On maximum-sum matchings of bichromatic points

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Abstract

Huemer et al. (Discrete Math, 2019) proved that for any two finite point sets R and B in the plane with $|R| = |B|$, the perfect matching that matches points of R with points of B, and maximizes the total *squared* Euclidean distance of the matched pairs, has the property that all the disks induced by the matching have a nonempty common intersection. A pair of matched points induces the disk that has the segment connecting the points as diameter. In this note, we characterize these maximum-sum matchings for any continuous (semi)metric, focusing on both the Euclidean distance and squared Euclidean distance. Using this characterization, we give a different but simpler proof for the common intersection property proved by Huemer et al..

1 Introduction

Let R and B be two point sets in the plane with $|R| = |B|$. The points in R are called *red*, and those in B are called *blue*. A *matching* $\mathcal{M}$ of $R \cup B$ is a partition of $R \cup B$ into n pairs such that each pair consists of a red point and a blue point. A point $p \in R$ and a point $q \in B$ are matched if and only if the (unordered) pair (p, q) is in the matching. For every $p, q \in \mathbb{R}^2$, we use pq to denote the segment connecting p and q, $\ell(pq)$ to denote the line through pq, and $\mathcal{B}(pq)$ to denote the disk with diameter equal to the length $\|p - q\|$ of pq, that is centered at the midpoint $(p + q)/2$ of pq. For any matching $\mathcal{M}$, we use $\mathcal{B}_{\mathcal{M}}$ to denote the set of the disks associated with, or induced by, the matching, that is, $\mathcal{B}_{\mathcal{M}} = \{\mathcal{B}(pq) : (p, q) \in \mathcal{M}\}$, where $\mathcal{B}(pq)$ is the disk induced by the edge (p, q).

Given a metric, or semi-metric function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$, we say that a matching $\mathcal{M}$ is *max-sum* if it maximizes $\sum_{(p,q)\in\mathcal{M}} d(p, q)$ among all matchings of R and B. Recall that a function is a semi-metric if it satisfies all the properties of a metric function except the triangle inequality.

Huemer et al. [8] proved that if $\mathcal{M}$ is a max-sum matching for $d(p, q) = \|p - q\|^2$ for all $p, q \in \mathbb{R}^2$, then all the disks of $\mathcal{B}_{\mathcal{M}}$ have a point in common. That is, d is the squared Euclidean distance, also called *quadrance*, and it is in fact a semi-metric.

For the Euclidean distance $d(p, q) = \|p - q\|$, Bereg et al. [4] proved that any max-sum matching $\mathcal{M}$ between R and B does not always satisfy that all disks in $\mathcal{B}_M$ have a common point, although the disks intersect pairwise. As their main result, they proved that the common intersection is always

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satisfied when $\mathcal{M}$ is a max-sum matching between $2n$ uncolored points. For an even point set in the plane, a matching is simply a partition of the set into pairs.

In this note, we consider max-sum matchings between planar colored point sets R and B with $|R| = |B| = n$, $n \geq 1$, for *any continuous* (semi)metric d of the Euclidean plane. In Section 2, we characterize the max-sum matchings between *three* red points and *three* blue points in terms of the common intersection of certain sets defined by the six points. Using this characterization, in the case where d is the Euclidean quadrance, we give in Section 3 an elementary proof to the main theorem by Huemer et al. [8]: The disks induced by a max-sum matching have a common point.

1.1 Related work

Motivated by the study of geometric matchings of planar point sets, Huemer et al. [8] proved that for any finite bichromatic point set $R \cup B$ with $|R| = |B|$, and a max-sum matching $\mathcal{M}$ of R and B according to the squared Euclidean distance, all disks in $\mathcal{B}_{\mathcal{M}}$ have a nonempty common intersection. Later, Bereg et al. [4] proved that all disks in $\mathcal{B}_{\mathcal{M}}$ have a nonempty common intersection for any point set P of $2n$ uncolored points in the plane, when $\mathcal{M}$ is a max-sum matching of P with respect to the Euclidean distance. This result has been slightly strengthened by Barabanshchikova and Polyanskii [3], who proved that the *interiors* of all disks in $\mathcal{B}_{\mathcal{M}}$ have a nonempty common intersection in the case where all elements of P are distinct.

