity, we only explicitly state in this introduction our results for spaces of smooth vectors. All these statements have their counterparts for ultradifferentiable vectors, but the formulations require introducing some more notation and concepts, which we choose to postpone for future sections. By a representation of a (real) Lie group G on a Fréchet space E we simply mean a group homomorphism $\pi : G \to \mathrm{GL}(E)$, where $\mathrm{GL}(E)$ stands for the group of topological isomorphisms of E (in general, we will not require π to be strongly continuous throughout the article). We refer to the preparatory Sections 2-4 for more information about representations and the associated spaces of smooth and ultradifferentiable vectors. In the smooth case, our main results may now be summarized as follows:

Theorem 1.1. *Let π be a locally equicontinuous representation of a Lie group on a Fréchet space E. Let E^∞ be the space of smooth vectors associated with π.*

- *(i) E^∞ is quasinormable if E is so.*
- *(ii) E^∞ satisfies (Ω) if E does so.*

In fact, we show that part *(i)* of Theorem 1.1 holds not only for Fréchet spaces but also for general sequentially complete locally convex Hausdorff spaces. Theorem 1.1 and its analogue for spaces of ultradifferentiable vectors are shown in Section 6. The proofs of Theorem 1.1 and its ultradifferentiable counterpart for quasinormability make use of a standard approximation procedure involving approximate identities [18, 20]. This procedure is revisited in Subsection 5.1. Our analysis of the property (Ω) for ultradifferentiable vectors however requires some new approximation tools. Our arguments are then based on the so-called parametrix method, a powerful technique that goes back to Schwartz [34] and was further developed by Komatsu within the theory of ultradifferentiable functions and ultradistributions [26]. In Subsection 5.2 we present an extension of the parametrix method to the setting of ultradifferentiable vectors associated with Lie group representations by adapting a key idea from the proof of the celebrated Dixmier-Malliavin factorization theorem [13].

It turns out that many Fréchet function spaces on a Lie group G may be identified with spaces of smooth and ultradifferentiable vectors associated with the left- or right regular representation of G on a suitably chosen Fréchet function space E (usually E is a weighted space of continuous or integrable functions). This makes Theorem 1.1 and its ultradifferentiable analogue into power devices to study the quasinormabiltiy and the property (Ω) for concrete Fréchet function spaces. We carry out this idea in a very general framework in Sections 7 and 8. We end the introduction by discussing an important instance of these ideas.

Let G be a Lie group and let $\mathcal{V} = (v_j)_{j\in\mathbb{N}}$ be a pointwise non-decreasing sequence of strictly positive continuous functions on G satisfying the mild regularity condition

$$\forall K \subseteq G \text{ compact } \forall i \in \mathbb{N}\, \exists j \geq i, C > 0\, \forall x \in G, y \in K \,:\, v_i(xy) \leq C v_j(x). \tag{1.1}$$

For $p \in [1, \infty]$ we define

$$\mathcal{D}_{L^p_{\mathcal{V}}}(G) = \{f \in C^\infty(G) \mid v_j D f \in L^p(G),\, \forall D \in U(\mathfrak{g}), j \in \mathbb{N}\}.$$

Here, the Lebesgue space $L^p(G)$ is defined with respect to a fixed right-invariant Haar measure on G, while the elements of the universal enveloping algebra $U(\mathfrak{g})$ are to be interpreted as left-invariant differential operators. We endow $\mathcal{D}_{L^p_{\mathcal{V}}}(G)$ with its natural locally convex topology, for which it becomes a Fréchet space. If $G = (\mathbb{R}^n, +)$ these spaces are weighted variants of the classical Schwartz spaces $\mathcal{D}_{L^p}(\mathbb{R}^n)$ [34] (c.f. [12]). The sequence space representation $\mathcal{D}_{L^p}(\mathbb{R}^n) \cong s\widehat{\otimes}\ell_p$ [35] implies that $\mathcal{D}_{L^p}(\mathbb{R}^n)$ satisfies (Ω) and thus is quasinormable. Our results from Section 8 yield the following characterization of the quasinormabiltiy and the property (Ω) for the spaces $\mathcal{D}_{L^p_{\mathcal{V}}}(G)$ in terms of $\mathcal{V}$:

Theorem 1.2. *Let $\mathcal{V} = (v_j)_{j\in\mathbb{N}}$ be a pointwise non-decreasing sequence of strictly positive continuous functions on a Lie group G satisfying (1.1) and let $p \in [1, \infty]$.*

(a) Consider the following statements:

(i) $\mathcal{V}$ satisfies the condition

$$\forall i \in \mathbb{N}\, \exists j \geq i\, \forall m \geq j\, \forall \varepsilon \in (0, 1]\, \exists C > 0\, \forall x \in G \,:\, \frac{1}{v_j(x)} \leq \frac{\varepsilon}{v_i(x)} + \frac{C}{v_m(x)}.$$

(ii) $\mathcal{D}_{L^p_{\mathcal{V}}}(G)$ is quasinormable.

Then, (i) $\Rightarrow$ (ii). If $p = \infty$ or G is unimodular, then also (ii) $\Rightarrow$ (i).

(b) Consider the following statements:

(i) $\mathcal{V}$ satisfies the condition

$$\forall i \in \mathbb{N}\, \exists j \geq i\, \forall m \geq j\, \exists C, s > 0\, \forall \varepsilon \in (0, 1]\, \forall x \in G \,:\, \frac{1}{v_j(x)} \leq \frac{\varepsilon}{v_i(x)} + \frac{C}{\varepsilon^s} \frac{1}{v_m(x)}.$$

(ii) $\mathcal{D}_{L^p_{\mathcal{V}}}(G)$ satisfies (Ω).

Then, (i) $\Rightarrow$ (ii). If $p = \infty$ or G is unimodular, then also (ii) $\Rightarrow$ (i).

2. Spaces of vector-valued ultradifferentiable functions on open subsets of $\mathbb{R}^n$

In this short section we recall some facts about vector-valued ultradifferentiable functions on subsets of $\mathbb{R}^n$. By a *weight function* [6] we mean a continuous increasing function $\omega : [0,\infty) \to [0,\infty)$ with $\omega_{|[0,1]} \equiv 0$ satisfying the following properties:

(α) $\omega(2t) = O(\omega(t))$.

(β) $\displaystyle\int_1^\infty \frac{\omega(t)}{t^2}dt < \infty$.

(γ) $\log t = o(\omega(t))$.

(δ) $\phi : [0,\infty) \to [0,\infty)$, $\phi(t) = \omega(e^t)$, is convex.

Since ω is increasing, condition (β) implies that $\omega(t) = o(t)$.

Example 2.1. The *Gevrey weight of order* s, $s > 0$, is defined as

$$\omega_s(t) = \max\{0, t^s - 1\}.$$

We shall always assume that $s < 1$, which ensures that ω_s is a weight function.

Throughout the rest of this article we fix a weight function ω and write $\phi(t) = \omega(e^t)$ (cf. condition (δ) above). We define

$$\phi^* : [0,\infty) \to [0,\infty),\ \phi^*(t) = \sup_{u \geq 0}\{tu - \phi(u)\}.$$

The function ϕ^* is increasing, convex, $\phi^*(0) = 0$, $(\phi^*)^* = \phi$, and $\phi^*(t)/t \nearrow \infty$ on $[0,\infty)$. We have that [24, Lemma 2.6]

$$\forall C_1, C_2, h > 0\, \exists C, k > 0\, \forall t \geq 0 \,:\, \frac{1}{k}\phi^*(k(t + C_1)) + C_2 t \leq \frac{1}{h}\phi^*(ht) + \log C. \tag{2.1}$$

The conditions $\omega(t) = o(t)$ and (2.1) imply that

$$\forall h, k > 0\, \exists C > 0\, \forall j \in \mathbb{N} \,:\, j! \leq C k^j \exp\left(\frac{1}{h}\phi^*(hj)\right). \tag{2.2}$$

Example 2.2. For the Gevrey weights ω_s, we set $\phi_s(t) = \omega_s(e^t)$. Then,

$$\exp(\phi_s^*(t)) = e\left(\frac{1}{se}\right)^{\frac{t}{s}} t^{\frac{t}{s}}.$$

Consequently, we have that for all $h > 0$

$$\exp\left(\frac{1}{h}\phi_s^*(ht)\right) = e^{\frac{1}{h}}\left(\frac{h}{se}\right)^{\frac{t}{s}} t^{\frac{t}{s}}.$$

Let $\Theta \subseteq \mathbb{R}^n$ be open and let E be a lcHs (= locally convex Hausdorff space). We denote by $\operatorname{csn}(E)$ the family of all continuous seminorms on E. For $h > 0$ we define $\mathcal{E}^{\omega,h}(\Theta; E)$ as the space consisting of all $f \in C^\infty(\Theta; E)$ such that for all $K \subseteq \Theta$ compact and $p \in \operatorname{csn}(E)$ it holds that

$$p_{K,\omega,h}(f) = \sup_{x \in K}\sup_{\alpha \in \mathbb{N}^n} p(f^{(\alpha)}(x)) \exp\left(-\frac{1}{h}\phi^*(h|\alpha|)\right) < \infty.$$

We endow $\mathcal{E}^{\omega,h}(\Theta;E)$ with the Hausdorff locally convex topology generated by the system of seminorms $\{p_{K,\omega,h} \,|\, K \subseteq \Omega \text{ compact}, p \in \operatorname{csn}(E)\}$. We set

$$\mathcal{E}^{(\omega)}(\Theta;E) = \varprojlim_{h\to 0^+} \mathcal{E}^{\omega,h}(\Theta;E).$$

We write $\mathcal{E}^{(\omega)}(\Theta) = \mathcal{E}^{(\omega)}(\Theta;\mathbb{C})$. The non-quasianalyticity condition (β) means that $\mathcal{E}^{(\omega)}(\Theta)$ contains non-zero compactly supported functions (see [6] for more information). Let $\mathcal{A}(\Theta)$ be the space of real analytic functions on Θ. The inequality (2.2) implies that $\mathcal{A}(\Theta) \subseteq \mathcal{E}^{(\omega)}(\Theta)$.

Lemma 2.3. *Let $\Theta, \Theta' \subseteq \mathbb{R}^n$ be open, and let E be a lcHs. Let $\varphi : \Theta \to \Theta'$ be real analytic. Then, $f \circ \varphi \in \mathcal{E}^{(\omega)}(\Theta;E)$ for all $f \in \mathcal{E}^{(\omega)}(\Theta';E)$.*

Proof. This can be shown by adapting the proof of [25, Proposition 8.4.1]; the details are left to the reader. □

3. Spaces of vector-valued smooth and ultradifferentiable functions on manifolds

We shall now discuss vector-valued smooth and ultradifferentiable functions on manifolds. In this section we fix a smooth manifold M of dimension n. In the ultradifferentiable case, we shall always tacitly assume the manifold to be real analytic. Throughout this article the term *regular* will mean smooth if the manifold under consideration is smooth and real analytic if it is real analytic.

Let E be a lcHs and $j \in \mathbb{N} \cup \{\infty\}$. We define $C^j(M,E)$ ($\mathcal{E}^{(\omega)}(M;E)$) as the space consisting of all $f : M \to E$ such that $f \circ \varphi^{-1} \in C^j(\varphi(U),E)$ ($f \circ \varphi^{-1} \in \mathcal{E}^{(\omega)}(\varphi(U),E)$) for all regular charts (φ,U) of M. We endow $C^j(M;E)$ ($\mathcal{E}^{(\omega)}(M;E)$) with the initial topology with respect to the mappings

$$C^j(M;E) \to C^j(\varphi(U),E),\ f \mapsto f \circ \varphi^{-1},$$
$$(\mathcal{E}^{(\omega)}(M;E) \to \mathcal{E}^{(\omega)}(\varphi(U),E),\ f \mapsto f \circ \varphi^{-1}),$$

where (φ,U) runs over all regular charts of M. We have that $C^j(M;E)$ and $\mathcal{E}^{(\omega)}(M;E)$ are sequentially complete if E is so. We write $C(M;E) = C^0(M;E)$, $C^j(M) = C^j(M;\mathbb{C})$, and $\mathcal{E}^{(\omega)}(M) = \mathcal{E}^{(\omega)}(M;\mathbb{C})$. Note that Lemma 2.3 guarantees that we could have defined $\mathcal{E}^{(\omega)}(M;E)$ just through a given regular atlas, and that then its definition does not depend on the choice of the particular altlas. The latter is obviously true as well for $C^j(M;E)$.

We denote by $C^j_c(M)$, $j \in \mathbb{N} \cup \{\infty\}$, the subspace of $C^j(M)$ consisting of elements with compact support. We write $C_c(M) = C^0_c(M)$ and $\mathcal{D}(M) = C^\infty_c(M)$. We set $\mathcal{D}^{(\omega)}(M) = \mathcal{E}^{(\omega)}(M) \cap C_c(M)$. The non-quasianalyticity assumption (β) again ensures the non-triviality of $\mathcal{D}^{(\omega)}(M)$. We endow the spaces $C^j_c(M)$, $j \in \mathbb{N} \cup \{\infty\}$, and $\mathcal{D}^{(\omega)}(M)$ with their natural (LF)-space topology. For $K \subseteq M$ compact we denote by $\mathcal{D}_K$ ($\mathcal{D}^{(\omega)}_K$) for the subspace of $C^\infty(M)$ ($\mathcal{E}^{(\omega)}(M)$) consisting of elements with support in K.

Let E be a lcHs and let $f : M \to E$. For $v' \in E'$ we define the mapping

$$\langle v', f\rangle : M \to \mathbb{C},\ x \mapsto \langle v', f(x)\rangle.$$

If $f \in C^\infty(M;E)$, then $\langle v', f\rangle \in C^\infty(M)$ and for each vector field X on M it holds that $X\langle v', f\rangle = \langle v', Xf\rangle$.

Lemma 3.1. *Let E be a sequentially complete lcHs and let $f : M \to E$. Then, $f \in C^\infty(M;E)$ if and only if $\langle v', f\rangle \in C^\infty(M)$ for all $v' \in E'$.*

Proof. The result is well-known if M is an open subset of $\mathbb{R}^n$ (cf. [33, Appendice Lemme II]). The general case follows from it by using local coordinates. □

Let $U \subseteq M$ be open. A system $\mathbf{X} = (X_1, \ldots, X_n)$ of vector fields on U is called a *frame* on U if $(X_1|_x, \ldots, X_n|_x)$ is a basis of T_xU at each point $x \in U$. The frame $\mathbf{X}$ is said to be regular if the vector fields $X_1, \ldots, X_n$ are regular on U. For $\alpha = (\alpha_1, \ldots, \alpha_j) \in \{1, \ldots, n\}^j$, $j \in \mathbb{N}$, we write $X^\alpha = X_{\alpha_1} \cdots X_{\alpha_j}$.

