

The Wilson-Fisher Fixed point revisited: importance of the form of the cutoff

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In this work we re-examine the Wilson Fisher fixed point. We study Wilsonian momentum space renormalization group (RG) flow for various forms of the cutoff. We show that already at order $(4-d)^1$, where d is the dimension of the ϕ^4 theory, there are changes to the position of the fixed point and the direction of irrelevant coupling parameters. We also show in a multi-flavor ϕ^4 model that symmetries of the Lagrange function can be destroyed if the different flavors have different cutoffs (that is the Lagrangian can flow to a non-symmetric fixed point). Some related comments are made about a similar situation in parquet RG (pRG). In future works [1] we will study Wilsonian RG to order $(4-d)^2$ and find non-universal critical exponents that depend on the cutoff.

I. INTRODUCTION

One of the most successful approaches to the study of both quantum and classical phase transitions is the momentum space renormalization group (RG) [2–13]. The momentum space renormalization group provides the mathematical underpinnings of both scaling and critical phenomena. These ideas were developed by Kadanoff [14], Wilson [5–7] and others [2–4, 15] into powerful mathematical tools for tackling the physics problems associated with scaling, criticality and phase transitions. The main idea supporting momentum space RG is due to Kadanoff [14] is that close to a phase transition a critical system becomes scale invariant. That is, there are very few relevant parameters (say the mass in ϕ^4 theory) which control the distance between the theory (system) and the scale invariant critical theory where all correlation lengths are infinite. Under a rescaling (integrating out) of various degrees of freedom these relevant parameters rescale in a specific way and the system looks like a scaled system with renormalized parameters. In the simplest case when there is only one relevant parameter, under a rescaling by a dilation factor of l , the relevant parameter t and the correlation length ξ transform as [2, 3, 15]:

$$t \rightarrow t \cdot l^\nu, \xi \rightarrow \xi \cdot l \quad (1)$$

For some exponent ν , which Wilsonian RG can compute. In this case we see that we may read off the correlation length from the value of the relevant parameter t :

$$tl^\nu \sim 1 \rightarrow \xi \sim 1, \Rightarrow \xi \sim t^{-1/\nu} \quad (2)$$

The idea due to Wilson [5–7] and others was to systematically, in momentum space, integrate out the short wavelength degrees of freedom thereby mimicking a rescaling transformation and follow mathematically how relevant and also irrelevant parameters transform. As such Wilson [3, 5–7] and others were able to compute ν and other critical parameters through perturbative momentum space RG calculation.

There are roughly three different types of momentum shell RG: 1) Wilsonian RG [2, 3, 5–7, 13, 15], 2) field theoretic RG [8–11], 3) functional RG [12]. In Wilsonian

RG there is a cutoff, it is rescaled and the effect of integrating out the high degrees of freedom in the effective Lagrange function is accounted for systematically up to some loop order [2, 3, 5–7, 13, 15]. In field theoretic RG the theory is regularized, often by dimensional regularization - where divergent integrals are analytically continued from dimensions where they converge, renormalization conditions are demanded - where specific values are set to Green's functions at certain momenta - and the Callan-Symanzik equation is set up to compute Green's functions at other momenta [8–11]. In FRG the propagators are infinitely massive below a certain cutoff which is then rescaled and the effective action is computed as a function of the cutoff [12]. In each case a regularization, cutoff is needed to make the theory finite and control the divergences of various Feynman diagrams. The effect of the nature of this cutoff on the RG procedure has been studied with mixed success. For the Kondo model at two loop order the Bethe Ansatz solution and the numerical Wilsonian RG solution seem to differ, perhaps because of the nature of the cutoff [16–18]. For field theoretic RG different regulators have been compared with favorable results indicating similar physical properties for all cutoffs [11, 12]. For FRG there have been numerous studies on the subject matter [19–22] with some universality observed [20–22] though higher loop corrections seem to dependent on the form of the cutoff [19].

