A Noisy Approach to Intrinsically Mixed-State Topological Order

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We propose a general framework for studying two-dimensional (2D) topologically ordered states subject to local correlated errors and show that the resulting mixed-state can display *intrinsically mixed-state topological order* (imTO)—topological order which is not expected to occur in the ground state of 2D local gapped Hamiltonians. Specifically, we show that decoherence, previously interpreted as anyon condensation in a doubled Hilbert space, is more naturally phrased as, and provides a physical mechanism for, "gauging out" anyons in the original Hilbert space. We find that gauging out anyons generically results in imTO, with the decohered mixed-state strongly symmetric under certain anomalous 1-form symmetries. This framework lays bare a striking connection between the decohered density matrix and *topological subsystem codes*, which can appear as anomalous surface states of 3D topological orders. Through a series of examples, we show that the decohered state can display a classical memory, encode logical qubits (i.e., exhibit a quantum memory), and even host chiral or non-modular topological order. We argue that the decohered states represent genuine mixed-state quantum phases of matter and that a partial classification of imTO is given in terms of braided fusion categories.

I. INTRODUCTION

Quantum many-body states with non-trivial entanglement serve as resource states for various tasks in quantum information processing. Quintessential amongst these are states with *topological order*, which support fractionalized excitations (anyons) and may serve as platforms for topological quantum computation [1, 2]. These states, which arise as locally indistinguishable degenerate ground-states of certain gapped Hamiltonians, form the code space for topological quantum error correcting codes (QECC), with the Hamiltonians provably robust to weak, local perturbations [3–5]: their utility as resource states thus extends to all states within the topologically ordered *phase*.

Recent years have witnessed remarkable progress in preparing and manipulating such states in programmable quantum simulators [6–10]. Decoherence is invariably present in these platforms and thus identifying a sharp notion of *mixed-state topological order* is not merely of fundamental interest, but also of immediate practical import. While any finite temperature is known to destroy topological order (TO) in two spatial dimensions (2D) [11, 12], for local decoherence below a certain threshold, the quantum information encoded in a topological QECC is recoverable. Indeed, the persistence and eventual breakdown of topological order in a pure state $|\psi\rangle$ subject to a local decoherence channel $\mathcal{E}$ has recently been studied through the lens of the entanglement properties of the "corrupted" density matrix $\mathcal{E}[|\psi\rangle\langle\psi|]$ [13–22]. While naïvely one expects that local errors destroy

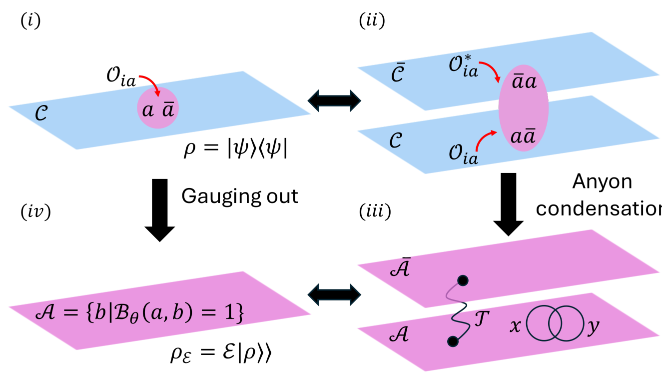

FIG. 1: (i) A pure state with TO described by a UMTC $\mathcal{C}$ subject to locally correlated noise can be represented as (ii) a vector in a doubled Hilbert space with $\mathcal{C} \times \bar{\mathcal{C}}$ TO undergoing anyon condensation. (iii) The decohered state in the doubled Hilbert space has TO given by $\mathcal{A}$ ($\bar{\mathcal{A}}$) in the bra (ket) space, along with some transparent anyons $\mathcal{T}$. (iv) In the physical Hilbert space, this process corresponds to gauging out the decohered anyon a, with the *quantum* TO in the mixed-state described by the anyon theory $\mathcal{A}$ which only includes anyons from $\mathcal{C}$ which braid non-trivially with a.

quantum correlations (and hence TO), the decohered state is not the Gibbs state and can in principle encode structured entanglement. Strikingly, for initial pure symmetry protected topological (SPT) phases, the decohered density matrix can display *intrinsically mixed* SPT order [23–26]. While Ref. [27] recently observed a non-trivial topological entanglement negativity in a decohered Toric code (indicating imTO), a systematic framework for addressing *intrinsically mixed-state topological order* (imTO) is lacking.

In this paper, we propose a general framework for characterizing the topological order in mixed-states that are obtained by subjecting arbitrary 2D topologically or-

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Author order is symmetric under exchange of bra and ket spaces.

dered pure states to local decoherence. For initial pure states with Abelian TO, locally correlated noise can induce imTO in the resulting mixed-state, which we show can be described by *topological subsystem codes* [28–31]. As special cases, we recover previous mixed-state topological orders [14, 27], which support only classical memories [1]. In general however, topological subsystem codes (TSSCs) describe zero correlation length phases supporting both classical and *quantum* memories i.e., they can encode logical qubits. We attribute this memory structure in the decohered state to certain anomalies in the 1-form symmetries of the TSSC; these in turn stem from the 1-form symmetries of the parent topological order and of the decoherence channel. These 1-form symmetries confer stability on the imTOs, which thus present genuine quantum phases of matter.

Remarkably, topological subsystem codes can describe *non-modular* topological orders—phases in which certain anyons are "invisible" to all other anyons—and even *chiral* topological phases, which admit no gapped boundary to vacuum. Thus, our work provides a physical mechanism for realizing such codes: indeed, we expect that all TSSCs can be realized by subjecting twisted quantum doubles [32] to locally correlated noise. Formally, this implies that the classification of imTO is *at least* as rich as that of TSSCs. On the practical side, we show that Abelian topological orders (which permit gapped boundaries) are resource states for preparing anomalous topological phases (including non-modular and chiral phases) under locality preserving quantum channels (LPQC) [33], where this would otherwise require sequential quantum circuits [34, 35] or measurements and feedback circuits (which suffer from post-processing bottlenecks).

As in prior works, an essential step in our analysis is mapping the decohered density matrix to a vector in a "doubled" Hilbert space. In this doubled Hilbert space, the decoherence channel can be understood as inducing anyon condensation across the two layers [14, 15]. Our key insight is that in the original (physical) Hilbert space, this process corresponds to the *"gauging out"* of an anyon in the original topological order [36] (see Fig. 1). While only bosonic anyons are permitted to condense [37], decoherence allows us to gauge out anyons with *any* spin (e.g., fermions or semions). This is intimately related to the fact that TSSCs appear as anomalous surface states of certain topological orders, lattice realizations of which are given by Walker-Wang (WW) models [38]: locally correlated errors hence provide a means of exfoliating surface states of 3D WW models, which can be anomalous [39, 40]. We further extend our framework to incorporate initial pure states with non-Abelian anyons though, strictly speaking, the resulting mixed-state is no longer a TSSC (these are characterized by Abelian anyon theories) but nevertheless corresponds to an anomalous WW surface state. This leads us to conclude that imTO is partially classified in terms of braided fusion categories.

