Von Neumann Algebras in Double-Scaled SYK

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Abstract: It's been argued that a finite effective temperature emerges and characterizes the thermal property of double-scaled SYK model in the infinite temperature limit [1]. On the other hand, in static patch of de Sitter, the maximally entangled state exhibits KMS condition of infinite temperature [2], suggesting the Type II_1 nature of the algebra formed by operators that are gravitationally dressed to the static patch observer. In the current work we study the double-scaled algebra generated by chord operators in double-scaled SYK model. We show that the algebra exhibits a behavior reminiscent of both perspectives. In particular, we prove that it's a Type II_1 factor, and the empty state with no chords satisfies the tracial property, thus aligning with the expectation in [3]. Furthermore, we show it's a cyclic separating state by exploring the modular structure of the algebra. We then study various limits of the theory and explore corresponding relations to JT gravity, theory of baby universe, and Brownian double-scaled SYK. We also present a full solution to the energy spectrum of 0- and 1-particle irreducible representations.

Contents

1 Introduction

Over the past few decades, remarkable advancement in understanding of quantum gravity has been made in the framework of the AdS/CFT correspondence [4–30]. The success lies in part due to the existence of an unambiguous specification of dynamics from the asymptotic boundary of AdS. More specifically, there is a putative boundary observer who can access the holographic degrees of freedom that are distinct from the

observer's own degree of freedom. In de Sitter, however, the situation changes because the observer is no longer external to the system. It's been argued that a physical clock made up of fundamental degrees of freedom is required in cosmological applications of quantum gravity for a long time, see for example [31–34]. This evolves into a fundamental challenge regarding the formulation of de Sitter gravity in an invariant manner.

Recently, several approaches [35–39] aimed at fundamental description of de Sitter gravity have been proposed, and notable advancements have been achieved in formulating static patch physics in terms of local operators that are gravitationally dressed to the observer's worldline [2, 40]. In particular, it's been shown in [40] those operators form a Type II_1 algebra $\mathcal{A}_{obs}$ with the a maximally entangled state $\Psi_{\max}$ consisting of the empty state Ψ_{dS} in static patch and the thermal equilibrium state of the observer with inverse temperature β_{dS}. Subsequently, it is verified that $\Psi_{\max}$ satisfies the KMS condition corresponding to infinite temperature, by explicitly examining the two point function of dressed operators [2]:

$$\langle\Psi_{\max}|\hat{a}\hat{b}|\Psi_{\max}\rangle = \langle\Psi_{\max}|\hat{b}\hat{a}|\Psi_{\max}\rangle, \quad \forall \hat{a}, \hat{b} \in \mathcal{A}_{obs}. \tag{1.1}$$

This can be viewed as a distinguished feature of observables within static patch. In addition, It is particularly noteworthy that there are no a priori assumptions made regarding any infinite temperature limit in the derivation of [2].

On the other hand, it's been observed that in double-scaled SYK model (DSSYK), a finite effective temperature emerges in the infinite temperature limit, and characterizes the thermal behavior of the system in this limit [1, 35]. This serves as the motivation for the proposal in [37], which claims that the infinite temperature limit of double-scaled SYK describes confined degree of freedom that lives on the stretched horizon of de Sitter. The bulk physics, in this scenario, emerges at finite effective temperature from holographic degrees of freedom at infinite temperature. A comprehensive analysis of the states and operators within the framework of DSSYK has been explored in [3, 41] and [42], uitilizing the chord language. Furthermore, various approaches from the representation theory of the quantum group have been proposed [43–45]. Specifically, it has been emphasized that the double-scaled algebra generated by chord operators manifests as a Type II_1 algebra, and the empty chord state Ω exhibits maximal entropy, drawing a parallel to the scenario observed in the static patch of de Sitter space.

The current paper aims to put various statements mentioned above on a firmer ground, by explicit construction of double-scaled algebra $\mathcal{A}$ formed by linear span of strings of chord operators in double-scaled SYK, and rigorously prove that it's indeed a Type II_1 Von Neumann factor. In presenting the proof, we declare that the empty state Ω with no chord is cyclic separating for $\mathcal{A}$, fulfilling the KMS condition of infinite

temperature and giving rise to the unique Type II_1 trace "Tr". This serves as an alternative motivation for the following equation:

$$Z_0(\beta) = \langle\Omega|e^{-\beta H_0}|\Omega\rangle \stackrel{?}{=} \mathrm{Tr}(e^{-\beta H_0}), \tag{1.2}$$

originally defined in an earlier paper [46], motivated by chord statistics, and interpreted as the partition function of the theory at the inverse temperature β.

The algebra $\mathcal{A}$ is the same as the double-scaled algebra defined in [3]. We will introduce a useful set of operator basis which can be viewed as field operators defined on classical chord configurations. They generate the corresponding state by acting on the empty chord state:

$$\Phi(n_0, \cdots, n_k) \equiv :H^{n_0} M H^{n_1} \cdots H^{n_{k-1}} M H^{n_k}:, \tag{1.3}$$

with carefully defined normal ordering for strings of Hamiltonian chord operator H and matter chord operator M. The generators M and H are Hermitian, making them suitable to be regarded as observables within the bulk Hilbert space description.

Organization of the Paper In section 2 we briefly review the construction of Hilbert space $\mathcal{H}_0$ in double-scaled SYK model without matter chords. We present both the chord number basis and energy basis and the overlap between them. We discuss two interesting scenarios where in the first case, the observer has access to all bounded operators $\mathcal{B}(\mathcal{H}_0)$ and in latter case the observer only has access to bounded functions of the Hamiltonian H_0, denoted as $\mathcal{A}_0$. In the first case the algebra is of Type I_∞ and the trace of this algebra is uniquely defined as summing over expectation values of all basis states. In the second case the algebra $\mathcal{A}_0$ is not a factor and there is no preferred definition of a trace. Consequently, in both scenarios, there is no justification for employing the expectation value in Ω as a trace, as was implicitly assumed in (1.2).

In section 3 we construct the double-scaled algebra $\mathcal{A}$ generated by chord operators and prove it's a Type II_1 factor. We specify the Hilbert space $\mathcal{H}$ of double-scaled SYK model with a single type of matter with weight Δ, and define operators of $\mathcal{A}$ by explicitly specifying their action on a generic state in $\mathcal{H}$. By construction, $\mathcal{H}$ contains $\mathcal{H}_0$ as a subspace. We then explore the modular structure of $\mathcal{A}$ and prove that the empty state Ω is cyclic separating for $\mathcal{A}$. Hence, an alternative and equally valid approach would be to initially define the operators' action on Ω and subsequently applying the GNS construction. We opt for the current presentation approach as it is directly motivated by the chord statistics established in [1, 46]. We assert the existence of an operator basis $\{\Phi_\xi|\ \forall|\xi\rangle \in \mathcal{H}\}$ of $\mathcal{A}$ by applying normal orderings to strings of chord operators. Subsequently, we formulate the cyclic property of Ω in terms of the operator-state

correspondence as follows:

$$\Phi_\xi|\Omega\rangle = |\xi\rangle. \tag{1.4}$$

We will utilize this operator language when addressing the finite emergent temperature in the following sections.

In section 4 we explore various limits of double-scaled SYK model and relations to JT gravity, theory of baby universe, and Brownian double-scaled SYK. In section 4.1 we revisit the triple scaling limit explored in [3] and extend the discussion of resulting Liouville quantum mechanics to one-particle sector. Various relations among gauge invariant wavefunctions in JT gravity can be derived by taking triple scaling limit of their counter part in DSSYK. We present two solvable scenarios for the one-particle wavefunction and provide insights on the expectations in a more general situation. In section 4.2 we consider the $q \to 1$ limit of DSSYK with r and r_V fixed as independent parameters. We present an explicit expression of inner product between states with arbitrary amount of matter and Hamiltonian chords. In a specific instance, we illustrate how chord dynamics closely resembles the behavior seen in the theory of baby universes in the semi-classical limit. This involves processes such as splitting and rejoining, or the direct evolution of baby universes from their initial to final states [9]. We conclude the section with an alternative presentation of the inner product, where the sum-over-matrices involved are achieved with help of an integral implementation of constraints. In section 4.3 we comment on the relation between $q \to 0$ limit with r and r_V fixed and Brownian DSSYK developed in [47]. We present the expression of the inner product in this limit and explore the corresponding relations to states and algebra in Brownian DSSYK.

In section 5 we discuss various future prospective of the algebraic study of DSSYK. In particular,We reformulate the results of the inner product by expressing them in terms of correlation functions of operators within $\mathcal{A}$ under the triple scaling limit. Note that it's a correlation function in an infinite temperature state $|\Omega\rangle$, but the result exhibits explicit dependence on a finite temperature parameter $c = c(\beta)$. We intend to view this fact as a preliminary manifestation of the idea that a finite effective emerges, serving to characterize the thermal behavior of the system within an infinite-temperature state. The dependence of this effective temperature is encoded in the operator algebra, despite the fact that the state Ω exhibits infinite temperature. We leave future exploration of this point and a potential algebraic characterization of hyper-fast scrambling to future work.

We present various details that are used in the main text in appendices. In appendix A, we present a full solution of energy spectrum in DSSYK with a generating function method. In particular, the one-particle irreducible representations can be solved in

terms of eigenstates of the left and right Hamiltonian. We further show that the inner product between 1-particle states can be reproduced by inserting an energy eigenbasis, and integrating the left and right energy with proper measure. In appendix B we comment on the relations among Lin-Stanford basis and Fock basis. We present an alternative formulation of 1-particle inner product in terms of matter correlators in the Fock basis. In appendix C we present a detailed derivation of left and right Liouville Hamiltonian with matter emerging from triple scaling limit.

Glossary

- $q \equiv e^{-\lambda}, \lambda > 0$ gives the penalty factor for crossing Hamiltonian chords.
- r_V gives the penalty factor for crossing between a Hamiltonian chord and a matter chord. We keep it independent of q in most context of the current paper and specify $r_V = q^{\Delta_V}$ only in certain context.
- r gives the penalty factor for crossing matter chords. We keep it independent of q in most context of the current paper and specify $r = q^{\Delta_V^2}$ only in certain context.
- $[A, B]_q \equiv AB - qBA$
- $[n]_q \equiv \frac{1-q^n}{1-q}$ is the q-integer.
- $(a;q)_n \equiv \prod_{k=1}^{n} \left(1 - aq^{k-1}\right)$ is the q-Pochhammer symbol. $(q;q)_0 \equiv 1$.
- $(a_1, a_2 \cdots, a_k; q)_\infty = \prod_{i=1}^{k} (a_i; q)_\infty$
- $[n]_q! = (q;q)_n / (1-q)$ is the q-factorial
- $\mathcal{H}_0$: The Hilbert space of DSSYK without matter chords.
- $\mathcal{H}$: The Hilbert space of DSSYK that contains matter chords. In the current paper we only consider a single type of matter, characterized by r_V and r.
- $|\Omega\rangle$ or Ω: Empty state with no open chords.
- $|n_0, n_1, \cdots, n_k\rangle$: A typical state in $\mathcal{H}$ consists of k matter chords, with n_{i-1} Hamiltonian chords positioned between the $(i-1)$-th and i-th matter chord, where i ranges from 1 to k.
- $a_L^\dagger$: creates a Hamiltonian chord from the left: $a_L^\dagger |n_0, \cdots, n_k\rangle = |n_0 + 1, \cdots, n_k\rangle$.
- a_L: defined as Hermitian conjugate of $a_L^\dagger$. Its action on state is defined in (3.7).

- $b_L^\dagger$: creates a matter chord from the left: $b_L^\dagger|n_0,\cdots,n_k\rangle = |0,n_0,\cdots,n_k\rangle$.
- b_L: defined as Hermitian conjugate of $b_L^\dagger$. Its action on state is defined in (3.16).
- $H_L \equiv a_L + a_L^\dagger$ is the left Hamiltonian chord operator.
- $M_L \equiv b_L + b_L^\dagger$ is the left matter chord operator.
- $\mathcal{A}_L$: The left double-scaled algebra generated by completion of finite linear span of strings with two letters: $H_L^{n_0} M_L H_L^{n_1} \cdots M_L H_L^{n_k}$.
- $\Phi_L(n_0,\cdots,n_k)$: The left chord field operator that satisfies $\Phi_L(n_0,\cdots,n_k)|\Omega\rangle = |n_0,\cdots,n_k\rangle$.
- $H_n(x|q)$: q-Hermite polynomial of order n.
- $H_{m,n}(x,y|q,r_V)$ bivariate r_V-weighted q-Hermite polynomial of order (m,n).
- $K_\nu(x)$: Bessel function of the first kind with order ν.
- $\mathcal{B}(\mathcal{H})$: The Von Neumann algebra of bounded linear operators acting on Hilbert space $\mathcal{H}$.
- $|\theta\rangle$: energy eigenbasis in $\mathcal{H}_0$.
- $\mu(\theta) \equiv (2\pi)^{-1}\left(e^{\pm 2i\theta},q;q\right)_\infty$ is the measure that defines inner product between energy basis. $\langle\theta_1|\theta_2\rangle = \mu(\theta_1)^{-1}\delta(\theta_1-\theta_2)$.

2 Warm up: States and Algebra in DSSYK without Matter Chords

We initiate our analysis of states and operator algebra of DSSYK without matter chords in this section. This is a theory that characterizes the dynamics of Hamiltonian chords[1]. The Hilbert space is generated by the linear span of states comprising a specific number of Hamiltonian chords. For a state with fixed chord number, The time evolution can either give rise to the creation of a new chord, resulting in a state with an additional chord, or lead to the annihilation of an existing chord within the state. Notably, the annihilation is sensitive to its location in the sequence of chords. When the annihilated

[1]We keep our language consistent with [46] and [1], where Hamiltonian chords are those generated or annihilated by H_0, and matter chords are generated by different operators. In the language of [37], Hamiltonian chords are named as chords while matter chords are named as cords.

chord intersects with another chord, it contributes to a factor of q. In subsequent discussion of this paper, we always assume $|q| < 1$.

The dynamics of chords is fully characterized by the following Hamiltonian that involves q-ladder operator:

$$H_0 = a + a^\dagger. \tag{2.1}$$

where a and $a^\dagger$ satisfy the q-commutator:

$$[a, a^\dagger]_q = aa^\dagger - qa^\dagger a = 1 \tag{2.2}$$

The spectrum of the theory is specified by states with different number of chords:

$$a^\dagger|n\rangle = |n+1\rangle, \quad a|n\rangle = [n]|n-1\rangle, \quad a|0\rangle = 0. \tag{2.3}$$

where $[n]$ is a q-deformed integer defined as follows:

$$[n] = q[n-1] + 1, \quad [n] = \frac{1-q^n}{1-q}, \quad [n]! = \frac{(q;q)_n}{(1-q)^n}, \tag{2.4}$$

where $[n]! = \prod_{k=1}^{n}[k]$ is the n-factorial. It's then straightforward to show

$$\langle n|m\rangle = \delta_{nm}[n]!. \tag{2.5}$$

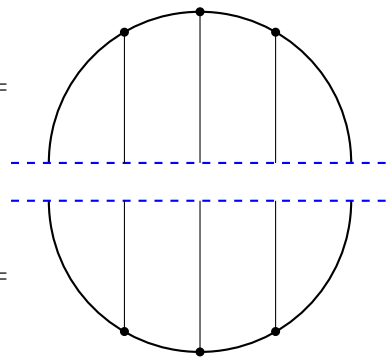
In chord language, the preparation of state $|n\rangle$ is achieved by slicing open a chord diagram with n non-intersecting open Hamiltonian chords, see (2.6) for an illustration. The inner product between a bra state $\langle m|$ and a ket state $|n\rangle$ is defined by sewing the two open diagrams into a joint one, and sum over all possible pairings of open chords between the initial and final state. For each given pairing, the result is weighted by q^c where c is the amount of crossings in the chord configuration determined by the pairing. Therefore, if $m \neq n$, one cannot pair all open chords, leading to vanishing result. When $n = m$, the result is $[n]!$ which correctly counts the q-weighted sum.