Let $\mathbf{X}$ be a regular frame on the open subset $U \subseteq M$ and let E be a lcHs. Let K be a compact subset of U and $p \in \operatorname{csn}(E)$. For $j \in \mathbb{N}$ we define

$$p_{\mathbf{X},K,j}(f) = \max_{i \leq j} \max_{\alpha \in \{1,\ldots,n\}^i} \sup_{x \in K} p(X^\alpha f(x)), \qquad f \in C^\infty(U;E).$$

For $h > 0$ we define

$$p_{\mathbf{X},K,\omega,h}(f) = \sup_{j \in \mathbb{N}} \max_{\alpha \in \{1,\ldots,n\}^j} \sup_{x \in K} p(X^\alpha f(x)) \exp\left(-\frac{1}{h}\phi^*(hj)\right), \qquad f \in C^\infty(U;E).$$

For $E = \mathbb{C}$ and $p = |\cdot|$ we write $p_{\mathbf{X},K,j} = \|\cdot\|_{\mathbf{X},K,j}$ and $p_{\mathbf{X},K,\omega,h} = \|\cdot\|_{\mathbf{X},K,\omega,h}$. Given $p \in \operatorname{csn}(E)$, we set $V_p = \{v \in E \,|\, p(v) \leq 1\}$ and write V_p° for its polar set in E'. the bipolar theorem yields that for all $f \in C^\infty(U;E)$

$$p_{\mathbf{X},K,j}(f) = \sup_{v' \in V_p^\circ} \|\langle v', f\rangle\|_{\mathbf{X},K,j}, \qquad p_{\mathbf{X},K,\omega,h}(f) = \sup_{v' \in V_p^\circ} \|\langle v', f\rangle\|_{\mathbf{X},K,\omega,h}. \tag{3.1}$$

The next result will be frequently used throughout this article.

Proposition 3.2. *Let $U \subseteq M$ be open, let $\mathbf{X}$ and $\mathbf{Y}$ be regular frames on U, and let E be a lcHs.*

(*i*) *For all $K \subseteq U$ compact and $j \in \mathbb{N}$ there is $C > 0$ such that for all $p \in \operatorname{csn}(E)$*

$$p(f)_{\mathbf{X},K,j} \leq Cp(f)_{\mathbf{Y},K,j}, \qquad \forall f \in C^j(U;E).$$

(*ii*) *For all $K \subseteq U$ compact and $h > 0$ there are $C, k > 0$ such that for all $p \in \operatorname{csn}(E)$*

$$p(f)_{\mathbf{X},K,\omega,h} \leq Cp(f)_{\mathbf{Y},K,\omega,k}, \qquad \forall f \in C^\infty(U;E).$$

Proof. It suffices to consider the case $E = \mathbb{C}$, the general case follows from it by (3.1). Furthermore, by using local coordinates and a compactness argument, we may assume that U is an open subset of $\mathbb{R}^n$. The statement (i) is clear. We now show (ii). Let K be an arbitrary compact subset of U. For $j \in \mathbb{N}$ we define

$$|f|_{\mathbf{X},K,j} = \max_{\alpha \in \{1,\ldots,n\}^j} \sup_{x \in K} |X^\alpha f(x)|, \quad |f|_{\mathbf{Y},K,j} = \max_{\alpha \in \{1,\ldots,n\}^j} \sup_{x \in K} |Y^\alpha f(x)|, \qquad f \in C^\infty(U).$$

There are $a_{\ell,m} \in \mathcal{A}(U)$, $\ell, m = 1, \ldots, n$, such that

$$X_\ell(f)(x) = \sum_{m=1}^n a_{\ell,m}(x) Y_m(f)(x), \qquad f \in C^\infty(U),$$

for all $\ell = 1, \ldots, n$. Since $a_{\ell,m} \in \mathcal{A}(U)$ there is $H \geq 1$ such that

$$|a_{\ell,m}|_{\mathbf{X},K,j} \leq H^{j+1} j!, \qquad j \in \mathbb{N},$$

for all $\ell, m = 1, \ldots, n$ [20, Theorem 3.1] (which is originally a result due to Nelson [30]). We claim that for all $j \in \mathbb{N} \setminus \{0\}$

$$|f|_{\mathbf{X},K,j} \leq (2nH)^j \sum_{i=1}^{j} \binom{j}{i} (j-i)! |f|_{\mathbf{Y},K,i}, \qquad f \in C^\infty(U). \tag{3.2}$$

Before we show the claim, let us show how the result follows from it. Let $h > 0$ be arbitrary. By (2.1) there are $C, k > 0$ such that

$$\frac{1}{k}\phi^*(kt) + t\log(4nH) \leq \frac{1}{h}\phi^*(ht) + \log C, \qquad t \geq 0. \tag{3.3}$$

The convexity of ϕ^*, (3.2), and (3.3) imply that for all for $j \in \mathbb{N} \setminus \{0\}$ and $f \in C^\infty(U)$

$$\begin{aligned} |f|_{\mathbf{X},K,j} \exp\left(-\frac{1}{h}\phi^*(hj)\right) &\leq \frac{1}{2^j} \sum_{i=1}^{j} \binom{j}{i} (4nH)^{j-i} (j-i)! \exp\left(-\frac{1}{h}\phi^*(h(j-i))\right) \times \\ &\qquad (4nH)^i |f|_{\mathbf{Y},K,i} \exp\left(-\frac{1}{h}\phi^*(hi)\right) \\ &\leq C' \|f\|_{\mathbf{Y},K,\omega,k}, \end{aligned}$$

where

$$C' = C \sup_{l \in \mathbb{N}} (4nH)^l l! \exp\left(-\frac{1}{h}\phi^*(hl)\right) < \infty,$$

see (2.2). This shows the result. We now prove (3.2) by induction. The case $j = 1$ is clear (recall that $H \geq 1$). Let $j \in \mathbb{N} \setminus \{0\}$ and assume that (3.2) holds for all $l = 1, \ldots, j$. Let $f \in C^\infty(U)$ be arbitrary. For all $\alpha \in \{1, \ldots, n\}^j$, $\ell \in \{1, \ldots, n\}$, and

$x \in K$ it holds that

$$
\begin{aligned}
|X^\alpha X_\ell f(x)| &\leq \sum_{m=1}^{n} |X^\alpha (a_{\ell,m}(x) Y_m(f)(x))| \\
&\leq \sum_{m=1}^{n} \sum_{l=0}^{j} \binom{j}{l} |a_{\ell,m}|_{\mathbf{X},K,j-l} |Y_m(f)|_{\mathbf{X},K,l} \\
&\leq n \left(H^{j+1} j! |f|_{\mathbf{Y},K,1} + \sum_{l=1}^{j} \binom{j}{l} H^{j-l+1} (j-l)! (2nH)^l \sum_{i=1}^{l} \binom{l}{i} (l-i)! |f|_{\mathbf{Y},K,i+1} \right) \\
&\leq (nH)^{j+1} \left(j! |f|_{\mathbf{Y},K,1} + \sum_{l=1}^{j} \sum_{i=1}^{l} \frac{j!}{i!} 2^l |f|_{\mathbf{Y},K,i+1} \right) \\
&\leq (2nH)^{j+1} \left(j! |f|_{\mathbf{Y},K,1} + \sum_{i=1}^{j} \binom{j+1}{i+1} (j-i)! |f|_{\mathbf{Y},K,i+1} \right) \\
&\leq (2nH)^{j+1} \sum_{i=1}^{j+1} \binom{j+1}{i} (j+1-i)! |f|_{\mathbf{Y},K,i}.
\end{aligned}
$$

This shows the induction step. □

Proposition 3.2 yields the following result.

Proposition 3.3. *Let $(U_i)_{i \in I}$ be an open covering of M, let $\mathbf{X}_i$ be a regular frame on U_i for each $i \in I$, and let E be a lcHs.*

- *(i) The locally convex topology of $C^\infty(M;E)$ is generated by the system of seminorms $\{p_{\mathbf{X}_i,K_i,j} \,|\, i \in I, K_i \subseteq U_i \text{ compact}, j \in \mathbb{N}, p \in \operatorname{csn}(E)\}$.*
- *(ii) $f \in C^\infty(M;E)$ belongs to $\mathcal{E}^{(\omega)}(M;E)$ if and only if $p_{\mathbf{X}_i,K_i,\omega,h}(f) < \infty$ for all $i \in I$, $K_i \subseteq U_i$ compact, $h > 0$, and $p \in \operatorname{csn}(E)$. Moreover, the locally convex topology of $\mathcal{E}^{(\omega)}(M;E)$ is generated by the system of seminorms $\{p_{\mathbf{X}_i,K_i,\omega,h} \,|\, i \in I, K_i \subseteq U_i \text{ compact}, h > 0, p \in \operatorname{csn}(E)\}$.*

Remark 3.4. All the results from Sections 2 and 3 remain valid if we replace the condition (β) on ω by the weaker assumption $\omega(t) = o(t)$.

Remark 3.5. The theorem of iterates from [16, Corollary 3.19] also yields the set equality part from Proposition 3.3(ii), that is, the equivalence of $f \in \mathcal{E}^{(\omega)}(M;E)$ to having $p_{\mathbf{X}_i,K_i,\omega,h}(f) < \infty$ for all of these seminorms. However, it should be noticed that the method from [16], being based on reducing the Beurling case to the Roumieu case (see the proof of [16, Proposition 3.2]), does not deliver the required continuity estimates needed to conclude that the system $\{p_{\mathbf{X}_i,K_i,\omega,h} \,|\, i \in I, K_i \subseteq U_i \text{ compact}, h > 0, p \in \operatorname{csn}(E)\}$ generates the locally convex topology of $\mathcal{E}^{(\omega)}(M;E)$.

4. Spaces of smooth and ultradifferentiable vectors associated with Lie group representations

Throughout the rest of this article we fix a (real) Lie group G of dimension n with identity element e. In the ultradifferentiable case, we shall always tacitly view the underlying manifold structure of G as a real analytic one (cf. Section 3). Given a function f on G and $x \in G$, we define

$$\check{f}(y) = f(y^{-1}), \qquad L_x f(y) = f(x^{-1}y), \qquad R_x f(h) = f(yx), \qquad y \in G.$$

A frame $\mathbf{X} = \{X_1, \ldots, X_n\}$ on G is said to be left-invariant (right-invariant) if the vector fields $X_1, \ldots, X_n$ are left-invariant (right-invariant). Every left-invariant (right-invariant) frame on G is automatically regular and corresponds to the left-invariant (right-invariant) vector fields associated with a basis in the Lie algebra of G. We fix a left Haar measure on G. Unless explicitly stated otherwise, all integrals on G will be be meant with respect to this measure. All vector-valued integrals in this article are to be interpreted in the weak sense. If E is a sequentially complete lcHs, the integral $\int_G f(x)dx$ exists for all $f \in C_c(G; E)$. Let E be a lcHs. We denote by $\mathrm{GL}(E)$ the group of topological isomorphisms of E. By a *representation* of G on E we mean a group homomorphism $\pi : G \to \mathrm{GL}(E)$. The representation π is called *locally equicontinuous* if $\{\pi(x) \,|\, x \in K\}$ is equicontinuous for each compact set $K \subseteq G$. For $v \in E$ the mapping

$$\gamma_v : G \to E,\ x \mapsto \pi(x)v$$

is called the *orbit* of v. We denote by E^0 the space consisting of all $v \in E$ such that $\gamma_v \in C(G, E)$. The representation is said to be *continuous* if $E = E^0$.

Remark 4.1. Each continuous representation of G on a barrelled lcHs E is automatically locally equicontinuous, as follows from the Banach-Steinhaus theorem. In particular, this holds if E is a Fréchet space or an (LF)-space.

Let π be a representation of G on a lcHs E. Let $j \in \mathbb{N} \cup \{\infty\}$, We define E^j ($E^{(\omega)}$) as the space consisting of all $v \in E$ such that $\gamma_v \in C^j(G, E)$ ($\gamma_v \in \mathcal{E}^{(\omega)}(G, E)$) and endow this space with the initial topology with respect to the mapping

$$E^\infty \to C^j(G, E),\ v \mapsto \gamma_v$$
$$(E^{(\omega)} \to \mathcal{E}^{(\omega)}(G, E),\ v \mapsto \gamma_v).$$

Note that E^j and $E^{(\omega)}$ are sequentially complete if E is so. Let $\mathbf{X}$ be a regular frame on G. Let $K \subseteq G$ be compact and $p \in \operatorname{csn}(E)$. We write $p_{\mathbf{X},K,j}(v) = p_{\mathbf{X},K,j}(\gamma_v)$, $j \in \mathbb{N}$, and $p_{\mathbf{X},K,\omega,h}(v) = p_{\mathbf{X},K,\omega,h}(\gamma_v)$, $h > 0$, for $v \in E^\infty$. Proposition 3.3 implies that the locally convex topology of E^∞ is generated by the system of seminorms $\{p_{\mathbf{X},K,j} \,|\, K \subseteq G \text{ compact}, j \in \mathbb{N}, p \in \operatorname{csn}(E)\}$. Moreover,

$$E^{(\omega)} = \{v \in E^\infty \,|\, p_{\mathbf{X},K,\omega,h}(v) < \infty \text{ for all } K \subseteq G \text{ compact}, h > 0, p \in \operatorname{csn}(E)\}$$

and the system of seminorms $\{p_{\mathbf{X},K,\omega,h} \,|\, K \subseteq G \text{ compact}, h > 0, p \in \operatorname{csn}(E)\}$ generates its locally convex topology.

Let π be again a representation of G on a lcHs E and let $\mathbf{X}$ be a left-invariant frame on G. For $\alpha \in \{1, \ldots, n\}^j$, $j \in \mathbb{N}$, we define

$$X^\alpha v = X^\alpha \gamma_v(e), \qquad v \in E^\infty.$$

Since $\mathbf{X}$ is left-invariant, we have that

$$\gamma_{X^\alpha v} = X^\alpha \gamma_v, \qquad v \in E^\infty. \tag{4.1}$$

Let $p \in \operatorname{csn}(E)$. For $j \in \mathbb{N}$ we define

$$p_{\mathbf{X},j}(v) = \max_{i \leq j} \max_{\alpha \in \{1,\ldots,n\}^i} p(X^\alpha v), \qquad v \in E^\infty,$$

and for $h > 0$ we define

$$p_{\mathbf{X},\omega,h}(v) = \sup_{j \in \mathbb{N}} \max_{\alpha \in \{1,\ldots,n\}^j} p(X^\alpha v) \exp\left(-\frac{1}{h}\phi^*(hj)\right), \qquad v \in E^\infty. \tag{4.2}$$

Proposition 4.2. *Let π be a locally equicontinuous representation of G on a lcHs E and let $\mathbf{X}$ be a left-invariant frame on G.*

- (i) *The locally convex topology of E^∞ is generated by the system of seminorms $\{p_{\mathbf{X},j} \,|\, j \in \mathbb{N}, p \in \operatorname{csn}(E)\}$.*
- (ii) *$v \in E^\infty$ belongs to $E^{(\omega)}$ if and only if $p_{\mathbf{X},\omega,h}(v) < \infty$ for all $h > 0$ and $p \in \operatorname{csn}(E)$. Moreover, the locally convex topology of $E^{(\omega)}$ is generated by the system of seminorms $\{p_{\mathbf{X},\omega,h} \,|\, h > 0, p \in \operatorname{csn}(E)\}$.*

Proof. In view of (4.1), this is a consequence of Proposition 3.3 and the fact that π is locally equicontinuous. □

Example 4.3. For the Gevrey weights ω_s, Example 2.2 implies that the system of seminorms $\{p_{\mathbf{X},\omega_s,h} \,|\, h > 0, p \in \operatorname{csn}(E)\}$ is equivalent to $\{\widetilde{p}_{\mathbf{X},\omega_s,h} \,|\, h > 0, p \in \operatorname{csn}(E)\}$, where

$$\widetilde{p}_{\mathbf{X},\omega_s,h} = \sup_{j \in \mathbb{N}} \max_{\alpha \in \{1,\ldots,n\}^j} \frac{p(X^\alpha v)}{h^j j!^{1/s}}, \qquad v \in E^\infty.$$

Hence, $\{\widetilde{p}_{\mathbf{X},\omega_s,h} \,|\, h > 0, p \in \operatorname{csn}(E)\}$ also generates the locally convex topology of $E^{(\omega)}$. These spaces and their Roumieu type variants were considered by Goodman and Wallach [20, 21, 22] (cf. the introduction).