The balance of this paper is organized as follows: in Sec. II, we discuss the set of local decoherence channels under consideration and show that, for maximal decoherence, the resulting mixed-state belongs to the code space of a TSSC. In particular, we show that decoherence provides a physical mechanism for "gauging out" certain anyons, whereby only those anyons which braid trivially with the decohered anyons remain as deconfined excitations in the resulting theory. We illustrate this framework through examples in Sec. III, where we show that the decohered state can host a quantum memory as well as chiral or even non-modular topological order. In Sec. IV, we argue that our framework can naturally be generalized to include parent non-Abelian theories, which leads to our claim that braided tensor categories provide a partial classification for imTO. We then state our results in the language of strong 1-form symmetries in Sec. V, where we also show an analogy between imTOs and surface states of Walker-Wang models. Finally, we conclude in Sec. VI with a discussion of open questions and future directions.

II. TOPOLOGICAL SUBSYSTEM CODES VIA DECOHERENCE

While our framework extends to generic TOs, we first illustrate our construction with Abelian TOs that admit gapped boundaries in the context of lattice models realizing topological stabilizer codes, which serve as parent states for topological subsystem codes. Consider a square lattice with periodic boundary conditions and place a d-dimensional qudit on each vertex. We define the Pauli operators X_i and Z_i acting on site i, which satisfy the $\mathbb{Z}_d$ algebra

$$Z_i X_i = \omega X_i Z_i, \qquad \omega = \exp(2\pi i/d). \tag{1}$$

We consider commuting projector translation-invariant Hamiltonians:

$$H_{\mathcal{C}} = \sum_{i,\alpha} \frac{1-\theta_i^\alpha}{2} + \text{H.c.}\,, \tag{2}$$

where θ_i^α are constructed from finite, local products of Pauli operators acting near site i and are mutually commuting: $[\theta_i^\alpha, \theta_j^\beta] = 0\,\forall, \alpha, \beta$. The θ_i^α are *stabilizers* which generate the stabilizer group $\mathcal{S} = \langle\{\theta_i^\alpha\}\rangle$. The index α labels different families of stabilizers acting at site i. Since the Hamiltonian is positive semi-definite, the ground state manifold, also known as the *code space* $\mathcal{H}_C$, is uniquely specified by the set of all states satisfying $\theta_i^\alpha \left|\psi\right\rangle = \left|\psi\right\rangle\ \forall\ i, \alpha$.

We are interested in topological stabilizer models, whose ground states are topologically ordered. Recall that a TO $\mathcal{C}$ is described by a braided unitary fusion category: a finite set of anyons $\{a, b, \dots\}$, their fusion rules,

[1] Note that the TO in these decohered states remains *quantum*, even though they only encode a classical memory.

and their braiding statistics $\mathcal{B}_\theta(a,b) \equiv \theta_{ab}$. It is generally believed that local gapped Hamiltonians in 2D can at most support unitary modular fusion categories, with the extra condition that the only excitation that braids trivially with itself and all other anyons is the vacuum superselection sector. Here, we will use the term "anyon theory" to refer generally to braided fusion categories i.e., without the modularity constraint and specify when the anyon theory is modular.

Now, given a topological stabilizer model realizing the UMTC $\mathcal{C}$, each anyon is associated with a "string-like" operator which violates the stabilizer conditions $\theta_i^\alpha = 1$ only at its endpoints; anyons correspond to *errors* that take one out of the code space. To each anyon, we also associate Wilson-loop operators, $W^a_{x,y}$, which wrap around the x and y cycles of the torus, respectively, and physically correspond to locally creating an anyon a and its conjugate $\bar{a}$, before transporting a around one cycle of the torus and then annihilating it with $\bar{a}$. These Wilson-loop operators commute with the stabilizers and thus preserve the code space, corresponding to a non-trivial ground state degeneracy of $H_\mathcal{C}$ on the torus. The non-trivial braiding of the anyons is encoded in the Wilson-loop algebra: $W^a_x W^b_y = e^{i\theta_{ab}} W^b_y W^a_x$. Since topological stabilizer models can only realize *modular* TOs [36], each anyon a braids non-trivially with at least one other anyon b. The Wilson-loop operators thus correspond to *logical operators*, in that they provide a representation of the Pauli algebra on the code space; topological orders hence provide quantum memories. The paradigmatic example is provided by the Toric code, for which the Wilson loops associated to the e and m anyon excitations satisfy the Pauli algebra, $\{W^e_x, W^m_y\} = \{W^m_x, W^e_y\} = 0$, such that the code space encodes two logical qubits.

In our framework, we will always take as input a topologically ordered pure state, which hosts TO described by a UMTC $\mathcal{C}$. To make an explicit connection with topological stabilizer codes and TSSCs, we discuss Abelian theories that admit gapped boundaries here, though we will later relax this restriction. We are now interested in the fate of the TO (equivalently, the code space) under locally correlated noise. Such error processes correspond to quantum channels, where we consider translation-invariant channels of the form

$$\mathcal{E}_a[\rho] = \prod_i \mathcal{E}_{a,i}[\rho], \qquad \mathcal{E}_{a,i}[\rho] = \sum_{m=0}^{k-1} p_m O^m_{i,a} \rho (O^m_{i,a})^\dagger , \tag{3}$$

where $\sum_m p_m = 1$. Here, $O_{i,a}$ is a local operator supported near site i corresponding to a "short" Wilson string creating an anyon a and its conjugate $\bar{a}$ near i. We take $O_{i,a}$ to be a product of Pauli operators such that $(O_{i,a})^k = 1$ for some integer $k \leq d$. We restrict ourselves to the case of maximal decoherence $p_m = 1/k$; as we will see, the resulting density matrices will have zero correlation length and correspond to fixed points of a genuinely mixed-state phase of matter [19]. These error channels thus have the physical interpretation of incoherently proliferating anyons of the type a^m, for $m = 0, \ldots, k-1$. For a general Abelian twisted quantum double model, which can be expressed in the general form Eq. (2), the set of anyons $\{a^m\}$ ($m = 0, \ldots, k-1$) can for instance be taken as the set of gauge charges of the model, which generate a Lagrangian subgroup [32]. These Pauli stabilizer models realize all Abelian quantum double models (equivalently, all Abelian TOs that admit gapped boundaries [41]), which is the class of systems we now consider.

Let ρ be an arbitrary density matrix in the ground state manifold (code space) of $H_\mathcal{C}$, such that $\theta_i^\alpha \rho = \rho(\theta_i^\alpha)^\dagger = \rho$. Naïvely, one expects decoherence will wash out any long-range entanglement present in the state, but we will now show that while the TO is indeed reduced (consistent with our error channels being LPQCs [33]), it can remain non-trivial and represent a genuine mixed-state quantum phase of matter, which is not expected to be realized as the gapped ground state of any local 2D Hamiltonian but instead does arise as an anomalous surface state of a 3D TO.