An alternative way of understanding the origin of $[n]!$ is to think of the q-weighted sum above as a summation over element π in permutation group S_n with a discrete measure $q^{\iota(\pi)}$, where $\iota(\pi)$ counts the inversions in π. A given configuration of chords with fixed initial and final states can be mapped to an element π in S_n, and the amount

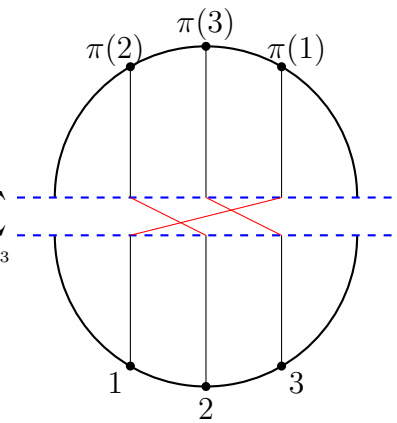
of parings is counted by the inversions in π. As an example, let's consider the case $n = 3$:

$\langle 3| =$

$|3\rangle =$



$= \sum_{\pi \in S_3}$



. (2.6)

There are six ways of pairing the chords in initial and final state, each corresponds to an element π in S_3. Different π corresponds to a 3-permutation, and we can specify it by its image: $(\pi(1), \pi(2), \pi(3)) \in \{(123), (231), (312), (213), (132), (321)\} = S_3$, with corresponding amount of inversions $\{0, 2, 2, 1, 1, 3\}$. Therefore we find in this case:

$$\sum_{\pi \in S_3} q^{\iota(\pi)} = (1+q)(1+q+q^2) = [3]_q!, \tag{2.7}$$

which reproduces the sum over intermediate crossings in (2.6). More generally, we have

$$\sum_{\pi \in S_n} q^{\iota(n)} = [n]!. \tag{2.8}$$

We shall adopt this formulation of inner product in the discussion involving matter. In conclusion, one can define the Hilbert space of DSSYK without matter as

$$\mathcal{H}_0 = \oplus_{n=0}^{\infty} \mathbb{C}|n\rangle. \tag{2.9}$$

with inner product specified above.

It's also helpful to introduce energy basis $|\theta\rangle$. The energy basis was originally found in [48] by diagonalizing H_0 with an infinite transfer matrix, and correctly evaluate the normalization factor. We briefly summarize the result as follows. The action of H_0 on $|\theta\rangle$ is given by:

$$H_0|\theta\rangle = E_0(\theta)|\theta\rangle = \frac{2\cos\theta}{\sqrt{1-q}}|\theta\rangle, \quad \theta \in [0, \pi]. \tag{2.10}$$

Clearly the energy spectrum is compactly supported in $[-\frac{2}{\sqrt{1-q}}, \frac{2}{\sqrt{1-q}}]$. The inner product between energy eigenstate is:

$$\langle\theta_1|\theta_2\rangle = \frac{\delta(\theta_1 - \theta_2)}{\mu(\theta_1)}, \quad \mu(\theta) = \frac{\left(e^{\pm 2i\theta}, q; q\right)_\infty}{2\pi}. \tag{2.11}$$

The state with n-chords in energy basis can now be expressed in terms of q-Hermite polynomial,

$$\langle\theta|n\rangle = \psi_n(\theta) = \frac{H_n(\cos\theta|q)}{(q;q)_n}, \quad \psi_0(\theta) = 1, \tag{2.12}$$

and with some algebra one can show that the inner product between $\langle n|$ and $|m\rangle$ can be consistently evaluated as:

$$\langle n|m\rangle = \int_0^\pi \mu(\theta)\,\mathrm{d}\theta\langle n|\theta\rangle\langle\theta|m\rangle = \delta_{nm}[n]!. \tag{2.13}$$

As a direct application, The partition function[2] can be evaluated as

$$\begin{aligned} Z_0(\beta) = \langle 0|e^{-\beta H}|0\rangle &= \int_0^\pi \mu(\theta)\,|\psi_0(\theta)|^2 \mathrm{d}\theta e^{-\frac{2\beta\cos\theta}{\sqrt{1-q}}} \\ &= \frac{\sqrt{1-q}}{\beta}\sum_{\nu=0}^{\infty}(-1)^p q^{\frac{\nu(\nu+1)}{2}}(2\nu+1)I_{2\nu+1}\left(\frac{2\beta}{\sqrt{1-q}}\right), \end{aligned} \tag{2.14}$$

where $I_\nu(x)$ is the modified Bessel function of the first kind. The result is valid for arbitrary inverse temperature β and $q : |q| < 1$.

In terms of energy basis, we can define $\mathcal{H}_0$ alternatively as all L^2-integrable functions in $[0,\pi]$ with measure $\mu(\theta)$:

$$\mathcal{H}_0 = L^2([0,\pi],\mu(\theta)). \tag{2.15}$$

Now we move on to the discussion of the operator algebra. One situation is that the observer has full access to all bounded operators that act on $\mathcal{H}_0$, namely, in this case $\mathcal{A}_{obs} = \mathcal{B}(\mathcal{H}_0)$. This is a Von Neumann algebra of Type I_∞ , and is equipped with a natural trace defined as summing over the expectation value in all basis vectors. Namely, for $a \in \mathcal{B}(\mathcal{H}_0)$, we have

$$\mathrm{tr}(a) := \sum_{n=0}^{\infty}\langle n|a|n\rangle = \int_0^\pi \mu(\theta)\,\langle\theta|a|\theta\rangle\mathrm{d}\theta. \tag{2.16}$$

In particular, the observer can measure the amount of chords in state $|n\rangle$ by looking at the expectation value of the size operator $q^{\hat{n}}$. This is a bounded operator with discrete spectrum in $[0,1]$, and is a trace-class operator with

$$\mathrm{tr}\left(q^{\hat{n}}\right) = \sum_{n=0}^{\infty} q^n = \frac{1}{1-q}. \tag{2.17}$$

[2]The rationale behind defining $Z_0(\beta)$ as the expectation value of $e^{-\beta H_0}$ in the state $|0\rangle$ remains unclear at the moment. We comment on this question at the end of this section and address it in subsequent sections.

There is another interesting situation where the observer has only access to H_0, or more concretely, all operators that are functions of H_0. In this case, the observer algebra $\mathcal{A}_{obs}$ is the maximal commutative Von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ that contains H_0, which we denote as $\mathcal{A}_0$.

With operators in H_0, the observer can access the state with arbitrary chords by adding chords into the empty state $|0\rangle$:

$$|1\rangle = H_0|0\rangle, \quad |2\rangle = H_0^2|0\rangle - |0\rangle, \ldots, \tag{2.18}$$

however, in this case the observer would not be able to know the trace defined in (2.16). This is because for a commutative algebra such as $\mathcal{A}_0$, any faithful positive linear functional $p : \mathcal{A}_0 \to \mathbb{C}$ satisfies

$$p(ab) = p(ba), \quad \forall a, b \in \mathcal{A}_0. \tag{2.19}$$

As a result, they are equally valid to the observer as a trace. This is similar to the situation in pure JT gravity where the only gauge invariant boundary operators are functions of the left or right Hamiltonian [49]. They are forced to be equivalent $H_L^{JT} = H_R^{JT} = H_0^{JT}$ after gauging the $SL(2,\mathbb{R})$ symmetry. Consequently, the theory lacks a preferred choice of trace unless an additional independent assumption is incorporated into its definition.

In either situation above, there is no natural reason to define the trace of the theory to be the expectation value in the vacuum state, as we did in defining the partition function $Z_0(\beta)$ as in (2.14). In the following section, we will enlarge $\mathcal{A}_{obs}$ by incorporating matter chord operators. As a result, the vacuum state characterized by the absence of both Hamiltonian chords and matter chords, becomes the tracial state unique up to constant rescaling for the extended algebra. Consequently, there is a preferred choice of trace Tr and one can reformulate (2.14) as

$$Z_0(\beta) = \langle\Omega|e^{-\beta H_0}|\Omega\rangle = \mathrm{Tr}(e^{-\beta H_0}). \tag{2.20}$$

3 States and Algebra in DSSYK with Matter Chords

In previous section we established Hilbert space description of the dynamics of Hamiltonian chords. In this section we introduce operators that generate matter chords in the state. For simplicity, we only consider one species of matter chord in the following discussion, and the analysis for multiple species follows in a similar manner.

3.1 Construction of the Double-Scaled Algebra

We construct the Hilbert space $\mathcal{H}$ by tensoring with the space of chord states that contains multiple particles, which is similar[3] to a Fock space construction of $\mathcal{H}_0$:

$$\oplus_{k=0}^{\infty}\text{span}_{\mathbb{C}}\{|n_0,\ldots,n_k\rangle|(n_0,\ldots,n_k)\in\mathbb{N}^{k+1}\}. \tag{3.1}$$

In literature, the state $|0\rangle$ is alternatively referred to as $|\Omega\rangle$ emphasizing its role as the vacuum state. In the following discussion, we will consistently employ the notation Ω to represent this state. A general state in $\mathcal{H}$ can then be denoted as $|n_0,\cdots,n_k\rangle$, which corresponds to a ket state with k open matter chords, which separates the half-chord diagram into $k+1$ divisions, and there are n_i Hamiltonian chords in the i-th division. The inner product for $\mathcal{H}$ is defined as

$$\langle n_0,n_1,\cdots,n_k|m_0,m_1,\cdots,m_l\rangle=\delta_{kl}\sum_{\pi\in S_k}r^{\iota(\pi)}\langle n_0,n_1,\cdots,n_k|m_0,m_1,\cdots,m_l\rangle^{\pi}, \tag{3.2}$$

where $\iota(\pi)$ is the number of inversion in permutation π, and r is the penalty factor for a crossing between two matter chords. In the following discussion, we keep r as a independent parameter of q and ranges from $r\in(0,1)$. The reason for summing over the permutations π in S_k in the inner product (3.2) is to include general configurations of matter chord intersections in the inner product. The permutation dependent inner product is defined recursively as:

$$\langle n_0,\cdots,n_k|m_0,\cdots,m_l\rangle^{\pi}=\sum_{j=0}^{l}[m_j]q^{\sum_{j'<j}m_j}r_V^{j}\langle n_0-1,\cdots,n_k|m_0,\cdots m_j-1,\cdots,m_k\rangle^{\pi}, \tag{3.3}$$

with boundary condition:

$$\langle 0,\cdots,n_i,\cdots,0|0,\cdots,n_i,\cdots,0\rangle^{\pi}=[n_i]_q!r_V^{2c_\pi(i)n_i},\quad i=0,1,\cdots k, \tag{3.4}$$

where the function of permutations $c_\pi(i)$ is defined as

$$c_\pi(i)=\#\{\pi(j)|j\le i,\quad i<\pi(j)\}, \tag{3.5}$$

which counts the extra crossings between Hamiltonian chords and matter chords in a given channel. To illustrate, let's consider the case $\pi:(1,2,3,4)\to(4,3,2,1)$, we know $c_\pi(1)=1,c_\pi(2)=2,c_\pi(3)=1$ and $c_\pi(0)=c_\pi(4)=0$. r_V in (3.4) is the penalty

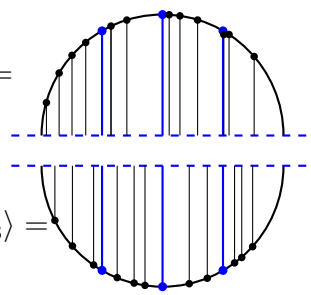
[3] This is not the standard Fock space construction because the states are not defined as simple tensor product of those in $\mathcal{H}_0$. To dinstinguish, we call the basis $|n_0,\cdots,n_k\rangle$ in $\mathcal{H}$ the Lin-Stanford basis and refer the reader to appendix B for exploration of its relation to the Fock basis $|n_0\rangle\otimes\cdots\otimes|n_k\rangle$.

factor for a single matter-Hamiltonian crossing, with its exponent counting the amount of such crossings. Similar to r, we treat r_V as an independent parameter that ranges from $(0,1)$ in the following discussion.

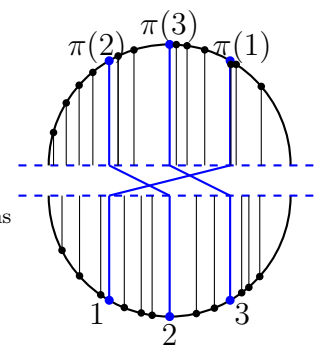
The evaluation of (3.3) can be understood as follows: For a fixed permutation $\pi \in S_k$, we have a chord diagram with a matter chord background corresponding to π, see (3.6) for an illustration. The amount of crossings among matter chords is counted by inversions of π. Then we insert Hamiltonian chords into this background, and prepare the bra and ket states by specifying the amount of Hamiltonian chords in between each matter chords. The inner product (3.2) then sums over all Hamiltonian chords configuration in a given matter chord background determined by π, and then sums over all permutations $\pi \in S_n$. The case with 3-particles can be visualized as:

$\langle n_0, n_1, n_2, n_3| =$

$|m_0, m_1, m_2, m_3\rangle =$



$$= \sum_{\pi \in S_3} \sum_{\text{configurations}}$$



. (3.6)

In the illustration above, the blue chords represent matter chords, whereas the black chords correspond to Hamiltonian chords.

Now we introduce the left and right ladder operators corresponding to Hamiltonian chords as:

$$a_L^\dagger \left|n_0, \cdots, n_k\right\rangle = \left|n_0+1, \cdots, n_k\right\rangle, \quad a_R^\dagger \left|n_0, \cdots, n_k\right\rangle = \left|n_0, \cdots, n_k+1\right\rangle, \tag{3.7}$$

$$a_L \left|n_0, \cdots, n_k\right\rangle = \sum_{j=0}^{k} [n_j]\, r_V^j q^{\sum_{l<j} n_l} \left|n_0, \cdots, n_j-1, \cdots, n_k\right\rangle, \tag{3.8}$$

$$a_R \left|n_0, \cdots, n_k\right\rangle = \sum_{j=0}^{k} [n_{k-j}]\, r_V^j q^{\sum_{l>k-j} n_l} \left|n_0, \cdots, n_{k-j}-1, \cdots, n_k\right\rangle. \tag{3.9}$$

It's straightforward to show that they satisfy the following commutation relations:

$$[a_L, a_L^\dagger]_q = [a_R, a_R^\dagger]_q = 1, \tag{3.10}$$

$$[a_L, a_R] = [a_L^\dagger, a_R^\dagger] = 0, \tag{3.11}$$

$$[a_L, a_R^\dagger] = [a_R, a_L^\dagger] = r_V^{\hat{n}_M} q^{\hat{n}_H}. \tag{3.12}$$

where the number operator $\hat{n}_H$ and $\hat{n}_M$ counts the number of Hamiltonian chords and matter chords correspondingly:

$$\hat{n}_H|n_0,\cdots,n_k\rangle = \sum_{i=0}^{k} n_i, \quad \hat{n}_M|n_0,\cdots,n_k\rangle = k. \tag{3.13}$$

Similar to (2.1), we introduce the left and right Hamiltonian chord operator as

$$H_{L/R} = a_{L/R} + a^\dagger_{L/R}. \tag{3.14}$$

They emerges as the double-scaled limit of Hamiltonian operator in SYK model. For a detailed explanation on this point, we refer the readers to [3]. In the current context, H_L and H_R are Hermitian operators that act on the chord Hilbert space $\mathcal{H}$. It's straightforward to examine that the left and right Hamiltonian chord operators commute:

$$[H_L, H_R] = 0. \tag{3.15}$$

We can similarly introduce ladder operators for matter chords as:

$$b^\dagger_L|n_0,\cdots,n_k\rangle = |0,n_0,\cdots,n_k\rangle, \quad b^\dagger_R|n_0,\cdots,n_k\rangle = |n_0,\cdots,n_k,0\rangle, \tag{3.16}$$

$$b_L|n_0,\cdots,n_k\rangle = \sum_{j=1}^{k} r^{j-1} r_V^{\sum_{l<j} n_l}|n_0,\cdots,n_{j-2},n_{j-1}+n_j,n_{j+1}\cdots,n_k\rangle, \tag{3.17}$$

$$b_R|n_0,\cdots,n_k\rangle = \sum_{j=1}^{k} r^{j-1} r_V^{\sum_{l>k-j} n_l}|n_0,\cdots,n_{k-j-1},n_{k-j}+n_{k-j+1},\cdots,n_k\rangle, \tag{3.18}$$

and they satisfy the following commutation relations:

$$[b_L, b^\dagger_L]_r = [b_R, b^\dagger_R]_r = 1, \tag{3.19}$$

$$[b_L, b_R] = [b^\dagger_L, b^\dagger_R] = 0, \tag{3.20}$$

$$[b_L, b^\dagger_R] = [b_R, b^\dagger_L] = r^{\hat{n}_M} r_V^{\hat{n}_H}. \tag{3.21}$$