Wilsonian RG is arguably the simplest of the three forms of RG. It has been used to study φ^4 theory, $O(n)$ models, non-linear σ models, numerically the Kondo problem, sine gordon theory, Hertz-Millis theory [2, 3, 15, 17, 23–25]. Here we study the effect of the nature of the cutoff on Wilsonian RG within the context of the φ^4 theory in the form of a $4-d$ expansion. Here we claim Wilson and others were somewhat careless as to the nature of what constitutes high frequency and high energy modes as they introduced a hard cutoff to separate high frequency and low frequency modes. Here following the paradigmatic example of the Wilson Fisher fixed point [7], which works well for ϕ^4 theory in the limit that we are working at nearly four dimensions $4-d \ll 1$, we show that the choice of hard cutoff is a poor unjustified choice by comparing it top other choices. By varying the form of the cutoff (see Eq. (4))

we explicitly show that we can obtain different values for the fixed point couplings (see Eq. (19)) and modify the various relevant and irrelevant directions as well as how they scale (see Eq. (24)). This invalidates, in part, Wilson's ideas as how to practically compute within the momentum space RG flow the scaling transformations introduced by Kadanoff as it is unclear which cutoff is best. This has far ranging implications for modern day condensed matter [2–4]. Condensed matter systems often comes with a practical cutoff associated with the lattice regularization (no modes with wave vector higher then the inverse lattice spacing are needed for most calculations). However this regularization is often considered far too difficult for practical calculations and a simpler “spherical cow” regularization is chosen whereupon momentum space RG is performed within this regularization [2–4]. The results of this work, where we choose a model (ϕ^4 theory) where it is practically possible to compare several different regularizations and obtain different results already at one loop level, casts strong doubts as to the validity of the procedure commonly carried out by many condensed matter physicists. Furthermore we show that variations in the cutoff for different field flavors, in multi-flavor ϕ^4 models, can lead to changes of the symmetry of the system whereby a symmetric (we chose the example of $O(2)$ symmetric) action can flow to non symmetric fixed points (see Eq. (28)). As such the symmetries and minimal Lagrange function relevant to the order parameter action can change due to different cutoffs [2, 4]. We also point out that a similar effect happens in parquet RG (pRG) and is arguably an even worse concern. In future works we will go to order $(4-d)^2$ to further confirm these results [1].

II. MAIN IDEA

We consider the ϕ^4 theory with the action being given by:

$$S = - \int d^d x \left[\phi(x) \frac{1}{2} [-\nabla^2 + r] \phi(x) + \frac{1}{4!} g \phi^4(x) \right] \quad (3)$$

We now regularize the theory by replacing:

$$\int d^d \mathbf{x} \rightarrow \int \frac{d^d \mathbf{k}}{(2\pi)^d} F^{1/2} \left(\frac{|\mathbf{k}|}{\Delta} \right) \quad (4)$$

For some large cutoff Δ with $F(0) = 1$ and $F(\infty) \rightarrow 0$ rapidly and the square root is for future convenience. It is straightforward to see that the main effect of this regulator on Feynman diagrams is to change the propagator:

$$\frac{1}{\mathbf{Q}^2 + r} \rightarrow \frac{F\left(\frac{|\mathbf{Q}|}{\Delta}\right)}{\mathbf{Q}^2 + r} \quad (5)$$

Here \mathbf{Q} is the momentum. This makes the procedure similar to FRG 16 [12]. The main interest in this work is how the form of the cutoff effects the Wilson Fisher fixed point [7] to order $(4-d)^1$, for order $(4-d)^2$ see 11 [1]. Now we see that the action is dimensionless so we may write that the scaling dimensions are given by $[\phi] = \frac{d-2}{2}$, and $[g] = 4-d$. As such we can rewrite Eq. (3) as:

$$S = \int d^d x \left[\phi^\dagger(x) [-\nabla^2 + \Delta^2 R] \phi(x) + \frac{1}{4!} \Delta^{4-d} G \phi^4(x) \right] \quad (6)$$