To characterize the topological order in the decohered density matrix $\rho_\mathcal{E} \equiv \mathcal{E}[\rho]$, it will prove conceptually fruitful to represent the density matrix as a vector in a doubled Hilbert space, through the Choi-Jamiołkowski isomorphism [42, 43]. Explicitly, we map $\rho = \sum_{a,b} \rho_{ab} |a\rangle \langle b|$ to the pure state $|\rho\rangle\rangle = \sum_{a,b} \rho_{ab} |a\rangle |b\rangle^* \in \mathcal{H}_+ \otimes \mathcal{H}_-$, where $|b\rangle^* \equiv K|b\rangle$, K is complex conjugation in the computational basis, and $\mathcal{H}_{\sigma=\pm}$ are the ket and bra spaces, respectively. In the doubled space, the stabilizer conditions become

$$\theta^\alpha_{i+} |\rho\rangle\rangle = (\theta^\alpha_{i-})^* |\rho\rangle\rangle = |\rho\rangle\rangle, \tag{4}$$

where $\theta^\alpha_{i\pm}$ is the action of θ^α_i on $\mathcal{H}_\pm$. Hence, $|\rho\rangle\rangle$ lies in the ground state manifold of the topological order $\mathcal{C} \times \overline{\mathcal{C}}$ in the doubled Hilbert space, with the two factors living on the ket and bra spaces, respectively. Consider the decohered density matrix $\rho_\mathcal{E}$ in the doubled Hilbert space:

$$|\rho_\mathcal{E}\rangle\rangle = \mathcal{E}_a |\rho\rangle\rangle = \prod_i \left(\sum_{m=0}^{k-1} \frac{1}{k} O^m_{i,a+} (O^m_{i,a-})^* \right) |\rho\rangle\rangle \, . \tag{5}$$

For maximal decoherence, the vectorized error channel thus has the effect of projecting $|\rho\rangle\rangle$ to the subspace satisfying $O_{i,a+}(O_{i,a-})^* = +1$. As previously noted (see e.g. Ref. [14]), the effect of local error channels of the form Eq. (3), associated with decohering the set of anyons $\hat{\mathcal{A}} = \{a^m\}$, can be understood as inducing anyon condensation of the anyon pair $a_+ a_-$ in the doubled Hilbert space representation of the initially pure density matrix [2]. Since the two anyons a_+ and a_- have opposite spins, their composite is a boson and can be condensed.

[2] While we consider only maximal decoherence here, for finite decoherence, the decoherence *transition* can be phrased in terms of anyon condensation on the boundary [14, 15].

We now turn to one of the key results of this work by providing a finer characterization of the resulting decohered state $|\rho_{\mathcal{E}}\rangle\rangle$. In particular, let us understand the effect of decoherence on the code space of the original stabilizer group $\mathcal{S}$. In the doubled Hilbert space, let $\mathcal{S}_{\pm}$ be the groups generated by the original stabilizers on the ket and bra spaces: $\mathcal{S}_{\mathcal{C},\sigma} = \langle\{\theta^{\alpha}_{i\sigma}\}_{i,\alpha}\rangle$. By assumption, $O_{i,a+}(O_{i,a-})^*$ do not commute with the original stabilizers (these create anyons and hence take us out of the code space). Thus, defining the group generated by the errors, $\mathcal{F}_{\mathcal{E}} = \langle\{O_{i,a+}(O_{i,a-})^*\}_{i,a}\rangle$, the state $|\rho_{\mathcal{E}}\rangle\rangle$ is stabilized by the group of mutually commuting elements in the union of the groups $\mathcal{F}_{\mathcal{E}}$, $\mathcal{S}_{\mathcal{C},+}$, and $\mathcal{S}_{\mathcal{C},-}$—that is to say, their centralizer $\mathcal{S}_{\mathcal{C}\times\bar{\mathcal{C}},\mathcal{E}} \equiv \mathcal{Z}(\mathcal{F}_{\mathcal{E}} \cup \mathcal{S}_{\mathcal{C},+} \cup \mathcal{S}_{\mathcal{C},-})$. Note that, in the doubled Hilbert space, $|\rho\rangle\rangle_{\mathcal{E}}$ is still an element of the code space of a topological stabilizer code [3].

Let us now restrict our attention to those elements $S_+ \in \mathcal{S}_{\mathcal{C}\times\bar{\mathcal{C}},\mathcal{E}}$ which act non-trivially only on $\mathcal{H}_+$. In the physical Hilbert space, these are precisely those stabilizers which commute with the errors: $[S, O_{i,a}] = 0$. In particular, if $S_+ \in \mathcal{S}_{\mathcal{C}\times\bar{\mathcal{C}},\mathcal{E}}$, then this implies $S_-^* \in \mathcal{S}_{\mathcal{C}\times\bar{\mathcal{C}},\mathcal{E}}$. These operators satisfy $S_+|\rho_{\mathcal{E}}\rangle\rangle = S_-^*|\rho_{\mathcal{E}}\rangle\rangle = |\rho_{\mathcal{E}}\rangle\rangle$ or, equivalently,

$$S\rho_{\mathcal{E}} = \rho_{\mathcal{E}}S^{\dagger} = \rho_{\mathcal{E}}\,, \forall S : [S, O_{i,a}] = 0\,. \tag{6}$$

Let $\mathcal{F} = \langle\{O_{i,a}\}_i\rangle$ be the group generated by all local errors and let $\mathcal{G} = \langle\mathcal{S}, \mathcal{F}\rangle$. Since the elements of $\mathcal{S}$ and $\mathcal{F}$ do not commute, the group $\mathcal{G}$ is in general non-Abelian. The set of stabilizers of $\rho_{\mathcal{E}}$ is then given by the center of $\mathcal{G}$, $\mathcal{S}_{\mathcal{E}} = \mathcal{Z}(\mathcal{G})$. In other words, $\rho_{\mathcal{E}}$ is an element of the code space of $\mathcal{S}_{\mathcal{E}}$, but this need not be the code space of a topological stabilizer code.

Some remarks are in order. First, note that Eq. (6) implies that $\rho_{\mathcal{E}}$ is *strongly symmetric* with respect to the stabilizers that correspond to Wilson loop operators for anyons that braid trivially with the set of decohered anyons $\hat{\mathcal{A}} = \{a^m\}$; thus $\rho_{\mathcal{E}}$ has (anomalous) strong 1-form symmetries. We will return to this point later in Sec. V. Second, since the logical information encoded in $\rho_{\mathcal{E}}$ is strictly less than that in the parent pure state ρ, this is sufficient to show that $\rho_{\mathcal{E}}$ cannot be in the same mixed-state phase as ρ [19] (since no finite-depth LPQC can recover logical information). Finally, since we are working with maximal decoherence, $\rho_{\mathcal{E}}$ is a state with zero-correlation length and represents a genuine mixed-state phase of matter, since finite-depth LPQCs satisfy an area law for the amount of correlations they can create [33]. The stability of the state to finite-depth LPQCs can also be attributed to its strong 1-form symmetries Eq. (6).