Let E be a sequentially complete lcHs. For $f \in C(G, E)$ and $\chi \in C_c(G)$ we define their (left-)convolution as

$$(f * \chi)(x) = \int_G f(y)\chi(y^{-1}x)dx = \int_G f(xy)\chi(y^{-1})dy, \qquad x \in G.$$

Note that

$$f * \chi = \int_G f(y)\, L_y\chi dy = \int_G R_y f\, \chi(y^{-1})dy.$$

Let $j \in \mathbb{N} \cup \{\infty\}$. If $\chi \in C^j_c(G)$ or $f \in C^j(G, E)$, then $f * \chi \in C^j(G, E)$. Let X be a vector field on G. If X is left-invariant, then

$$X(f * \chi) = f * (X\chi), \qquad f \in C(G, E), \chi \in C^j_c(G),$$

while if X is right-invariant, then

$$X(f * \chi) = (Xf) * \chi, \qquad f \in C^j(G,E), \chi \in C_c(G).$$

If $E = \mathbb{C}$, these statements hold under the weaker assumption that $f \in L^1_{\mathrm{loc}}(G)$.

Let π be a representation of G on a sequentially complete lcHs E. For $v \in E^0$ and $\chi \in C_c(G)$ we define

$$\Pi(\chi)v = \int_G \gamma_v(x)\chi(x^{-1})dx$$

Note that $\gamma_{\Pi(\chi)v} = \gamma_v * \chi$. Consequently, for $j \in \mathbb{N} \cup \{\infty\}$, it holds that $\Pi(\chi)v \in E^j$ if $\chi \in C^j_c(G)$.

Let $\mathbf{X}$ be a regular frame on G. For $j \in \mathbb{N}$ we define

$$\|f\|_{\mathbf{X},j} = \sup_{x \in G} \max_{i \leq j} \max_{\alpha \in \{1,\ldots,n\}^i} |X^\alpha f(x)|, \qquad f \in C^\infty(G).$$

For $h > 0$ we define

$$\|f\|_{\mathbf{X},\omega,h} = \sup_{x \in G} \sup_{j \in \mathbb{N}} \max_{\alpha \in \{1,\ldots,n\}^j} |X^\alpha f(x)| \exp\left(-\frac{1}{h}\phi^*(hj)\right), \qquad f \in C^\infty(G).$$

Lemma 4.4. *Let π be a locally equicontinuous representation of G on E and let $\mathbf{X}$ be a regular frame on G. Let $K, L \subseteq G$ be compact and $p \in \operatorname{csn}(E)$.*

(*i*) *There is $q \in \operatorname{csn}(E)$ such that for all $j \in \mathbb{N}$ there is $C > 0$ such that*

$$p_{\mathbf{X},K,j}(\Pi(\chi)v) \leq C\|\chi\|_{\mathbf{X},j}q(v), \qquad \forall v \in E^0, \chi \in C^j_c(G) \textit{ with } \operatorname{supp}\chi \subseteq L.$$

(*ii*) *There is $q \in \operatorname{csn}(E)$ such that for all $h > 0$ there is $C > 0$ such that*

$$p_{\mathbf{X},K,\omega,h}(\Pi(\chi)v) \leq C\|\chi\|_{\mathbf{X},\omega,h}q(v), \qquad \forall v \in E^0, \chi \in \mathcal{D}_L.$$

Proof. By Proposition 3.2 we may assume that $\mathbf{X}$ is left-invariant. Hence, for all $v \in E^0$, $\chi \in \mathcal{D}(G)$, and $\alpha \in \{1,\ldots,n\}^j$, $j \in \mathbb{N}$, it holds that

$$X^\alpha \gamma_{\Pi(\chi)v} = X^\alpha(\gamma_v * \chi) = \gamma_v * (X^\alpha \chi) = \int_G R_y\gamma_v\, X^\alpha\chi(y^{-1})dy.$$

The result now follows from the fact that π is locally equicontinuous. □

5. Some auxiliary results

5.1. **An approximation result.** We fix a regular chart (φ, U) on G with $\varphi(e) = 0$ and $\overline{B}(0,1) \subseteq \varphi(U)$. We define $S_\varepsilon = \varphi^{-1}(\overline{B}(0,\varepsilon))$ for $\varepsilon \in (0,1]$.

Lemma 5.1. *Let $\mathbf{X} = (X_1,\ldots,X_n)$ be a smooth frame on G and let E be a lcHs. For all $K \subseteq G$ compact there is $C > 0$ such that for all $p \in \operatorname{csn}(E)$ and $\varepsilon \in (0,1]$ it holds that*

$$\sup_{x \in K}\sup_{y \in S_\varepsilon} p(f(x) - f(xy)) \leq \varepsilon C \sup_{x \in K}\sup_{y \in S_1} \max_{i=1,\ldots,n} p(X_if(xy)), \qquad f \in C^\infty(G;E).$$

Proof. It suffices to consider the case $E = \mathbb{C}$, the general case follows from it by (3.1). Furthermore, we may assume that $\mathbf{X}$ is left-invariant. In this case, we will show the following stronger property: There is $C > 0$ such that for all $\varepsilon \in (0,1]$ it holds that

$$\sup_{y \in S_\varepsilon} |f(x) - f(xy)| \leq \varepsilon C \sup_{y \in S_1} \max_{i=1,\ldots,n} |X_i f(xy)|, \qquad f \in C^\infty(G), x \in G.$$

Set $\psi = \varphi^{-1}$ and denote by $\psi_* \partial_i$ the pushforward of ∂_i by ψ, $i = 1, \ldots, n$. There is $C > 0$ such that

$$\max_{i=1,\ldots,n} |(\psi_* \partial_i) f(y)| \leq C \max_{i=1,\ldots,n} |X_i f(y)|, \qquad f \in C^\infty(G), y \in S_1. \tag{5.1}$$

Let $f \in C^\infty(G)$ and $x \in G$ be arbitrary. The mean value theorem implies that for all $\varepsilon \in (0,1]$

$$\begin{aligned}
\sup_{y \in S_\varepsilon} |f(x) - f(xy)| &= \sup_{y \in S_\varepsilon} |(L_{x^{-1}} f)(e) - (L_{x^{-1}} f)(y)| \\
&= \sup_{x \in \overline{B}(0,\varepsilon)} |((L_{x^{-1}} f) \circ \psi)(0) - ((L_{x^{-1}} f) \circ \psi)(x)| \\
&\leq \varepsilon \sqrt{n} \sup_{t \in \overline{B}(0,1)} \max_{i=1,\ldots,n} |\partial_i ((L_{x^{-1}} f) \circ \psi)(t)| \\
&= \varepsilon \sqrt{n} \sup_{y \in S_1} \max_{i=1,\ldots,n} |(\psi_* \partial_i)(L_{x^{-1}} f)(y)|.
\end{aligned}$$

Applying (5.1) to $L_{x^{-1}} f$ and using the fact that $\mathbf{X}$ is left-invariant, we find that

$$\sup_{y \in S_\varepsilon} |f(x) - f(xy)| \leq \varepsilon C \sqrt{n} \sup_{y \in S_1} \max_{i=1,\ldots,n} |X_i (L_{x^{-1}} f)(y)| = \varepsilon C \sqrt{n} \sup_{y \in S_1} \max_{i=1,\ldots,n} |X_i f(xy)|.$$

$\square$

We are ready to show the main result of this subsection. For the Gevrey weights ω_s it may be considered as a quantified version of [20, Theorem 3.2 and Corollary 3.1].

Proposition 5.2. *Let π be a representation of G on a sequentially complete lcHs E and let $\mathbf{X}$ be a regular frame on G. Let $(\chi_\varepsilon)_{\varepsilon \in (0,1]} \subseteq \mathcal{D}(G)$ be such that $\chi_\varepsilon \geq 0$, $\operatorname{supp} \check{\chi}_\varepsilon \subseteq S_\varepsilon$, and $\int_G \check{\chi}_\varepsilon(x) dx = 1$ for all $\varepsilon \in (0,1]$.*

- *(i) For all $K \subseteq G$ compact and $j \in \mathbb{N}$ there is $C > 0$ such that for all $p \in \operatorname{csn}(E)$ and $\varepsilon \in (0,1]$*
$$p_{\mathbf{X},K,j}(v - \Pi(\chi_\varepsilon)(v)) \leq \varepsilon C p_{\mathbf{X},KS_1,j+1}(v), \qquad v \in E^\infty.$$
- *(ii) Suppose that $(\chi_\varepsilon)_{\varepsilon \in (0,1]} \subseteq \mathcal{D}^{(\omega)}(G)$. For all $h > 0$ there is $k > 0$ such that for all $K \subseteq G$ compact there is $C > 0$ such that for all $p \in \operatorname{csn}(E)$ and $\varepsilon \in (0,1]$*
$$p_{\mathbf{X},K,\omega,h}(v - \Pi(\chi_\varepsilon)(v)) \leq \varepsilon C p_{\mathbf{X},KS_1,\omega,k}(v), \qquad v \in E^{(\omega)}.$$

Proof. We will only show (ii) as the proof of (i) is similar but simpler. By Proposition 3.2(ii) we may assume that $\mathbf{X} = (X_1, \ldots, X_n)$ is right-invariant. Condition (2.1) implies that there are $C', k > 0$ such that

$$\frac{1}{k} \phi^*(k(t+1)) \leq \frac{1}{h} \phi^*(ht) + \log C', \qquad t \geq 0. \tag{5.2}$$

Let K be an arbitrary compact subset of G.

$$\sup_{x\in K}\sup_{y\in S_\varepsilon} p(f(x)-f(xy)) \leq \varepsilon C \sup_{x\in K}\sup_{y\in S_1}\max_{i=1,\dots,n} p(X_i f(xy)), \qquad \forall f \in C^\infty(G;E). \tag{5.3}$$

Let $p \in \operatorname{csn}(E)$, $\varepsilon \in (0,1]$, and $v \in E^{(\omega)}$ be arbitrary. Since $\mathbf{X}$ is right-invariant, we find that for all $x \in G$ and $\alpha \in \{1,\dots,n\}^j$, $j \in \mathbb{N}$,

$$\begin{aligned} X^\alpha(\gamma_v - \gamma_{\Pi(\chi_\varepsilon)v})(x) = X^\alpha(\gamma_v - \gamma_v * \chi_\varepsilon)(x) &= X^\alpha\gamma_v(x) - (X^\alpha\gamma_v) * \chi_\varepsilon(x) \\ &= \int_G (X^\alpha\gamma_v(x) - X^\alpha\gamma_v(xy))\check{\chi}_\varepsilon(y)dy. \end{aligned}$$

Hence, by applying (5.3) to $X^\alpha\gamma_v$, we obtain that

$$\sup_{x\in K} p(X^\alpha(\gamma_v - \gamma_{\Pi(\chi_\varepsilon)v})(x)) \leq \varepsilon C \sup_{x\in K}\sup_{y\in S_1}\max_{i=1,\dots,n} p(X_i X^\alpha \gamma_v(xy)).$$

By combining the latter inequality with (5.2), we find that

$$p_{\mathbf{X},K,\omega,h}(v - \Pi(\chi_\varepsilon)(v)) \leq \varepsilon C C' p_{\mathbf{X},KS_1,\omega,k}(v).$$

□

5.2. **The parametrix method.** Let E be a sequentially complete lcHs. Given a power series $P(z) = \sum_{i=0}^\infty a_i z^i$, $a_i \in \mathbb{C}$, we formally define

$$P(D)f = \sum_{i=0}^\infty a_i f^{(i)}, \qquad f \in C^\infty(\mathbb{R};E).$$

An entire function $P(z) = \sum_{i=0}^\infty a_i z^i$ is called an *ultrapolynomial of class* (ω) if there is $H > 0$ such that

$$\sup_{z\in\mathbb{C}} |P(z)| e^{-H\omega(|z|)} < \infty.$$

In such a case, by using the Cauchy estimates, we find that

$$\sup_{i\in\mathbb{N}} |a_i| \exp\left(H\phi^*\left(\frac{i}{H}\right)\right) < \infty.$$

Hence, (2.1) and the convexity of ϕ^* imply that for all $h > 0$ there is $k > 0$ such that

$$\sup_{j\in\mathbb{N}} \exp\left(-\frac{1}{h}\phi^*(hj)\right)\sum_{i=0}^\infty |a_i| \exp\left(\frac{1}{k}\phi^*(k(i+j))\right) < \infty. \tag{5.4}$$

Given an open set $\Theta \subseteq \mathbb{R}^n$ and $h > 0$, we set $\mathcal{D}^{\omega,h}(\Theta) = \mathcal{D}(\Theta)\bigcap \mathcal{E}^{\omega,h}(\mathbb{R}^n)$. We denote by $\mathcal{D}'(\mathbb{R})$ ($\mathcal{D}'^{(\omega)}(\mathbb{R})$) the dual of $\mathcal{D}(\mathbb{R})$ ($\mathcal{D}^{(\omega)}(\mathbb{R})$). We will assume the reader is familiar with the basic aspects of the spaces $\mathcal{D}'(\mathbb{R})$ and $\mathcal{D}'^{(\omega)}(\mathbb{R})$; see [6, 34] for more information. The following standard result is fundamental for us.

Proposition 5.3. *Let* $r > 0$.

(*i*) *For all* $j \in \mathbb{N}$ *there are a polynomial* P *and* $\psi_0, \psi_1 \in C^j_c((-r,r))$ *such that* $P(D)\psi_0 + \psi_1 = \delta$ *in* $\mathcal{D}'(\mathbb{R})$.