with R and G dimensionless. Now we perform a step of RG whereby:

$$\Delta \rightarrow \Delta(1-\varepsilon) \quad (7)$$

To compute the new action we first we do the tree level RG step (that is to order $(4-d)^0$) where:

$$\begin{aligned} \Delta^2 R &= \Delta^2 (1-\varepsilon)^2 R' \\ \Rightarrow R' &= R(1+2\varepsilon) \end{aligned} \quad (8)$$

$$\begin{aligned} \Delta^{4-d} G &= \Delta^{4-d} (1-\varepsilon)^{4-d} G' \\ \Rightarrow G' &= G(1+\varepsilon(4-d)) \end{aligned} \quad (9)$$

Now we perform RG to one loop level where we have that:

$$\begin{aligned} \Delta^2 R &\rightarrow \Delta^2 R - \varepsilon \Delta^{4-d} G \frac{V(S^{d-1})}{2(2\pi)^d} \int_0^\infty dk \frac{\partial}{\partial \varepsilon} \left[F\left(\frac{k}{\Delta}(1+\varepsilon)\right) \right] \frac{1}{k^2 + \Delta^2 R} k^{d-1} \\ \frac{1}{4!} \Delta^{4-d} G &\rightarrow \frac{1}{4!} \Delta^{4-d} G + 3\varepsilon (\Delta^{4-d} G)^2 \frac{V(S^{d-1})}{2(2\pi)^d} \int_0^\infty dk \frac{\partial}{\partial \varepsilon} \left[F^2\left(\frac{k}{\Delta}(1+\varepsilon)\right) \right] \left[\frac{1}{k^2 + \Delta^2 R} \right]^2 k^{d-1} \end{aligned} \quad (10)$$

As such we have that:

$$\begin{aligned} & - \int_0^\infty dk \frac{\partial}{\partial \varepsilon} \left[F\left(\frac{k}{\Delta}(1+\varepsilon)\right) \right] \frac{1}{k^2 + \Delta^2 R} k^{d-1} \\ & = \int_0^\infty dk \left(\frac{k}{\Delta}\right) \frac{1}{k^2 + \Delta^2 R} k^{d-1} \frac{\partial}{\partial \left(\frac{k}{\Delta}\right)} F\left(\frac{k}{\Delta}\right) \end{aligned} \quad (11)$$

$$\begin{aligned} & \int_0^\infty dk \frac{\partial}{\partial \varepsilon} \left[F^2\left(\frac{k}{\Delta}(1+\varepsilon)\right) \right] \left[\frac{1}{k^2 + \Delta^2 R} \right]^2 k^{d-1} \\ & = - \int_0^\infty dk \frac{k}{\Delta} \left[\frac{1}{k^2 + \Delta^2 R} \right]^2 k^{d-1} \frac{\partial}{\partial \left(\frac{k}{\Delta}\right)} F\left(\frac{k}{\Delta}\right) \end{aligned} \quad (12)$$

We now introduce $K = \frac{k}{\Delta}$ and obtain:

$$\begin{aligned} & \int_0^\infty dk \left(\frac{k}{\Delta} \right) \frac{1}{k^2 + \Delta^2 R} k^{d-1} \frac{\partial}{\partial \left(\frac{k}{\Delta} \right)} F \left(\frac{k}{\Delta} \right) \\ &= \Delta^{d-2} \int_0^\infty dK \frac{\partial}{\partial K} F(K) \frac{1}{K^2 + R} K^d \end{aligned} \quad (13)$$

$$\begin{aligned} & \int_0^\infty dk \frac{k}{\Delta} \left[\frac{1}{k^2 + \Delta^2 R} \right]^2 k^{d-1} \frac{\partial}{\partial \left(\frac{k}{\Delta} \right)} F^2 \left(\frac{k}{\Delta} \right) \\ &= \Delta^{d-4} \int_0^\infty dK \frac{\partial}{\partial K} F^2(K) \left[\frac{1}{K^2 + R} \right]^2 K^d \end{aligned} \quad (14)$$