Remarkably, the group structure given by the stabilizers $\mathcal{S}_{\mathcal{E}}$ and $\mathcal{G}$, the latter of which is known as a "gauge group" (not to be confused with the gauge group of a gauge theory), precisely realizes the structure of a *topological subsystem code*, which leads to one of our main results: the set of decohered states $\rho_{\mathcal{E}}$ on the torus form the code space for a TSSC. We briefly discuss the structure of TSSCs here, but refer the reader to Ref. [36] for a thorough exposition. As in the case of stabilizer codes discussed above, the Hilbert space for a TSSC can be written as a direct sum of the code space and its orthogonal complement $\mathcal{H} = \mathcal{H}_C \oplus \mathcal{H}_C^{\perp}$. For a subsystem code, the code space further factorizes $\mathcal{H}_C = \mathcal{H}_{\mathcal{L}} \otimes \mathcal{H}_{\mathcal{G}}$ such that the logical information is only encoded in the logical subsystem $\mathcal{H}_{\mathcal{L}}$, while $\mathcal{H}_{\mathcal{G}}$ is referred to as the gauge subsystem [29]. The gauge group $\mathcal{G}$ comprises a set of Pauli operators which preserve the code space (commute with the stabilizers), but their action within the code space induces the factorization of the code space. For a gauge group $\mathcal{G}$ that is proportional to the stabilizer group $\mathcal{S}$, the gauge subsystem $\mathcal{H}_{\mathcal{G}}$ is trivial and one again has a topological stabilizer code. If the gauge group is non-trivial, TSSCs can support non-local stabilizers which cannot be generated by local stabilizers; moreover, TSSCs must satisfy the constraint that there should be no non-local stabilizers or logical operators on an infinite plane.

Recently, Ref. [36] discussed a general procedure for generating TSSCs from parent topological stabilizer codes by "gauging out" appropriate anyons (see also Ref. [29]). In brief, given a parent Abelian TO with UMTC $\mathcal{C}$ that admits a gapped boundary (equivalently a topological stabilizer group $\mathcal{S}$), gauging out the set of anyons $\hat{\mathcal{A}} = \{a^m\}$ proceeds as follows: denote by $\mathcal{F}$ the group of short string operators for the set of anyons $\hat{\mathcal{A}}$. Note that $\mathcal{F}$ is only Abelian if a is a boson. Gauging out then takes the stabilizer group $\mathcal{S}$ to the gauge group $\mathcal{G} = \langle\mathcal{S}, \mathcal{F}\rangle$. This means that the short string operators for the anyons in $\hat{\mathcal{A}}$ get appended to the original Abelian gauge group ($\propto \mathcal{S}$). Physically, this means that any anyon $c \in \mathcal{C}$ that braids non-trivially with the anyons in $\hat{\mathcal{A}}$ is confined, since the short string operators for the gauged out anyons do not commute with the Wilson loop operators for c. Further, the Wilson loop operators for anyons in $\hat{\mathcal{A}}$ are now given by products of gauge operators and if an anyon $x \in \hat{\mathcal{A}}$ is transparent in $\hat{\mathcal{A}}$, it becomes a transparent anyon in the TSSC. This procedure is distinct from anyon condensation in that the gauged out anyons are not necessarily identified with the vacuum and the excitations that only differ up to fusion with anyons in $\hat{\mathcal{A}}$ are not identified.

With that brief review, let us return to the decohered mixed-state $\rho_{\mathcal{E}}$. Following our discussion, it is clear that locally correlated errors induced by the short string operators for the set $\hat{\mathcal{A}}$ have the effect of gauging out these anyons, since the decohered density matrix $\rho_{\mathcal{E}}$ (which is an element of the code space) is stabilized by precisely those stabilizers which commute with $O_{i,a}$ (see Eq. 6). As violations of these stabilizers correspond to those anyons from the parent TO $\mathcal{C}$ that braid trivially with anyons in $\hat{\mathcal{A}}$, we see that the decohered state has TO defined by a

[3] A quick way to see this is that condensing a boson in a UMTC always results in another UMTC which, for Abelian TOs, can always be described by a topological stabilizer model [36].

proper subset of anyons

$$\mathcal{A} \equiv \{b \in \mathcal{C} | \mathcal{B}_\theta(a,b) = 1\} \tag{7}$$

where $\mathcal{B}_\theta(x,y)$ denotes the braiding statistics between anyons x and y (encoded in the parent UMTC $\mathcal{C}$). By definition, $\mathcal{G}$ contains the short string operators for the set of anyons in $\hat{\mathcal{A}}$: thus, their Wilson loop operators are given by products of gauge operators. If an anyon $x \in \hat{\mathcal{A}}$ is invisible to all other anyons in that set, its logical operator corresponds to a non-local stabilizer [36], such that it becomes a transparent anyon in the decohered theory. Crucially, here $\mathcal{A}$ is an Abelian anyon theory which can be non-modular and corresponds to a TSSC. If $\mathcal{A}$ has no opaque (i.e., detectable via braiding with anyons in $\mathcal{A}$) anyons but still has transparent anyons, then the resulting mixed-state is a *classical* self-correcting memory [44, 45]. In contrast, on the torus Wilson loops for opaque anyons in the (generally non-modular) Abelian anyon theory $\mathcal{A}$ correspond to logical operators for the TSSC, and $\rho_\mathcal{E}$ can hence encode a *quantum* memory.

The identification of the decohered mixed-state $\rho_\mathcal{E}$ with a TSSC provides a powerful framework within which to study imTO, since we can leverage several known results about the former to characterize the latter. Indeed, a key message of our work is that TSSCs provide the natural language for (partially) classifying quantum phases with imTO. First, we have shown that decohering anyons in topological stabilizer model provides a physical mechanism for gauging out anyons. Since decoherence $\cong$ gauging out, the results of Ref. [36], which established that every Abelian anyon theory (not necessarily modular) can be obtained by gauging out anyons from some twisted quantum doubles, immediately imply that the classification of imTO is at least as rich as that of Abelian anyon theories. This also proves our claim that the decohered states host *intrinsically mixed* TO: $\rho_\mathcal{E}$ belongs to the code space of a TSSC which, unlike topological stabilizer codes, can realize *non-modular* and even *chiral* topological order, the latter of which is believed to not occur in the ground state of a locally commuting parent Hamiltonian in 2D [46]. It is also widely accepted that local gapped Hamiltonians in 2D only support modular anyon theories, and it stands within reason that the non-modular imTOs we obtain through decoherence are in fact intrinsically mixed.