(*ii*) *For all* $k' > 0$ *there are an ultrapolynomial* $P(z) = \sum_{i=0}^\infty a_i z^i$ *of class* (ω) *and* $\psi_0, \psi_1 \in \mathcal{D}^{\omega,k'}((-r,r))$ *such that* $P(D)\psi_0 + \psi_1 = \delta$ *in* $\mathcal{D}'^{(\omega)}(\mathbb{R})$.

Proof. (i) Let H be the Heaviside function, that is, the indicator function of $[0,\infty)$. Choose $\psi \in \mathcal{D}((-r,r))$ such that $\psi = 1$ on a neighborhood of 0. Then, $P(z) = z^{j+2}$, $\psi_0(x) = \frac{x^{j+1}}{(j+1)!}H(x)\psi(x)$, and $\psi_1(x) = P(D)(\frac{x^{j+1}}{(j+1)!}H(x)(1-\psi(x)))$ verify all requirements.
(ii) This is shown in [19, Corollary 2.6]. □

Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a left-invariant frame on G and let E be a sequentially complete lcHs E. For $h > 0$ we define $E^{\omega,h}_{\mathbf{X}}$ as the space consisting of all $v \in E^\infty$ such that $p_{\mathbf{X},\omega,h}(v) < \infty$ for all $p \in \operatorname{csn}(E)$. We endow $E^{\omega,h}_{\mathbf{X}}$ with the locally convex topology generated by the system of seminorms $\{p_{\mathbf{X},\omega,h} \,|\, p \in \operatorname{csn}(E)\}$. Note that $E^{\omega,h}_{\mathbf{X}}$ is a sequentially complete lcHs. Let $P(z) = \sum_{i=0}^\infty a_i z^i$ be an ultrapolynomial of class (ω). Let $h > 0$ be arbitrary and choose $k > 0$ such that (5.4) holds. Then, for all $j = 1, \ldots, n$ the linear mapping

$$P(X_j) : E^{\omega,k}_{\mathbf{X}} \to E^{\omega,h}_{\mathbf{X}}, \quad P(X_j)v = \sum_{i=0}^\infty a_i X_j^i v$$

is continuous, and the series $P(X_j)v = \sum_{i=0}^\infty a_i X_j^i v$ converges absolutely in $E^{\omega,h}_{\mathbf{X}}$ for each $v \in E^{\omega,k}_{\mathbf{X}}$. Our goal in this subsection is to show the following parametrix type result.

Theorem 5.4. *Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a left-invariant frame on G and let U be an open neigbhorhood of e.*

(i) *For all $j \in \mathbb{N}$ there are a polynomial P and $\chi_\theta \in C^j_c(U)$, $\theta = (\theta_1, \ldots, \theta_n) \in \{0,1\}^n$, such that for all $v \in E^\infty$*

$$v = \sum_{\theta \in \{0,1\}^n} \Pi(\chi_\theta)(P_{\theta_n}(X_n) \cdots P_{\theta_1}(X_1)v), \tag{5.5}$$

where $P_0 = P$ and $P_1 = 1$.

(ii) *For all $h, h' > 0$ there are $k > 0$, an ultrapolynomial P of class (ω), $\chi_\theta \in \mathcal{D}(U)$ with $\|\chi_\theta\|_{\mathbf{X},\omega,h'} < \infty$, $\theta \in \{0,1\}^n$ such that (5.5) holds for all $v \in E^{\omega,k}_{\mathbf{X}}$, where $P_0 = P$ and $P_1 = 1$, and the linear mapping*

$$P_{\theta_n}(X_n) \cdots P_{\theta_1}(X_1) : E^{\omega,k}_{\mathbf{X}} \to E^{\omega,h}_{\mathbf{X}}$$

is continuous for each $\theta \in \{0,1\}^n$.

We will show Theorem 5.4 by combining Proposition 5.3 with the aid of a technique due to Dixmier and Malliavin [13]. We need various notions and results in preparation.

Lemma 5.5. *Let E be a sequentially complete lcHs.*

(i) *Let P be a polynomial and $\psi_0, \psi_1 \in C_c(\mathbb{R})$ such that $P(D)\psi_0 + \psi_1 = \delta$ in $\mathcal{D}'(\mathbb{R})$. Then, for all $f \in C^\infty(\mathbb{R}; E)$*

$$f = (P(D)f) * \psi_0 + f * \psi_1. \tag{5.6}$$

(ii) *Let $P(z) = \sum_{i=0}^\infty a_i z^i$ be an ultrapolynomial of class (ω) and $\psi_0, \psi_1 \in C_c(\mathbb{R})$ such that $P(D)\psi_0 + \psi_1 = \delta$ in $\mathcal{D}'^{(\omega)}(\mathbb{R})$. Let $f \in C^\infty(\mathbb{R}; E)$ be such that $P(D)f = \sum_{i=0}^\infty a_i f^{(i)}$ converges absolutely in $C(\mathbb{R}; E)$. Then, (5.6) holds.*

Proof. We only show (ii) as the proof of (i) is similar. Recall that $f_{v'} = \langle v', f\rangle \in C^\infty(\mathbb{R})$ for $v' \in E'$. Note that $P(D)f_{v'} = \sum_{n=0}^\infty a_n f_{v'}^{(n)}$ converges absolutely in $C(\mathbb{R})$ and that $P(D)f_{v'} = \langle v', P(D)f\rangle$. In $\mathcal{D}'^{(\omega)}(\mathbb{R})$, we have that

$$f_{v'} = f_{v'} * \delta = f_{v'} * (P(D)\psi_0) + f_{v'} * \psi_1 = (P(D)f_{v'}) * \psi_0 + f_{v'} * \psi_1.$$

Since $P(D)f_{v'}, f_{v'} \in C(\mathbb{R})$ and $\psi_0, \psi_1 \in C_c(\mathbb{R})$, the equality $f_{v'} = (P(D)f_{v'})*\psi_0 + f_{v'}*\psi_1$ in fact holds pointwise. We have that

$$(P(D)f_{v'}) * \psi_0 = \langle v', P(D)f\rangle * \psi_0 = \langle v', P(D)f * \psi_0\rangle$$

and

$$f_{v'} * \psi_1 = \langle v', f * \psi_1\rangle.$$

Consequently, we find that for all $v' \in E'$

$$\langle v', f\rangle = f_{v'} = (P(D)f_{v'}) * \psi_0 + f_{v'} * \psi_1 = \langle v', (P(D)f) * \psi_0 + f * \psi_1\rangle,$$

which implies that $f = (P(D)f) * \psi_0 + f * \psi_1$ by the Hahn-Banach theorem. □

We denote by $\mathfrak{g}$ the Lie algebra of G and by $\exp : \mathfrak{g} \to G$ the exponential mapping. We identify each $X \in \mathfrak{g}$ with its associated left-invariant vector field on G, that is,

$$Xf(x) = \frac{d}{dt} f(x\exp(tX))|_{t=0}, \qquad f \in C^\infty(G), x \in G. \tag{5.7}$$

Let π be a representation of G on a sequentially complete lcHs E. For $X \in \mathfrak{g}$ we define the representation

$$\pi_X : (\mathbb{R}, +) \to \mathrm{GL}(E), \qquad \pi_X(t) = \pi(\exp(tX)).$$

We denote the orbit of $v \in E$ under π_X by $\gamma_{X,v}$. Note that $\gamma_{X,v}(t) = \gamma_v(\exp(tX))$. Hence, $\gamma_{X,v} \in C(\mathbb{R}; E)$ ($\gamma_{X,v} \in C^\infty(\mathbb{R}; E)$) if $v \in E^0$ ($v \in E^\infty$). In accordance to Section 4, we set

$$\Pi_X(f)v = \int_{\mathbb{R}} \gamma_{X,v}(t) f(-t)dt, \qquad f \in C_c(\mathbb{R}), v \in E^0.$$

It holds that

$$\gamma_{X,\Pi_X(f)v} = \gamma_{X,v} * f, \qquad f \in C_c(\mathbb{R}), v \in E^0.$$

For $i \in \mathbb{N}$ we write $X^i = X \cdots X$, where X occurs i times, and set

$$X^i v = X^i \gamma_v(e), \qquad v \in E^\infty.$$

We have that

$$\gamma_{X,X^i v} = \gamma_{X,v}^{(i)}, \qquad v \in E^\infty.$$

Given a power series $P(z) = \sum_{i=0}^\infty a_i z^i$, we formally define

$$P(X)v = \sum_{i=0}^\infty a_i X^i v, \qquad v \in E^\infty.$$

Lemma 5.6. *Let $X \in \mathfrak{g}$ and let π be a locally equicontinuous representation of G on a sequentially complete lcHs E.*

(*i*) *Let P be a polynomial and $\psi_0, \psi_1 \in C_c(\mathbb{R})$ such that $P(D)\psi_0 + \psi_1 = \delta$ in $\mathcal{D}'(\mathbb{R})$. Then, for all $v \in E^\infty$*

$$v = \Pi_X(\psi_0)P(X)v + \Pi_X(\psi_1)v. \tag{5.8}$$

(*ii*) *Let $P(z) = \sum_{i=0}^\infty a_i z^i$ be an ultrapolynomial of class (ω) and $\psi_0, \psi_1 \in C_c(\mathbb{R})$ such that $P(D)\psi_0 + \psi_1 = \delta$ in $\mathcal{D}'^{(\omega)}(\mathbb{R})$. Let $v \in E^\infty$ be such that $P(X)v = \sum_{i=0}^\infty a_i X^i v$ converges absolutely in E. Then, (5.8) holds*

Proof. Since π is locally equicontinuous, our assumption yields

$$P(D)\gamma_{X,v} = \sum_{i=0}^\infty a_i \gamma_{X,v}^{(i)} = \sum_{i=0}^\infty a_i \gamma_{X,X^i v}$$

converges absolutely in $C(\mathbb{R}; E)$. Moreover, it holds that $P(D)\gamma_{X,v} = \gamma_{X,P(X)v}$. Hence, Lemma 5.5 implies that

$$\gamma_{X,v} = (P(D)\gamma_{X,v}) * \psi_0 + \gamma_{X,v} * \psi_1 = \gamma_{X,P(X)v} * \psi_0 + \gamma_{X,v} * \psi_1 = \gamma_{X,\Pi_X(\psi_0)P(X)v} + \gamma_{X,\Pi_X(\psi_1)v}.$$

The result now follows by evaluating the above equality at 0. □

Let $\mathbf{X} = (X_1, \dots, X_n)$ be a basis of $\mathfrak{g}$ and consider the mapping

$$\Phi : \mathbb{R}^n \to G, \quad \Phi(t_1, \dots, t_n) = \exp(t_1 X_1) \cdots \exp(t_n X_n).$$

Then, there is $r_0 > 0$ such $\Phi : (-r_0, r_0)^d \to G$ is a regular diffeomorphism onto its image.

Lemma 5.7. *Let $\mathbf{X}$ be a basis of $\mathfrak{g}$ and let Φ, r_0 be as above, let π be a representation of G on a sequentially complete lcHs E, and let $r < r_0$.*

(*i*) *Let $j \in \mathbb{N}$. For all $\psi_1, \dots, \psi_n \in C_c^j((-r, r))$ there is $\chi \in C^j(\Phi((-r, r)^n))$ such that*

$$\Pi_{X_1}(\psi_1) \cdots \Pi_{X_n}(\psi_n)v = \Pi(\check{\chi})v, \qquad v \in E^0. \tag{5.9}$$

(*ii*) *For all $h' > 0$ there is $k' > 0$ such that for all $\psi_1, \dots, \psi_n \in \mathcal{D}^{\omega,k'}((-r, r))$ there is $\chi \in \mathcal{D}(\Phi((-r, r)^n))$ with $\chi \circ \Phi \in \mathcal{D}^{\omega,h'}((-r, r)^n)$ such that (5.9) holds.*

Proof. We only show (*ii*) as the proof of (*i*) is similar. Condition (2.1) yields that there are $C, k' > 0$ such that

$$\frac{1}{k'}\phi^*(k't) + (\log 2)t \le \frac{1}{h'}\phi^*(h't) + \log C, \qquad t \ge 0.$$

This inequality together with the convexity of ϕ^* implies that

$$f\psi \in \mathcal{D}^{\omega,h'}((-r, r)^n), \qquad f \in \mathcal{E}^{\omega,k'}((-r, r)^n), \psi \in \mathcal{D}^{\omega,k'}((-r, r)^n). \tag{5.10}$$

Let $\psi_1, \dots, \psi_n \in \mathcal{D}^{\omega,k'}((-r, r))$ be arbitrary. Then $\psi_1 \otimes \cdots \otimes \psi_n \in \mathcal{D}^{\omega,k'}((-r, r)^n)$ because ϕ^* is convex. Note that there is a real analytic function $J : \Phi((-r_0, r_0)^n) \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^n} f(t)\psi(t)dt = \int_G f(\Phi^{-1}(x))\psi(\Phi^{-1}(x))J(x)dx, \quad f \in C(\mathbb{R}^n; E), \psi \in \mathcal{D}((-r_0, r_0)^n).$$

Set $\chi(x) = (\psi_1 \otimes \cdots \otimes \psi_n)(-\Phi^{-1}(x))J(x)$, $x \in G$. Since $\psi_1 \otimes \cdots \otimes \psi_n \in \mathcal{D}^{\omega,k'}((-r,r)^n)$ and $J \circ \Phi \in \mathcal{A}((-r,r)^n) \subseteq \mathcal{E}^{\omega,k'}((-r,r)^n)$, (5.10) implies that $\chi \circ \Phi \in \mathcal{D}^{\omega,h'}((-r,r)^n)$. For all $v \in E^0$ it holds that

$$\begin{aligned}\Pi_{X_1}(\psi_1)\cdots\Pi_{X_n}(\psi_n)v &= \int_{\mathbb{R}^n} \gamma_v(\Phi(t))(\psi_1 \otimes \cdots \otimes \psi_n)(-t)dt \\ &= \int_G \gamma_v(x)(\psi_1 \otimes \cdots \otimes \psi_n)(-\Phi^{-1}(x))J(x)dx \\ &= \int_G \gamma_v(x)\chi(x)dx = \Pi(\check{\chi})v.\end{aligned}$$

□

Proof of Theorem 5.4. This follows from Proposition 5.3, and Lemmas 5.6 and 5.7 (for (ii) we also use Proposition 3.2(ii)). □

Remark 5.8. Cartier also employed the parametrix method to study smooth vectors in [7]. Our applications to be discussed in the following two sections will require the more explicit form we have treated in this subsection.

6. Main results

We are ready to study quasinormability and the property (Ω) for the spaces E^∞ and $E^{(\omega)}$.

6.1. **Quasinormability.** Given a lcHs F, we denote by $\mathcal{U}_0(F)$ the family of all absolutely convex neighborhoods of 0 in F and by $\mathcal{B}(F)$ the family of all bounded subsets of F. Given $p \in \operatorname{csn}(F)$ we write $V_p = \{v \in F \mid p(v) \leq 1\} \in \mathcal{U}_0(F)$. The space F is said to be quasinormable [29, p. 313] if

$$\forall U \in \mathcal{U}_0(F)\, \exists V \in \mathcal{U}_0(F)\, \forall \varepsilon \in (0,1]\, \exists B \in \mathcal{B}(F) \, : V \subseteq \varepsilon U + B.$$

We are ready to prove the first main result of this article.