Soberón and Tang [11] considered this kind of geometric intersection problems in the more general context of geometric graphs: Given a geometric graph with vertex set a finite point set in the plane (where the edges are straight segments connecting points), they define the graph as a *Tverberg graph* if the disks induced by the edges of the graph have a nonempty common intersection. They show that for any odd planar point set there exists a Hamiltonian cycle which is a Tverberg graph, and that for any even planar point set there exists a Hamiltonian path with the same property. Notice that the previous mentioned results can be stated in terms of Tverberg graphs. For example, the result of Huemer et al. [8]: For any finite bichromatic point set $R \cup B$ with $|R| = |B|$, any max-sum matching of R and B according to the squared Euclidean distance is a Tverberg graph.

Pirahmad et al. [10] refined and extended these results. They proved, for example, that: for any finite point set in the plane there exists a Hamiltonian cycle that is a Tverberg graph; for any even point set in $\mathbb{R}^d$ there exists a matching that is a Tverberg graph; and for any red and blue points in $\mathbb{R}^d$, $d \geq 3$, there exists a perfect red-blue matching that is a Tverberg graph. This last result generalizes the initial result of Huemer et al. [8] to higher dimensions.

In the same direction, Abu-Affash et al. [1] proved that for any finite planar point set P, the maximum-weight spanning tree of P is a Tverberg graph. In fact, they proved that the center of the smallest enclosing circle of P is contained in all the disks induced by the tree.

Fingerhut (see Eppstein [5]), motivated by a problem in designing communication networks (see Fingerhut et al. [6]), conjectured that given a set P of $2n$ uncolored points in the plane, not necessarily distinct, and a max-sum matching $\{(a_i, b_i), i = 1, \ldots, n\}$ of P, there exists a point o of the plane, not required to be a point of P and called the center of the matching, such that

$$\|a_i - o\| + \|b_i - o\| \;\leq\; \frac{2}{\sqrt{3}}\,\|a_i - b_i\| \quad \text{for all } i \in \{1, \ldots, n\}, \text{ where } 2/\sqrt{3} \approx 1.1547. \tag{1}$$

Bereg et al. [4], by proving that for any point set P of $2n$ uncolored points in the plane and a

max-sum matching $\mathcal{M} = \{(a_i, b_i), i = 1, \ldots, n\}$ of P we have that all disks in $\mathcal{B}_{\mathcal{M}}$ have a nonempty common intersection, obtained an approximation to this conjecture. Indeed, any point o in the common intersection satisfies

$$\|a_i - o\| + \|b_i - o\| \ \leq \ \sqrt{2}\ \|a_i - b_i\|, \text{ where } \sqrt{2} \approx 1.4142.$$

Recently, Barabanshchikova and Polyanskii [2] confirmed the conjecture of Fingerhut. After that, Pérez-Lantero and Seara [9] proved the bichromatic version of the main result obtained by Bereg et al. [4]: If R and B are two point sets in the plane with $|R| = |B| = n$, and $\mathcal{M} = \{(r_i, b_i) \in R \times B : i = 1, 2, \ldots, n\}$ is a max-sum matching for the Euclidean distance, there exists a point o of the plane such that

$$\|r_i - o\| + \|b_i - o\| \leq \sqrt{2}\ \|r_i - b_i\| \quad \text{for all } i \in \{1, 2, \ldots, n\}. \tag{2}$$

These results about the conjecture of Fingerhut are related to the common intersection of convex sets induced by the pairs of the max-sum matchings, ellipses in this case. Indeed, for two points $p, q \in \mathbb{R}^2$ and a real number $\lambda \geq 1$, let $\mathcal{E}_\lambda(pq)$ denote the (region bounded by) the ellipse with foci p and q and major axis length $\lambda\|p - q\|$. That is, $\mathcal{E}_\lambda(pq) = \{x \in \mathbb{R}^2 : \|p - x\| + \|q - x\| \leq \lambda\|p - q\|\}$. The statement of Equation (1) is equivalent to stating that the common intersection of the ellipses $\mathcal{E}_{2/\sqrt{3}}(a_1b_1), \mathcal{E}_{2/\sqrt{3}}(a_2b_2), \ldots, \mathcal{E}_{2/\sqrt{3}}(a_nb_n)$ is nonempty. Furthermore, the statement of Equation (2) is equivalent to stating that the common intersection of the ellipses $\mathcal{E}_{\sqrt{2}}(r_1b_1)$, $\mathcal{E}_{\sqrt{2}}(r_2b_2), \ldots, \mathcal{E}_{\sqrt{2}}(r_nb_n)$ is also nonempty [2, 5, 9].