The matter chord operator is defined as:

$$M_{L/R} = b_{L/R} + b^\dagger_{L/R}. \tag{3.22}$$

and the left and right matter chord operator commutes with each other:

$$[M_L, M_R] = 0. \tag{3.23}$$

Furthermore, one can show by applying the definition of ladder operators that the left and right generators commute with each other:

$$[H_L, H_R] = [H_L, M_R] = [M_L, H_R] = [M_L, M_R] = 0. \tag{3.24}$$

We now define the left/right algebra of chord observable to be the von Neumann algebra generated by all $H_{L/R}$ and $M_{L/R}$:

$$\mathcal{A}_{L/R} = \mathrm{vN}\left(H_{L/R}, M_{L/R}\right). \tag{3.25}$$

More precisely, the finite linear span of polynomials of generators in (3.14) and (3.23) form a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, which we denote as $\tilde{\mathcal{A}}_L$ and $\tilde{\mathcal{A}}_R$. They are not yet Von Neumann algebra because they do not necessarily contain the unit operator. We then complete them to Von Neumann algebra by taking double commutant $\mathcal{A}_{L/R} := \tilde{\mathcal{A}}''_{L/R}$. This gives a mathematically rigorous construction of the double-scaled algebra introduced in [3]. A typical operator in $\mathcal{A}_L$ can be written in the following form:

$$H_L^{n_0} M_L H_L^{n_1} \cdots M_L H_L^{n_k}. \tag{3.26}$$

Operators in form of (3.26) forms a dense subspace of $\mathcal{A}_L$ and similarly for $\mathcal{A}_R$. However, their action on Ω is complicated, as they generate superposition of states with different amount of chords. To illustrate, let's consider the action of H_L^k, which yields

$$H_L^k|\Omega\rangle = |k\rangle + \text{States with amount of chords less than } k. \tag{3.27}$$

It's more convenient in many situations to work with the normal ordered operator basis. Let's consider the situation without matter chords at first. The normal ordered operator basis can be defined recursively as:

$$:H_L^{k+1}:= H_L :H_L^k: - \overbracket{H_L :H_L^k}: \equiv H_L :H_L^k: - [k]_q :H_L^{k-1}:, \quad :H_L:\equiv H_L. \tag{3.28}$$

They generate state with definite amount of Hamiltonian chords:

$$: H_L^k : |\Omega\rangle = |k\rangle, \quad \forall k \in \mathbb{N}. \tag{3.29}$$

This can be shown by induction. One can assume (3.29) holds for $n \leq k$, and then show it holds for $k+1$ by acting the two sides of (3.28) on Ω. Note that the normal-ordering here is different from the conventional one defined by moving all creation operators to the left of all annihilation operators. Here it's defined with respect to the contraction rule of H_L in (3.28). As for matter chord operator M_L, we can define the normal ordering in a parallel manner:

$$: M_L^{k+1} := M_L : M_L^k : - [k]_r : M_L^{k-1} :, \quad : M_L := M_L \tag{3.30}$$

It's straightforward to show

$$: M_L^k : |\Omega\rangle = |0, 0, \cdots, 0\rangle, \quad \forall k \in \mathbb{N}. \tag{3.31}$$

We now define the normal ordering for a general operator basis in (3.26) as:

$$
\begin{aligned}
&:H_L^{n_0+1} M_L H_L^{n_1} \cdots M_L H_L^{n_k}: \equiv H_L :H_L^{n_0} M_L H_L^{n_1} \cdots M_L H_L^{n_k}: \\
&- \sum_{j=0}^{k} [n_j]_q \, r_V^j q^{\sum_{l<j} n_l} :H_L^{n_0} M_L \cdots M_L H_L^{n_j-1} M_L \cdots M_L H_L^{n_k}:,
\end{aligned}
\tag{3.32}
$$

where the terms subtracted are all possible contractions between H_L and $H_L^{n_0} M_L \cdots M_L H_L^{n_k}$. For example, we have:

$$
\overbrace{H_L :H:_L^{n_0} M_L \cdots M_L H_L^{n_j}} M_L \cdots = [n_j]_q \, r_V^j q^{\sum_{l<j} n_l} :H_L^{n_0} M_L \cdots M_L H_L^{n_j-1} M_L \cdots: \tag{3.33}
$$

We can apply similar rules to matter chord operators M_L, which gives rise to:

$$
\begin{aligned}
&:M_L H_L^{n_0} M_L \cdots M_L H_L^{n_k}: \equiv M_L :H_L^{n_0} M_L \cdots M_L H_L^{n_k}: \\
&- \sum_{j=1}^{k} r^{j-1} r_V^{\sum_{l<j} n_l} :H_L^{n_0} \cdots M_L H_L^{n_{j-1}+n_j} M_L \cdots M_L H_L^{n_k}: .
\end{aligned}
\tag{3.34}
$$

Equation (3.32) and (3.34) completely defines the normal ordering for any strings of operators M_L and $H_L^k, \forall k \in \mathbb{N}$.

Now, let's streamline the notation by introducing the chord field operator Φ_L as

$$
\begin{aligned}
&\Phi_L(n_0, \cdots, n_k) \equiv :H_L^{n_0} M_L H_L^{n_1} \cdots H_L^{n_{k-1}} M_L H_L^{n_k}:, \\
&\Phi_L(\Omega) \equiv \mathbf{1}, \quad \Phi_L(0, 0, \cdots, 0) =: M^k:, \quad \Phi(k) =: H^k: .
\end{aligned}
\tag{3.35}
$$

One can show following the same strategy as above that a general state $|n_0, \cdots, n_k\rangle$ can be generated by acting $\Phi_L(n_0, \cdots, n_k)$ on the empty state Ω:

$$
\Phi_L(n_0, \cdots, n_k) |\Omega\rangle = |n_0, \cdots, n_k\rangle. \tag{3.36}
$$

We can draw parallels with conventional quantum field theory by incorporating a classical configuration space of chords into the framework:

$$
\mathcal{C} := \cup_{k=0}^{\infty} \left\{ (n_0, \cdots, n_k) \in \mathbb{N}^{k+1} \right\}, \quad \Omega \equiv (0). \tag{3.37}
$$

Then $\Phi_L : \mathcal{C} \to \mathcal{A}_L$ is an operator-valued distribution on $\mathcal{C}$. For a given field configuration $x = (n_0, \cdots, n_k) \in \mathcal{C}$, $\Phi_L(x)$ is a field operator that generates a state $|x\rangle = |n_0, \cdots, n_k\rangle \in \mathcal{H}$ from the vacuum Ω. By construction, $\Phi_L(x), \forall x \in \mathcal{C}$ forms an operator basis that densely spans $\mathcal{A}_L$. Its equivalence to the original basis in (3.26) can be verified by the following observation:

We can introduce an operator order in the set of operator basis (3.26) by comparing a pair of numbers $(k, n_0, n_1, \ldots, n_k)$ in order, where k is the amount of M involved in

the operator monomial and $(n_0, \ldots, n_k)$ is the amount of Hs in between the matter chords. $(k, n_0, \ldots, n_k) > (l, m_0, \ldots, m_l)$ if $k > l$ or $k = l, (n_0, \ldots, n_k) > (m_0, \ldots, m_k)$ in order. If $\Phi(x)$ has larger operator order than $\Phi(y)$, we shall simply denote it as $\Phi(x) > \Phi(y)$. By construction, the left hand side of (3.32) and (3.34) are equal to the corresponding first term in the right hand side, where terms subtracted off are all smaller terms in the operator order. By induction on the operator order one can deduce that the normal ordered basis $\{\Phi_L(x), \forall x \in \mathcal{C}\}$ is equivalent to the original basis (3.26).

One can introduce the normal ordered field operator for the right operators in $\mathcal{A}_R$. We list corresponding definitions as follows:

$$\begin{aligned} &:H_R^{k+1}:\equiv H_R :H_R^k: -[k]_q :H_R^{k-1}:, \quad :H_R:\equiv H_R, \\ &:M_R^{k+1}:\equiv M_R :M_R^k: -[k]_q :M_R^{k-1}:, \quad :M_R:\equiv M_R. \end{aligned} \tag{3.38}$$

and:

$$\begin{aligned} &:H_R^{n_0+1} M_R H_R^{n_1} \cdots M_R H_R^{n_k}:\equiv H_R :H_R^{n_0} M_R H_R^{n_1} \cdots M_R H_R^{n_k}: \\ &- \sum_{j=0}^{k} [n_{k-j}]_q \, r_V^j q^{\sum_{l>k-j} n_l} :H_R^{n_0} M_R \cdots M_R H_R^{n_{k-j}-1} M_R \cdots M_R H_R^{n_k}: . \end{aligned} \tag{3.39}$$

$$\begin{aligned} &:M_R H_R^{n_0} M_R \cdots M_R H_R^{n_k}:\equiv M_R :H_R^{n_0} M_R \cdots M_R H_R^{n_k}: \\ &- \sum_{j=1}^{k} r^{j-1} r_V^{\sum_{l>k-j} n_l} :H_R^{n_0} \cdots M_R H_R^{n_{k-j}+n_{k-j+1}} M_R \cdots M_R H_R^{n_k}: . \end{aligned} \tag{3.40}$$

The right chord field operator $\Phi_R : \mathcal{C} \to \mathcal{A}_R$ is then defined as

$$\Phi_R(n_0, \cdots, n_k) \equiv :H_R^{n_0} M_R H_R^{n_1} \cdots M_R H_R^{n_k}: . \tag{3.41}$$

Different from (3.36), it generates a state with reversed ordering from empty state:

$$\Phi_R\left(n_0, \cdots, n_k\right) |\Omega\rangle = |n_k, \cdots, n_0\rangle. \tag{3.42}$$

In conclusion, we can view $\mathcal{A}_{L/R}$ as generated by field operators $\Phi_{L/R}(x)$ on all classical chord configuration $x \in \mathcal{C}$. We explore the structure of the algebra in subsequent sections with help of the operator-state correspondence (3.36) and (3.42).

3.2 Exploring Modular Structure of the Double-Scaled Algebra

We explore the modular structure of the double-scaled algebra in section 3.1, and point out the fact that the left and right algebra are commutants of each other. We begin by introducing the Tomita operator S_Ω [50] of $\mathcal{A}_L$:

$$S_\Omega \Psi |\Omega\rangle = \Psi^\dagger |\Omega\rangle, \quad \forall \Psi \in \mathcal{A}_L. \tag{3.43}$$

Note the above equation can be defined for any operator basis $\Phi(x)$, which means S_Ω can be defined in a dense subspace of $\mathcal{H}$. We now examine the action of S_Ω on a general state $|x\rangle = |n_0, \cdots, n_k\rangle \in \mathcal{H}$. Note that this state can be generated by acting an operator $\Phi_L(x)$ on Ω as

$$\Phi_L(x)|\Omega\rangle = |x\rangle = |n_0, \cdots, n_k\rangle. \tag{3.44}$$

We observe that $\Phi_L^\dagger(x)$ reverse the ordering of the operators in each term of operator products involved in its definition. More concretely, we have

$$\Phi_L(n_0, \cdots, n_k)^\dagger = \Phi_L(n_k, \cdots, n_0). \tag{3.45}$$

Therefore, we know for a general state $|n_0, \cdots, n_k\rangle \in \mathcal{H}$,

$$S_\Omega|n_0, \cdots, n_k\rangle = S_\Omega \Phi_L(x)|\Omega\rangle = \Phi_L^\dagger(x)|\Omega\rangle = |n_k, \cdots, n_0\rangle. \tag{3.46}$$

This matches the reflection operator R in [42]. In that context, it's an automorphism of the symmetry algebra that leads to an extra double trace irreducible representation. We then show in the following discussion it induces an isomorphism between $\mathcal{A}_L$ and $\mathcal{A}_R$.

The polar decomposition of S_Ω is given by:

$$S_\Omega = J\Delta_\Omega^{1/2}, \quad J^2 = \mathbf{1}, \quad \Delta_\Omega = S_\Omega^\dagger S_\Omega, \tag{3.47}$$

where J is anti-unitary and $\Delta_\Omega^{1/2}$ is hermitian and positive definite. In the current context, it follows from (3.46) that $\Delta_\Omega = \mathbf{1}$ and $J = S_\Omega$.

Now let's prove that the left and right algebras are commutants of each other. It's straightforward to show that $\mathcal{A}_L \subseteq \mathcal{A}_R'$ and $\mathcal{A}_R \subseteq \mathcal{A}_L'$ by the fact that the generators of the two algebras commute with each other, as shown in (3.24). A direct application of Tomita-Takesaki theory shows that $J\mathcal{A}_L J = \mathcal{A}_L'$. Hence, to prove the equivalence between $\mathcal{A}_R$ and $\mathcal{A}_L'$, we only need to show $J\mathcal{A}_L J \subseteq \mathcal{A}_R$. This can be verified by examing the action of $JH_L J$ and $JM_L J$ on a generic state $|n_0, \cdots, n_k\rangle \in \mathcal{H}$. we find

$$\begin{aligned}
JH_L J|n_0, \cdots, n_k\rangle &= JH_L|n_k, \cdots, n_0\rangle \\
&= J\left(|n_k+1, \cdots, n_0\rangle + \sum_{j=0}^{k}[n_{k-j}]r_V^j q^{\sum_{l>k-j} n_l}|n_k, \cdots, n_j - 1, \cdots, n_0\rangle\right) \\
&= |n_0, \cdots, n_k+1\rangle + \sum_{j=0}^{k}[n_{k-j}]r_V^j q^{\sum_{l>k-j} n_l}|n_0, \cdots, n_{k-j} - 1, \cdots, n_k\rangle \\
&= H_R|n_0, \cdots, n_k\rangle,
\end{aligned} \tag{3.48}$$

which implies that $JH_LJ = H_R$. Simiarly one can verify that $JM_LJ = M_R$. These relations among generators can then be extended to a dense subset of $\mathcal{A}_L$ consisting of finite linear span of strings of H_L and M_L. By taking closure of the subset, this confirms that $J\mathcal{A}_LJ \subseteq \mathcal{A}_R$. We conclude that $\mathcal{A}_L$ and $\mathcal{A}_R$ are indeed commutants of each other, with the following relations satisfied:

$$\mathcal{A}_L' = \mathcal{A}_R, \quad \mathcal{A}_R' = \mathcal{A}_L. \tag{3.49}$$

We conclude this section by highlighting the following observation: the empty state Ω serves as a cyclic separating state for both $\mathcal{A}_L$ and $\mathcal{A}_R$. The cyclic property of Ω can be directly inferred from the operator-state correspondence:

$$|n_0, \cdots, n_k\rangle = \Phi_L(n_0, \ldots, n_k)|\Omega\rangle = \Phi_R(n_k, \ldots, n_0)|\Omega\rangle. \tag{3.50}$$

Given that $\mathcal{A}_L$ and $\mathcal{A}_R$ are each other's commutants, it follows that the empty state Ω is separating for both algebras. Consequently, we establish that the empty state Ω is cyclically separating for the double-scaled algebra.

3.3 Tracial Property of Ω and The Type of the Double-Scaled Algebra

In following discussion, we shall denote $\mathcal{A}_L = \mathcal{A}$, $\mathcal{A}_R = \mathcal{A}'$, and $\Phi(x) = \Phi_L(x)$ for simplicity. We explain why $\langle\Omega| \cdot |\Omega\rangle$ serves as a natural trace in $\mathcal{A}$ and provide a proof that $\mathcal{A}$ is a Type II_1 factor. In particular, this means that any operator in $\mathcal{A}$ is trace-class and the trace of identity operator is unit.

We observe that the vacuum state $|\Omega\rangle$ satisfies a simplified version of KMS condition with infinite temperature[4]:

$$\begin{aligned}\langle\Omega|\Phi(x)\Phi(y)|\Omega\rangle &= \left\langle \Phi(x)^\dagger\Omega \mid \Phi(y)\Psi \right\rangle = \left\langle \Phi(x)^\dagger\Omega \left| S_\Omega^\dagger S_\Omega \right| \Phi(y)\Omega \right\rangle \\ &= \left\langle S_\Omega\Phi\left(y\right)\Omega \mid S_\Omega\Phi(x)^\dagger\Psi \right\rangle = \left\langle \Phi(y)^\dagger\Omega \mid \Phi(x)\Omega \right\rangle \\ &= \langle\Omega|\Phi(y)\Phi(x)|\Omega\rangle,\end{aligned} \tag{3.51}$$

where in the first line we used the fact that $\Delta_\Omega = \mathbf{1} = S_\Omega^\dagger S_\Omega$.[5] In the second line we used the definition of S_Ω and the fact that it's anti-linear. Since the finite linear

[4] There is no shift by $-i\beta$, and the two operators simply exchange.

[5] An alternative proof without reference to the modular operator can be derived from the following equation for every $\Phi_L(x)$:

$$\Phi_L\left(x\right)|\Omega\rangle = \Phi_R^\dagger\left(x\right)|\Omega\rangle. \tag{3.52}$$

As a result, we have

$$\begin{aligned}\langle\Omega|\Phi_L\left(x\right)\Phi_L\left(y\right)|\Omega\rangle &= \langle\Omega|\Phi_L\left(x\right)\Phi_R^\dagger\left(y\right)|\Omega\rangle = \langle\Omega|\Phi_R^\dagger\left(y\right)\Phi_L\left(x\right)|\Omega\rangle \\ &= \langle\Phi_R\left(y\right)\Omega|\Phi_L\left(x\right)\Omega\rangle = \langle\Phi_L^\dagger\left(y\right)\Omega|\Phi_L\left(x\right)|\Omega\rangle \\ &= \langle\Omega|\Phi_L\left(y\right)\Phi_L\left(x\right)|\Omega\rangle.\end{aligned} \tag{3.53}$$

combinations of field operators $\Phi(x)$ are dense in $\mathcal{A}$, we can uniquely extend the map $\langle\Omega|\cdot|\Omega\rangle$ to the whole algebra. In particular, we normalize the state so that $\langle\Omega|\mathbf{1}|\Omega\rangle = 1$.