Now we have that:

$$R \rightarrow R - \varepsilon G \frac{V(S^{d-1})}{(2\pi)^d} \int_0^\infty dK \frac{\partial}{\partial K} F(K) \frac{1}{K^2 + R} K^d$$

$$\cong R + \varepsilon G \frac{V(S^{d-1})}{(2\pi)^d} I_1^{F,d}(R)$$

$$G \rightarrow G + 3\varepsilon G^2 \frac{V(S^{d-1})}{(2\pi)^d} \int_0^\infty dK \frac{\partial}{\partial K} F^2(K) \left[\frac{1}{K^2 + R} \right]^2 K^d$$

$$\cong G - 3\varepsilon G^2 \frac{V(S^{d-1})}{(2\pi)^d} I_2^{F,d}(R) \quad (15)$$

As such combining with Eq. (??) we obtain:

$$\begin{aligned} \frac{\partial R}{\partial \varepsilon} &= 2R + \frac{1}{2} G \frac{V(S^3)}{(2\pi)^4} I_1^{F,d}(R) \\ \frac{\partial G}{\partial \varepsilon} &= (4-d)G - \frac{3}{2} G^2 \frac{V(S^3)}{(2\pi)^4} I_2^{F,d}(R) \end{aligned} \quad (16)$$

Where

$$\begin{aligned} I_1^{F,d}(R) &= - \int_0^\infty dK \frac{\partial}{\partial K} F(K) \frac{1}{K^2 + R} K^d \\ I_2^{F,d}(R) &= - \int_0^\infty dK \frac{\partial}{\partial K} F^2(K) \left[\frac{1}{K^2 + R} \right]^2 K^d \end{aligned} \quad (17)$$

Working only to order $4-d \ll 1$ and working only to order $(4-d)$ (that is ignoring order $(4-d)^2$ terms) we need only consider the integrals:

$$\begin{aligned} I_1^{F,4}(0) &= - \int_0^\infty dK \frac{\partial}{\partial K} F(K) K^2 = 2 \int_0^\infty dK F(K) K \\ I_2^{F,4}(0) &= - \int_0^\infty dK \frac{\partial}{\partial K} F^2(K) = 1 \end{aligned} \quad (18)$$

See supplementary online information [26]. We now look for a fixed point and again work only to order $4-d \ll 1$ then:

$$G^{fix} = \frac{2(4-d)(2\pi)^4}{3V(S^3)}, \quad R^{fix} = -\frac{(4-d)I_1^{F,4}(0)}{6} \quad (19)$$

We note that this is not the value of the Willson Fisher fixed point given by [2, 3, 7]:

$$G_{WF}^{fix} = \frac{2(4-d)(2\pi)^4}{3V(S^3)}, \quad R_{WF}^{fix} = -\frac{(4-d)}{6} \quad (20)$$

We see that the exact position of the fixed point has moved base on the value of $I_1^{F,4}(0)$ [26] as such the critical theory will change [3].

III. CRITICAL EXPONENTS

Let us recall Eq. (16). Now we write

$$\begin{aligned} G &= G_{fix} + \delta G \\ R &= R_{fix} + \delta R \end{aligned} \quad (21)$$

From which we see that:

$$\frac{\partial}{\partial \varepsilon} \begin{pmatrix} \delta R \\ \delta G \end{pmatrix} \cong \begin{pmatrix} 2 + \frac{2(4-d)}{3} & \frac{1}{2} \frac{V(S^3)}{(2\pi)^4} I_1^{F,4}(0) \\ 0 & -(4-d) \end{pmatrix} \begin{pmatrix} \delta R \\ \delta G \end{pmatrix} \quad (22)$$