2 Characterization of max-sum matchings of 6 points

Let $R = \{a, b, c\}$ be a set of three red points and $B = \{a', b', c'\}$ a set of three blue points. Let d be a continuous (semi)metric function on $\mathbb{R}^2$. Let $\{(a, a'), (b, b'), (c, c')\}$ be a matching of R and B. Let us define the following six sets:

$$\begin{aligned}
H(a, b) =& \{x \in \mathbb{R}^2 : d(a, b') - d(b, b') \leq d(a, x) - d(b, x)\},\\
H(b, c) =& \{x \in \mathbb{R}^2 : d(b, c') - d(c, c') \leq d(b, x) - d(c, x)\},\\
H(c, a) =& \{x \in \mathbb{R}^2 : d(c, a') - d(a, a') \leq d(c, x) - d(a, x)\},
\end{aligned}$$

and

$$\begin{aligned}
h(a, b) =& \{x \in \mathbb{R}^2 : d(a, x) - d(b, x) \leq d(a, a') - d(b, a')\},\\
h(b, c) =& \{x \in \mathbb{R}^2 : d(b, x) - d(c, x) \leq d(b, b') - d(c, b')\},\\
h(c, a) =& \{x \in \mathbb{R}^2 : d(c, x) - d(a, x) \leq d(c, c') - d(a, c')\}.
\end{aligned}$$

In other words, $H(a, b)$ is the set of the (blue) points $x \in \mathbb{R}^2$ such that $d(a, b') + d(b, x) \leq d(a, x) + d(b, b')$. That is, $\{(a, x), (b, b')\}$ is a max-sum matching of $\{a, b\}$ and $\{x, b'\}$. Similarly, $h(a, b)$ is the set of the (blue) points $x \in \mathbb{R}^2$ such that $\{(a, a'), (b, x)\}$ is a max-sum matching of $\{a, b\}$ and $\{a', x\}$. For example, if d is the Euclidean distance, then the boundaries of the sets $H(a, b)$ and $h(a, b)$ are arcs of hyperbolas with foci a and b (see Figure 1a). Similarly, if d is the *squared* Euclidean distance, then $H(a, b)$ and $h(a, b)$ are half-planes whose boundaries are perpendicular to the line $\ell(ab)$ through the segment ab (see Figure 1b).

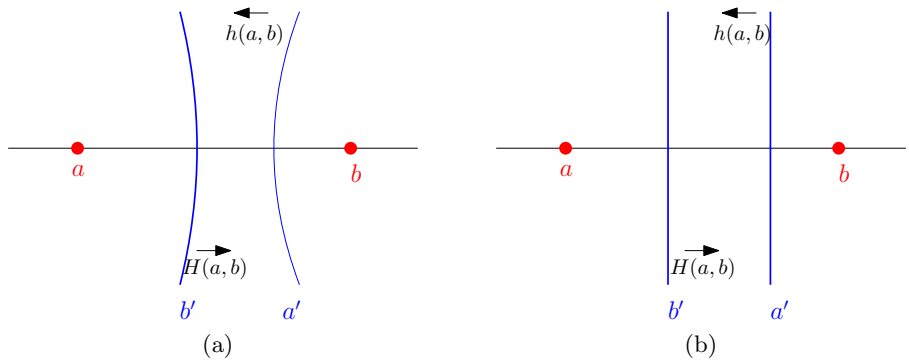

Figure 1: Definition of the sets $H(a,b)$ and $h(a,b)$ for (a) $\|\cdot\|$, and (b) $\|\cdot\|^2$ (the labels b' and a' are used to indicate that they are in the boundaries of $H(a,b)$ and $h(a,b)$, respectively).

Lemma 2.1. *If the five intersections $H(a,b)\cap H(b,c)\cap H(c,a)$, $h(a,b)\cap h(b,c)\cap h(c,a)$, $H(a,b)\cap h(a,b)$, $H(b,c)\cap h(b,c)$, and $H(c,a)\cap h(c,a)$ are all nonempty, then $\{(a,a'),(b,b'),(c,c')\}$ is a max-sum matching of $\{a,b,c\}$ and $\{a',b',c'\}$.*

Proof. Let $z_1 \in H(a,b)\cap H(b,c)\cap H(c,a)$. Then,

$$\begin{aligned} d(a,b') - d(b,b') &\le d(a,z_1) - d(b,z_1), \\ d(b,c') - d(c,c') &\le d(b,z_1) - d(c,z_1), \\ d(c,a') - d(a,a') &\le d(c,z_1) - d(a,z_1). \end{aligned}$$