Theorem 6.1. *Let π be a locally equicontinuous representation of G on a sequentially complete lcHs E.*

(i) E^∞ is quasinormable if E is so.
(ii) $E^{(\omega)}$ is quasinormable if E is so.

Proof. We only show (ii) as the proof of (i) is similar. Let $\mathbf{X}$ be a real analytic frame on G. By Proposition 3.3(ii) it suffices to show that

$$\begin{gathered}\forall K \subseteq G \text{ compact}, h > 0, p \in \operatorname{csn}(E)\, \exists L \subseteq G \text{ compact}, k > 0, q \in \operatorname{csn}(E) \\ \forall \varepsilon \in (0,1]\, \exists B \in \mathcal{B}(E^{(\omega)}) \, : \, V_{q_{\mathbf{X},L,\omega,k}} \subseteq \varepsilon V_{p_{\mathbf{X},K,\omega,h}} + B.\end{gathered}$$

Let $K \subseteq G$ compact, $h > 0$, and $p \in \operatorname{csn}(E)$ be arbitrary. Take $(\chi_\varepsilon)_{\varepsilon \in (0,1]} \subseteq \mathcal{D}^{(\omega)}(G)$ as in Proposition 5.2. Set $L = KS_1$. Hence, there are $C, k > 0$ such that for all $\varepsilon \in (0,1]$

$$p_{\mathbf{X},K,\omega,h}(v - \Pi(\chi_\varepsilon)(v)) \leq \varepsilon C p_{\mathbf{X},L,\omega,k}(v), \qquad v \in E^{(\omega)}.$$

Furthermore, Lemma 4.4(ii) implies that there is $r \in \operatorname{csn}(E)$ such that for all $\varepsilon \in (0,1]$

$$p_{\mathbf{X},K,\omega,h}(\Pi(\chi_\varepsilon)v) \leq \|\chi_\varepsilon\|_{\mathbf{X},\omega,h} r(v), \qquad v \in E^0.$$

Since E is quasinormable, there is $s \in \operatorname{csn}(E)$ such that

$$\forall \delta > 0 \, \exists A \in \mathcal{B}(E) \, : \, V_s \subseteq \delta V_r + A. \tag{6.1}$$

Set $q = \max\{s, Cp\} \in \operatorname{csn}(E)$. Let $\varepsilon \in (0,1]$ be arbitrary. Choose $A \in \mathcal{B}(E)$ according to (6.1) with $\delta = \varepsilon/(2\|\chi_{\varepsilon/2}\|_{\mathbf{X},\omega,h})$. Set $B = \{\Pi(\chi_{\varepsilon/2})(u) \mid u \in A\}$. Note that $B \in \mathcal{B}(E^{(\omega)})$ by Lemma 4.4(ii). Let $v \in V_{q_{\mathbf{X},L,\omega,k}}$ be arbitrary. There is some $u \in A$ such that $r(v-u) \leq \varepsilon/(2\|\chi_{\varepsilon/2}\|_{\mathbf{X},\omega,h})$. For $w = \Pi(\chi_{\varepsilon/2})(u) \in B$ we obtain that

$$\begin{aligned} p_{\mathbf{X},K,\omega,h}(v-w) &\leq p_{\mathbf{X},K,\omega,h}((v - \Pi(\chi_{\varepsilon/2})(v)) + p_{\mathbf{X},K,\omega,h}(\Pi(\chi_{\varepsilon/2})(v-u)) \\ &\leq \frac{\varepsilon C}{2} p_{\mathbf{X},L,\omega,k}(v) + \|\chi_{\varepsilon/2}\|_{\mathbf{X},\omega,h} r(v-u) \leq \varepsilon, \end{aligned}$$

which shows the result. $\square$

6.2. **The condition** (Ω)**.** A Fréchet space F is said to satisfy the condition (Ω) [29, p. 367] if

$$\forall U \in \mathcal{U}_0(F) \, \exists V \in \mathcal{U}_0(F) \, \forall W \in \mathcal{U}_0(F) \, \exists C, s > 0 \, \forall \varepsilon \in (0,1] \, : \, V \subseteq \varepsilon U + \frac{C}{\varepsilon^s} W.$$

Remark 6.2. By [29, Lemma 26.14] a Fréchet space F is quasinormable if and only if

$$\forall U \in \mathcal{U}_0(F) \, \exists V \in \mathcal{U}_0(F) \, \forall W \in \mathcal{U}_0(F) \, \forall \varepsilon \in (0,1] \, \exists C > 0 \, : \, V \subseteq \varepsilon U + CW.$$

Hence, (Ω) may be considered as a quantified version of quasinormability.

The second main result of this article reads as follows.

Theorem 6.3. *Let π be a locally equicontinuous representation of G on a Fréchet space E.*

(i) *E^∞ satisfies (Ω) if E does so.*
(ii) *$E^{(\omega)}$ satisfies (Ω) if E does so.*

We first prove Theorem 6.3(i). This shall be done by using a refinement of the technique employed in the proof of Theorem 6.1. For $j \in \mathbb{N}$ we write

$$\|f\|_j = \sup_{t \in \mathbb{R}^n} \max_{|\alpha| \leq j} |f^{(\alpha)}(t)|, \qquad f \in C^\infty(\mathbb{R}^n).$$

Proof of Theorem 6.3(i). We use the same notation as in Subsection 5.1. Let $\mathbf{X}$ be a smooth frame on G. Pick $\psi \in \mathcal{D}(B(0,1))$ such that $\psi \geq 0$ and $\int_{\mathbb{R}^n} \psi(t)dt = 1$. Set $\psi_\varepsilon(t) = \varepsilon^{-n}\psi(t/\varepsilon)$ for $\varepsilon \in (0,1]$. Define $\chi_\varepsilon \in \mathcal{D}(G)$ via

$$\check{\chi}_\varepsilon = \frac{\psi_\varepsilon \circ \varphi}{\|\psi_\varepsilon \circ \varphi\|_{L^1(G)}}.$$

Then, $(\chi_\varepsilon)_{\varepsilon \in (0,1]}$ satisfies the assumptions of Proposition 5.2. There is a positive smooth function J on U such that

$$\int_{\mathbb{R}^n} f(t)dt = \int_G f(\varphi(x))J(x)dx, \qquad f \in \mathcal{D}(\varphi(U)).$$

Hence, we obtain that for all $\varepsilon \in (0,1]$

$$\|\psi_\varepsilon \circ \varphi\|_{L^1(G)} \geq \frac{1}{\sup_{g\in\varphi^{-1}(\overline{B}(0,1))} J(g)}.$$

Proposition 3.2(i) therefore implies that for all $j \in \mathbb{N}$ there is $C_j > 0$ such that for all $\varepsilon \in (0,1]$

$$\|\chi_\varepsilon\|_{\mathbf{X},j} \leq C_j \|\psi_\varepsilon\|_j \leq \frac{C_j\|\psi\|_j}{\varepsilon^j}. \tag{6.2}$$

We are ready to show that E^∞ satisfies (Ω). By Proposition 3.3(i) and rescaling, it suffices to show that

$$\forall K \subseteq G \text{ compact}, j \in \mathbb{N}, p \in \operatorname{csn}(E)\, \exists q \in \operatorname{csn}(E)\, \forall L \subseteq G \text{ compact}, m \in \mathbb{N}, r \in \operatorname{csn}(E)$$
$$\exists C_1, C_2, s > 0\, \forall \varepsilon \in (0,1] \,:\, V_{q_{\mathbf{X},KS_1,j+1}} \subseteq \varepsilon C_1 V_{p_{\mathbf{X},K,j}} + \frac{C_2}{\varepsilon^s} V_{r_{\mathbf{X},L,m}}$$

Let $K \subseteq G$ compact, $j \in \mathbb{N}$, $p \in \operatorname{csn}(E)$ be arbitrary. Proposition 5.2(i) implies that there is $C > 0$ such that

$$p_{\mathbf{X},K,j}(v - \Pi(\chi_\varepsilon)v) \leq \varepsilon C p_{\mathbf{X},KS_1,j+1}(v), \qquad v \in E^\infty.$$

By Lemma 4.4(i), there is $p' \in \operatorname{csn}(E)$ such that for all $\varepsilon \in (0,1]$

$$p_{\mathbf{X},K,j}(\Pi(\chi_\varepsilon)v) \leq \|\chi_\varepsilon\|_{\mathbf{X},j} p'(v), \qquad v \in E^0.$$

Since E satisfies (Ω), there is $q \in \operatorname{csn}(E)$, $q \geq \max\{p, p'\}$, such that

$$\forall r' \in \operatorname{csn}(E)\, \exists C', s' > 0\, \forall \delta \in (0,1] \,:\, V_q \subseteq \delta V_{p'} + \frac{C'}{\delta^{s'}} V_{r'}. \tag{6.3}$$

Let $L \subseteq G$ compact, $m \in \mathbb{N}$, and $r \in \operatorname{csn}(E)$ be arbitrary. Lemma 4.4(i) implies that there is $r' \in \operatorname{csn}(E)$ such that for all $\varepsilon \in (0,1]$

$$r_{\mathbf{X},L,m}(\Pi(\chi_\varepsilon)v) \leq \|\chi_\varepsilon\|_{\mathbf{X},m} r'(v), \qquad v \in E^0.$$

Choose $C', s' > 0$ according to (6.3). Let $v \in V_{q_{\mathbf{X},KS_1,j+1}}$ be arbitrary. By (6.3) we have that for all $\delta \in (0,1]$ there is $w_\delta \in C'\delta^{-s'} V_{r'}$ such that $v - w_\delta \in \delta V_{p'}$. For all $\varepsilon, \delta \in (0,1]$ it holds that

$$\begin{aligned} p_{\mathbf{X},K,j}(v - \Pi(\chi_\varepsilon)w_\delta) &\leq p_{\mathbf{X},K,j}(v - \Pi(\chi_\varepsilon)v) + p_{\mathbf{X},K,j}(\Pi(\chi_\varepsilon)(v - w_\delta)) \\ &\leq \varepsilon C + \|\chi_\varepsilon\|_{\mathbf{X},j} p'(v - w_\delta) \\ &\leq \varepsilon C + \frac{C_j\|\psi\|_j}{\varepsilon^j}\delta \end{aligned}$$

and

$$r_{\mathbf{X},L,m}(\Pi(\chi_\varepsilon)w_\delta) \leq \|\chi_\varepsilon\|_{\mathbf{X},m} r'(w_\delta) \leq \frac{C'C_l\|\psi\|_m}{\varepsilon^m \delta^{s'}}.$$

We set $s = m + (j+1)s'$ and $\delta_\varepsilon = \varepsilon^{j+1}$ for $\varepsilon \in (0,1]$. Then, for all $\varepsilon \in (0,1]$

$$v = (v - \Pi(\chi_\varepsilon)w_{\delta_\varepsilon}) + \Pi(\chi_\varepsilon)w_{\delta_\varepsilon} \in \varepsilon(C + C_j\|\psi\|_j)V_{p_{\mathbf{X},K,j}} + \frac{C'C_l\|\psi\|_m}{\varepsilon^s} V_{r_{\mathbf{X},L,l}}.$$

□

Next, we show Theorem 6.3(*ii*). Our proof is based on the parametrix method presented in Subsection 5.2. We need two results in preparation.

Lemma 6.4. *Let $F_1 \supseteq F_2 \supseteq \cdots$ be a decreasing sequence of Fréchet spaces with continuous inclusion mappings. Set $F = \bigcap_{j\in\mathbb{N}} F_j$ and endow F with its natural Fréchet space topology, i.e., the initial topology with respect to the inclusion mappings $F \to F_j$, $j \in \mathbb{N}$. Suppose that*

$$\forall i \in \mathbb{N}, U \in \mathcal{U}_0(F_i)\, \exists j \geq i, V \in \mathcal{U}_0(F_j)\, \forall m \geq j, W \in \mathcal{U}_0(F_m)$$
$$\exists C, s > 0\, \forall \varepsilon \in (0,1] \,:\, V \subseteq \varepsilon U + \frac{C}{\varepsilon^s} W.$$

Then, F satisfies (Ω).

Proof. This follows from the Mittag-Leffler theorem [39, Theorem 3.2.8], see [10, Lemma 2.4] for details. □

Our next step is to show a quantified approximation result for compactly supported ultradifferentiable functions on G. We first prove an analogous result for periodic ultradifferentiable functions on $\mathbb{R}^n$. Let $a > 0$. We denote by $C^\infty_a(\mathbb{R}^n)$ the Fréchet space consisting of all smooth $a\mathbb{Z}^n$-periodic functions. For $h > 0$ we write

$$\|f\|_{\omega,h} = \sup_{t\in\mathbb{R}^n} \sup_{\alpha\in\mathbb{N}^n} |f^{(\alpha)}(t)| \exp\left(-\frac{1}{h}\phi^*(h|\alpha|)\right), \qquad f \in C^\infty(\mathbb{R}^n).$$

Lemma 6.5. *Let $a > 0$. Then,*

$$\forall h > 0\, \exists k > 0\, \forall l > 0\, \exists C, s > 0\, \forall f \in C^\infty_a(\mathbb{R}^n), \|f\|_{\omega,k} \leq 1$$
$$\forall \varepsilon \in (0,1]\, \exists f_\varepsilon \in C^\infty_a(\mathbb{R}^n) \,:\, \|f - f_\varepsilon\|_{\omega,h} \leq \varepsilon \quad \text{and} \quad \|f_\varepsilon\|_{\omega,l} \leq \frac{C}{\varepsilon^s}.$$

Proof. Define the Fréchet space

$$s(\mathbb{Z}^n) = \{c = (c_\alpha)_{\alpha\in\mathbb{Z}^n} \in \mathbb{C}^{\mathbb{Z}^n} \mid \sup_{\alpha\in\mathbb{Z}^n} |c_\alpha||\alpha|^k < \infty,\, \forall k \in \mathbb{N}\}.$$

Given $f \in C^\infty_a(\mathbb{R}^n)$, we define its Fourier coefficients as

$$\widehat{f}(\alpha) = \frac{1}{a}\int_0^a f(t) e^{-\frac{2\pi i}{a}\alpha\cdot t} dt, \qquad \alpha \in \mathbb{Z}^n.$$

Then

$$\mathcal{F} : C^\infty_a(\mathbb{R}^n) \to s(\mathbb{Z}^n),\, f \mapsto \widehat{f} = (\widehat{f}(\alpha))_{\alpha\in\mathbb{Z}^n}$$

is a topological isomorphism whose inverse is given by

$$\mathcal{F}^{-1}(c)(t) = \sum_{\alpha\in\mathbb{Z}^n} c_\alpha e^{\frac{2\pi i}{a}\alpha\cdot t}, \qquad c = (c_\alpha)_{\alpha\in\mathbb{Z}^n} \in s(\mathbb{Z}^n), \tag{6.4}$$

and the series in the right-hand side is absolutely convergent in $C^\infty_a(\mathbb{R}^n)$. For $h > 0$ we define

$$|c|_{\omega,h} = \sup_{\alpha\in\mathbb{Z}^n} |c_\alpha| e^{\frac{1}{h}\omega(|\alpha|)}, \qquad c \in s(\mathbb{Z}^n).$$