Thus, subjecting a 2D pure state to a locally correlated noise channel induces topological order that would otherwise require a sequential quantum circuit [35] or measurement with feedback, and we take this to be a defining feature of imTO. An appealing perspective is then that ground states of (non-chiral) topological stabilizer codes furnish resource states for the dissipative-preparation of chiral (or non-modular) topological order under locally correlated noise. Given that a large class of topologically ordered gapped ground states can in principle be realized in quantum simulators using single-shot measurement and feedback [47], our results suggest that engineered dissipation can play a crucial role in the preparation of quantum states with imTO. As noted above, the decohered state $\rho_\mathcal{E}$ can in principle encode a quantum memory (see examples below), and the lifetime of this encoded quantum information will remain infinite in the presence of any noise that respects the stabilizer symmetry of the TSCC (see Eq. (6)). Indeed, an appealing interpretation of the decohered code space is that of a *noiseless* [48] or *decoherence free* subspace [49], where the noise only acts within the gauge subsystem, leaving the logical subspace intact.

III. EXAMPLES: PARENT ABELIAN TOS

With the general framework for imTO established, we now analyze several concrete examples that illustrate our results. In the process, we also discuss how anyon condensation in the doubled Hilbert space is equivalent to gauging out in the physical Hilbert space. As mentioned earlier, in principle one can straightforwardly obtain any (non-modular) Abelian anyon theory by decohering the gauge charges of the twisted quantum double models presented in Ref. [36] (which also furnishes the appropriate short-string operators and verifies that these satisfy the required braiding and fusion properties).

A. $\mathbb{Z}_2^{(0)}$ and $\mathbb{Z}_2^{(1)}$ TSSC from $\mathbb{Z}_2$ Toric code

As the paradigmatic example of a topological stabilizer code, the stability of the $\mathbb{Z}_2$ Toric code to decoherence has been extensively investigated [14, 17, 19]. We revisit this problem here in light of our interpretation of the decohered state as a TSSC. Consider a system of qubits placed on the edges of a square lattice with periodic boundary conditions, with the Hamiltonian given by

$$H_{\mathbb{Z}_2} = \sum_s \frac{1 - A_s}{2} + \sum_p \frac{1 - B_p}{2},\ A_s = \prod_{i \in s} X_i,\ B_p = \prod_{i \in p} Z_i, \tag{8}$$

where s and p denote stars and plaquettes, as usual. This Hamiltonian exhibits $\mathbb{Z}_2$ topological order ($\mathcal{C} = \mathbb{Z}_2 \times \mathbb{Z}_2$), with anyons given by the electric charge e, the magnetic charge m, and their fermionic composite $f = e \times m$. As a quantum memory, the Toric code supports two logical qubits, with logical operators given by the Wilson loops of the e and m anyons: $W^e_{x,y} = \prod_{i \in \Gamma_{x,y}} Z_i$ and $W^m_{x,y} = \prod_{i \in \hat{\Gamma}_{x,y}} X_i$, which satisfy $\{W^e_x, W^m_y\} = \{W^e_y, W^m_x\} = 0$. Here, $\Gamma_{x,y}$ and $\hat{\Gamma}_{x,y}$ are the corresponding non-contractible paths on the direct and dual lattices, respectively.

We now consider two distinct error channels of the form Eq. (3), which proliferate errors associated with the e and f anyons, respectively:

$$\mathcal{E}_{i,e}[\rho] = \frac{\rho + Z_i \rho Z_i}{2},\ \mathcal{E}_{i,f}[\rho] = \frac{\rho + Z_i X_{i+\boldsymbol{\delta}} \rho Z_i X_{i+\boldsymbol{\delta}}}{2}, \tag{9}$$

where $\boldsymbol{\delta} = (\frac{1}{2}, -\frac{1}{2})$ [4]. Here the short string operators O are given by the operators Z_i and $Z_i X_{i+\delta}$ for e and f respectively.

Given an arbitrary state ρ in the ground state manifold of Eq. (8), we wish to characterize the decohered states $\rho_{e,f} \equiv \mathcal{E}_{e,f}[\rho]$. Clearly, $\mathcal{E}_e[W^e_{x,y}] = W^e_{x,y}$ while $\mathcal{E}_e[W^m_{x,y}] = 0$, and so ρ_e only forms a classical memory with a single bit of information encoded in each of $W^e_{x,y}$. Likewise, one also finds that ρ_f forms a classical memory, with classical bits stored in the f Wilson loops, defined as $W^f_{x,y} = \prod_{i\in\Gamma_{x,y}} X_i Z_{i+\boldsymbol{\delta}}$. While superficially it appears that errors have rendered the state "trivial," we now show that $\rho_{e,f}$ exhibit richer structure.

Recall that since $A_s\rho = \rho A_s = \rho$ and $B_p\rho = \rho B_p = \rho$, ground states of Eq. (8) can be interpreted as closed loop condensates of the e, m, and f anyons. After maximal decoherence, we instead only have $B_p\rho_e = \rho_e B_p = \rho_e$ and $A_s\rho_e A_s = \rho_e$. Physically, the e-noise has the effect of "freezing" the m-loops (and hence also the f loops) into a classical ensemble, while leaving the "quantum" condensate of e-loops untouched. More precisely, e-errors break the strong 1-form magnetic symmetry of the original pure state down to a weak 1-form symmetry, while leaving the strong 1-form electric symmetry intact. We will later provide a general discussion of the role 1-form symmetries play in characterizing generic imTOs (see Sec. V).

One might thus be inclined to view ρ_e as describing a topologically ordered state in which the only deconfined anyon excitation is the bosonic e anyon of the parent Toric code. Indeed, in the notation of Ref. [50], a phase with anyon content given by the vacuum and a single e anyon corresponds to the $\mathbb{Z}_2^{(0)}$ topological order [5]. Notably, this topological order is *non-modular*: since e is the only non-trivial anyon in the theory, it cannot be detected by braiding with any other anyons i.e., it is transparent. While non-modular topological orders cannot be realized by topological stabilizer models, they do arise in the aforementioned topological stabilizer codes. We can in fact make the correspondence with TSSCs precise, following the preceding general analysis in Sec. II. The gauge group for ρ_e is given by [6]

$$\mathcal{G}_e = \langle i, Z_i, A_s\rangle, \tag{10}$$

such that the stabilizer group is given by $\mathcal{Z}(\mathcal{G}_e) = \langle B_p, W^e_{x,y}\rangle$. This precisely describes the $\mathbb{Z}_2^{(0)}$ topological subsystem code, which is shown to exhibit the $\mathbb{Z}_2^{(0)}$ topological order in Ref. [36]. Since e is transparent in this theory, there are no logical operators and ρ_e does not encode any qubits.