Because the eigenvalues of the matrix in Eq. (22) do not depend on F we recover the Wilson Fisher critical exponents to order $4-d$, however because of the changes to G_{fix} , R_{fix} there will be changes to order $(4-d)^2$ [1]. However the irrelevant eigenvector changes to:

$$\begin{pmatrix} 2 + \frac{2(4-d)}{3} & \frac{1}{2} \frac{V(S^3)}{(2\pi)^4} I_1^{F,4}(0) \\ 0 & -(4-d) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -(4-d) \begin{pmatrix} a \\ b \end{pmatrix} \quad (23)$$

whereby:

$$\frac{a}{b} = -\frac{\frac{1}{2} \frac{V(S^3)}{(2\pi)^4} I_1^{F,4}(0)}{2 + \frac{5}{3}(4-d)} \quad (24)$$

so there is an explicit change to the irrelevant vector depending on the values of $\frac{\frac{1}{2} \frac{V(S^3)}{(2\pi)^4} I_1^{F,4}(0)}{2 + \frac{5}{3}(4-d)}$. As such the critical flow changes due to various types of cutoffs already at leading order and not just the position of the fixed point.

IV. EMERGENT SYMMETRIES

We would like to show that emergent symmetries can also be affected by the softness of the cutoff not just the value of the fixed point. We consider the two species ϕ^4 theory given by:

$$\begin{aligned} S &= -\frac{1}{2} \int d^d x \phi_1(x) [-\nabla^2 + r_1] \phi_1(x) \\ &\quad -\frac{1}{2} \int d^d x \phi_2(x) [-\nabla^2 + r_2] \phi_2(x) \\ &\quad -\frac{1}{4!} \int d^d x \sum_{ab} g_{ab} \phi_a^2(x) \phi_b^2(x) \end{aligned} \quad (25)$$

where $a, b = 1, 2$ with $g_{12} = g_{21}$. Now we regularize the theory in a different way for each of the fields:

$$\begin{aligned} a = 1 : \int d^d x &\rightarrow \int \frac{d^d k}{(2\pi)^d} F_1^{1/2} \left(\frac{k}{\Delta} \right) \\ a = 2 : \int d^d x &\rightarrow \int \frac{d^d k}{(2\pi)^d} F_2^{1/2} \left(\frac{k}{\Delta} \right) \end{aligned} \quad (26)$$

Now we write a dimensionless variable action:

$$\begin{aligned} S = & -\frac{1}{2} \int d^d x \phi_1(x) [-\nabla^2 + \Delta^2 R_1] \phi_1(x) \\ & -\frac{1}{2} \int d^d x \phi_2(x) [-\nabla^2 + \Delta^2 R_2] \phi_2(x) \\ & -\frac{1}{4!} \int d^d x \Delta^{4-d} G_{ab} \phi_a^2(x) \phi_b^2(x) \end{aligned} \quad (27)$$

Now we perform a step of RG as in Eq. (7). We now find the RG equations of motion to one loop order: keeping only terms of order $4-d$ (similarly to Eq. (16)):

$$\begin{aligned} \frac{\partial}{\partial \varepsilon} R_1 &= 2R_1 + \frac{1}{2} \frac{V(S^3)}{(2\pi)^4} \left[G_{11} I_1^{F_1,4}(0) + \frac{1}{3} G_{12} I_1^{F_2,4}(0) \right] \\ \frac{\partial}{\partial \varepsilon} R_2 &= 2R_2 + \frac{1}{2} \frac{V(S^3)}{(2\pi)^4} \left[G_{22} I_1^{F_2,4}(0) + \frac{1}{3} G_{12} I_1^{F_1,4}(0) \right] \\ \frac{\partial}{\partial \varepsilon} G_{11} &= (4-d) G_{11} - \frac{V(S^3)}{(2\pi)^4} \left[\frac{3}{2} G_{11}^2 + \frac{1}{6} G_{12}^2 \right] \\ \frac{\partial}{\partial \varepsilon} G_{22} &= (4-d) G_{22} - \frac{V(S^3)}{(2\pi)^4} \left[\frac{3}{2} G_{22}^2 + \frac{1}{6} G_{12}^2 \right] \\ \frac{\partial}{\partial \varepsilon} G_{12} &= (4-d) G_{12} \\ & - \frac{V(S^3)}{(2\pi)^4} \left[\frac{3}{4} G_{22} G_{12} + \frac{3}{4} G_{11} G_{12} + \frac{1}{6} G_{12}^2 \right] \end{aligned} \quad (28)$$