We then show that $\langle\Omega|\cdot|\Omega\rangle$ is a faithful, normal and semifinite weight. Faithful means for any non-zero operator $\Psi \in \mathcal{A}$, $\langle\Omega|\Psi^\dagger\Psi|\Omega\rangle$ cannot be 0. This can be verified by noticing that

$$\langle\Omega|\Psi^\dagger\Psi|\Omega\rangle = \langle\Psi\Omega|\Psi\Omega\rangle. \tag{3.54}$$

If the above equation becomes zero, it implies $\Psi|\Omega\rangle = 0$. Note that $|\Omega\rangle$ is separating for $\mathcal{A}$, as a result, $\Psi|\Omega\rangle = 0$ leads to $\Psi = 0$. This contradicts the initial assumption of Ψ being non-zero. Therefore, we know $\langle\Omega|\Psi^\dagger\Psi|\Omega\rangle > 0, \Psi \neq 0$, and the state is indeed faithful.

Now we move on to show that the state is normal[6], which means for some increasing[7] net of positive operators $\Psi_\nu \in \mathcal{A}$ for ν in some directed index set $\mathcal{J}$, we have [51]

$$\langle\Omega|\sup_\nu \Psi_\nu|\Omega\rangle = \sup_\nu \langle\Omega|\Psi_\nu|\Omega\rangle. \tag{3.55}$$

Here increasing means $\Psi_\nu \leq \Psi_{\nu'}$ whenever $\nu \leq \nu'$. For a given sequence of operators $\{\Psi_\nu\}$, we can expand each Ψ_ν in terms of the operator basis as $\Psi_\nu = \sum_{x\in\mathcal{C}} c_x(\nu)\,\Phi(x)$, then it follows that

$$\langle\Omega|\sup_\nu \Psi_\nu|\Omega\rangle = \langle\Omega|\sup_\nu \sum_{x\in\mathcal{C}} c_x(\nu)\Phi(x)\,|\Omega\rangle = \langle\Omega|\sum_{x\in\mathcal{C}}(\sup_\nu c_x(\nu))\Phi(x)|\Omega\rangle = \sup_\nu c_\Omega(\nu)\,, \tag{3.56}$$

where we exchanged the order of the supremum over ν and sum over x in the second step because the sum is consistently convergent for all ν, in order for every Ψ_ν to be an element in $\mathcal{A}$. In the last step, only the identity component c_Ω survives projection. On the other hand, we know

$$\sup_\nu \langle\Omega|\Psi_\nu|\Omega\rangle = \sup_\nu \langle\Omega|\sum_{x\in\mathcal{C}} c_x(\nu)\,\Phi(x)\,|\Omega\rangle = \sup_\nu c_\Omega(\nu)\,. \tag{3.57}$$

This confirms that (3.55) holds true for Ω.

Finally, a weight $\varphi : \mathcal{A} \to \mathbb{C}$ is said to be semifinite if for every nonzero positive operator $\Psi \in \mathcal{A}$, there exists a positive operator $\Psi' \leq \Psi$ with finite $\varphi(\Psi')$. In our case, we can introduce a projection operator P_0 that projects onto states within an energy

[6]Normality is a direct consequence of the ultraweak continuity of the weight $\langle\Omega|\cdot|\Omega\rangle$. For a more detailed discussion, we refer the reader to section 4 of [49]. Here, we provide a proof based on the definition of normality.

[7]In this context, increasing refers to the ordering of positive operators. Specifically, for two positive operators A and B, we say that A is larger than B if their difference $A - B$ yields a non-zero positive operator.

window: $-E_0 < E < E_0$ with some energy cutoff $E_0 > 0$. For any positive $\Psi \in \mathcal{A}$, the operator $\Psi^{1/2} P_0 \Psi^{1/2}$ converges to Ψ in the strong operator topology as $E_0 \to \infty$[8]. As a result, $\Psi^{1/2} P_0 \Psi^{1/2}$ is nonzero for sufficiently large E_0, and

$$\langle \Omega | \Psi^{1/2} P_0 \Psi^{1/2} | \Omega \rangle = \langle \Omega | P_0 \Psi P_0 | \Omega \rangle \tag{3.58}$$

is finite. This shows that $\langle \Omega | \cdot | \Omega \rangle$ is indeed semifinite.

Combined with results all above, we know Ω is a faithful semifinite normal tracial state[9]. Consequently, we have

$$\mathrm{Tr}\Psi \equiv \langle \Omega | \Psi | \Omega \rangle, \forall \Psi \in \mathcal{A}, \quad \mathrm{Tr}(\mathbf{1}) = 1. \tag{3.59}$$

Now let's prove that $\mathcal{A}$ is a factor by showing that its center is trivial. The idea is that for a non-trivial operator $\mathcal{O}$ in $\mathcal{A}$, commuting with M or H generally increases its operator ordering. Let's make this clear by expanding $\mathcal{O}$ in terms of the operator basis, with

$$\mathcal{O} = \sum_{x \in \mathcal{C}} c_x(\mathcal{O}) \Phi(x). \tag{3.60}$$

The fact that $\mathcal{O}$ belongs to the algebra $\mathcal{A}$ implies that

$$\sum_{x \in \mathcal{C}} |c_x(\mathcal{O})|^2 < \infty. \tag{3.61}$$

Now let's consider a specific $x = (n_0, \cdots, n_k)$ with $c_x(\mathcal{O}) \neq 0$. The commutator between $\Phi(x)$ and M yields

$$[\Phi(x), M] = \Phi(x, 0) - \Phi(0, x) + \sum_{y \in \mathcal{C}, |y| < |x|} r_y \Phi(y), \tag{3.62}$$

where we have introduced $\Phi(x, 0) \equiv \Phi(n_0, \ldots, n_k, 0)$ and similarly for $\Phi(0, x)$. One can deduce from (3.34) that the residual terms are smaller in the operator ordering, with coefficient r_y given by product of r_V and r to some power. Since $|r| < 1$ and $|r_V| < 1$, we conclude that coefficients r_y in the residual terms are generally no less than 1.

Equation (3.62) shows that the commutator between $\Phi(x)$ and M contains larger terms in the operator order unless $x = \Omega$ or $x = (0, \ldots, 0)$. If this is not the case, then the fact that $[\mathcal{O}, M] = 0$ requires the existence of x' with $\Phi(x') > \Phi(x)$ and $c_{x'}(\mathcal{O}) \neq 0$, such that the residual terms in $[\Phi(x'), M]$ cancel out with $\Phi(x, 0)$ and $\Phi(0, x)$. Combined with the fact that the coefficients of residual terms are no less

[8] In fact, within the range $|q| < 1, |r| < 1, |r_V| < 1$, the energy spectrum of double-scaled SYK has been observed to be finite [46], which means $\Psi^{1/2} P_0 \Psi^{1/2}$ converges to Ψ at some finite E_0.

[9] Here, a state ω is defined as a weight with $\omega(\mathbf{1}) = 1$.

than 1, this requires $c_{x'}(\mathcal{O}) \geq c_x(\mathcal{O})$. We can continue with the same analysis for x', which then indicates the existence of an infinite amount of terms in (3.60) with non-decreasing coefficients as the operator order goes larger, violating the requirement $\sum_{x\in\mathcal{C}} |c_x(\mathcal{O})|^2 < \infty$. Consequently, x can only be of the form $(0,\ldots,0)$ or Ω, which means $\mathcal{O}$ can be expanded as

$$\mathcal{O} = c_\Omega(\mathcal{O})\mathbf{1} + \sum_{k=1}^{\infty} c_k(\mathcal{O}) :M^k: . \tag{3.63}$$

It's straightforward to show that all c_ks must vanish because there is no sequence of non-zero $\{c_k\}$s that can make $\mathcal{O}$ commute with H. Therefore, $\mathcal{O}$ has to be a multiple of identity operator. This means the center of $\mathcal{A}$ is trivial, thereby confirming that it is a factor.

We can now determine the type of $\mathcal{A}$ by the following observation: $\mathcal{A}$ cannot be of Type III because there exists a faithful normal semifinite trace and $\mathrm{Tr}(\mathbf{1}) = 1$. $\mathcal{A}$ cannot belong to Type I, because of the presence of a cyclic separating tracial state Ω and the algebra being of infinite dimensionality. Finally, since $\mathrm{Tr}(\mathbf{1}) = 1$ we conclude that $\mathcal{A}$ is a Type II_1 factor. This also demonstrates that $\mathrm{Tr}(\cdot) = \langle\Omega|\cdot|\Omega\rangle$ is the unique trace of $\mathcal{A}$ up to constant rescaling.

4 Exploring Various Limits of Double-Scaled SYK

In this section, we delve into various limits of the double-scaled SYK model, examining its connections to other theories in detail.

4.1 Revisiting the Triple Scaling Limit and its Connection to JT Gravity

In this section we discuss the triple scaling limit of DSSYK and examine the limiting result of 0- and 1-particle wavefunctions. The triple scaling limit is characterized by setting the parameter $q = e^{-\lambda}$ in (3.3) to 1, while maintaining a constant value for $\lambda n = l$. Here, the variable n is related to the total number of chords in a typical state, and we will provide specific details when addressing the triple scaling limit of an individual state. In the following discussion, we briefly review the derivation of the emergent Liouville Hamiltonian within this limit observed in [3], and extend the discussion to 1-particle case. We establish a dictionary between explicit expression of wavefunctions in DSSYK and their corresponding triple scaling limit. In addition, we provide interpretation in the context of JT gravity.

0-Particle Wavefunction The action of the Hamiltonian on the 0-particle spectrum gives rise to the following recursion relation for states with a fixed chord number:

$$\frac{2\cos\theta}{\sqrt{1-q}}\psi_n(\theta) = \sqrt{[n+1]_q}\psi_{n+1}(\theta) + \sqrt{[n]_q}\psi_n(\theta), \tag{4.1}$$

where $\psi_n(\theta) := \langle\theta|n\rangle$ defined in section 2. Now we take triple scaling limit of (4.1) with length and energy given by:

$$\begin{aligned}(1-q)[n]_q &= 1 - e^{-\lambda n} = 1 - \lambda^2 e^{-\tilde{l}},\\ \cos\theta(s) &= \cos\lambda s \simeq 1 - \frac{1}{2}\lambda^2 s^2 + O\left(\lambda^3\right),\end{aligned} \tag{4.2}$$

where in presenting chord number, we defined the renormalized length $\tilde{l}$, which is related to l by subtracting a divergent constant: $\tilde{l} = \log l + 2\log\lambda$. We also zoom in to the edge of the energy spectrum by introducing $\theta(s) = \lambda s$, where λ is a small parameter, and $s \geq 0$ serves as a new parameter for energy. We now introduce the "bulk" wavefunction by switching the energy and position in the original wave function, and express it in terms of the new parameters. More concretely, we introduce $\Psi_s(\tilde{l})$ as

$$\Psi_s(\tilde{l}) := \psi_{n(\tilde{l})}(\theta(s)). \tag{4.3}$$

Now let's reformulate (4.1) in terms of $\Psi_s(\tilde{l})$. We find it an identity at leading order in $O(\lambda)$. At next leading order, it becomes

$$\left(-\partial_l^2 + e^{-l}\right)\Psi_s(l) = s^2\Psi_s(l), \tag{4.4}$$

which turns out to be the equation satisfied by an energy eigenstate in Liouville quantum mechanics with energy $E_{\text{Liouville}}(s) = s^2$. The explicit solution to (4.4) can be expressed in terms of Bessel function as

$$\Psi_s(l) = 2K_{2is}\left(2e^{-l/2}\right), \tag{4.5}$$

where we have deduced the normalization constant of Ψ_s by taking triple scaling limit of the normalization condition of $\psi_n(\theta)$:

$$\int_0^\pi (\mu(\theta)\,\mathrm{d}\theta)\,\psi_n(\theta)\,\psi_m(\theta) = \delta_{nm}. \tag{4.6}$$

The measure $\mu(\theta)$ is specified in (A.17), and can be represented in terms of q-Gamma function as

$$\mu(\theta) = \frac{(q;q)_\infty^3(1-q)^2}{2\pi}\frac{1}{\Gamma_q(\pm 2i\theta/\lambda)}. \tag{4.7}$$

We drop out the θ-independent diverging constant in front when taking triple scaling limit, and the integral measure becomes [46]

$$\int_0^{\pi} \mathrm{d}\theta \mu(\theta) = C(\lambda) \int_0^{\pi/\lambda} \frac{\mathrm{d}s}{2\pi} \frac{1}{\Gamma_q(\pm 2is)} \xrightarrow{\lambda \to 0} \int_0^{\infty} \frac{\mathrm{d}s}{2\pi} \frac{1}{\Gamma(\pm 2is)}. \tag{4.8}$$

This reproduces the density of states in pure JT gravity:

$$\rho(s) = \frac{1}{2\pi\Gamma(\pm 2is)} = \frac{s}{2\pi^2} \sinh(2\pi s), \tag{4.9}$$

and by identifying $\psi_n(\theta)$ with $2K_{2is}\left(2e^{-l/2}\right)$, As a result, the normalization condition yields the appropriate normalization condition in the triple scaling limit:

$$\int_0^{\infty} \mathrm{d}s \rho(s) \left(2K_{2is}\left(2e^{-\tilde{l}/2}\right)\right) \left(2K_{2is}\left(2e^{-\tilde{l}'/2}\right)\right) = \delta\left(\tilde{l} - \tilde{l}'\right). \tag{4.10}$$

As a cross check, we can examine the generating function:

$$\frac{1}{(e^{\pm i\theta}; q)_{\infty}} = \sum_{n=0}^{\infty} \frac{H_n(\cos\theta|q)}{(q;q)_n}. \tag{4.11}$$

Continuing with a similar strategy, one can show that this relation accurately reproduces the following identity in the triple scaling limit:

$$\int_{-\infty}^{\infty} d\tilde{l} \left(2K_{2is}(2e^{-\tilde{l}/2})\right) = \Gamma(\pm is). \tag{4.12}$$

In conclusion, we establish the following correspondence table between expressions in DSSYK and their counter part in the triple scaling limit:

$$\begin{aligned} \psi_n(\theta) &\longleftrightarrow 2K_{2is}\left(2e^{-\tilde{l}/2}\right), \\ \int_0^{\pi} \mu(\theta)\,\mathrm{d}\theta &\longleftrightarrow \int_0^{\infty} \rho(s)\,\mathrm{d}s, \\ \sum_{n=0}^{\infty} \frac{H_n(\cos\theta|q)}{(q;q)_n} &\longleftrightarrow \int_{-\infty}^{\infty} \mathrm{d}\tilde{l} \left(2K_{2is}\left(2e^{-\tilde{l}/2}\right)\right). \end{aligned} \tag{4.13}$$

Matrix Components of Matter Operator Let's now move on to discuss the matter operator $\mathcal{O}$ with weight Δ. In chord formulation of double-scaled SYK , two point matter insertions with weight Δ is simply rephrased as insertion of $q^{\Delta\hat{n}}$ in the chord Hilbert space, where $\hat{n}$ is the number operator of Hamiltonian chords. Note that this does not belong to $\mathcal{A}_0$ defined in section 2 because it cannot be represented as a

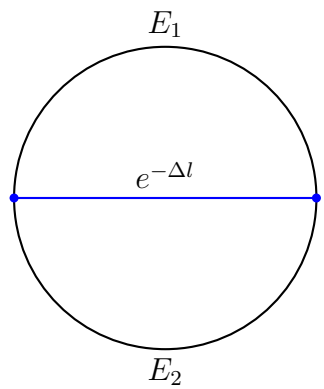
bounded function of H_0. This can be seen from the observation that $q^{\Delta\hat{n}}$ is not diagonal in the energy basis. Instead, its components in the energy basis is given by:

$$\left\langle \theta_1 \left| q^{n\Delta} \right| \theta_2 \right\rangle = \sum_{n=0}^{\infty} q^{n\Delta} \psi_n\left(\theta_1\right) \psi_n\left(\theta_2\right) = \frac{\left(q^{2\Delta}; q\right)_\infty}{\left(q^{\Delta} e^{\pm i\theta_1 \pm i\theta_2}; q\right)_\infty}. \tag{4.14}$$

In evaluating the formula, we have inserted the identity $\mathbf{1}_{\mathcal{H}_0} = \sum_{n=0}^{\infty} |n\rangle\langle n|$. We can then study the triple scaling limit of the above expression. Note that $q^{\Delta} \to \lambda e^{-\Delta \tilde{l}}$ within the limit, and combine it with the dictionary (4.13), (4.14) becomes:

$$\int_{-\infty}^{\infty} d\ell e^{-\Delta l} \left(2K_{2is_1}\left(2e^{-\ell/2}\right)\right)\left(2K_{2is_2}\left(2e^{-\ell/2}\right)\right) = \frac{\Gamma\left(\Delta \pm is_1 \pm is_2\right)}{\Gamma(2\Delta)}. \tag{4.15}$$

This precisely matches the 2 point function $|\langle E_1|\mathcal{O}|E_2\rangle|^2$ in energy basis at disk level of JT gravity, with identification $E_{1/2} = s_{1/2}^2$. The same formula can be obtained from boundary particle formalism after fixing the $SL\left(2,\mathbb{R}\right)$ gauge [52], illustrated as:

$|\langle E_1|\mathcal{O}|E_2\rangle|^2 =$ 

(4.16)

In the current context, the theory lacks gauge redundancies in its description. Observables are unambiguously defined through their action on the physical Hilbert space. The outcome of the triple scaling limit, applied to relationships among observables in DSSYK, transforms into the corresponding relationships among gauge-invariant quantities in JT gravity. We adopt this perspective as a guiding principle in our subsequent exploration of the limit for 1-particle wavefunctions.