By using (2.1), a standard argument shows that for all $h > 0$ there are $C, k > 0$ such that

$$|\widehat{f}|_{\omega,h} \leq C\|f\|_{\omega,k}, \qquad f \in C^\infty_a(\mathbb{R}^n),$$

and that for all $h > 0$ there are $C, k > 0$ such that

$$\|\mathcal{F}^{-1}(c)\|_{\omega,h} \leq C|f|_{\omega,k}, \qquad f \in C^\infty_a(\mathbb{R}^n).$$

Since $\mathcal{F} : C^\infty_a(\mathbb{R}^n) \to s(\mathbb{Z}^n)$ is an isomorphism whose inverse is given by (6.4), it therefore suffices to show that

$$\forall h > 0\, \exists k > 0\, \forall l > 0\, \exists C, s > 0\, \forall c \in s(\mathbb{Z}^n), |c|_{\omega,k} \leq 1$$
$$\forall \varepsilon \in (0, 1]\, \exists c_\varepsilon \in s(\mathbb{Z}^n) : |c - c_\varepsilon|_{\omega,h} \leq \varepsilon \quad \text{and} \quad |c_\varepsilon|_{\omega,l} \leq \frac{C}{\varepsilon^s}.$$

Let $h > 0$ be arbitrary. Set $k = h/2$. Let $l > 0$ be arbitrary. Let $c \in s(\mathbb{Z}^n)$ with $|c|_{\omega,k} \leq 1$ be arbitrary. For $\varepsilon \in (0, 1]$ we choose a number $N_\varepsilon \in [0, \infty)$ such that $\omega(N_\varepsilon) = h \log(1/\varepsilon)$. We define $c_\varepsilon = (c_{\varepsilon,\alpha})_{\alpha\in\mathbb{Z}^n} \in s(\mathbb{Z}^n)$ by $c_{\varepsilon,\alpha} = c_\alpha$ if $|\alpha| \leq N_\varepsilon$ and $c_{\varepsilon,\alpha} = 0$ otherwise. Set $s = h/l$. Then, for all $\varepsilon \in (0, 1]$ it holds that $|c - c_\varepsilon|_{\omega,h} \leq \varepsilon$ and $|c_\varepsilon|_{\omega,l} \leq \varepsilon^{-s}$. □

Lemma 6.6. *Let* $\mathbf{X}$ *be a real analytic frame on* G*. Let* K *and* L *be compact subsets of* G *such that* $K \subseteq \operatorname{int} L$*. Then,*

$$\forall h > 0\, \exists k > 0\, \forall l > 0\, \exists C, s > 0\, \forall f \in \mathcal{D}_K, \|f\|_{\mathbf{X},\omega,k} \leq 1$$
$$\forall \varepsilon \in (0, 1]\, \exists f_\varepsilon \in \mathcal{D}_L \ : \|f - f_\varepsilon\|_{\mathbf{X},\omega,h} \leq \varepsilon \quad \textit{and} \quad \|f_\varepsilon\|_{\mathbf{X},\omega,l} \leq \frac{C}{\varepsilon^s}.$$

Proof. By using local coordinates, a partition of the unity argument, and Proposition 3.2(ii), we may assume that $G = \mathbb{R}^n$ and that $\mathbf{X} = \{\partial_1, \ldots, \partial_n\}$. Choose $a > 0$ such that $L \subseteq (-a, a)^n$. Let $h > 0$ be arbitrary. Condition (2.1) yields that there are $C', h' > 0$ such that

$$\frac{1}{h'}\phi^*(h't) + (\log 2)t \leq \frac{1}{h}\phi^*(ht) + \log C', \qquad t \geq 0.$$

Pick $k > 0$ according to Lemma 6.5 with $h = h'$. Let $l > 0$ be arbitrary. By (2.1) there are $C'', l' > 0$ such that

$$\frac{1}{l'}\phi^*(l't) + (\log 2)t \leq \frac{1}{l}\phi^*(lt) + \log C'', \qquad t \geq 0.$$

Lemma 6.5 implies that there are $C, s > 0$ such that

$$\forall f \in C^\infty_a(\mathbb{R}^n), \|f\|_{\omega,k} \leq 1\, \forall \varepsilon \in (0, 1]\, \exists f_\varepsilon \in C^\infty_a(\mathbb{R}^n) : \tag{6.5}$$
$$\|f - f_\varepsilon\|_{\omega,h'} \leq \varepsilon \quad \text{and} \quad \|f_\varepsilon\|_{\omega,l'} \leq \frac{C}{\varepsilon^s}.$$

Let $f \in \mathcal{D}_K$ with $\|f\|_{\omega,k} \leq 1$ be arbitrary. Denote by f_a the $a\mathbb{Z}^n$-periodic extension of f and note that $\|f_a\|_{\omega,k} = \|f\|_{\omega,k} \leq 1$. Hence, (6.5) yields that for all $\varepsilon \in (0, 1]$ there is $f_{a,\varepsilon} \in C^\infty_a(\mathbb{R}^n)$ such that $\|f_a - f_{a,\varepsilon}\|_{\omega,h'} \leq \varepsilon$ and $\|f_{a,\varepsilon}\|_{\omega,l'} \leq C\varepsilon^{-s}$. Let $\chi \in \mathcal{D}^{(\omega)}_L$ be such that $\chi = 1$ on a neighborhood of K. Set $f_\varepsilon = \chi f_{a,\varepsilon} \in \mathcal{D}_L$. Then,

$$\|f - f_\varepsilon\|_{\omega,h} = \|\chi(f_a - f_{a,\varepsilon})\|_{\omega,h} \leq C'\|\chi\|_{\omega,h'}\|f_a - f_{a,\varepsilon}\|_{\omega,h'} \leq C'\|\chi\|_{\omega,h'}\varepsilon$$

and

$$\|f_\varepsilon\|_{\omega,l} = \|\chi f_{a,\varepsilon}\|_{\omega,l} \leq C''\|\chi\|_{\omega,l'}\|f_{a,\varepsilon}\|_{\omega,l'} \leq \frac{CC''\|\chi\|_{\omega,l'}}{\varepsilon^s}.$$

The result now follows by rescaling. □

Proof of Theorem 6.3(ii). Fix a left-invariant frame $\mathbf{X}$ on G. By Corollary 4.2(ii), we have that $E^{(\omega)} = \bigcap_{h>0} E^{\omega,h}_{\mathbf{X}}$ as locally convex spaces, where we endow the right-hand side with its natural Fréchet space topology. By Lemma 6.4 and rescaling, it therefore suffices to show that

$$\forall h > 0, p \in \operatorname{csn}(E)\,\exists k > 0, q \in \operatorname{csn}(E)\,\forall l > 0, r \in \operatorname{csn}(E)\,\exists C_1, C_2, s > 0$$
$$\forall v \in E^{\omega,k}_{\mathbf{X}}, q_{\mathbf{X},\omega,k}(v) \leq 1\,\forall \varepsilon \in (0,1]\,\exists v_\varepsilon \in E^{\omega,l}_{\mathbf{X}} :$$
$$p_{\mathbf{X},\omega,h}(v - v_\varepsilon) \leq C_1\varepsilon \quad \text{and} \quad r_{\mathbf{X},\omega,l}(v_\varepsilon) \leq \frac{C_2}{\varepsilon^s}.$$

Fix compact subsets K and L of G with $e \in \operatorname{int} K$ and $K \subseteq \operatorname{int} L$. Let $h > 0$ and $p \in \operatorname{csn}(E)$ be arbitrary. By Corollary 6.6, there is $h' > 0$ such that

$$\forall l > 0\,\exists C', s' > 0\,\forall f \in \mathcal{D}_K, \|f\|_{\mathbf{X},\omega,h'} \leq 1\,\forall \varepsilon \in (0,1]\,\exists f_\varepsilon \in \mathcal{D}_L : \tag{6.6}$$
$$\|f - f_\varepsilon\|_{\mathbf{X},\omega,h} \leq \varepsilon \quad \text{and} \quad \|f_\varepsilon\|_{\mathbf{X},\omega,l} \leq \frac{C'}{\varepsilon^{s'}}.$$

By Proposition 5.4(ii), there are $k > 0$, $\chi_j \in \mathcal{D}_K$ with $\|\chi_j\|_{\mathbf{X},\omega,h'} < \infty$, and continuous linear mappings $T_j : E^{\omega,k}_{\mathbf{X}} \to E$, $j = 1, \ldots, 2^n$, such that

$$v = \sum_{j=1}^{2^n} \Pi(\chi_j)T_j(v), \qquad v \in E^{\omega,k}_{\mathbf{X}}. \tag{6.7}$$

Lemma 4.4(ii) implies that there is $p' \in \operatorname{csn}(E)$ such that

$$p_{\mathbf{X},\omega,h}(\Pi(\chi)w) \leq \|\chi\|_{\mathbf{X},\omega,h}p'(w), \qquad w \in E, \chi \in \mathcal{D}_L.$$

Since E satisfies (Ω), there is $q' \in \operatorname{csn}(E)$, $q' \geq p'$, such that

$$\forall r' \in \operatorname{csn}(E)\,\exists C'', s'' > 0\,\forall w \in E, q'(w) \leq 1\,\forall \delta \in (0,1]\,\exists\, w_\delta \in E : \tag{6.8}$$
$$p'(w - w_\delta) \leq \delta \quad \text{and} \quad r'(w_\delta) \leq \frac{C''}{\delta^{s''}}.$$

Pick $q \in \operatorname{csn}(E)$ such that for all $j = 1, \ldots, 2^n$ it holds that

$$q'(T_j(v)) \leq q_{\mathbf{X},\omega,k}(v), \qquad v \in E^{\omega,k}_{\mathbf{X}}.$$

Let $l > 0$ and $r \in \operatorname{csn}(E)$ be arbitrary. We may assume that $l \leq h$. Lemma 4.4(ii) implies that there is $r' \in \operatorname{csn}(E)$ such that

$$r_{\mathbf{X},\omega,l}(\Pi(\chi)w) \leq \|\chi\|_{\mathbf{X},\omega,l}r'(w), \qquad w \in E, \chi \in \mathcal{D}_L.$$

Choose $C', s' > 0$ and $C'', s'' > 0$ according to (6.6) and (6.8), respectively. Let $v \in E^{\omega,k}_{\mathbf{X}}$ with $q_{\mathbf{X},\omega,k}(v) \leq 1$ be arbitrary. Condition (6.6) and a rescaling argument imply that there is $C''' > 0$ such that for all $j = 1, \ldots, 2^n$ and $\varepsilon \in [0,1)$ there is $\chi_{j,\varepsilon} \in \mathcal{D}_L$ with $\|\chi_j - \chi_{j,\varepsilon}\|_{\mathbf{X},\omega,h} \leq \varepsilon$ and $\|\chi_{j,\varepsilon}\|_{\mathbf{X},\omega,l} \leq C'''\varepsilon^{-s'}$. Note that $q'(T_j(v)) \leq q_{\mathbf{X},\omega,k}(v) \leq 1$ for all $j = 1, \ldots, 2^n$. Hence, by (6.8), we have that for all $j = 1, \ldots, 2^n$ and $\delta \in [0,1)$ there is $w_{j,\delta} \in E$ with $p'(T_j(v) - w_{j,\delta}) \leq \delta$ and $r'(w_{j,\delta}) \leq C''\delta^{-s''}$. Equation (6.7) gives that for all $\varepsilon, \delta \in (0,1]$

$$\begin{aligned}
&p_{\mathbf{X},\omega,h}(v - \sum_{j=1}^{2^n} \Pi(\chi_{j,\varepsilon}) w_{j,\delta}) \\
&= p_{\mathbf{X},\omega,h}(\sum_{j=1}^{2^n} \Pi(\chi_j) T_j(v) - \Pi(\chi_{j,\varepsilon}) w_{j,\delta}) \\
&\leq \sum_{j=1}^{2^n} p_{\mathbf{X},\omega,h}(\Pi(\chi_j - \chi_{j,\varepsilon}) T_j(v)) + p_{\mathbf{X},\omega,h}(\Pi(\chi_{j,\varepsilon})(T_j(v) - w_{j,\delta})) \\
&\leq \sum_{j=1}^{2^n} \|\chi_j - \chi_{j,\varepsilon}\|_{\mathbf{X},\omega,h} p'(T_j(v)) + \|\chi_{j,\varepsilon}\|_{\mathbf{X},\omega,h} p'(T_j(v) - w_{j,\delta}) \\
&\leq 2^n \varepsilon + \frac{2^n C'''}{\varepsilon^{s'}} \delta
\end{aligned}$$

and

$$r_{\mathbf{X},\omega,l}(\sum_{j=1}^{2^n} \Pi(\chi_{j,\varepsilon}) w_{j,\delta}) \leq \sum_{j=1}^{2^n} \|\chi_{j,\varepsilon}\|_{\mathbf{X},\omega,l} r'(w_{j,\delta}) \leq \frac{2^n C'' C'''}{\varepsilon^{s'} \delta^{s''}}.$$

We set $s = s' + (s'+1)s''$ and $\delta_\varepsilon = \varepsilon^{s'+1}$ for $\varepsilon \in (0,1]$. Then, for all $\varepsilon \in (0,1]$ we have that $\sum_{j=1}^{2^n} \Pi(\chi_{j,\varepsilon}) w_{j,\delta_\varepsilon} \in E_{\mathbf{X}}^{\omega,l}$ by Lemma 4.4(ii). Moreover,

$$p_{\mathbf{X},\omega,h}(v - \sum_{j=1}^{2^n} \Pi(\chi_{j,\varepsilon}) w_{j,\delta_\varepsilon}) \leq 2^n(1 + C''')\varepsilon$$

and

$$r_{\mathbf{X},\omega,l}(\sum_{j=1}^{2^n} \Pi(\chi_{j,\varepsilon}) w_{j,\delta_\varepsilon}) \leq \frac{2^n C'' C'''}{\varepsilon^{s}}.$$

□

Remark 6.7. We do not know whether it is possible to use the same more elementary technique as in the proof of Theorem 6.3(i) to show Theorem 6.3(ii). The problem is that we are unable to find a family $(\chi_\varepsilon)_{\varepsilon \in (0,1]} \subseteq \mathcal{D}^{(\omega)}(G)$ that satisfies the assumptions of Proposition 5.2 and suitable polynomial type bounds in ε with respect to the scale of norms $(\| \cdot \|_{\mathbf{X},\omega,h})_{h>0}$ (cf. (6.2)).