By adding the above three equations, we obtain that

$$d(a,b') + d(b,c') + d(c,a') \le d(a,a') + d(b,b') + d(c,c').$$

Let $z_2 \in h(a,b)\cap h(b,c)\cap h(c,a)$. Then,

$$\begin{aligned} d(a,z_2) - d(b,z_2) &\le d(a,a') - d(b,a'), \\ d(b,z_2) - d(c,z_2) &\le d(b,b') - d(c,b'), \\ d(c,z_2) - d(a,z_2) &\le d(c,c') - d(a,c'). \end{aligned}$$

By adding the above three equations, we obtain that

$$d(a,c') + d(b,a') + d(c,b') \le d(a,a') + d(b,b') + d(c,c').$$

Let $z_3 \in H(a,b)\cap h(a,b)$. By the definitions of $H(a,b)$ and $h(a,b)$, we have that

$$d(a,b') - d(b,b') \le d(a,z_3) - d(b,z_3) \le d(a,a') - d(b,a').$$

Then, we obtain $d(a,b') + d(b,a') \le d(a,a') + d(b,b')$, which implies that

$$d(a,b') + d(b,a') + d(c,c') \le d(a,a') + d(b,b') + d(c,c').$$

Analogously, we obtain that

$$d(a,a') + d(b,c') + d(c,b') \le d(a,a') + d(b,b') + d(c,c'), \quad \text{and}$$

$$d(a,c') + d(b,b') + d(c,a') \leq d(a,a') + d(b,b') + d(c,c').$$

Therefore, $\{(a,a'),(b,b'),(c,c')\}$ is a max-sum matching of $\{a,b,c\}$ and $\{a',b',c'\}$, since its total sum is never smaller than that of any other matching. $\square$

Lemma 2.2. *If $\{(a,a'),(b,b'),(c,c')\}$ is a max-sum matching of $\{a,b,c\}$ and $\{a',b',c'\}$, then the five intersections $H(a,b)\cap H(b,c)\cap H(c,a)$, $h(a,b)\cap h(b,c)\cap h(c,a)$, $H(a,b)\cap h(a,b)$, $H(b,c)\cap h(b,c)$, and $H(c,a)\cap h(c,a)$ are all nonempty.*

Proof. Observe that $\{(a,a'),(b,b')\}$ must be a max-sum matching of $\{a,b\}$ and $\{a',b'\}$. Then, we have that $d(a,b') + d(b,a') \leq d(a,a') + d(b,b')$, which is equivalent to $d(a,b') - d(b,b') \leq d(a,a') - d(b,a')$. This implies that $a',b' \in H(a,b)\cap h(a,b)$, which means that $H(a,b)\cap h(a,b)$ is not empty. Analogously, $H(b,c)\cap h(b,c)$ and $H(c,a)\cap h(c,a)$ are not empty.

Let us prove now that $H(a,b)\cap H(b,c)\cap H(c,a) \neq \emptyset$. The proof for $h(a,b)\cap h(b,c)\cap h(c,a)$ is analogous. We divide the proof into two cases.

As the first case, suppose that $a' \in H(b,c)$, $b' \in H(c,a)$, or $c' \in H(a,b)$. Assume w.l.o.g. that $a' \in H(b,c)$.

By the definition and properties of the sets $H(\cdot,\cdot)$, we have that $a' \in H(a,b)\cap H(c,a)$, which implies that $a' \in H(a,b)\cap H(b,c)\cap H(c,a)$, and the result thus follows.

It is a known result from point-set topology that given two closed sets U and V such that the three sets $U\setminus V$, $U\cap V$, and $V\setminus U$ are all nonempty, the boundaries ∂U and ∂V are not empty, and there exists a point $x \in U\cap V$ which is in both boundaries ∂U and ∂V.