1-Particle Wavefunction We now move on to study the triple scaling limit of 1-particle wavefunctions. A typical 1-particle state in this case is labeled as $|n_L, n_R\rangle$ with left and right chord number specified as n_L and n_R. In appendix A, we provide a comprehensive derivation of the 1-particle energy spectrum. In this section, we leverage the outcomes obtained in the appendix and explore the triple scaling limit of them.

We denote the wavefunction in energy eigenbasis as $\psi_{n_L,n_R}\left(\theta_L,\theta_R\right) = \langle\theta_L,\theta_R|n_L,n_R\rangle$, where $|\theta_L,\theta_R\rangle$ labels an eigenstate of both the left and right Hamitonian, with corresponding

eigenvalue $2\cos\theta_L/\sqrt{1-q}$ or $2\cos\theta_R/\sqrt{1-q}$. Therefore, the action of left and right Hamiltonian $H_{L/R}$ yields the following two recursion relations of ψ_{n_L,n_R}:

$$\begin{aligned}
\frac{2\cos\theta_L}{\sqrt{1-q}}\psi_{n_L,n_R} &= \sqrt{[n_L+1]_q}\psi_{n_L+1,n_R} + \sqrt{[n_L]_q}\psi_{n_L-1,n_R} + q^{\Delta+n_L}\sqrt{[n_R]_q}\psi_{n_L,n_R-1}, \\
\frac{2\cos\theta_R}{\sqrt{1-q}}\psi_{n_L,n_R} &= \sqrt{[n_R+1]_q}\psi_{n_L,n_R+1} + \sqrt{[n_R]_q}\psi_{n_L,n_R-1} + q^{\Delta+n_R}\sqrt{[n_L]_q}\psi_{n_L-1,n_R}.
\end{aligned} \tag{4.17}$$

They are related to the bi-variate q-Hermite functions in (A.8) via

$$\psi_{n_L,n_R}(\theta_1,\theta_2) = H_{n_L,n_R}\left(\cos\theta_L,\cos\theta_R|q,q^{\Delta}\right)/\sqrt{(q;q)_n\,(q;q)_m}. \tag{4.18}$$

Now we discuss the triple scaling limit of (4.17). As before, we introduce the new set of parameters that stay finite in the limit as

$$\begin{aligned}
&(1-q)\,[n_{L/R}]_q = 1-e^{-\lambda n_{L/R}} = 1-\lambda e^{-\tilde{l}_{L/R}}, \\
&\cos\theta_{L/R}(s_{L/R}) = \cos\lambda s_{L/R} \simeq 1-\frac{1}{2}\lambda^2 s_{L/R}^2 + O\left(\lambda^3\right).
\end{aligned} \tag{4.19}$$

where the renormalized left and right length is defined as $\tilde{l}_{L/R} = \lambda n_{L/R} + \log\lambda$. We then introduce the one particle wavefunction in terms of the new parameters by switching the energy and position in $\psi_{n,m}$ as:

$$\Psi^{\Delta}_{s_1,s_2}(\tilde{l}_L,\tilde{l}_R) = \psi_{n_L(\tilde{l}_L),n_R(\tilde{l}_R)}\left(\theta_1(s_1),\theta_2(s_2)\right). \tag{4.20}$$

By keeping the first non-trivial order of the two sides in the recursion equation (4.17), we find that $\Psi^{\Delta}_{s_1,s_2}$ satisfies the following equations:

$$\begin{aligned}
\left(-\partial_{\mathrm{L}}^2 + e^{-\tilde{l}_L}\left(\Delta+\partial_R-\partial_L\right) + e^{-\tilde{\ell}_{\mathrm{L}}-\tilde{\ell}_{\mathrm{R}}}\right)\Psi^{\Delta}_{s_1,s_2}(\tilde{l}_L,\tilde{l}_R) &= s_1^2\Psi^{\Delta}_{s_1,s_2}(\tilde{l}_L,\tilde{l}_R), \\
\left(-\partial_{\mathrm{R}}^2 + e^{-\tilde{l}_R}\left(\Delta-\partial_R+\partial_L\right) + e^{-\tilde{\ell}_{\mathrm{L}}-\tilde{\ell}_{\mathrm{R}}}\right)\Psi^{\Delta}_{s_1,s_2}(\tilde{l}_L,\tilde{l}_R) &= s_2^2\Psi^{\Delta}_{s_1,s_2}(\tilde{l}_L,\tilde{l}_R),
\end{aligned} \tag{4.21}$$

where $\partial_{R/L}$ are derivatives with respect to $\tilde{l}_{R/L}$. Note that in deriving the equation we only assumed that Δ is $O(1)$ in λ. Therefore, the wavefunction $\Psi^{\Delta}_{s_1,s_2}$ is only valid in the probing limit, where the matter does not back-react to the background geometry. Specifically, the geodesic length $\tilde{l}_{L/R}$ remains independent of the value of Δ, which is a continuous parameter that does not scale with λ. This observation is further substantiated by noting that Δ does not contribute to the energy spectrum. The energy of the gravitational state is characterized by two continuous parameters, denoted as (s_1,s_2), ranging from 0 to infinity. In the following discussion, we consider two limits of (4.21) where $\Psi^{\Delta}_{s_1,s_2}$ reduces to the familiar cases.

The first limit is $\Delta = 0$, where one expects recovery of 0-particle wavefunction. This assertion is supported by observing that:

$$\lim_{\Delta\to 0} \Psi^{\Delta}_{s_1,s_2}\left(\tilde{l}_L, \tilde{l}_R\right) = 2\delta\left(s_1 - s_2\right) K_{2is_1}\left(2e^{-\left(\tilde{l}_L+\tilde{l}_R\right)/2}\right), \tag{4.22}$$

which says the 1-particle wavefunction becomes the 0-particle wavefunction with total length $\tilde{l} = \tilde{l}_L + \tilde{l}_R$, and the energy spectrum is supported in the diagonal set with equal left and right energy $H_L = H_R$. This aligns with the observation in pure JT gravity. In this situation, there are no local energy fluctuations in bulk, and the left and right Hamiltonian are equal after imposing the constraints [49].

The validity of the above equation can be confirmed by examining the two limits in a reversed ordering. Let's consider taking $\Delta \to 0$ limit first in DSSYK by taking $r_V \to 1$ in (B.2) with equal left and right energy. This leads to the linearization formula for q-Hermite polynomials

$$\frac{H_{n_L}(\cos\theta \mid q) H_{n_R}(\cos\theta \mid q)}{(q;q)_{n_L}(q;q)_{n_R}} = \sum_{k=0}^{\min(n_L,n_R)} \frac{H_{n_L+n_R-2k}(\cos\theta \mid q)}{(q;q)_{n_L-k}(q;q)_{n_R-k}(q;q)_k}. \tag{4.23}$$

The triple scaling limit of the two sides results in the following identity

$$\begin{aligned}\left(2K_{2is}\left(2e^{-\tilde{l}_1/2}\right)\right)\left(2K_{2is}\left(2e^{-\tilde{l}_2/2}\right)\right) = \int_{-\infty}^{\infty} \mathrm{d}\tilde{l} \exp\left(-e^{\left(\tilde{l}-\tilde{l}_1\right)/2} - e^{\left(\tilde{l}-\tilde{l}_2\right)/2} - e^{-\tilde{l}/2}\right) \\ \times 2K_{2is}\left(2e^{\left(2\tilde{l}-\tilde{l}_1-\tilde{l}_2\right)/2}\right).\end{aligned} \tag{4.24}$$

This is equivalent to the equation (261) of [53]. From the boundary particle prospective, (4.24) can be interpreted as the composition law of propagators of the boundary particle. Here we obtain it as a fundamental relation between the $\Delta \to 0$ limit of 1-particle state and 0-particle state.

Now we move on towards another solvable case where we require $n_L = n_R$ and keep Δ as $O(1)$ when taking the triple scaling limit. We have $s_L = s_R$ simultaneously because of the left/right symmetry. We then denote the wavefunction in this case as $\Psi^{\Delta}_s(\tilde{l})$ where the total length is defined as: $\tilde{l} = 2\tilde{l}_{L/R}$. We find $\Psi^{\Delta}_s(\tilde{l})$ satisfies the following equation:

$$\left(-\tilde{\partial}^2 + \Delta e^{-\tilde{l}} + e^{-2\tilde{l}}\right)\Psi^{\Delta}_s(\tilde{l}) = s^2 \Psi^{\Delta}_s(\tilde{l}). \tag{4.25}$$

This corresponds to a particle moving in the Morse potential, and was obtained in [54] by quantizing JT gravity with end of world brane boundary conditions. In that context,

Δ is related to the radial derivative of dilaton field in the end-of-world brane boundary. The solution to (4.25) can be explicitly written in terms of Whittaker function as:

$$\Psi_s^{\Delta}(\tilde{l}) = e^{\tilde{l}/2} W_{-\frac{\Delta}{2}, is}(e^{-\tilde{l}}). \tag{4.26}$$

The normalization condition for Ψ_s^{Δ} can also be obtained by taking triple scaling limit of its counter part in DSSYK. It was observed in [55] that the triple scaling limit of big-continuous q-Hermite function $\psi_n^{\Delta}(\theta)$ defined by the following recursion:

$$\frac{2\cos\theta}{\sqrt{1-q}}\psi_n^{\Delta}(\theta) = \sqrt{[n+1]_q}\psi_{n+1}^{\Delta}(\theta) + q^{n+\frac{1+\Delta}{2}}\psi_n^{\Delta}(\theta) + \sqrt{[n]_q}\psi_{n-1}^{\Delta}(\theta), \tag{4.27}$$

leads to the same equation as (4.25). They are orthogonal with respect to a Δ dependent measure as:

$$\int_0^{\pi} \frac{d\theta}{2\pi}\mu^{\Delta}(\theta)\,\psi_n^{\Delta}(\theta)\psi_m^{\Delta}(\theta) = \delta_{n,m} \quad \mu^{\Delta}(\theta) = \frac{\left(e^{\pm 2i\theta}, q; q\right)_{\infty}}{\left(q^{\frac{\Delta+1}{2}} e^{\pm i\theta}; q\right)_{\infty}}. \tag{4.28}$$

Taking the triple scaling limit of the two sides, and with help of the correspondence table (4.13), we obtain

$$\int_0^{\infty} \mathrm{d}s \rho^{\Delta}(s)\,\Psi_s(\tilde{l})\Psi_s(\tilde{l}') = \delta\left(\tilde{l}-\tilde{l}'\right), \quad \rho^{\Delta}(s) = \rho(s)\,\Gamma\left(\frac{\Delta+1}{2} \pm is\right). \tag{4.29}$$

For state $\Psi_{s_L,s_R}^{\Delta}(\tilde{l}_L, \tilde{l}_R)$ with general configuration in energy and length, we expect a derivation for an analytic expression by taking triple scaling limit of (B.3). By taking triple scaling limit of (B.2), we expect the following relation to hold,

$$\left(2K_{2is_1}\left(2e^{-\tilde{l}_1/2}\right)\right)\left(2K_{2is_2}\left(2e^{-\tilde{l}_2/2}\right)\right) = \int_{-\infty}^{\infty} \mathrm{d}\tilde{l} I^{\Delta}\left(\tilde{l}, \tilde{l}_1, \tilde{l}_2\right) \Psi_{s_1,s_2}^{\Delta}\left(\tilde{l}-\tilde{l}_1, \tilde{l}-\tilde{l}_2\right), \tag{4.30}$$

which relates the 1-particle wavefunction with a product of two 0-particle wavefunctions, and reduces to (4.24) in the $\Delta \to 0$ limit. The current results suggest that a rich amount of relationships for gauge-invariant quantities in JT gravity, particularly in the probing limit, can be derived from the triple scaling limit of double-scaled SYK. The systematic exploration of extracting triple scaling limits from double-scaled SYK wavefunctions is deferred to future research.

4.2 The $q \to 1$ Limit and its Connection to Dynamical Baby Universes

We study the $q \to 1$ limit with r, r_V fixed, and its connection to theory of baby universe in this seciton.

In this limit, we can solve the recursive definition of the inner product explicitly, and represent the result of (3.2) as:

$$\begin{aligned}\langle n_0,\cdots,n_l|m_0,\cdots,m_l\rangle|_{q\to 1} &= \sum_{\pi\in S_l}\widetilde{\sum_{k_{ij}}} r^{\iota(\pi)} r_V^{\mathrm{tr}(\mathbb{D}^\pi\mathbb{K})}\cdot\frac{\prod_{i=0}^l n_i!m_i!}{\prod_{i,j=0}k_{ij}!}\\ &= \sum_{\pi\in S_l} r^{\iota(\pi)}\widetilde{\sum_{k_{ij}}} r_V^{\mathrm{tr}(\mathbb{D}^\pi\mathbb{K})}\cdot\prod_{i=0}^l\binom{n_i}{k_{i1},\cdots,k_{il}}\prod_{j=0}^l m_j!,\end{aligned} \tag{4.31}$$

where the sum $\widetilde{\sum}_{\{k_{ij}\}}$ is defined as summing over matrices $\mathbb{K}_{ij}=k_{ij}$ with the following constraint:

$$\sum_{j=0}^l k_{ij}=n_i,\quad \sum_{i=0}^l k_{ij}=m_j,\quad i,j=0,1,\cdots,l. \tag{4.32}$$

The distance matrix $\mathbb{D}^\pi_{ij}$ in the exponent of r_V is defined by

$$\mathbb{D}^\pi_{ij}=|i-j|+2c_\pi(i)\,\delta_{ij},\quad i,j=0,1,\cdots,l. \tag{4.33}$$

The intuition for the formula is as follows: $q\to 1$ means that the crossings among Hamiltonian chords do not give rise to any penalty factor, so the inner product (4.31) really counts the amount of ways of reassigning an ensemble of $(n_0,\cdots,n_l)$ Hamiltonian chords into another ensemble $(m_0,\cdots,m_l)$. This leads to the constraint sum over k_{ij} and the product of combinatoric factor. The matrix element k_{ij} is the number of chords at the i-th site in the initial state that have evolved into the j-th site in the final state. However, note that when this evolution is implemented, the Hamiltonian chords must intersect with matter chords with $|i-j|$ times, which is the distance between the two sites and appears in the distance matrix. The second term in (4.33) counts the extra crossings due to the intersecting configuration of matter chords.