7. Right-invarant spaces of smooth and ultradifferentiable functions

A lcHs E is said to be a *right-invariant locally convex Hausdorff function space (= right-invariant lcHfs) on* G if E is non-trivial and satisfies the following four properties:

- $(A.1)$ E is continuously embedded into $L^1_{\mathrm{loc}}(G)$.
- $(A.2)$ $R_x(E) \subseteq E$ for all $x \in G$.

$(A.3)$ For all $K \subseteq G$ compact and $p \in \operatorname{csn}(E)$ there is $q \in \operatorname{csn}(E)$ such that

$$\sup_{x \in K} p(R_x f) \leq q(f), \qquad \forall f \in E.$$

$(A.4)$ $E * C_c(G) \subseteq E$ and for each $f \in E$ the mapping

$$C_c(G) \to E,\, \chi \mapsto f * \chi$$

is continuous.

Conditions $(A.1)$-$(A.3)$ imply that the right-regular representation

$$\pi_R : G \to \mathrm{GL}(E),\, g \mapsto \pi_R(g) := R_g$$

is well-defined and locally equicontinuous. Condition $(A.4)$ means that E is a right-module over the (left-)convolution algebra $C_c(G)$. A Banach space that is a right-invariant lcHfs on G is simply called a *right-invariant (Bf)-space on G*. Our definition of a right-invariant lcHfs is inspired by the Banach function spaces used in the coorbit theory of Feichtinger and Gröchening [15] as well as the translation-invariant Banach spaces of distributions from [12].

Example 7.1. Let ρ be a right-invariant Haar measure on G. The Banach spaces spaces $L^p(G) = L^p(G, \rho)$, $1 \leq p \leq \infty$, are right-invariant (Bf)-spaces. We define $L^0(G)$ as the space of all $f \in L^\infty(G)$ such that for all $\varepsilon > 0$ there is a compact subset K of G such that

$$\operatorname*{ess\,sup}_{x \in K} |f(x)| \leq \varepsilon$$

and endow $L^0(G)$ with the subspace topology induced by $L^\infty(G)$. Then, $L^0(G)$ is a right-invariant (Bf)-space as well. Weighted variants of the spaces $L^p(G)$, $p \in \{0\} \cup [1, \infty]$ (cf. Introduction) will be discussed in the next section.

Remark 7.2. If E is ultrabornological and $(A.1)$ and $(A.2)$ hold, then $(A.3)$ is automatically satisfied, as follows from the closed graph theorem of De Wilde [29, Theorem 24.31] and the Banach-Steinhaus theorem.

Lemma 7.3. *Let E be a sequentially complete lcHs satisfying* $(A.1)$-$(A.3)$ *and consider the right-regular representation π_R of G on E. Then,*

$$\Pi_R(\chi) f = f * \chi, \qquad f \in E^0, \chi \in C_c(G),$$

Proof. It suffices to show that

$$\int_G \Pi_R(\chi) f(x) \psi(x) dx = \int_G (f * \chi)(x) \psi(x) dx, \qquad \forall \psi \in C_c(G).$$

Let $\psi \in C_c(G)$ be arbitrary. As E is continuously embedded into $L^1_{\mathrm{loc}}(G)$ (condition $(A.1)$), we may view ψ as an element of E' via

$$E \to \mathbb{C}, f \mapsto \int_G f(x) \psi(x) dx.$$

We obtain that

$$\begin{aligned}\int_G \Pi_R(\chi)f(x)\psi(x)dx &= \langle \Pi_R(\chi)f, \psi\rangle \\ &= \int_G \langle R_x f, \psi\rangle \chi(x^{-1})dx \\ &= \int_G \left(\int_G f(yx)\psi(y)dy\right)\chi(x^{-1})dx \\ &= \int_G \left(\int_G f(yx)\chi(x^{-1})dx\right)\psi(y)dy \\ &= \int_G (f*\chi)(y)\psi(y)dy.\end{aligned}$$

□

Remark 7.4. Let E be a sequentially complete lcHs satisfying $(A.1)$-$(A.3)$. Suppose that the right-regular representation of G on E is continuous (this particularly holds true if $C_c(G)$, or more generally $\mathcal{D}^{(\sigma)}(G)$ for some weight function σ, is continuously and densely embedded in E). Lemma 4.4(i) and Lemma 7.3 yield that E satisfies $(A.4)$ and therefore is a right-invariant lcHfs on G. However, there are right-invariant lcHfs E such that the right-regular representation of G on E is not continuous, consider e.g. $E = L^\infty(G)$.

Let E be a sequentially complete right-invariant lcHfs on G. Our goal is to the determine the spaces E^∞ and $E^{(\omega)}$ for the right-regular representation of G on E. Let $\mathbf{X}$ be a left-invariant frame on G, let $f \in C^\infty(G)$, and let $\alpha \in \{1,\ldots,n\}^j$, $j \in \mathbb{N}$. To avoid confusion, we momentarily write $\widetilde{X}^\alpha f = X^\alpha f$, where we interpret X^α as an operator acting on $C^\infty(G)$. The next lemma shows that for $f \in E^\infty \cap C^\infty(G)$ it holds that $\widetilde{X}^\alpha f$ coincides with $X^\alpha f$, where we now interpret X^α as an operator acting on E^∞.

In fact, Lemma 7.8 below tells us that $E^\infty \subseteq C^\infty(G)$, so that the operator $X^\alpha : E^\infty \to E^\infty$ just becomes the classical derivative with respect to the left-invariant vector field X.

Lemma 7.5. *Let* $\mathbf{X}$ *be a left-invariant frame on* G*. Consider the right-regular representation of* G *on* $L^1_{\mathrm{loc}}(G)$*. Then,* $C^\infty(G) \subseteq L^1_{\mathrm{loc}}(G)^\infty$*. Moreover, for all* $f \in C^\infty(G)$ *and* $\alpha \in \{1,\ldots,n\}^j, j \in \mathbb{N}$*, it holds that* $X^\alpha f = \widetilde{X}^\alpha f$*, where we we interpret* X^α *as an operator acting on* $L^1_{\mathrm{loc}}(G)^\infty$*.*

Proof. The dual of $L^1_{\mathrm{loc}}(G)$ may be identified with the space $L^\infty_c(G)$ consisting of all elements in $L^\infty(G)$ whose essential support is compact. Since $L^1_{\mathrm{loc}}(G)$ is sequentially complete, Lemma 3.1 implies that $(L^1_{\mathrm{loc}}(G))^\infty$ consists precisely of all those $f \in L^1_{\mathrm{loc}}(G)$ such that $f * \check{\chi} \in C^\infty(G)$ for all $\chi \in L^\infty_c(G)$. Consequently, $C^\infty(G) \subseteq L^1_{\mathrm{loc}}(G)^\infty$. Let $f \in C^\infty(G)$ be arbitrary. Formula (5.7) implies that $X_j\gamma_f = \gamma_{\widetilde{X}_j f}$ for all $j = 1,\ldots,n$. Hence, by using induction, we find that $X^\alpha\gamma_f = \gamma_{\widetilde{X}^\alpha f}$ for all $\alpha \in \{1,\ldots,n\}^j$, $j \in \mathbb{N}$.

By evaluating this equality at e, we obtain that $X^\alpha f = \widetilde{X}^\alpha f$ for all $\alpha \in \{1, \ldots, n\}^j$, $j \in \mathbb{N}$. $\square$

In view of Lemmas 7.3 and 7.5, Theorem 5.4(i) yields the following result.

Lemma 7.6. *Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a left-invariant frame on G. For all $j \in \mathbb{N}$ there are a polynomial P and $\chi_\theta \in C^j_c(G)$, $\theta = (\theta_1, \ldots, \theta_n) \in \{0, 1\}^n$, such that for all $f \in C^\infty(G)$*

$$f = \sum_{\theta \in \{0,1\}^n} (P_{\theta_n}(X_n) \cdots P_{\theta_1}(X_1) f) * \chi_\theta,$$

where $P_0 = P$ and $P_1 = 1$.

Let E be a sequentially complete right-invariant lcHfs on G and let $\mathbf{X}$ be a left-invariant frame on G. We denote by $\mathcal{D}_{E,\mathbf{X}}(G)$ the space consisting of all $f \in C^\infty(G)$ such that $X^\alpha f \in E$ for all $\alpha \in \{1, \ldots, n\}^j$, $j \in \mathbb{N}$. For $j \in \mathbb{N}$ and $p \in \operatorname{csn}(E)$ we set

$$p_{\mathbf{X},j}(f) = \max_{i \leq j} \max_{\alpha \in \{1,\ldots,n\}^i} p(X^\alpha f) < \infty, \qquad f \in \mathcal{D}_E(G).$$

We endow $\mathcal{D}_{E,\mathbf{X}}(G)$ with the locally convex topology generated by the system of seminorms $\{p_{\mathbf{X},j} \,|\, j \in \mathbb{N}, p \in \operatorname{csn}(E)\}$. Likewise, we write $\mathcal{D}^{(\omega)}_{E,\mathbf{X}}(G)$ for the space consisting of all $f \in \mathcal{D}_{E,\mathbf{X}}(G)$ such that for all $h > 0$ and $p \in \operatorname{csn}(E)$ it holds that

$$p_{\mathbf{X},\omega,h}(f) = \sup_{j \in \mathbb{N}} \max_{\alpha \in \{1,\ldots,n\}^j} p(X^\alpha f) \exp\left(-\frac{1}{h}\phi^*(hj)\right) < \infty.$$

and endow $\mathcal{D}^{(\omega)}_{E,\mathbf{X}}(G)$ with the locally convex topology generated by the system of seminorms $\{p_{\mathbf{X},\omega,h} \,|\, h > 0, p \in \operatorname{csn}(E)\}$.

Remark 7.7. Proposition 3.3 and condition $(A.4)$ imply that $f * \chi \in \mathcal{D}_{E,\mathbf{X}}(G)$ $(\mathcal{D}^{(\omega)}_{E,\mathbf{X}}(G))$ for all $f \in E$ and $\chi \in \mathcal{D}(G)$ $(\chi \in \mathcal{D}^{(\omega)}(G))$. In particular, the spaces $\mathcal{D}_{E,\mathbf{X}}(G)$ and $\mathcal{D}^{(\omega)}_{E,\mathbf{X}}(G)$ are non-trivial (recall that E is assumed to be non-trivial).

Lemma 7.8. *Let E be a sequentially complete right-invariant lcHfs on G and let $\mathbf{X}$ be a left-invariant frame on G. Then, $E^\infty = \mathcal{D}_{E,\mathbf{X}}(G)$ as sets.*

Proof. We first prove that $E^\infty \subseteq \mathcal{D}_{E,\mathbf{X}}(G)$. Let $f \in E^\infty$ be arbitrary. By Theorem 5.4(i) and Lemma 7.3 it holds that for all $j \in \mathbb{N}$ there are $\chi_i \in C^{j+1}_c(G)$ and $f_i \in E^\infty \subseteq E^0$, $i = 1, \ldots, 2^n$, such that

$$f = \sum_{i=1}^{2^n} \Pi_R(\chi_i) f_i = \sum_{i=1}^{2^n} f_i * \chi_i \in C^{j+1}(G).$$

Moreover, for all $\alpha \in \{1, \ldots, n\}^j$ we have that

$$X^\alpha f = \sum_{i=i}^{2^n} f_i * X^\alpha \chi_i \in E,$$

where the last inclusion follows from the fact that $E * C_c(G) \subseteq E$ (condition $(A.4)$). Hence, $f \in \mathcal{D}_{E,\mathbf{X}}(G)$. Next we show that $\mathcal{D}_{E,\mathbf{X}}(G) \subseteq E^\infty$. Let $f \in \mathcal{D}_{E,\mathbf{X}}(G)$ be arbitrary. We first prove that $f \in E^0$. By Lemma 7.6 there are $\chi_i \in C_c(G)$ and $f_i \in E$, $i = 1, \dots, 2^n$, such that

$$f = \sum_{i=1}^{2^n} f_i * \chi_i.$$

Hence,

$$R_x f = \sum_{i=1}^{2^n} f_i * R_x \chi_i, \qquad x \in G.$$

The orbit mapping $G \to E$, $x \mapsto R_x f$ is now continuous because for all $i = 1, \dots, 2^n$ the mappings $G \to C_c(G)$, $g \mapsto R_g \chi_i$ and $C_c(G) \to E$, $\chi \mapsto f_i * \chi$ are continuous (condition $(A.4)$). Another application of Lemma 7.6 together with Lemma 7.3 shows that for all $j \in \mathbb{N}$ there are $\chi_i \in C_c^j(G)$ and $f_i \in E$, $i = 1, \dots, 2^n$, such that

$$f = \sum_{i=1}^{2^n} f_i * \chi_i = \sum_{i=1}^{2^n} \Pi_R(\chi_i)(f_i) \in E^j$$

Hence, $f \in E^\infty$. $\square$

Remark 7.9. Lemma 7.8 for the particular case $E = L^p(G)$, $1 \leq p < \infty$, was shown by Poulsen [31] via completely different methods.

Lemma 7.8 and Proposition 4.2 yield the following result.

Theorem 7.10. *Let E be a sequentially complete right-invariant lcHfs on G and let $\mathbf{X}$ be a left-invariant frame on G.*

(*i*) *$E^\infty = \mathcal{D}_{E,\mathbf{X}}(G)$ as locally convex spaces.*
(*ii*) *$E^{(\omega)} = \mathcal{D}^{(\omega)}_{E,\mathbf{X}}(G)$ as locally convex spaces.*

Let E be a sequentially complete right-invariant lcHfs on G. We write $\mathcal{D}_E(G) = E^\infty$ and $\mathcal{D}^{(\omega)}_E(G) = E^{(\omega)}$. By Theorem 7.10 we have that $\mathcal{D}_E(G) = \mathcal{D}_{E,\mathbf{X}}(G)$ and $\mathcal{D}^{(\omega)}_E(G) = \mathcal{D}^{(\omega)}_{E,\mathbf{X}}(G)$ for any left-invariant frame $\mathbf{X}$ on G. Theorems 6.1 and 6.3 imply the following two results.

Theorem 7.11. *Let E be a sequentially complete right-invariant lcHfs on G.*

(*i*) *$\mathcal{D}_E(G)$ is quasinormable if E is so.*
(*ii*) *$\mathcal{D}^{(\omega)}_E(G)$ is quasinormable if E is so.*

Theorem 7.12. *Let E be a Fréchet space that is a right-invariant lcHfs on G.*

(*i*) *$\mathcal{D}_E(G)$ satisfies (Ω) if E does so.*
(*ii*) *$\mathcal{D}^{(\omega)}_E(G)$ satisfies (Ω) if E does so.*

8. Examples of right-invarant Fréchet spaces of smooth and ultradifferentiable functions

In this final section, we discuss the quasinormability and the property (Ω) for a particular class of weighted Fréchet spaces of smooth and ultradifferentiable functions.