Now we see that the fixed point corresponds to

$$\begin{aligned} G_{11}^{fix} = G_{12}^{fix} = G_{22}^{fix} &= \frac{3(4-d)}{5 \frac{V(S^3)}{(2\pi)^4}} \\ R_1^{fix} &= -\frac{3(4-d)}{20} \left[I_1^{F_1,4}(0) + \frac{1}{3} I_1^{F_2,d}(0) \right] \\ R_2^{fix} &= -\frac{3(4-d)}{20} \left[I_1^{F_2,4}(0) + \frac{1}{3} I_1^{F_1,d}(0) \right] \end{aligned} \quad (29)$$

We see that the $O(2)$ symmetric point is no longer a fixed point of the low energy theory. Furthermore, we see that we can start with a $O(2)$ symmetric action and under RG flow to a non-symmetric fixed point.

V. COMMENT ON PRG

The situation with pRG is worse than with the Wilson Fisher fixed point with respect to cutoff dependence. We recall that in pRG one introduces various couplings

between different patches [27] and computes to a certain loop level the changes of these couplings when integrating out high energy degrees of freedom. The one loop equations of motion for the various couplings g_i are given by [27]:

$$\frac{\partial g_i}{\partial \Delta} = \sum_{jk} A_{jk}^i(\Delta) g_j g_k \quad (30)$$

Here $A_{jk}(\Delta)$ are parameters that depend on the density of states associated with the various polarization bubbles [27] and Δ is an overall flow parameter which controls how these density of states for the relevant patches is integrated out. Now we look for blow up solutions [27]

$$g_i = \frac{c_i}{\Delta - \Delta_0} + \dots \quad (31)$$

Now we note then that [27] we have that:

$$c_i = \sum_{jk} A_{jk}^i(\Delta_0) c_j c_k \quad (32)$$

Now we may perform pRG with both smooth and sharp cutoffs and obtain very different Δ_0 and $A_{jk}^i(\Delta_0)$ much like in the main part of the paper [1]. As such the various c_i which control how the various couplings blow up are modified due to different cutoff dependent $A_{jk}^i(\Delta_0)$. Therefore the various instabilities which depend on the signs and magnitudes of the c_i (see Eq. (31)) now depend on the shape of the cutoff. As such pRG has reliability problems as different cutoffs can predict different instabilities [1].

VI. CONCLUSIONS

In this work we have re-examined the Wilson Fisher fixed point [7] in the context of more general cutoffs than the hard cutoff considered in the original Wilson Fisher work [7]. We find that both the position of the fixed point the direction of irrelevant perturbations depends explicitly on the form of the cutoff already at order $(4-d)$ or one loop level and explicit checks will show that the critical exponents depend on the form of the cutoff at the two loop order $(4-d)^2$ level [1]. We have shown that emergent symmetries where the theory flows to actions with high level of symmetry such as $O(2)$ depend on the form of the cutoff and if different flavors of fields have different cutoffs the high symmetry fixed point can be destroyed so a $O(2)$ symmetric theory may flow to a non-symmetric fixed point already at order $4-d$. Since the nature of the order parameter and other properties of the system depend on the symmetries of the low energy effective action [2–4, 15] this calculation shows that the form of the cutoff can influence the form of the order parameter for the effective theory. We have shown that the situation with pRG is even more difficult to control than with the Wilson Fisher fixed point. Indeed for pRG

divergences show up at finite times in the flow parameter [27] where depending on the nature of the cutoff and how it flows the density of states $A_{jk}^i(\Delta_0)$ can significantly change whereby which couplings diverge how fast (see Eqs. (31) and (32)) can change which leads to different instabilities being dominant [27]. In the eyes of the author this makes pRG highly unreliable as a method to predict which instabilities dominate depending on which density of states and which initial couplings are present. In future works it would be of great interest to see what can be salvaged from momentum space RG type ideas in a highly controlled reliable way and to study the Wilson Fisher fixed point for arbitrary cutoffs to order $(4-d)^2$.