Note that the above intuition is very similar to the theory of Baby universe developed in [9], and later in [56]. When calculating the amplitude from evolving n_i initial baby universes to n_f finial universes, one needs to sum over all interpolating geometries. The dynamics of baby universes allow some number m of the initial ones to evolve into m of the final one, or there could be any number k baby universes that split off and then rejoin and results in a big universe. Such feature of baby universes can be modelled in DSSYK in the $q\to 1$ limit. Let's consider the amplitude in (4.31) with 1-particle:

$$\langle n_0,n_1|m_0,m_1\rangle=\sum_{k=0}^{\min(n_1,m_0)} r_V^{n_0-m_0+2k}\frac{n_0!n_1!m_0!m_1!}{k!\,(n_0-m_0+k)!\,(n_1-k)!\,(m_0-k)!}, \tag{4.34}$$

where we have set $k_{01}=k$ and solved other k_{ij}s in terms of it. In particular, we have $k_{10}=n_0-m_0+k$. Now let's try to sum over k_{10} from 0 to infinity of the above result,

which leads to [10].

$$\widehat{\sum_{n_0-m_0+k}} \langle n_0, n_1|m_0, m_1\rangle = e^{r_V} \sum_{k=0}^{\min(n_1,m_0)} r_V^k \frac{n_0!n_1!m_0!m_1!}{k!\,(n_1-k)!\,(m_0-k)!}. \tag{4.35}$$

This matches equation 3.6 of [9] up to normalization constant, with identification $r_V^{-1} = e^{-2S_0}\,(VT)^2$. In that context, the factor r_V^{-1} measures the suppression in the Euclidean path integral by higher genus. We can reformulate (4.35) in the following suggestive form:

$$\widehat{\sum_{n_0-m_0+k}} \langle n_0, n_1|m_0, m_1\rangle|_{\mathrm{DSSYK}_{q\to 1}} \simeq \langle n_1|e^{-H_{\mathrm{BU}}T}|m_0\rangle|_{\mathrm{BU}}. \tag{4.36}$$

where $\simeq$ means up to normalization of states and identification of parameters. The reason why the chord statistics agrees with the dynamics of baby universe can be understood as follows: the sum over k_{01} basically counts the number of configurations with k_{01} of m_0 initial chords to k_{01} of n_1 final chords. Note that they all crosses the matter Hamiltonian, this produces the factor r_V^k in (4.35). The summation over k_{10} mimics the dynamics of splitting and rejoining. Summing over it creates the factor e^{r_V} in (4.35).

Based on the observations, we conclude that the $q \to 1$ limit of DSSYK might serve as certain completion of the Baby universe model, by incorporating a matter degree of freedom. In the following we present an alternative formulation of (4.31) by completing the sum over k_{ij}s. This is done by implementing the constrained sum as a free sum with delta functions as:

$$\prod_i \delta\left(n_i - \sum_j k_{ij}\right) = \int_{-\infty}^{\infty} \prod_i \left(\frac{\mathrm{d}\phi_i}{2\pi} e^{i\phi_i n_i - i\phi_i \sum_j k_{ij}}\right). \tag{4.37}$$

Now for fixed permutation π, the sum over $\{k_{ij}\}$ becomes

$$\sum_{\{k_{ij}\}} r_V^{\mathrm{Tr}(\mathbb{D}^\pi \mathbb{K})} \cdot \frac{\prod_{i=0}^l m_i!}{\prod_{i,j=0} k_{ij}!} = \int_{-\infty}^{\infty} \prod_i \left(\frac{\mathrm{d}\phi_i}{2\pi} e^{i\phi_i n_i}\right) \times \left(\sum_{k_{ij}:\sum_i k_{ij}=m_j} e^{-i\sum_{ij}\left(\phi_i k_{ij} - 2\lambda\Delta \mathbb{D}^\pi_{ij} k_{ij}\right)} \prod_{i=0}^l \binom{m_i}{k_{i0}, k_{i1}, \cdots, k_{il}}\right), \tag{4.38}$$

[10] We added a hat in the sum because there is no k dependence in the left hand side of the equation. Summing over k_{10} really means release one of the constraint in (4.34)

where we have set $r_V = e^{-\lambda\Delta}$. This then provides the following grouping of terms in terms of a product of polynomial expansion:

$$
\begin{aligned}
e^{-\sum_{ij}\left(i\phi_i k_{ij}+2\lambda\Delta\mathbb{D}^{\pi}_{ij}k_{ij}\right)}\prod_{i=0}^{l}\binom{m_i}{k_{0i},k_{1i},\cdots,k_{li}} = \\
\prod_{i=0}^{l}\left[\binom{m_i}{k_{0i},k_{1i},\cdots,k_{li}}\prod_{j=0}^{l}\left(e^{-i\phi_j-2\lambda\Delta\mathbb{D}^{\pi}_{ij}}\right)^{k_{ji}}\right].
\end{aligned}
\tag{4.39}
$$

Therefore, the sum over k_{ij}s can be easily implemented and leads to the following compact formula, with a product of sum over phases:

$$
\sum_{k_{ij}:\sum_i k_{ij}=m_j}\prod_{i=0}^{l}\left[\binom{m_i}{k_{i0},k_{i1},\cdots,k_{il}}\prod_{j=0}^{l}\left(e^{i\phi_i-2\lambda\Delta\mathbb{D}^{\pi}_{ij}}\right)^{k_{ij}}\right] = \prod_{i=0}^{l}\left(\sum_{j=0}^{l}e^{-i\phi_j-2\lambda\Delta\mathbb{D}^{\pi}_{ij}}\right)^{m_i}.
\tag{4.40}
$$

Plug this back into (4.31), the inner product can be evaluated as

$$
\begin{aligned}
\langle n_0,\cdots,n_l|m_0,\cdots,m_l\rangle|_{q\to 1} = \\
\sum_{\pi\in S_l} r^{\iota(\pi)}\prod_{i=0}^{l}\left[\int_{-\infty}^{\infty}\left(\frac{n_i!\mathrm{d}\phi_i}{2\pi}e^{i\phi_i(n_i-m_i)}\right)\left(\sum_{j=0}^{l}e^{i(\phi_i-\phi_j)-2\lambda\Delta\mathbb{D}^{\pi}_{ij}}\right)^{m_i}\right].
\end{aligned}
\tag{4.41}
$$

The form of (4.41) allows the potentiality of certain saddle point approximation, and extends beyond the regime where $\Delta \simeq O(1)$ in $\lambda \to 0$ limit. It would be interesting to explore the saddle point of (4.41) with fixed value of $\lambda\Delta$ in taking $\lambda \to 0$.

4.3 The $q \to 0$ limit and its Connection to Brownian DSSYK

The authors in [47] studied the algebra of Brownian DSSYK (BDSSYK) by explicitly constructing the algebra from the combined rules of chord statistics in BDSSYK and Schwinger-Keldysh path integral. A typical feature of the chord rules is that Hamiltonian chords are prohibited from intersecting. This implies considering the limit $q \to 0$ when defining the inner product. In this section we study the $q \to 0$ limit with r, r_V fixed of the algebra and states in section 3, and comment on its relation to BDSSYK.

The $q \to 0$ limit of $\mathcal{H}_0$ Let's first look at the $q \to 0$ limit in the 0-particle sector. This is a limit where crossings among Hamiltonian chords are not allowed. Therefore, it's easy to deduce that

$$
\langle n|m\rangle = \delta_{mn},
\tag{4.42}
$$

since for given amount of Hamiltonian chords, there is only one chord diagram that survives the limit. This is consistent with the fact that $\lim_{q\to 0}[n]_q = 1, \forall n \in \mathbb{Z}_{>0}$. The action of H_0 in this case yields the following recursion relation

$$2\cos\theta\psi_n(\theta) = \psi_{n+1}(\theta) + \psi_n(\theta), \quad \psi_0(\theta) = 1, \psi_{-1}(\theta) = 0, \tag{4.43}$$

where we have used the fact that $[n]_{q=0} = 1$. The solution to this recursion is given by Chebyshev polynomials of the second kind:

$$\psi_n(\theta) = U_n(\cos\theta) = \langle\theta|n\rangle. \tag{4.44}$$

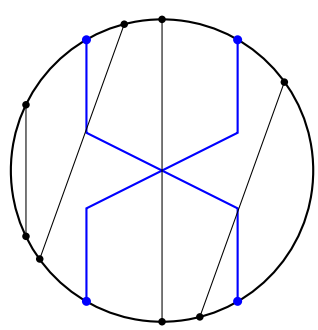
They are orthogonal under the Wigner measure:

$$\int_0^\pi \mu_0(\theta)\mathrm{d}\theta\psi_n(\theta)\psi_m(\theta) = \delta_{nm}, \quad \mu_0(\theta) = \frac{2}{\pi}\sin^2\theta, \tag{4.45}$$

and we conclude that $\mathcal{H}_0$ can be viewed as L^2-integrable functions in $[0,\pi]$ with this measure:

$$\mathcal{H}_0 = L^2([0,\pi], \mu_0(\theta)). \tag{4.46}$$

The $q \to 0$ limit of $\mathcal{H}$ We now consider taking $q \to 0$ limit with r, r_V fixed of the inner product (3.2). In this limit, crossings between Hamiltonian chords are forbidden, and we only need to sum over configurations that involves Hamiltonian-matter crossings and matter-matter crossings. A typical chord diagram is depicted as follows:



$$\subseteq \langle 1,2,1|2,2,0\rangle \tag{4.47}$$

We present the resulting inner product as follows:

$$\langle i_0,\cdots,i_k|j_0,\cdots j_l\rangle = \delta_{kl}\sum_{\pi\in S_k} r^{\iota(\pi)} r_V^{d_0(I,J)+d_\pi(I,J)}. \tag{4.48}$$

where $d_0(I,J)$ and $d_\pi(I,J)$ are two discrete metric on the space of $k+1$-partitions $I = \{i_0,\cdots,i_k\}$ and $J = \{j_0,\cdots,j_k\}$ of integer $n = i_0+\cdots i_k = j_0+\cdots+j_l$. They are

defined as

$$
\begin{aligned}
d_0\left(I,J\right) &= \sum_{m=0}^{k} \left|\left(i_0-j_0\right)+\left(i_1-j_1\right)+\cdots+\left(i_m-j_m\right)\right|, \\
d_\pi\left(I,J\right) &= 2\sum_{m=0}^{k} c_\pi\left(m\right)\min\{0, i_m - \left|\sum_{n=0}^{m-1}\left(i_n-j_n\right)\right|, j_m - \left|\sum_{n=0}^{m-1}\left(i_n-j_n\right)\right|\}.
\end{aligned}
\tag{4.49}
$$

The sum d_0+d_π correctly counts the total amount of intersections between Hamiltonian chords and matter chords. d_0 counts intersections arising from the evolution of Hamiltonian chords from one site in the initial state to a distinct site in the final state. For k-th matter chord, the amount of such intersections are given by the difference between $i_0+\cdots+i_{k-1}$ and $j_0+\cdots+j_{k-1}$, which can be understood as Hamiltonian chords that leak from the left to the right of the k-th matter chord. Therefore, summing over k counts the total number of intersections of this particular kind. The other term d_π counts the amount of Hamiltonian chords that remain at the same site in the evolution. They intersects with matter chords due to the intricate arrangement of matter chords. It's then easy to deduce that the amount of those crossings are given by the second equation of (4.49), with explicit dependence on matter configuration determined by π.

connection to Brownian double-scaled SYK We conclude this subsection by pointing out potential connections to the Brownian model introduced in [47]. Clearly, the fact that $q \to 0$ models the situation where none of Hamiltonian chords can intersect among themselves. However, the Brownian model could live in a different representation of the algebra. To illustrate, let's consider the 0-particle sector. Note that the above limit of DSSYK yields a Hilbert space $\mathcal{H}_0|_{q=0}$ with infinite dimension. However, in BDSSYK the Hilbert space $\mathcal{H}_0^B$ associated with a single timefold without matter insertion is 1 dimensional, as one can collapse any amount of Hamiltonian chords to the vacuum state without creating any physical significance:

$$
|\Omega\rangle = \supset, \quad H_0|\Omega\rangle = \supset\!\!\supset = |\Omega\rangle. \tag{4.50}
$$

This difference arises because of an emergent 1-dimensional representation of the algebra generated by a and $a^\dagger$. That's because the q-commutator becomes $aa^\dagger = 1$ in the limit, and therefore one can construct a one-dimensional representation of the algebra $\mathcal{A}_0|_{q=0}$ by defining

$$
a|\Omega\rangle = a^\dagger|\Omega\rangle = H_0|\Omega\rangle = |\Omega\rangle. \tag{4.51}
$$

This differs from $\mathcal{H}_0|_{q=0}$ in DSSYK. Yet by incorporating matter chord operators one can accommodate more states in the Hilbert space $\mathcal{H}$ with matter, as studied in [47] and

[57]. We expect similar form of inner product as in (4.48) to appear in the context of Brownian DSSYK. The discussions presented in this paper are expected to be directly applicable in the context of Brownian DSSYK as well.

5 Discussions and Future Prospective

We conclude with future prospective as follows:

Emergent temperature and Hyper-fast Scrambling We have shown that the empty state Ω satisfies a simplified version of KMS condition at infinite temperature, however, this does not mean that the theory is insensitive to the finite temperature effect. Instead, the temperature dependence is encoded in the operator algebra $\mathcal{A}$. In [42] the authors proposed a geometrical realization of such dependence by incorporating it into coordinates that parameterise the fake disk, which serves as a natural space for the symmetry algebra to act on.

In our context, we can consider semi-classical limit of the operator algebra and examine the correlation functions. As an example, let's consider the operator $\Phi\left(n_L, n_R\right)$ with

$$l = \lambda\left(n_L + n_R\right) = \lambda n = -2\log c, \tag{5.1}$$

where c is fixed to be a constant when we take $\lambda \to 0$ and is related to the inverse temperature β as $c = \cos \pi v/2 = \pi v/\beta$. Now we consider the two point function of operators $\Phi\left(n_L, n_R\right)$ and $\Phi\left(n'_L, n'_R\right)$ in this limit. Note that the two-point function is vanishing unless $n_L + n_R = n'_L + n'_R$, therefore, we introduce

$$x = \frac{\lambda}{2}\left(n_L - n_R\right), \quad x' = \frac{\lambda}{2}\left(n'_L - n'_R\right), \tag{5.2}$$

together with the following operators in the semi-classical limit as:

$$\varphi_\beta\left(x\right) := \lim_{\lambda\to 0}\left[\frac{\lambda}{-2\log c}\right]^{1/2}\Phi\left(x - \frac{\log c}{\lambda}, -x - \frac{\log c}{\lambda}\right), \tag{5.3}$$

where the β dependence in $\varphi_\beta\left(x\right)$ comes from c. Then the result of two point function can be expressed as

$$\langle\Omega|\varphi_\beta\left(x\right)\varphi_\beta\left(x'\right)|\Omega\rangle = \left[\frac{\left(1-c^2\right)/2}{\cosh\frac{x-x'}{2} - c\cosh\frac{x+x'}{2}}\right]^{2\Delta}, \tag{5.4}$$

which exhibits explicit dependence on c, even though $|\Omega\rangle$ is an infinite temperature state with respect of the operator algebra. This aligns with the observation in [1],[35]

that a finite effective temperature can emerge and characterizes the thermal behavior of the system. In particular, the scrambling time depends on this emergent temperature instead of $\log N$, which is referred to as hyperfast and is conjectured to be a key feature of a putative holographic description of de Sitter gravity [58],[59],[37]. It's natural to extend the above results to the entire algebra $\mathcal{A}$. Another natural question is whether the current discussion can be generalized to systems that exhibit similar emergence of temperature behavior. It would be helpful to formulate an algebraic formalism that characterizes emergent effective temperature and hyperfast scrambling.

Hagedorn Transition and Emergent Type III$_1$ Algebra There is an alternative semi-classical regime of DSSYK that one expects the algebra of one-sided operators to be of Type III$_1$. This is the regime where one fixes $\beta\mathcal{J}$ and let $\lambda = 2p^2/N$ goes to 0. This is the regime where the collective field analysis applies [60] and the chord statics can be correctly reproduced by Liouville field theory on a compactified causal wedge [42]. In this case, the partition function is given by

$$Z(\lambda) = \int Dg(\tau_1, \tau_2)\, e^{-I(\lambda)},$$
$$I(\lambda) = \frac{\beta\mathcal{J}}{\lambda} \int_{[0,1]^2/\mathbb{Z}_2} \mathrm{d}^2\tau \left(\partial_{\tau_1} g \partial_{\tau_2} g - e^{g(\tau_1,\tau_2)}\right),$$

which behaves similar to the phase above the Hagedorn temperature described in [61],[62], with $Z|_{\lambda\to 0} = \infty$. It was pointed out in [63],[64] that the transition to Type III$_1$ can be probed by the real two point function of single trace operators and the recent paper [65] raises a class of theories that exhibit such transitions[11]. It would be interesting to understand the phase structure of double-scaled SYK and characterizes the transition of the algebra constructed in the current work.

The Switched Role of Energy and Position To obtain the bulk wavefunction in JT gravity, we have switched the position and energy in the wavefunction of double-scaled SYK. We want to point out that this is not by accident but a generic feature of emergent gravitational interpretation. The fact that one needs to switch energy and position to extract gravitational physics was also observed in the double scaling limit of matrix models [67],[68]. In the double scaling limit, orthogonal polynomials in the matrix theory become continuous functions, and by zooming in on the edge of the string equation simultaneously, one obtains a dual quantum mechanical system. The energy and position operators of this system are obtained by carefully taking the leading

[11] In that context, however, the transition happens from Type I$_\infty$ to Type III$_1$. See also [66] which suggests that distinct types of von Neumann algebra emerge, each accounting for different phases within a model of Majorana chains.

order fluctuating part of the position and energy in the limiting recursion relations of orthogonal polynomials. For a detailed derivation, We refer the readers to equations (42) – (45) in [69] We leave a general discussion on this point to future work [70].