Similar considerations hold for ρ_f. We have that $A_v B_{v-\mathbf{y}}\rho_f = \rho_f A_v B_{v-\mathbf{y}} = \rho_f$, where $A_v B_{v-\mathbf{y}}$ generates a closed f loop and $v - \mathbf{y}$ denotes the plaquette to the south-east of vertex v, while we only have that $A_v\rho_f A_v = \rho_f$. Thus, f-noise freezes both the e and m loops but leaves the f loops untouched, such that ρ_f describes a quantum condensate of *fermionic excitations* (stated otherwise, ρ_f retains a strong 1-form symmetry): this is not expected to occur in the ground state of a gapped, local 2D Hamiltonian. Remarkably, despite starting with a *bosonic* topological order, decoherence has resulted in a state in which the sole deconfined quantum excitation is a *fermion*. In a rough sense, local decoherence allows one to "peel off" half of the original state.

Again, this heuristic interpretation can be formalized by following our general analysis in Sec. II—the gauge group of ρ_f is given by

$$\mathcal{G}_f = \langle i, Z_i X_{i+\delta}, A_s\rangle \tag{11}$$

which yields the stabilizer group $\mathcal{S}_f = \langle A_v B_{v-\mathbf{y}}, W^f_{x,y}\rangle$. This precisely describes a topological subsystem code describing the $\mathbb{Z}_2^{(1)}$ topological order which, again, is *non-modular* [36]. Like the previous case, f is transparent (it braids trivially with itself) and hence the decohered state encodes no quantum memory, consistent with Ref. [27].

It will be instructive to study these mixed-states through the complementary perspective of the doubled Hilbert space. As discussed earlier, the vectorized initial density matrix $|\rho\rangle\rangle$ lies in the ground state manifold of a bilayer Toric code, with anyon content

$$\mathcal{C} \times \overline{\mathcal{C}} = \{1_+, e_+, m_+, f_+\} \times \{1_-, e_-, m_-, f_-\}, \tag{12}$$

where $\pm$ subscripts denote the ket and bra spaces respectively. In this picture, the e and f noise channels have the effect of condensing the anyons e_+e_- and f_+f_- respectively, which for maximal decoherence lead to the resulting daughter topological orders

$$\mathcal{C}_e = \{1_+1_-, e_+, m_+m_-, f_+m_-\}, \tag{13}$$
$$\mathcal{C}_f = \{1_+1_-, e_+e_-, e_+m_-, f_+\}. \tag{14}$$

It is readily apparent that the resulting TO in either case is that of a single $\mathbb{Z}_2$ Toric code, with the fusion group of the Abelian anyons given by $\mathbb{Z}_2 \times \mathbb{Z}_2$. This can also be directly verified with the explicit forms of $|\rho_{e,f}\rangle\rangle$ in the lattice model. In light of our above stabilizer analysis however, we note a key distinction between the mixed-states $\mathcal{C}_e$ and $\mathcal{C}_f$. Restricting attention to anyons with support on only the ket or bra space, we see that both orders support a single such anyon. For $\mathcal{C}_e$, this is the *boson* $e_+ \sim e_-$, while for $\mathcal{C}_f$, this is the *fermion* $f_+ \sim f_-$, where the equivalences are up to fusion with the condensed anyon. This is consistent with our observation in the stabilizer analysis that under e and f noise, the sole remaining deconfined quantum excitations are simply the original e and f excitations, respectively.

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[4] As noted in Ref. [27], while Y-errors locally create f anyons, arbitrary configurations of such errors can create unbalanced numbers of e and m anyons. It is hence crucial to consider "framed" short-string operators to generate strictly f type errors.

[5] In general, $\mathbb{Z}_N^{(p)}$ topological order is generated by a single anyon a such that $a^N = 1$ and a has spin $\theta_a = \exp(2\pi i p/N)$.

[6] $\mathcal{G}$ must include appropriate roots of unity to ensure that it generates a representation of the Pauli group.

We now show that this anyon condensation across the ket and bra spaces, at the level of the density matrix in the original Hilbert space, corresponds to gauging out an anyon. Recall that anyon condensation proceeds in two steps: to condense an anyon a, one first (i) projects out from the theory those anyons which braid non-trivially with a (i.e. they become confined excitations) and then (ii) identifies those anyon types which differ by fusion with a. For instance, in the usual Toric code, condensing e confines the m and f anyons while e becomes identified with the vacuum: the resulting state has no remaining anyon excitations and is trivial. Gauging out an anyon however corresponds to only performing step (i) of this process. For instance, gauging out e still confines m and f, but leaves e distinct from the vacuum, such that one is left with the anyon content $\{1, e\}$—precisely that of the $\mathbb{Z}_2^{(0)}$ non-modular TO realized via decoherence of e.

Crucially, one can also gauge out anyons that *cannot* be condensed (non-bosonic anyons); analogously, one can decohere non-bosonic anyons a since this corresponds to the conventional condensation of the bosonic pair (a_+a_-) in the doubled Hilbert space. For instance, one may gauge out f from the $\mathbb{Z}_2$ Toric code to obtain the $\mathbb{Z}_2^{(1)}$ TO, precisely replicating the effect of f errors. Surprisingly, as this simple example illustrates, locally correlated errors (which correspond to anyon condensation under the Choi map) provide a *physical* implementation of the gauging out procedure, which thus far remained a conceptual device for generating topological subsystem codes from topological stabilizer codes [28–31, 36].

Let us pause to recapitulate our observations in the context of the Toric code. We found that anyonic decoherence, previously shown to correspond to anyon condensation across the ket and bra spaces, implements the gauging out of anyons in the original Hilbert space, including those which are forbidden from condensing under purely unitary evolution. This process led to mixed-states supporting non-modular topological order, corresponding to topological subsystem codes, which is not believed to be allowed in the ground state of a locally gapped Hamiltonian in 2D. Moreover, the resulting mixed-state corresponds to a zero-correlation length state with strictly lower logical information than the parent state and thus represents a distinct mixed-state quantum phase of matter [19] i.e., it represents an *intrinsically mixed* topological phase of matter. Here, the stability of the maximally decohered state can be attributed to the fact that LPQCs can create at most area-law entanglement in finite time. For the specific case of the $\mathbb{Z}_2$ Toric code subject to f errors, Ref. [27] numerically verified the robustness of the resulting imTO against finite noise channels that explicitly break the strong 1-form symmetry. However, we generically expect states with strong 1-form symmetries to remain robust up to some finite noise rate.

We now explore the above principles through examples that illustrate the general structures permitted in the decohered mixed-states. Specifically, we choose parent states with Abelian TO (that admit gapped boundaries) and identify appropriate error channels that lead to imTO with mixed quantum-classical memories as well as chiral anyon theories. Note that the presence of a classical memory is tied to non-local stabilizers in the decohered theory, which correspond to transparent anyons.