That said, suppose now that $a' \notin H(b,c)$, $b' \notin H(c,a)$, and $c' \notin H(a,b)$, as the second case of the proof. We then have that $a' \in H(a,b)\setminus H(b,c)$, $b' \in H(a,b)\cap H(b,c)$, and $c' \in H(b,c)\setminus H(a,b)$. Given that $H(a,b)$ and $H(b,c)$ are closed regions of the plane, due to the fact that d is continuous, the topological result above implies that there exists a point $z \in H(a,b)\cap H(b,c)$ in the boundary of $H(a,b)$ and also in the boundary of $H(b,c)$. Assume by contradiction that $z \notin H(c,a)$. Then, we have that

$$\begin{aligned}
d(c,z) - d(a,z) &< d(c,a') - d(a,a'), && (z \notin H(c,a))\\
d(a,z) - d(b,z) &= d(a,b') - d(b,b'), && (z \in \partial H(a,b))\\
d(b,z) - d(c,z) &= d(b,c') - d(c,c'). && (z \in \partial H(b,c))
\end{aligned}$$

By adding the above three equations, we obtain that $d(a,a')+d(b,b')+d(c,c') < d(a,b')+d(b,c')+d(c,a')$, which means that $\{(a,a'),(b,b'),(c,c')\}$ is not a max-sum matching since the matching $\{(a,b'),(b,c'),(c,a')\}$ has larger sum, which is a contradiction. Hence, we must have that $z \in H(c,a)$, which implies $z \in H(a,b)\cap H(b,c)\cap H(c,a)$. The result thus follows. $\square$

By combining Lemma 2.1 and Lemma 2.2, we obtain the main result of this section:

Theorem 2.3. *The matching $\{(a,a'),(b,b'),(c,c')\}$ is a max-sum matching of the red points $\{a,b,c\}$ and the blue points $\{a',b',c'\}$ if and only if the five intersections $H(a,b)\cap H(b,c)\cap H(c,a)$, $h(a,b)\cap h(b,c)\cap h(c,a)$, $H(a,b)\cap h(a,b)$, $H(b,c)\cap h(b,c)$, and $H(c,a)\cap h(c,a)$ are all nonempty.*

3 Common intersection for the Euclidean quadrance

Let R and B be two point sets in the plane, with $|R| = |B| = n \geq 2$, and let $\mathcal{M}$ be a max-sum matching of R and B with respect to the Euclidean quadrance. Huemer et al. [8] first proved that the disks in $\mathcal{B}_{\mathcal{M}}$ intersect pairwise, so the common intersection of the disks in $\mathcal{B}_{\mathcal{M}}$ holds for $n = 2$. To give our new proof of this property for $n \geq 3$, we will use Helly's Theorem [7] in $\mathbb{R}^2$, that is, we will prove the common intersection for $n = 3$, by using the characterization of Section 2. For completeness, we state Helly's Theorem:

Theorem 3.1 (Helly [7])**.** *Let $\mathcal{F}$ be a finite family of closed convex sets in $\mathbb{R}^m$ such that every subfamily of $m + 1$ sets of $\mathcal{F}$ has nonempty intersection. Then, all sets in $\mathcal{F}$ have nonempty intersection.*

Let $\mathcal{M} = \{(a, a'), (b, b'), (c, c')\}$ be a max-sum matching of $R = \{a, b, c\}$ and $B = \{a', b', c'\}$, and let us define the three sets $S(a, b) = H(a, b) \cap h(a, b)$, $S(b, c) = H(b, c) \cap h(b, c)$, and $S(c, a) = H(c, a) \cap h(c, a)$. Given that $S(a, b)$, $S(b, c)$, and $S(c, a)$ are nonempty strips perpendicular to the lines $\ell(ab)$, $\ell(bc)$, and $\ell(ca)$, respectively, Theorem 2.3 asserts that $S(a, b) \cap S(b, c) \cap S(c, a) \neq \emptyset$ (see Figure 2a). So, let z be a point of $S(a, b) \cap S(b, c) \cap S(c, a)$.

For two distinct points $p, q \in \mathbb{R}^2$, let $\tau(pq)$ be the ray with apex p that goes through q. For three distinct points $p, q, r \in \mathbb{R}^2$, let $\omega(pqr)$ denote the convex wedge bounded by $\tau(pq)$ and $\tau(pr)$ and Δpqr denote the triangle with vertices p, q, and r.