A right-invariant (Bf)-space E on G is said to be *solid* [14, 15] if for all $f \in E$ and $g \in L^1_{\mathrm{loc}}(G)$ it holds that

$$|g(x)| \leq |f(x)| \text{ for almost all } x \in G \implies g \in E \text{ and } \|g\|_E \leq \|f\|_E.$$

We will also use the following assumptions on a right-invariant (Bf)-space E (cf. [14]):

$(A.5)$ $L_x(E) \subseteq E$ for all $x \in G$ and there is $B > 0$ such that

$$\|L_x f\|_E \leq B\|f\|_E, \qquad \forall f \in E, x \in G.$$

Example 8.1. The spaces $L^p(G)$, $p \in \{0\} \cup [1, \infty]$, are solid right-invariant (Bf)-spaces. The spaces $L^0(G)$ and $L^\infty(G)$ satisfy $(A.5)$, whereas $L^p(G)$, $1 \leq p < \infty$, satisfy $(A.5)$ if and only if G is unimodular.

By a *weight function system on* G we mean a pointwise non-decreasing sequence $\mathcal{V} = (v_j)_{j\in\mathbb{N}}$ of strictly positive continuous functions on G satisfying the condition (1.1) from the introduction. Given a solid right-invariant (Bf)-space E on G, we denote by $E_\mathcal{V}$ the space consisting of all $f \in L^1_{\mathrm{loc}}(G)$ such that $fv_j \in E$ for all $j \in \mathbb{N}$. For $j \in \mathbb{N}$ we set

$$\|f\|_{E,v_j} = \|fv_j\|_E, \qquad f \in E_\mathcal{V},$$

and endow $E_\mathcal{V}$ with the locally convex topology generated by the norms $\{\| \cdot \|_{E,v_j} \mid j \in \mathbb{N}\}$. Then, $E_\mathcal{V}$ is a Fréchet space that is a right-invariant lcHfs on G.

Example 8.2. Suppose that G is connected. Fix a left- or right-invariant Riemannian metric on G and consider the associated distance function $d : G \times G \to [0, \infty)$ [2, 18]. Set $d(x) = d(e, x)$ for $x \in G$. Then, $d : G \to [0, \infty)$ is continuous and subadditive, i.e., $d(xy) \leq d(x) + d(y)$ for all $x, y \in G$. Let $(w_j)_{j\in\mathbb{N}}$ be a pointwise non-decreasing sequence of strictly positive continuous functions on $[0, \infty)$ such that

$$\forall i \in \mathbb{N}\, \exists j \in \mathbb{N}, C > 0\, \forall t \geq 0 \,:\, w_i(t + 1) \leq Cw_j(t).$$

Then, $(w_j \circ d)_{j\in\mathbb{N}}$ is a weight function system on G.

Let E be a solid right-invariant (Bf)-space on G and let $\mathcal{V}$ be a weight function system on G. In the next two results, we characterize the quasinormability and the property (Ω) for $\mathcal{D}_{E_\mathcal{V}}(G)$ and $\mathcal{D}^{(\omega)}_{E_\mathcal{V}}(G)$ in terms of $\mathcal{V}$.

Theorem 8.3. *Let E be a solid right-invariant (Bf)-space on G and let $\mathcal{V} = (v_j)_{j\in\mathbb{N}}$ be a weight function system on G. Consider the following statements:*

(i) $\mathcal{V}$ satisfies the condition

$$\forall i \in \mathbb{N}\, \exists j \geq i\, \forall m \geq j\, \forall \varepsilon \in (0, 1]\, \exists C > 0\, \forall x \in G \,:\, \frac{1}{v_j(x)} \leq \frac{\varepsilon}{v_i(x)} + \frac{C}{v_m(x)}. \tag{8.1}$$

(ii) $E_\mathcal{V}$ is quasinormable.

(iii) $\mathcal{D}_{E_\mathcal{V}}(G)$ ($\mathcal{D}^{(\omega)}_{E_\mathcal{V}}(G)$) is quasinormable.

Then, $(i) \Rightarrow (ii) \Rightarrow (iii)$. If in addition E satisfies $(A.5)$, then $(iii) \Rightarrow (i)$.

Proof. $(i) \Rightarrow (ii)$ By Remark 6.2 it suffices to show that

$$\forall i \in \mathbb{N}\, \exists j \geq i\, \forall m \geq j\, \forall \varepsilon \in (0,1]\, \exists C > 0\, :\, V_{\|\cdot\|_{E,v_j}} \subseteq \varepsilon V_{\|\cdot\|_{E,v_i}} + C V_{\|\cdot\|_{E,v_m}}.$$

Let $i \in \mathbb{N}$ be arbitrary. Choose $j \geq i$ according to (8.1). Let $m \geq j$ and $\varepsilon \in (0,1]$ be arbitrary. By (8.1) there is $C > 0$ such that

$$\frac{1}{v_j(x)} \leq \frac{\varepsilon}{2v_i(x)} + \frac{C}{v_m(x)}, \qquad \forall x \in G.$$

Note that

$$v_i(x) \geq \varepsilon v_j(x) \implies v_m(x) \leq 2Cv_j(x), \qquad x \in G. \tag{8.2}$$

Let χ_ε be the indicator function of the set

$$\{x \in G \mid v_i(x) \leq \varepsilon v_j(x)\}.$$

Let $f \in V_{\|\cdot\|_{E,v_j}}$ be arbitrary. Since E is solid we have that $f\chi_\varepsilon \in E_\mathcal{V}$. For all $x \in G$ it holds that

$$|f(x)\chi_\varepsilon(x)|v_i(x) \leq \varepsilon|f(x)|v_j(x)$$

and, by (8.2),

$$|f(x)(1-\chi_\varepsilon(x))|v_m(x) \leq 2C|f(x)|v_j(x).$$

Since E is solid and $\|f\|_{E,v_j} \leq 1$, we obtain that $\|f\chi_\varepsilon\|_{E,v_i} \leq \varepsilon$ and $\|f(1-\chi_\varepsilon)\|_{E,v_m} \leq 2C$, whence

$$f = f\chi_\varepsilon + f(1-\chi_\varepsilon) \in \varepsilon V_{\|\cdot\|_{E,v_i}} + 2CV_{\|\cdot\|_{E,v_m}}.$$

$(ii) \Rightarrow (iii)$ This follows from Theorem 7.11.
$(iii) \Rightarrow (i)$ *(under the extra assumption* $(A.5)$*)* We will show that $\mathcal{V}$ satisfies (8.1) if $\mathcal{D}^{(\omega)}_{E_\mathcal{V}}(G)$ is quasinormable (the proof that $\mathcal{V}$ satisfies (8.1) if $\mathcal{D}_{E_\mathcal{V}}(G)$ is quasinormable is very similar). Let $\mathbf{X}$ be a left-invariant frame on G. For $h > 0$ and $j \in \mathbb{N}$ we write

$$\|f\|_{j,h} = \sup_{l\in\mathbb{N}} \max_{\alpha\in\{1,\ldots,n\}^l} \|X^\alpha f\|_{E,v_j} \exp\left(-\frac{1}{h}\phi^*(hl)\right), \qquad f \in \mathcal{D}^{(\omega)}_{E_\mathcal{V}}(G).$$

As $\mathcal{D}^{(\omega)}_{E_\mathcal{V}}(G)$ is quasinormable, Remark 6.2 and Theorem 7.10(ii) imply that

$$\begin{gathered}\forall i' \in \mathbb{N}\, \exists j' \geq i', h > 0\, \forall m' \geq j'\, \forall \varepsilon \in (0,1]\, \exists C > 0 \\ \forall f \in \mathcal{D}^{(\omega)}_{E_\mathcal{V}}(G), \|f\|_{j',h} \leq 1\, \exists f_1, f_2 \in E_\mathcal{V} : f = f_1 + f_2, \\ \|f_1\|_{E,v_{i'}} \leq \varepsilon \quad \text{and} \quad \|f_2\|_{E,v_{m'}} \leq C.\end{gathered} \tag{8.3}$$

Let $f \in \mathcal{D}^{(\omega)}_{E_\mathcal{V},\mathbf{X}}(G)\backslash\{0\}$ (Remark 7.9). Choose $\chi \in \mathcal{D}^{(\omega)}(G)$ such that $f\chi \neq 0$. Set $K = \operatorname{supp}\chi$. Condition (2.1) and the fact that E is solid imply that $g = f\chi \in \mathcal{D}^{(\omega)}_{E_\mathcal{V},\mathbf{X}}(G)$. Let $B > 0$ be as in condition $(A.5)$. Let $i \in \mathbb{N}$ be arbitrary. By (1.1) there are $i' \geq i$ and $C' > 0$ such that

$$v_i(xy^{-1}) \leq C'v_{i'}(x), \qquad x \in G, y \in K. \tag{8.4}$$

Choose $j' \geq i'$ and $h > 0$ according to (8.3). Condition (1.1) yields that there are $j \geq j'$ and $C'' > 0$ such that

$$v_{j'}(xy) \leq C''v_j(x), \qquad x \in G, y \in K. \tag{8.5}$$

Let $m \geq j$ be arbitrary. Another application of (1.1) gives us that there are $m' \geq m$ and $C''' > 0$ such that

$$v_m(xy^{-1}) \leq C''' v_{m'}(x), \qquad x \in G, y \in K. \tag{8.6}$$

Let $\varepsilon > 0$ be arbitrary and choose $C > 0$ according to (8.3). Let $x \in G$ be arbitrary. The left-invariance of $\mathbf{X}$, the fact that E is solid, condition $(A.5)$, and (1.1) imply that $L_x g \in \mathcal{D}^{(\omega)}_{E_\mathcal{V}, \mathbf{X}}(G)$ and

$$\|L_x g\|_{j',h} \leq B \sup_{y \in K} v_{j'}(xy) \|g\|_{j',h}.$$

Hence, by (8.5),

$$\|L_x g\|_{j',h} \leq BC'' \|g\|_{j',h} v_j(x).$$

Set $C_0 = BC''\|g\|_{j',h}$. Applying (8.3) to $f = L_x g/(C_0 v_j(x))$, we find that there are $g_1, g_2 \in E_\mathcal{V}$ with $\|g_1\|_{E,v_{i'}} \leq \varepsilon$ and $\|g_2\|_{E,v_{m'}} \leq C$ such that

$$\frac{L_x g}{C_0 v_j(x)} = g_1 + g_2.$$

Since $\operatorname{supp} g \subseteq K$, we have that

$$\frac{g}{C_0 v_j(x)} = (L_{x^{-1}} g_1) 1_K + (L_{x^{-1}} g_2) 1_K,$$

where 1_K is the indicator function of K. The fact that E is solid, condition $(A.5)$, and (8.4) imply that $(L_{x^{-1}} g_1) 1_K \in E$ and

$$\|(L_{x^{-1}} g_1) 1_K\|_E = \|L_{x^{-1}}(g_1 v_{i'}) L_{x^{-1}} \left(\frac{1}{v_{i'}}\right) 1_K\|_E \leq B \|g_1\|_{E,v_{i'}} \sup_{y \in K} \frac{1}{v_{i'}(xy)} \leq \frac{BC'\varepsilon}{v_i(x)}.$$

Similarly, by using (8.6) instead of (8.4), we have that $(L_{x^{-1}} g_2) 1_K \in E$ and

$$\|(L_{x^{-1}} g_2) 1_K\|_E \leq \frac{BCC'}{v_m(x)}.$$

Hence,

$$\frac{\|g\|_E}{C_0 v_j(x)} \leq \|(L_{x^{-1}} g_1) 1_K\|_E + \|(L_{x^{-1}} g_2) 1_K\| \leq \frac{BC'\varepsilon}{v_i(x)} + \frac{BCC'}{v_m(x)}.$$

The result now follows from a simple rescaling argument (note that $\|g\|_E > 0$ as $g \neq 0$). □

Theorem 8.4. *Let E be a solid right-invariant (Bf)-space on G and let $\mathcal{V} = (v_j)_{j \in \mathbb{N}}$ be a weight function system on G. Consider the following statements:*

(i) $\mathcal{V}$ satisfies the condition

$$\forall i \in \mathbb{N}\, \exists j \geq i\, \forall m \geq j\, \exists C, s > 0\, \forall \varepsilon \in (0,1]\, \forall x \in G \,:\, \frac{1}{v_j(x)} \leq \frac{\varepsilon}{v_i(x)} + \frac{C}{\varepsilon^s} \frac{1}{v_m(x)}. \tag{8.7}$$

(ii) $E_\mathcal{V}$ satisfies (Ω).

(iii) $\mathcal{D}_{E_\mathcal{V}}(G)$ ($\mathcal{D}^{(\omega)}_{E_\mathcal{V}}(G)$) satisfies (Ω).

Then, $(i) \Rightarrow (ii) \Rightarrow (iii)$. If in addition E satisfies $(A.5)$, then $(iii) \Rightarrow (ii)$.

Proof. The implication $(ii) \Rightarrow (iii)$ follows from Theorem 7.12. The proofs of $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (i)$ (under the extra assumption $(A.5)$) are similar to those of the same implications in Theorem 8.3 and are therefore left to the reader. □

Theorem 1.2 in the introduction now follows from Example 8.1 and Theorems 8.3 and 8.4.

Remark 8.5. We believe the implications $(iii) \Rightarrow (i)$ in Theorems and 8.3 and 8.4 hold without the additional assumption $(A.5)$ but were unable to show this.

Remark 8.6. Let $\mathcal{V} = (v_j)_{j\in\mathbb{N}}$ be a weight function system on G.
(i) The condition (8.1) is always fulfilled if for all $i \in \mathbb{N}$ there is $j \geq i$ such that v_i/v_j vanishes at infinity.
(ii) The condition (8.1) means that the sequence $(1/v_n)_{n\in\mathbb{N}}$ is regularly decreasing in the sense of [4]. We refer to [4] for more information and various characterizations of (8.1). A slight variant of the condition (8.1) is considered in [32], where the quasinormability of certain weighted Fréchet spaces of measurable functions is studied.
(iii) The condition (8.7) is equivalent to

$$\forall i \in \mathbb{N}\, \exists j \geq i\, \forall m \geq j\, \exists C > 0, \theta \in (0,1)\, \forall g \in G\, :\, v_i^{\theta}(g) v_m^{1-\theta}(g) \leq C v_j(g). \tag{8.8}$$

This follows by taking the infimum over ε in the right-hand side of the inequality in (8.7).
(iv) Let G be connected and non-compact. Let d and $(w_j)_{j\in\mathbb{N}}$ be as in Example 8.2. Set $(v_j)_{j\in\mathbb{N}} = (w_j \circ d)_{j\in\mathbb{N}}$. Then, $(v_j)_{j\in\mathbb{N}}$ satisfies (8.8) if and only if

$$\forall i \in \mathbb{N}\, \exists j \geq i\, \forall m \geq j\, \exists C > 0, \theta \in (0,1)\, \forall t \geq 0\, :\, w_i^{\theta}(t) w_m^{1-\theta}(t) \leq C w_j(t).$$

We refer to [11, Section 6] for various examples of sequences $(w_j)_{j\in\mathbb{N}}$ that do (not) satisfy this condition.

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