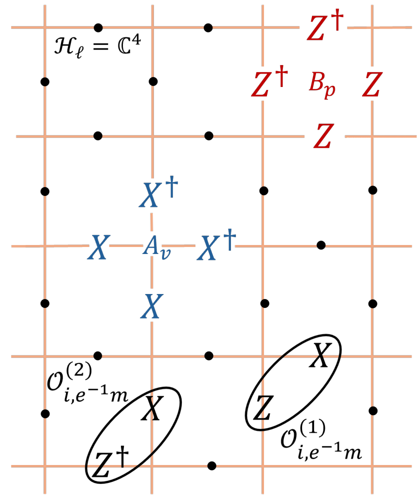

FIG. 2: The $\mathbb{Z}_4$ Toric code is defined on the 2D square lattice with a $d = 4$ qudit (black) on each link. The star (plaquette) stabilizers are shown here in blue (red). Short string operators for the $e^{-1}m$ anyon are also shown.

B. $\mathbb{Z}_4^{(1)}$ TSSC from $\mathbb{Z}_4$ Toric code

We next consider a square lattice with $d = 4$ qudits on each edge. A Hamiltonian realizing $\mathbb{Z}_4$ TO is given by

$$H_{\mathbb{Z}_4} = -\sum_s (A_s + A_s^\dagger) - \sum_p (B_p + B_p^\dagger), \qquad (15)$$

with the star and plaquette operators defined in Fig. 2. The ground state manifold is determined by the constraints $A_s = B_p = 1$, violations of which indicate the presence of electric e and magnetic m excitations, respectively. Explicitly, Z_i applied to the ground state excites an e and an e^{-1} anyon at vertices connected by the edge i. Likwise, applying X_i creates an m and m^{-1} on plaquettes separated by the edge i. These anyons satisfy $\mathbb{Z}_4 \times \mathbb{Z}_4$ fusion rules $e^4 = m^4 = 1$ and the braiding statistics between two composite objects $e^a m^b$ and $e^c m^d$ is given by $\mathcal{B}_\theta(ab, cd) = i^{ad+bc}$. On the torus, the non-contractible Wilson loops $W^e_{x,y} = \prod_{i\in\Gamma_{x,y}} Z_i$ and $W^m_{x,y} = \prod_{i\in\hat{\Gamma}_{x,y}} X_i$ serve as the logical operators and satisfy the algebra $W^e_{x/y} W^m_{y/x} = i W^m_{y/x} W^e_{x/y}$, such that the code space stores two $d = 4$ qudits.

Here, we consider local errors for the set of anyons generated by the $e^{-1}m$ anyon, $\hat{\mathcal{A}} = \{1, e^{-1}m, e^2m^2, em^3\}$. The corresponding decoherence channel is given by Eq. (3) with the generating short string operators $O^{(1)}_{i,e^{-1}m}$, $O^{(2)}_{i,e^{-1}m}$ for $e^{-1}m$ shown in Fig. 2. Here, the group of local errors $\mathcal{F}$ is precisely the group generated by the short string operators of $e^{-1}m$ (see Fig. 2).

Now, for an arbitrary state ρ in the ground state manifold of the $\mathbb{Z}_4$ Toric code, we wish to characterize $\rho_{e^{-1}m} \equiv \mathcal{E}_{e^{-1}m}[\rho]$. Let us proceed formally first in this case. Following the general prescription in Sec. II, the gauge group $\mathcal{G}_{e^{-1}m} = \langle \mathcal{S}, \mathcal{F} \rangle$ is given by

$$\mathcal{G}_{e^{-1}m} = \langle e^{i\pi/2}, A_v, B_p, O^{(1)}_{i,e^{-1}m}, O^{(2)}_{i,e^{-1}m} \rangle \,. \tag{16}$$

The stabilizer group for the decohered density matrix is then $\mathcal{S}_{e^{-1}m} = \langle A_v B_{v+\mathbf{y}}, W^{e^2m^2}_{x,y} \rangle$, where $v + \mathbf{y}$ denotes the plaquette to the north-east of vertex v, $W^{e^2m^2}_{x,y} = \prod_{i \in \Gamma_{x,y}} X_i^2 Z_i^2$ is the Wilson loop operator for the e^2m^2 anyons, and $A_v B_{v+\mathbf{y}}$ generates a closed loop of em anyons. This is precisely the topological subsystem code corresponding to the $\mathbb{Z}_4^{(1)}$ topological order, which is given by the Abelian anyon theory $\mathcal{A} = \{1, em, e^2m^2, e^{-1}m^{-1}\}$, in which both em and e^3m^3 are semions and e^2m^2 is a transparent boson (it braids trivially with all other anyons in $\mathcal{A}$). This stems from the fact that the open Wilson line operator for the e^2m^2 is built out of gauge operators and commutes with all of the stabilizers in $\mathcal{S}_{e^{-1}m}$ at its endpoints. The code space, stabilized by $\mathcal{S}_{e^{-1}m}$ has two logical operators on the torus, which are the Wilson loop operators of the two semions, and encodes a single logical qubit in its logical subystem. Thus, $\rho_{e^{-1},m}$ realizes imTO as it is a *non-modular* Abelian anyon theory that cannot be the ground state of a gapped local Hamiltonian in 2D and also realizes a quantum memory.

Recall that the original pure state ρ satisfies $A_s\rho = \rho A_s = \rho$ and $B_p\rho = \rho B_p = \rho$ and can be thought of as a closed loop condensate of all non-trivial anyons $e^a m^b$. Clearly, $\mathcal{E}[W^{\alpha}_{x,y}] = 0$ for any anyons α that braid non-trivially with anyons in $\hat{\mathcal{A}}$. Since only the anyons in $\mathcal{A} = \{1, em, e^2m^2, e^{-1}m^{-1}\}$ braid trivially with those in $\hat{\mathcal{A}}$, decoherence does not affect their Wilson loops: $\mathcal{E}[W^{b\in\mathcal{A}}_{x,y}] = W^{b\in\mathcal{A}}_{x,y}$. Intuitively, decoherence has thus frozen out the loops for any anyons $\notin \mathcal{A}$ into a classical ensemble, while the quantum condensate of anyons in $\mathcal{A}$ is left untouched.

Said more formally, $\hat{\mathcal{A}}$ errors break most of the strong 1-form symmetries of ρ down to weak 1-form symmetries, while leaving the strong 1-form symmetries corresponding to $\mathcal{A}$ anyons intact. This is encoded in Eq. (6) and the fact that the stabilizer group $\mathcal{S}_{e^{-1}m}$ for the TSSC is generated by small loops for the em anyon (which generates $\mathcal{A}$). Finally, since e^2m^2 is transparent in $\hat{\mathcal{A}}$, it remains transparent in $\mathcal{A}$ by definition. We thus obtain the same result as above: the set of decohered density matrices on the torus form the code space for a TSSC which describes a non-modular Abelian anyon theory $\mathcal{A}$, whose non-trivial anyons are two semions and a transparent boson. The mutual statistics of the semions result in this mixed-state encoding a logical qubit in its logical subsystem, and since it is again a zero-correlation length mixed-state (we are working at maximal dephasing), we expect it will be stable up to some finite noise threshold (for local noise).