Proposition 3.2. *Let z^* be the orthogonal projection of the point z into the line $\ell(ab)$. The following statements are satisfied: (i) If $z^* \in ab$, then $z^* \in \mathcal{B}(aa') \cap \mathcal{B}(bb')$. (ii) If $z^* \in \tau(ab) \setminus ab$, then $z^*, b \in \mathcal{B}(aa')$. (iii) If $z^* \in \tau(ba) \setminus ab$, then $z^*, a \in \mathcal{B}(bb')$.*

Proof. Let a^* and b^* be the orthogonal projections of a' and b' into $\ell(ab)$, respectively. Since $z, z^* \in H(a, b) \cap h(a, b)$, $a' \in \partial h(a, b)$, and $b' \in \partial H(a, b)$, we have that z^* belongs to the segment b^*a^*. Note that if $z^* \in aa^*$, then $z^* \in \mathcal{B}(aa')$ by Thales' Theorem. Similarly, if $z^* \in bb^*$, then $z^* \in \mathcal{B}(bb')$. Hence: (i) If $z^* \in ab$, then $z^* \in aa^* \cap bb^*$, which implies $z^* \in \mathcal{B}(aa') \cap \mathcal{B}(bb')$ (see Figure 2b). (ii) If $z^* \in \tau(ab) \setminus ab$, then $z^*, b \in aa^*$, which implies $z^*, b \in \mathcal{B}(aa')$ (see Figure 2c). (iii) If $z^* \in \tau(ba) \setminus ab$, then $z^*, a \in bb^*$, which implies $z^*, a \in \mathcal{B}(bb')$. □

We prove first the extreme case in which a, b, and c are collinear points. Assume w.l.o.g. that b belongs to the segment ac. Since $S(a, b) \cap S(b, c) \cap S(c, a) \neq \emptyset$, this intersection must be a line through b' perpendicular to $\ell(ac)$. So, let z^* be the intersection of this line with $\ell(ac)$, and note that $z^* \in \mathcal{B}(bb')$. If $z^* \in ac$, then $z^* \in \mathcal{B}(aa') \cap \mathcal{B}(cc')$ by Proposition 3.2 (i). This implies that $z^* \in \mathcal{B}(aa') \cap \mathcal{B}(bb') \cap \mathcal{B}(cc')$. If $z^* \in \tau(ac) \setminus ac$, then $c \in \mathcal{B}(bb')$ since $c \in bz^*$, and $c \in \mathcal{B}(aa')$ by Proposition 3.2 (ii). Hence, $c \in \mathcal{B}(aa') \cap \mathcal{B}(bb') \cap \mathcal{B}(cc')$. Similarly, if $z^* \in \tau(ca) \setminus ac$, then $a \in \mathcal{B}(bb')$ since $a \in bz^*$, and $a \in \mathcal{B}(cc')$ by Proposition 3.2 (iii). Hence, $a \in \mathcal{B}(aa') \cap \mathcal{B}(bb') \cap \mathcal{B}(cc')$. In all the cases, we have that $\mathcal{B}(aa') \cap \mathcal{B}(bb') \cap \mathcal{B}(cc') \neq \emptyset$.

From this point forward, assume that a, b, and c are not collinear points.

Proposition 3.3. *If the point z belongs to the wedge $\omega(abc)$ (resp. $\omega(bca)$, $\omega(cab)$), then z is contained in $\mathcal{B}(aa')$ (resp. $\mathcal{B}(bb')$, $\mathcal{B}(cc')$).*

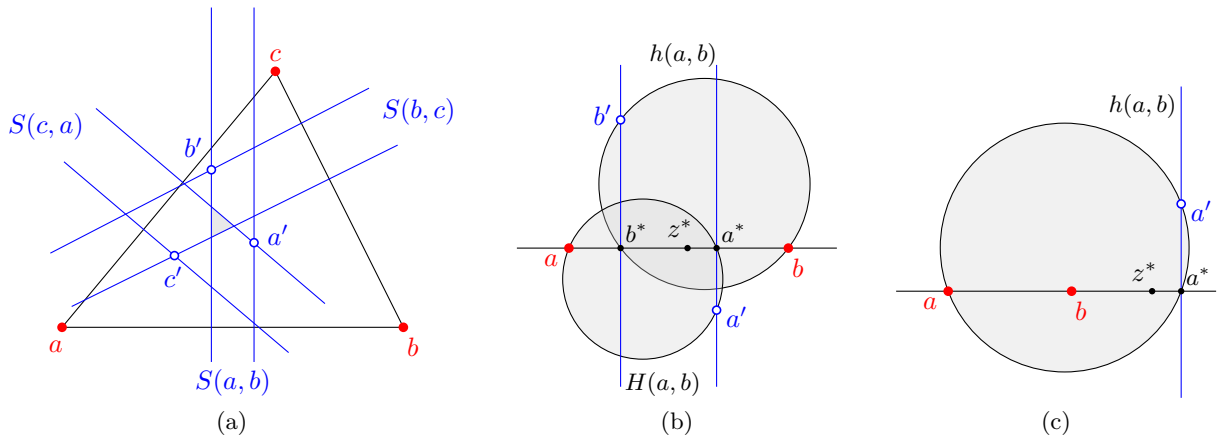

Figure 2: (a) Strips $S(a,b)$, $S(b,c)$, and $S(c,a)$ with their common intersection. (b-c) Proof of Proposition 3.2.