Entropy and Emergent Dilaton Profile A straightforward application of statements in [40],[71] and [49] shows that there is a notion of entropy for the matter chords algebra unique up to constant rescaling. However, it's not clear how such an algebraic entropy would agree with the result in JT gravity in the semi-classical limit. Specifically, a clear understanding of the emergence of a dilaton profile from the algebra $\mathcal{A}$ in this limit is not avaliable yet. Investigating this aspect further is a goal we aim to pursue in future research.

Delayed Scrambling and Hierarchy of Chaos It was found in [72] that at time scale $t_* \sim \beta_{\rm GH} \log(S)$ the light propagating fields in static patch starts to contribute significantly to the 2-point function of operators localized at the stretched horizon. A further study in [73] shows that the fact the two point function remains large for a relatively long time signatures a delay in scrambling[12], which is conjectured to happen for singlets in (charged-)DSSYK$_\infty$. The vast majority of entropy-carrying degrees of freedom exhibits hyper-fast scrambling without ever escaping the stretched horizon. One natural question arises: can we establish an algebraic framework for de Sitter that distinguishes these two distinct scrambling behaviors? If such a formulation exists, what potential connections might it unveil regarding the Hierarchy of chaos in Von Neumann algebra recently revisited in [74]?

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[12] I thank Leonard Susskind for pointing this out.

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A Towards a Full Solution of Energy Spectrum of 0- and 1-Particle Sector

In this section we present a full solution to the energy spectrum by first constructing the generating function of wavefunctions of fixed length state in energy basis in 0- and 1-particle sector of $\mathcal{H}$. We then show that the inner product in [42] are reproduced by integrating over the energy basis. This can be viewed as an independent derivation compared to the original q-weighted random walk approach.

A.1 The generating function of wavefunctions

We start with the 0-particle case. The action of H_0 on state $\psi_n(\theta_1)$ is:

$$\frac{2\cos\theta}{\sqrt{1-q}}\psi_n(\theta) = \psi_{n+1}(\theta) + [n]\psi_{n-1}(\theta). \tag{A.1}$$

We now introduce $x = \cos\theta$ and $H_n(x|q) = \sqrt{(q;q)_n}\psi_n$, we find the above relation becomes the standard recursion of q-Hermite polynomials:

$$2xH_n\left(x;q\right) = H_{n+1}\left(x;q\right) + \left(1-q^n\right)H_{n-1}\left(x;q\right), \tag{A.2}$$

with boundary condition that $H_{-1}\left(x;q\right) = 0, H_0\left(x;q\right) = 1$. We present a detailed derivation for its generating function and the strategy aligns with the latter solution of 1-particle case in later discussion. we introduce the following generating function:

$$F_0\begin{bmatrix} x \\ s \end{bmatrix} = \sum_{n=0}^{\infty} \frac{H_n\left(x;q\right)s^n}{\left(q;q\right)_n}. \tag{A.3}$$

The above recursion relation can then be written as

$$2xF_0\begin{bmatrix} x \\ s \end{bmatrix} = \frac{1}{s}\left(F_0\begin{bmatrix} x \\ s \end{bmatrix} - F_0\begin{bmatrix} x \\ qs \end{bmatrix}\right) + sF_0\begin{bmatrix} x \\ s \end{bmatrix}, \tag{A.4}$$

which can be presented in the following way

$$F_0\begin{bmatrix} x \\ s \end{bmatrix} = \frac{1}{1-2xs+s^2}F_0\begin{bmatrix} x \\ qs \end{bmatrix}. \tag{A.5}$$

This is a q-difference equation. If we introduce $x = \cos\theta$, we find the denominator factorizes into

$$1 - 2xs + s^2 = \left(1 - se^{i\theta}\right)\left(1 - se^{-i\theta}\right), \tag{A.6}$$

and we can use the above recursion for infinite time, yielding:

$$\begin{aligned} F_0\begin{bmatrix} x \\ s \end{bmatrix} &= \lim_{k\to\infty} \frac{1}{\prod_{j=0}^{k}\left(1 - q^j se^{i\theta}\right)\left(1 - q^j se^{i\theta}\right)} F_0\begin{bmatrix} x \\ q^k s \end{bmatrix} \\ &= \frac{1}{\left(se^{\pm i\theta}; q\right)_\infty} F_0\begin{bmatrix} x \\ 0 \end{bmatrix} = \frac{1}{\left(se^{\pm i\theta}; q\right)_\infty}, \quad |q| < 1. \end{aligned} \tag{A.7}$$

where we have used the fact that $F_0 = 1$ for $s = 0$, and we keep $|q| < 1$ so that the infinite q-Pochhammer symbol is finite.

Now we move on and solve for the following recursion relation induced from the action of H_L on $|m, n\rangle$:

$$\begin{aligned} 2xH_{m,n}(x, y; q, r_V) = {} & H_{m+1,n}(x, y; q, r_V) \\ & + \left(1 - q^m\right) H_{m-1,n}(x, y; q, r_V) \\ & + q^m \left(1 - q^n\right) r_V H_{m,n-1}(x, y; q, r_V), \end{aligned} \tag{A.8}$$

with $H_{m,0}\left(x, y; q, r_V\right) = H_m\left(x; q\right)$ and $H_{0,n}\left(x, y; q, r_V\right) = H_n\left(y\right)$. $H_{m,n}$ is symmetric under left-right exchange:

$$H_{m,n}\left(x, y; q, r_V\right) = H_{n,m}\left(y, x; q, r_V\right). \tag{A.9}$$

Therefore, we only need to solve (A.8) with above boundary conditions, and the result will satisfy the recursion for y automatically. In the following, we shall leave the q, r_V dependence of $H_{m,n}$ implicit for simplicity. We introduce the following generating function

$$F_1\begin{bmatrix} x & y \\ s & t \end{bmatrix} := \sum_{m,n=0}^{\infty} \frac{H_{m,n}(x, y) s^m t^n}{(q; q)_m (q; q)_n}. \tag{A.10}$$

Similar derivation shows that the recursion of $H_{m,n}$ translates into the following q-difference equation of F_1:

$$F_1\begin{bmatrix} x & y \\ s & t \end{bmatrix} = \frac{1 - r_V st}{1 - 2xs + s^2} F_1\begin{bmatrix} x & y \\ qs & t \end{bmatrix} \tag{A.11}$$

keep using the above recursion for $|q| < 1$ we end up with:

$$F_1\begin{bmatrix} x & y \\ s & t \end{bmatrix} = \frac{\left(r_V st; q\right)_\infty}{\left(se^{\pm i\theta}; q\right)_\infty} F_1\begin{bmatrix} x & y \\ 0 & t \end{bmatrix} = \frac{\left(r_V st; q\right)_\infty}{\left(se^{\pm i\theta}, te^{\pm i\phi}; q\right)_\infty}, \tag{A.12}$$

where we have set $x = \cos\theta$ and $y = \cos\phi$, and used the fact that

$$F_1 \begin{bmatrix} x & y \\ 0 & t \end{bmatrix} = \frac{1}{\left(te^{\pm i\phi}; q\right)_\infty} \tag{A.13}$$

in presenting the final result. In conclusion, we have:[13]

$$\frac{\left(r_V st; q\right)_\infty}{\left(se^{\pm i\theta}, te^{\pm i\phi}; q\right)_\infty} = \sum_{m,n=0}^{\infty} \frac{H_{m,n}\left(x, y\right) s^m t^n}{\left(q; q\right)_m \left(q; q\right)_n}, \tag{A.14}$$

which plays an important role in the computation of inner product in subsequent discussion.

A.2 Evaluation of Inner Product with Energy Basis

We show how to compute the inner product between fixed chord number states by inserting the energy eigenbasis. For 0-particle state, we show that

$$\langle n_1 | n_2 \rangle = \int_0^\pi \mathrm{d}\theta \mu\left(\theta\right) \langle n_1 | \theta \rangle \langle \theta | n_2 \rangle, \tag{A.15}$$

where the normalized wavefunction is given by

$$\psi_{n_1}\left(\theta\right) = \langle n_1 | \theta \rangle = \frac{H_{n_1}\left(x\right)}{\sqrt{\left(1-q\right)^{n_1}}}, \quad x = \cos\theta, \tag{A.16}$$

and the 0-particle measure

$$\mu\left(\theta\right) = \left(2\pi\right)^{-1} \left(e^{\pm 2i\theta}, q; q\right)_\infty, \tag{A.17}$$

is deduced from the Jacobian when one moves from chord number basis to energy basis.

To evaluate the integral (A.15), we consider the integral of the generating function as follows:

$$\int_0^\pi \mathrm{d}\theta \mu\left(\theta\right) \frac{1}{\left(se^{\pm i\theta}; q\right)_\infty} \cdot \frac{1}{\left(te^{\pm i\theta}; q\right)_\infty} = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \frac{\left(e^{\pm 2i\theta}, q; q\right)_\infty}{\left(se^{\pm i\theta}, te^{\pm i\theta}; q\right)_\infty}. \tag{A.18}$$

One can evaluate this integral by the Askey-Wilson integral given by equation (3.1.2) in [76], the result reads

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \frac{\left(e^{\pm 2i\theta}, q; q\right)_\infty}{\left(se^{\pm i\theta}, te^{\pm i\theta}; q\right)_\infty} = \frac{1}{\left(st; q\right)_\infty} = \sum_{n=0}^{\infty} \frac{s^n t^n}{\left(q; q\right)_n}. \tag{A.19}$$

[13] The same generating function has been found in an earlier literature [75], however, as we shall see in the following discussion, the measure in the current context differs from that in the paper and correctly reproduces the inner product in [42].

On the other hand, we can expand the integrand in (A.18) as a double series in s and t, and integrate term by term, which yields

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \frac{\left(e^{\pm 2i\theta}, q; q\right)_\infty}{\left(se^{\pm i\theta}, te^{\pm i\theta}; q\right)_\infty} = \sum_{n,m=0}^{\infty} \int_0^{2\pi} \mu\left(\theta\right) \frac{H_n\left(x\right) H_m\left(x\right)}{\left(q;q\right)_n \left(q;q\right)_m} \mathrm{d}\theta. \tag{A.20}$$

Therefore, we know

$$\int_0^{2\pi} \mu\left(\theta\right) H_n\left(x\right) H_m\left(x\right) = \delta_{n,m} \left(q;q\right)_m, \tag{A.21}$$

and the integral in (A.15) becomes

$$\langle n_1 | n_2 \rangle = \delta_{n_1,n_2} [n_1]!, \tag{A.22}$$

where we have introduced the q-factorial

$$[n]_q! := \frac{\left(q;q\right)_n}{\left(1-q\right)^n}. \tag{A.23}$$

The result matches the one derived by recursively using the q-commutation relation. We now move on to the one-particle case, and we show the following equation holds:

$$\langle n_L, n_R | n'_L, n'_R \rangle = \int \prod_{i=1}^{2} \mu\left(\theta_i\right) \mathrm{d}\theta_i |\langle \theta_1 | \mathcal{O} | \theta_2 \rangle|^2 \langle n_L, n_R | \theta_1, \theta_2 \rangle \langle \theta_1, \theta_2 | n'_L, n'_R \rangle, \tag{A.24}$$

where the matter matrix element and wavefunction is defined as [46]

$$\begin{aligned} |\langle \theta_1 | \mathcal{O} | \theta_2 \rangle|^2 &= \frac{\left(r_V^2; q\right)_\infty}{\left(r_V e^{\pm i\theta_1 \pm i\theta_2}; q\right)_\infty}, \\ \langle \theta_1, \theta_2 | n_L, n_R \rangle &= \frac{H_{n_1,n_2}\left(x,y\right)}{\sqrt{\left(1-q\right)^{n_1+n_2}}}, \quad x = \cos\theta_1, y = \cos\theta_2, \end{aligned} \tag{A.25}$$

where $r_V = q^{\Delta_V}$, Δ_V is the conformal weight of the matter operator V. In the following discussion, we keep r_V and q as two independent parameters that ranges from 0 to 1. Following the same strategy, we evaluate (A.24) by considering the integral of the generating function:

$$\begin{aligned} I\left(s_1, s_2, t_1, t_2\right) &:= \int_0^{\pi} \left(\prod_{i=1}^{2} \mu\left(\theta_i\right) \mathrm{d}\theta_i \right) |\langle \theta_1 | \mathcal{O} | \theta_2 \rangle|^2 \cdot \frac{\left(r_V s_1 t_1, r_V s_2 t_2; q\right)_\infty}{\left(s_1 e^{\pm i\theta_1}, s_2 e^{\pm i\theta_1}, t_1 e^{\pm i\theta_2}, t_2 e^{\pm i\theta_2}; q\right)_\infty} \\ &= \int_{[0,\pi]^2} \frac{\mathrm{d}\theta_1 \mathrm{d}\theta_2}{\left(2\pi\right)^2} \frac{\left(r_V s_1 t_1, r_V s_2 t_2, r_V^2, e^{\pm 2i\theta_1}, e^{\pm 2i\theta_2}, q, q; q\right)_\infty}{\left(s_1 e^{\pm i\theta_1}, s_2 e^{\pm i\theta_1}, t_1 e^{\pm i\theta_2}, t_2 e^{\pm i\theta_2}, r_V e^{\pm i\theta_1 \pm i\theta_2}; q\right)_\infty}. \end{aligned} \tag{A.26}$$

We evaluate the θ_1 integral first by Askey-Wilson formula, we find

$$\int_0^\pi \frac{\mathrm{d}\theta_1}{2\pi} \frac{\left(e^{\pm 2i\theta_1}, q; q\right)_\infty}{\left(s_1 e^{\pm i\theta_1}, s_2 e^{\pm i\theta_1}, r_V^{\pm i\theta_1 \pm i\theta_2}; q\right)_\infty} = \frac{\left(s_1 s_2 r_V^2; q\right)_\infty}{\left(s_1 s_2, r_V^2; q\right)_\infty \left(s_1 r_V e^{\pm i\theta_2}, s_2 r_V e^{\pm i\theta_2}; q\right)_\infty}. \tag{A.27}$$

The subsequent integral over θ_2 gives:

$$\begin{aligned} &\int_0^\pi \frac{\mathrm{d}\theta_2}{2\pi} \frac{\left(e^{\pm 2i\theta_2}, q; q\right)_\infty}{\left(s_1 r_V e^{\pm i\theta_2}, s_2 r_V e^{\pm i\theta_2}, t_1 e^{\pm i\theta_2}, t_2 e^{\pm i\theta_2}; q\right)_\infty} \\ &= \frac{\left(s_1 s_2 t_1 t_2 r_V^2; q\right)_\infty}{\left(s_1 s_2 r_V^2, s_1 t_1 r_V, s_1 t_2 r_V, s_2 t_1 r_V, s_2 t_2 r_V, t_1 t_2; q\right)_\infty}. \end{aligned} \tag{A.28}$$

Therefore, we know

$$I\left(s_1, s_2, t_1, t_2\right) = \frac{\left(s_1 s_2 t_1 t_2 r_V^2; q\right)_\infty}{\left(s_1 s_2, t_1 t_2, s_1 t_2 r_V, s_2 t_1 r_V; q\right)_\infty}. \tag{A.29}$$

We can expand the integrand of (A.26) and exchange the integral and sum, which yields

$$\begin{aligned} I(s_1, s_2, t_1, t_2) = &\sum_{n_L, n_R, n'_L, n'_R = 0}^{\infty} \int_0^\pi \left(\prod_{i=1}^{2} \mu\left(\theta_i\right) \mathrm{d}\theta_i \right) \left|\langle \theta_1 | \mathcal{O} | \theta_2 \rangle\right|^2 \\ &\times \frac{H_{n_L, n_R} H_{n'_L, n'_R}}{(q,q)_{n_L} (q,q)_{n_R} (q,q)_{n'_L} (q,q)_{n'_R}} s_1^{n_L} t_1^{n_R} s_2^{n'_L} t_2^{n'_R}. \end{aligned} \tag{A.30}$$

by matching order by order in (s_1, t_1, s_2, t_2), we conclude that for $n_L \geq n'_L$, we have

$$\begin{aligned} &\int_{[0,\pi]^2} \left(\prod_{i=1}^{2} \mu\left(\theta_i\right) \mathrm{d}\theta_i \left|\langle \theta_1 | \mathcal{O} | \theta_2 \rangle\right|^2 H_{n_L, n_R} H_{n'_L, n'_R} = \delta_{n_L + n_R, n'_L + n'_R} \times \right. \\ &\left. \left(\sum_{k=0}^{\min\left(n_R, n'_L\right)} q^{k^2 + k\left(n_L - n'_L\right)} r_V^{2k + n_L - n'_L} \frac{(q;q)_{n_L} (q;q)_{n_R} (q;q)_{n'_L} (q;q)_{n_L + n_R - n'_L}}{(q;q)_{n'_L - k} (q;q)_{n_R - k} (q;q)_{n_L - n'_L + k} (q;q)_k} \right) \right. . \end{aligned} \tag{A.31}$$

It's straightforward to show that the result matches [42] by converting H_{n_L, n_R} to the corresponding normalized wavefunction ψ_{n_L, n_R}.