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Supplementary online information

SOME EXPLICIT EXAMPLES

We will now explicitly show there are no relations for the function $I_1^{F,4}(0)$. Now consider the functions cutoffs:

$$\begin{aligned} F_1(K) &= \Theta(K-1) \\ F_2(K) &= \exp(-K) \\ F_3(K) &= \exp(-K^2) \\ F_4(K) &= \exp(-K^4) \end{aligned} \quad (33)$$

Here Θ is the heavy-side function. Then we have that:

$$\begin{aligned} I_1^{F_1,4}(0) &= 1 \\ I_1^{F_2,4}(0) &= 2 \\ I_1^{F_3,4}(0) &= 1 \\ I_1^{F_4,4}(0) &= \frac{\sqrt{\pi}}{2} \end{aligned} \quad (34)$$

As such $I^{F,4}(0) > 0$ is an arbitrary non-universal constant effecting the fixed point and flow of RG.

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- [1] G. Goldstein in preparation.
 - [2] M. Continentino, *Quantum Scaling in Many Body Systems: An Approach to Quantum Phase Transitions* (Cambridge University Press, Cambridge, 2017).
 - [3] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, 2011).
 - [4] S. Sachdev, *Quantum Phases of Matter* (Cambridge University Press, Cambridge, 2023).
 - [5] K. G. Wilson, Phys. Rev. B **4**, 3174 (1971).
 - [6] K. G. Wilson, Phys. Rev. B **4**, 3184 (1971).
 - [7] K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. **28**, 240 (1972).
 - [8] G. Parisi, *Statistical Field Theory* (Addison Wesley Publishing Company, Reading MA, 1988).
 - [9] P. Ramond, *Field Theory: A Modern Primer* (Westview Press, Boulder, 2001).
 - [10] J. Zinn-Justin, *Phase Transitions and Renormalization Group* (Oxford University Press, Oxford, 2007).
 - [11] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Oxford University Press, Oxford, 2002).
 - [12] P. Kopietz, L. Bartosch, and F. Schutz, *Introduction to the Functional Renormalization Group* (Springer, Berlin, 2010).
 - [13] K. G. Wilson and G. Kogut, Phys. Rep. **12**, 75 (1974).
 - [14] L. P. Kadanoff, Physics **2**, 263 (1966).
 - [15] R. K. Pathria and P. D. Beale, *Statistical Mechanics* (Elsevier, Amsterdam, 2011).
 - [16] N. Andrei, K. Furuya, and J. H. Lowenstein, Rev. Mod. Phys. **55**, 331 (1983).
 - [17] A. C. Hewson, *The Kondo Problem to Heavy Fermions* (Cambridge University Press, Cambridge, 1993).
 - [18] P. Coleman, *Introduction to Many-Body Physics* (Cambridge University Press, Cambridge, 2016).
 - [19] J. Gaité, Universe **9**, 409 (2023).
 - [20] S. Liao, J. Polonyi, and M. Strickland, Nuc. Phys. B **567**, 493 (2000).
 - [21] D. F. Litim, Phys. Lett. B **486**, 92 (2000).
 - [22] D. F. Litim, Phys. Rev. D **64**, 105007 (2001).
 - [23] T. Giamarchi, *Quantum Physics in One Dimension* (Clarendon Press, Oxford, 2003).
 - [24] X. G. Wen, *Quantum Field Theory of Many Body Systems* (Oxford University Press, Oxford, 2004).
 - [25] E. Fradkin, *Field Theories of Condensed Matter Physics* (Cambridge University Press, Cambridge, 2013).
 - [26] See supplementary online information.
 - [27] S. Maiti and A. V. Chubukov, arXiv 1305.4609.