Proof. We prove the case $z \in \omega(abc)$. The proofs for the other cases are analogous. Given that $z \in h(a,b) \cap H(c,a)$ and that $z \in \omega(abc)$, the lines $\partial h(a,b)$ and $\partial H(c,a)$ cut perpendicularly the rays $\tau(ab)$ and $\tau(ac)$, respectively. Let $u = \tau(ab) \cap \partial h(a,b)$ and $v = \tau(ac) \cap \partial H(c,a)$, which are the orthogonal projections of a' into $\tau(ab)$ and $\tau(bc)$, respectively (see Figure 3a). Furthermore, z is in one of the right triangles $\Delta aa'u$ and $\Delta aa'v$, both contained in $\mathcal{B}(aa')$. Hence, $z \in \mathcal{B}(aa')$. $\square$

The proof is now divided into three cases.

Case 1: The point z is contained in Δabc. We then have that $z \in \mathcal{B}(aa') \cap \mathcal{B}(bb') \cap \mathcal{B}(cc')$ by applying Proposition 3.3 three times.

Case 2: The point z is contained in a wedge among $\omega(abc)$, $\omega(bca)$, and $\omega(cab)$, but not in Δabc. Assume w.l.o.g. that $z \in \omega(abc)$. Let z^* be the orthogonal projection of z into $\ell(bc)$, and let u and v be the orthogonal projections of a' into $\tau(ab)$ and $\tau(ac)$, respectively. If $z^* \in bc$, then $z^* \in \mathcal{B}(bb') \cap \mathcal{B}(cc')$ by Proposition 3.2. We also have that $z^* \in \mathcal{B}(aa')$ by using Proposition 3.3. In fact, z^* belongs to the convex quadrilateral of vertices a, u, a', and v (see Figure 3b). Hence, we have that $z^* \in \mathcal{B}(aa') \cap \mathcal{B}(bb') \cap \mathcal{B}(cc')$. Otherwise, if $z^* \notin bc$, say $z^* \in \tau(bc) \setminus bc$ w.l.o.g., then $c \in av$ which implies $c \in \mathcal{B}(aa')$ (see Figure 3c). Furthermore, $c \in \mathcal{B}(bb')$ by Proposition 3.2. Hence, $c \in \mathcal{B}(aa') \cap \mathcal{B}(bb') \cap \mathcal{B}(cc')$.

Case 3: The point z is not contained in any of the wedges $\omega(abc)$, $\omega(bca)$, and $\omega(cab)$. Assume w.l.o.g. that z is such that $c \in \Delta abz$ (see Figure 3d). The orthogonal projection of z into $\ell(ac)$ belongs to $\tau(ac) \setminus ac$, so $c \in \mathcal{B}(aa')$ by Proposition 3.2. Similarly, the orthogonal projection of z into $\ell(bc)$ belongs to $\tau(bc) \setminus bc$, so $c \in \mathcal{B}(bb')$ by Proposition 3.2. Hence, $c \in \mathcal{B}(aa') \cap \mathcal{B}(bb') \cap \mathcal{B}(cc')$.

In all of the cases, we can find a point in the intersection $\mathcal{B}(aa') \cap \mathcal{B}(bb') \cap \mathcal{B}(cc')$.