A general state with arbitrary amount of matter chords can be constructed from 0- and 1-particle states through block decomposition. For example, a general 2-particle state $|n_L, n_1, n_R\rangle$ can be decomposed into chord irreducible representations as [42]:

$$|n_L, n_1, n_R\rangle = \sum_{m_L + m_R + k = n_1} \psi_{k, m_L, m_R} \left|[\mathrm{VW}]_k; n_L + m_L, n_R + m_R\right\rangle, \tag{A.32}$$

with Clebsch-Gordon coefficients ψ_{k,m_L,m_R}. Therefore, we expect that the results present in this section to extend to the entire Hilbert space $\mathcal{H}$ and determines the full energy spectrum of the theory.

B The Fock-Decomposition of Lin-Stanford Basis

In this section we try to clarify the relationship between Lin-Stanford basis and Fock space basis, and point out a reformulation of the inner product in terms of a doubled Hilbert space, as suggested in [77]. For better clearance, in the following discussion we denote Lin-Stanford basis with k matter chords as $|n_1, \cdots, n_k\rangle$ and the Fock basis as $|n_1\rangle \otimes \cdots \otimes |n_k\rangle$. From the discussion in section 2, it's clear that the 0-particle states are equivalent. We then move on to discuss 1-particle states.

We observe that the generating function of 1-particle states can be re-expressed in terms of 0-particle states as:

$$\frac{(r_V st; q)_\infty}{|(se^{i\theta}, te^{i\phi}; q)_\infty|^2} = \sum_{k,m,n=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} r_V^k s^{m+k} t^{m+k}}{(q;q)_k (q;q)_m (q;q)_n} H_m(x) H_n(y). \tag{B.1}$$

Therefore, by comparing (B.1) with (A.14), we conclude that the tensor product of two 0-particle wavefunction can be expressed in terms of superposition of 1-particle wavefunctions as:

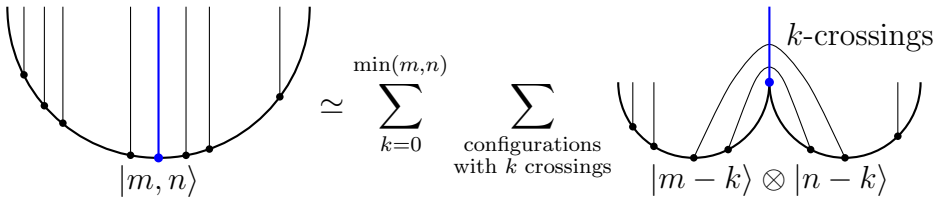
$$\frac{H_n(x|q)\, H_m(y|q)}{(q;q)_n (q;q)_m} = \sum_{k=0}^{\min(m,n)} \frac{r_V^k H_{n+m-2k}(x, y|q, r_V)}{(q;q)_{n-k} (q;q)_{m-k} (q;q)_k}. \tag{B.2}$$

The inverse of the above formula then presents a way to express $H_{m,n}$ in terms of a weighted summation of product of H_ns:

$$H_{m,n}(x,y) = \sum_{k=0}^{\min(m,n)} \frac{(-1)^k q^{k(k-1)/2} (q;q)_m (q;q)_n r_V^k}{(q;q)_{m-k} (q;q)_{n-k} (q;q)_k} H_{m-k}(x;q) H_{n-k}(y;q) \tag{B.3}$$

The formula can be understood as mapping $|m,n\rangle$ to a Fock space of $\mathcal{H}_0$, where the resulting state sums over all possible pairings between open chords in the left and right. Each crossing between Hamiltonian chords contributes to a weight of q and each crossing between Hamiltonian chords and matter chords contributes to a weight of $-r_V$. The coefficient in (B.3) correctly counts the result of the weighted sum. This map can

be illustrated as:[14]



. (B.4)

Now we extrapolate the above relations a little bit and try to formulate the inner product (A.31) in a doubled Hilbert space[?]. We do this by rewriting (B.3) as:

$$\begin{aligned} |m,n\rangle &= \sum_{k=0}^{\min(m,n)} \frac{(-1)^k q^{\binom{k}{2}} (q;q)_m (q;q)_n r_V^k}{(q;q)_{m-k}(q;q)_{n-k}(q;q)_k} |m-k\rangle \otimes |n-k\rangle \\ &= |m\rangle \otimes |n\rangle + (\text{states with total chord number less than } m+n)\,, \end{aligned} \tag{B.5}$$

from which one can deduce the linear-independence of Lin-Stanford basis. We explore further this relation and denote (B.5) systematically as

$$|m,n\rangle = \sum_{k=0}^{\min(m,n)} c_{m,n}(k)\, |m-k\rangle \otimes |n-k\rangle, \quad c_{0,0}(k) = 1. \tag{B.6}$$

The inner product discussed in the previous section can be expressed as:

$$\begin{aligned} \langle m,n|m',n'\rangle = \int \prod_{i=1}^{2} \mu(\theta_i)\, \mathrm{d}\theta_i |\langle\theta_1|\mathcal{O}|\theta_2\rangle|^2 \sum_{k=0}^{\min(m,n)} \sum_{k'=0}^{\min(m',n')} c_{m,n}(k)\, c_{m',n'}(k') \\ \times \left(\langle m-k|\theta_1\rangle\langle n-k|\theta_2\rangle\langle\theta_1|m'-k'\rangle\langle\theta_2|n'-k'\rangle\right). \end{aligned} \tag{B.7}$$

The matter density of state can be interpreted as a two point function in the double Hilbert space as:

$$|\langle\theta_1|\mathcal{O}|\theta_2\rangle|^2 = \langle\theta_1,\theta_2|\mathcal{O}^L\mathcal{O}^R|\theta_2,\theta_1\rangle, \tag{B.8}$$

where we have embedded the original operator $\mathcal{O}$ as an 2-sided operator as $\mathcal{O}^L := \mathcal{O}\otimes 1_R$ and similar for $\mathcal{O}^R$. Therefore, we find the integral becomes

$$\begin{aligned} &\int \prod_{i=1}^{2} \left(\mu(\theta_i)\, \mathrm{d}\theta_i\right) \times \\ &\qquad \left(\langle m-k| \otimes \langle n-k|\right) |\theta_1,\theta_2\rangle\langle\theta_1,\theta_2|\mathcal{O}^L\mathcal{O}^R|\theta_1,\theta_2\rangle\langle\theta_1,\theta_2| \left(|m'-k'\rangle \otimes |n'-k'\rangle\right). \end{aligned} \tag{B.9}$$

[14] I thank Ahmed Almheiri for helpful discussions on this point.

The diagrammatic rules developed in [46] suggest that

$$\langle\theta_1,\theta_2|\mathcal{O}^L\mathcal{O}^R|\theta_3,\theta_4\rangle \propto \mu^{-1}(\theta_3)\,\mu^{-1}(\theta_4)\,\delta(\theta_3-\theta_2)\,\delta(\theta_4-\theta_1)\,. \tag{B.10}$$

Therefore, we can rewrite the above equation as

$$\int\prod_{i=1}^{4}(\mu(\theta_i)\,\mathrm{d}\theta_i)\times \\ (\langle m-k|\otimes\langle n-k|)\,|\theta_1,\theta_2\rangle\langle\theta_1,\theta_2|\mathcal{O}^L\mathcal{O}^R|\theta_3,\theta_4\rangle\langle\theta_3,\theta_4|\,(|m'-k'\rangle\otimes|n'-k'\rangle)\,. \tag{B.11}$$

Now with the completeness relation, we know this is equivalent to

$$(\langle m-k|\otimes\langle n-k|)\mathcal{O}^L\mathcal{O}^R(|m'-k'\rangle\otimes|n'-k'\rangle). \tag{B.12}$$

Combined with (B.6), we conclude with

$$\langle m,n|m',n'\rangle = \sum_{k=0}^{\min(m,n)}\sum_{k'=0}^{\min(m',n')} c_{m,n}(k)c_{m',n'}(k')\times \\ \big(\langle m-k|\otimes\langle n-k|\mathcal{O}^L\mathcal{O}^R|m'-k'\rangle\otimes|n'-k'\rangle\big)\,. \tag{B.13}$$

Therefore, we deduce that the inner product between Lin-Stanford 1-particle states can be mapped to two point correlators in Fock basis. The resulting Fock states are obtained by summing over all possible pairings of open chords between the two sides in the original state.

C Detailed Derivation of (4.21)

In this section we present a detailed derivation of (4.21). The derivation strategy closely largely follows with [3], emphasizing precise normalization of states and keeping track of all approximations to ensure their validity. Due to the left/right symmetry, in the following derivation we shall only focus on the left Hamiltonian H_L.

The one-particle spectrum is fully specified by the action of $H_{L/R}$ on the states. The left Hamiltonian can be represented in terms of q-ladder operators as:

$$H_L = a_L + a_L^\dagger, \tag{C.1}$$

where $a_L a_L^\dagger - q a_L^\dagger a_L = 1$. Its action on a one-particle state is used as defining property of wavefunctions in A. We have:

$$H_L(\theta_L)\,|n_L,n_R\rangle = \psi_{n_L,n_R} + [n_L]\psi_{n_L,n_R} + q^{\Delta+n_L}[n_R]\psi_{n_L,n_R-1}, \tag{C.2}$$

where we have set $r = q^{\Delta}$. The matrix elements of H_L should be defined in normalized states. In 0-particle case this is simple to implement as states with different number chords are orthogonal. In 1-particle case, however, since the overlap between states $|n_L, n_R\rangle$ with equal $n_L + n_R$ is complicated, we do need to be careful when evaluating the matrix elements of H_L in such basis. As an illustration, let's consider the overlap between $H_L|n_L, n_R\rangle$ with $|n_L + 1, n_R\rangle$. We find that

$$\frac{\langle n_1+1, n_2|H_L|n_1, n_2\rangle}{\sqrt{\langle n_1, n_2|n_1, n_2\rangle\langle n_1+1, n_2|n_1+1, n_2\rangle}} = \sqrt{\frac{\langle n_1+1, n_2|n_1+1, n_2\rangle}{\langle n_1, n_2|n_1, n_2\rangle}}. \tag{C.3}$$

The numerator can be evaluated by the recursive definition of inner product (3.2) as:

$$\langle n_L+1, n_R|n_L+1, n_R\rangle = [n_L+1]\langle n_L, n_R|n_L, n_R\rangle + q^{n_L+\Delta+1}[n_R]\langle n_L, n_R|n_L+1, n_R-1\rangle. \tag{C.4}$$

In the triple scaling limit we take both n_L and n_R to infinity, in this limit therefore the overlap between state $|n_L, n_R\rangle$ and $|n_L+1, n_R-1\rangle$ becomes equal to the overlap of $|n_L, n_R\rangle$ with itself. Therefore, we have

$$\langle n_L, n_R \mid n_L+1, n_R-1\rangle \simeq \langle n_L, n_R|n_L, n_R\rangle. \tag{C.5}$$

Therefore, we find

$$\frac{\langle n_1+1, n_2|H_L|n_1, n_2\rangle}{\sqrt{\langle n_1, n_2|n_1, n_2\rangle\langle n_1+1, n_2|n_1+1, n_2\rangle}} \simeq \sqrt{\frac{1-(1-q^{\Delta})\,q^{n_L+1} - q^{n_L+n_R+\Delta+1}}{1-q}}. \tag{C.6}$$

Similar strategy applies to the overlap with states in second and third term in (C.2):

$$\begin{aligned}
\frac{\langle n_L-1, n_R\,|H_L|\,n_L, n_R\rangle}{\sqrt{\langle n_L, n_R \mid n_L, n_R\rangle\,\langle n_L-1, n_R \mid n_L-1, n_R\rangle}} &= [n_L]\sqrt{\frac{\langle n_L-1, n_R \mid n_L-1, n_R\rangle}{\langle n_L, n_R \mid n_L, n_R\rangle}} \\
\frac{\langle n_L, n_R-1\,|H_L|\,n_L, n_R\rangle}{\sqrt{\langle n_L, n_R \mid n_L, n_R\rangle\,\langle n_L, n_R-1 \mid n_L, n_R-1\rangle}} &= q^{n_L+\Delta}\,[n_R]\sqrt{\frac{\langle n_L, n_R-1 \mid n_L, n_R-1\rangle}{\langle n_L, n_R \mid n_L, n_R\rangle}}.
\end{aligned} \tag{C.7}$$

Applying similar approximation as (C.5), we find

$$[n_L]\sqrt{\frac{\langle n_L-1, n_R \mid n_L-1, n_R\rangle}{\langle n_L, n_R \mid n_L, n_R\rangle}} \simeq \sqrt{\frac{1-q^{n_L}}{1-q}} \times \left(1-\left(1-q^{\Delta}\right)q^{n_L} - q^{n_L+n_R+\Delta}\right)^{-1/2}, \tag{C.8}$$

and

$$q^{n_L+\Delta}\,[n_R]\sqrt{\frac{\langle n_L, n_R-1 \mid n_L, n_R-1\rangle}{\langle n_L, n_R \mid n_L, n_R\rangle}} = \sqrt{\frac{q^{n_L+\Delta}-q^{n_R+n_L+\Delta}}{1-q}} \times \left(1-\left(1-q^{\Delta}\right)q^{n_L}-q^{n_L+n_R+\Delta}\right)^{-1/2}. \tag{C.9}$$

Now let's consider the triple scaling limit, where we introduce the renomarlized length $l_{L/R}$ as

$$q^{n_{L/R}} = \lambda e^{-\tilde{l}_{L/R}}. \tag{C.10}$$

The states $\lambda|n_L, n_R\rangle$ can now be labeled with the length parameter, and we have

$$|n_L+1, n_R\rangle \simeq e^{\lambda\partial_L}|\tilde{l}_L, \tilde{l}_R\rangle. \tag{C.11}$$

We introduce the distance function $d(\tilde{l}_L, \tilde{l}_R)$ as:

$$\mathcal{L}\left(\tilde{l}_L, \tilde{l}_R\right) = \lambda\left(1-e^{-\lambda\Delta}\right)e^{-\tilde{l}_L} + \lambda^2 e^{-\tilde{l}_L-\tilde{l}_R-\lambda\Delta}. \tag{C.12}$$

Note that in $\Delta \to 0$ limit it produces the Liouville potential with total length $\tilde{l} = \tilde{l}_L + \tilde{l}_R$. Combining (C.6), (C.8) and (C.9), we find the Hamiltonian can be represented in terms of the new parameters as

$$\begin{aligned}\tilde{H}_L = \lambda^{-1/2}H_L = &-\left(\sqrt{\frac{1-\mathcal{L}\left(\tilde{l}_L, \tilde{l}_R\right)}{(1-q)\lambda}}e^{\lambda\tilde{\partial}_L} + e^{\lambda\tilde{\partial}_L}\sqrt{\frac{1-\mathcal{L}\left(\tilde{l}_L, \tilde{l}_R\right)}{(1-q)\lambda}}\right) \\ &-\frac{1}{\sqrt{(1-q)\lambda}} \times \sqrt{\frac{e^{-\tilde{l}_L-\lambda\Delta}-e^{-\tilde{l}_L-\tilde{l}_R-\lambda\Delta}}{1-\mathcal{L}\left(\tilde{l}_L, \tilde{l}_R\right)}}\left(e^{\lambda\tilde{\partial}_R}-e^{\lambda\tilde{\partial}_L}\right),\end{aligned} \tag{C.13}$$

where we have rescaled H_L by a factor of $\lambda^{1/2}$. The leading order of the right hand side in (C.13) gives a constant, which we denoted as E_0. Then by keeping terms up to $O(\lambda)$, we find

$$\tilde{H}_L - E_0 = \lambda\left(-\tilde{\partial}_{\mathrm{L}}^2 + e^{-\tilde{l}_L}\left(\Delta+\partial_R-\partial_L\right) + e^{-\tilde{\ell}_{\mathrm{L}}-\tilde{\ell}_{\mathrm{R}}}\right) + O(\lambda^2). \tag{C.14}$$

This matches up to a constant normalization with the Hamiltonian in the first equation in (4.21).

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