Theorem 3.4 (Huemer et al. [8])**.** *Let $|R| = |B|$. Any max-sum matching $\mathcal{M}$ of R and B with respect to $\|.\|^2$ is such that all the disks of $\mathcal{B}_{\mathcal{M}}$ have a common intersection.*

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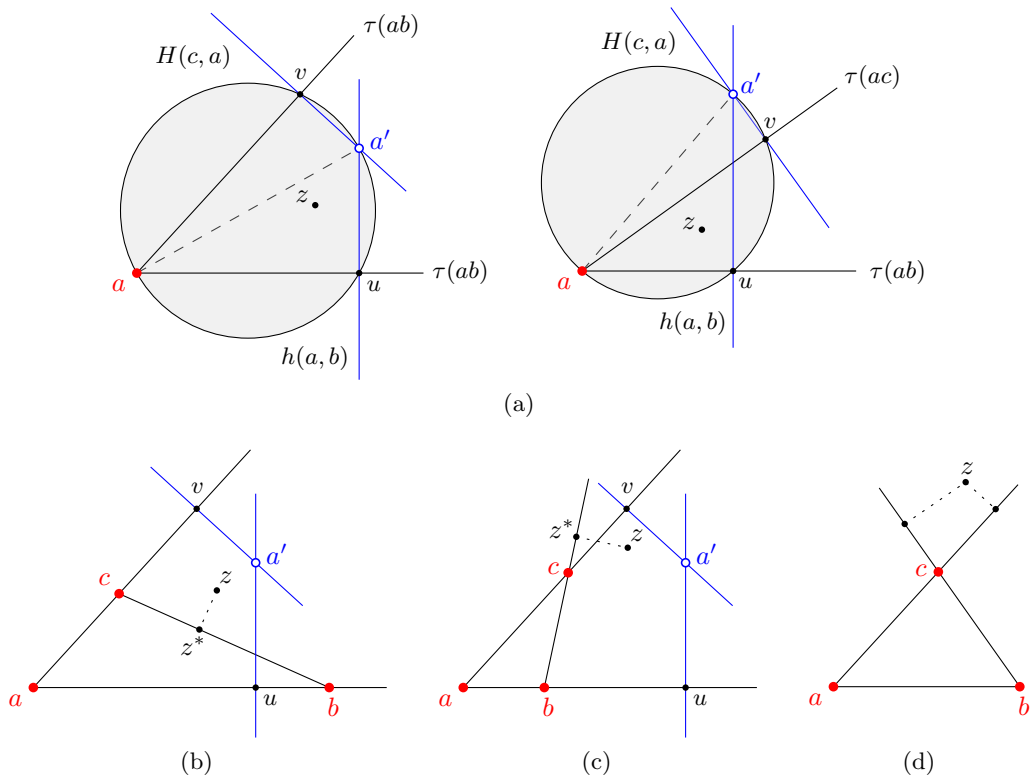

Figure 3: (a) Proof of Proposition 3.3. (b-c) Proof of **Case 2**. (d) Proof of **Case 3**.

Data and materials availability statement: Data and materials sharing not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] A. K. Abu-Affash, P. Carmi, and M. Maman. Piercing diametral disks induced by edges of maximum spanning trees. In *WALCOM*, pages 71–77. Springer, 2023.

[2] P. Barabanshchikova and A. Polyanskii. Intersecting ellipses induced by a max-sum matching. *Journal of Global Optimization*, pages 1–13, 2023.

[3] P. Barabanshchikova and A. Polyanskii. Intersecting diametral balls induced by a geometric graph II. *Discrete Mathematics*, 347(1):113694, 2024.

[4] S. Bereg, O. P. Chacón-Rivera, D. Flores-Peñaloza, C. Huemer, P. Pérez-Lantero, and C. Seara. On maximum-sum matchings of points. *Journal of Global Optimization*, 85(1):111–128, 2023.

[5] D. Eppstein. Geometry Junkyard. https://www.ics.uci.edu/~eppstein/junkyard/maxmatch.html.

[6] J. A. Fingerhut, S. Suri, and J. S. Turner. Designing least-cost nonblocking broadband networks. *J. Algorithms*, 24(2):287–309, 1997.

[7] E. Helly. Über Mengen konvexer Körper mit gemeinschaftlichen Punkte. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 32:175–176, 1923.

[8] C. Huemer, P. Pérez-Lantero, C. Seara, and R. I. Silveira. Matching points with disks with a common intersection. *Discrete Mathematics*, 342(7):1885–1893, 2019.

[9] P. Pérez-Lantero and C. Seara. Center of maximum-sum matchings of bichromatic points. *Discrete Mathematics*, 347(3):113822, 2024.

[10] O. Pirahmad, A. Polyanskii, and A. Vasilevskii. Intersecting diametral balls induced by a geometric graph. *Discrete & Computational Geometry*, pages 1–18, 2022.

[11] P. Soberón and Y. Tang. Tverberg's theorem, disks, and Hamiltonian cycles. *Annals of Combinatorics*, 25(4):995–1005, 2021.