It is instructive to once again consider the gauging out procedure from the perspective of the doubled Hilbert space. In this picture, decoherence of the anyons in $\hat{\mathcal{A}}$ corresponds to condensing $\{[1]_+[1]_-, [e^{-1}m]_+[e^{-1}m]_-, [e^2m^2]_+[e^2m^2]_-, [em^3]_+[em^3]_-\}$, which form a Lagrangian subgroup of the TO in the doubled space $\mathcal{C} \times \bar{\mathcal{C}}$, where $\mathcal{C} = \mathbb{Z}_4 \times \mathbb{Z}_4$. Anyon condensation proceeds in the usual way: each anyon from $\hat{\mathcal{A}}$ is identified with the vacuum. Next, any excitation which braids non-trivially with any condensed anyon becomes confined and, of the remaining deconfined excitations, any that differ only up to fusion by anyons in $\hat{\mathcal{A}}$ are identified. A simple calculation shows that the resulting topological order is that of a $\mathbb{Z}_4$ gauge theory, with only the following anyons supported solely on the ket space: $[em]_+$, and $[e^{-1}m^{-1}]_+$, while $[e^2m^2]_+ \sim [e^2m^2]_-$ can move freely between the ket and bra spaces, and is a transparent anyon. These are the deconfined excitations in the decohered theory (which correspond to strong 1-form symmetries of the resulting state), which unsurprisingly match those we obtained by working directly in the Hilbert space, where the anyons in $\hat{\mathcal{A}}$ were gauged out rather than condensed.

This example already displays much of the rich structure that emerges when anyonic errors are introduced into a pure topologically ordered state, with the most striking features being the presence of a robust quantum memory alongside a non-modular Abelian anyon theory that is generally believed to not occur in the ground state of a locally gapped Hamiltonian.

Equipped with the preceding understanding of the correspondence between decoherence, gauging out, and anyon condensation in the doubled Hilbert space, we now briefly discuss two other examples which illustrate the breadth of Abelian anyon theories that can be "peeled off" via decoherence. Moreover, we have established a mapping from the space of imTOs that result from decohering a set of anyons $\hat{\mathcal{A}}$ when starting from a parent topological stabilizer code to the space of TSSCs that results from gauging out $\hat{\mathcal{A}}$ from the same parent topological stabilizer code. Thus, we can directly use results from Ref. [36], which provides a thorough exploration of TSSCs. In particular, once we specify the parent TO and the set of decohered anyons $\hat{\mathcal{A}}$, we can immediately read off the gauge group and the structure of the code space from the results contained in Ref. [36].

C. Chiral Semion from Double Semion

As an instance of this mapping, let us take the doubled semion anyon theory as our parent TO. This theory can be realized as a Pauli stabilizer Hamiltonian [32] and its anyons form a $\mathbb{Z}_2 \times \mathbb{Z}_2$ group under fusion, with elements $\{1, s, \bar{s}, s\bar{s}\}$. Here, s is a semion (it has self-statistics $\theta(s) = i$), $\bar{s}$ is an anti-semion ($\theta(\bar{s}) = -i$), and $s\bar{s}$ is a boson. Now, we subject a ground state of this system to an error channel that incoherently proliferates the semion s i.e., $\hat{\mathcal{A}} = \{1, s\}$. This corresponds to gauging out s, which braids trivially with $\bar{s}$. The resulting anyon theory for the decohered mixed-state is given by $\mathcal{A} = \{1, \bar{s}\}$ i.e., it is the *chiral* (anti)-semion Abelian anyon theory. This zero-correlation length state represents an intrinsically mixed-state phase of matter as chiral UMTCs are widely held to be forbidden in the ground states of locally commuting Hamiltonians [46] and are not expected to be realized in fixed point wave-functions. Since $\bar{s}$ has non-trivial self-statistics, the decohered code space encodes exactly one logical qubit; this example represents the minimal model in which one obtains a chiral anyon theory with an encoded logical qubit.

D. Three-Fermion from $\mathbb{Z}_2 \times \mathbb{Z}_2$ Toric code

Take the initial pure state to be a ground state of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ Toric code. The anyons in this theory form a $\mathbb{Z}_2^4$ group under fusion, with elements $\{1, e_1, m_1, f_1\} \times \{1, e_2, m_2, f_2\}$. As noted in Ref. [30], the anyon types can be relabeled $\{1, f_1, e_1 f_2, m_1 f_2\} \times \{1, f_2, f_1 e_2, f_1 m_2\}$ which is equivalent to two copies of the *Three-Fermion* (3F) anyon theory ($f_1 = e_1 m_1$ and $f_2 = e_2 m_2$ are fermions). The 3F anyon theory is a chiral Abelian UMTC which contains the anyons $\{1, \psi_1, \psi_2, \psi_3\}$ where $\theta(\psi_i) = -1$ for $i = 1, 2, 3$, and with the braiding between the fermions given by $\mathcal{B}_\theta(\psi_i, \psi_j) = -1$ for any $i, j = 1, 2, 3$.

We now wish to "peel off" a single 3F theory (which is a chiral Abelian UMTC) from the parent topological order. For this, we need to identify a set of anyons $\hat{\mathcal{A}}$ that braid trivially only with three fermions in the parent TO. One can choose the set $\hat{\mathcal{A}} = \{1, f_1, f_2 e_1, f_1 f_2 e_1\}$. Maximally decohering the initial density matrix with respect to these error channels amounts to gauging these anyons out, with the resulting anyon theory $\mathcal{A} = \{1, f_2, e_2 f_1, e_2 f_1 f_2\}$ identical to a single 3F anyon theory. Due to the self and mutual braiding statistics of this theory, its logical subspace encodes 2 logical qubits.

As a final remark, finite temperature mixed-states also provide simple instances of our general framework. For example, consider the $D = 2, 3, 4$ Toric code at finite temperature [12, 51]. For $D = 2$, any finite-T state corresponds to both the e, m anyons being condensed: the resulting state hosts no deconfined anyons and is hence trivial. Now, in $D = 3$, the e charges proliferate at any non-zero temperature, but below a critical T_c, the flux-loops of the 3D Toric code remain deconfined but are now transparent. This corresponds to a TSSC that does not encode any logical qubits in its logical subsystem but still has a non-trivial classical memory due to the transparent loops. Finally, for the 4D Toric code which has only loop-like excitations, there exist two critical temperatures: below the first, none of the excitations proliferate and the finite-T mixed-state is a TSSC that is equivalent to a topological stabilizer code (i.e., its gauge group is proportional to the stabilizer group). Hence, it represents finite-T topological order. Above the first, but below the second critical temperature, only one of the loop excitations proliferate and one obtains a TSSC with a classical memory. Above the second critical temperature, all anyons are condensed and the mixed-state is topologically trivial. Thus, prior results on topological order at finite temperature are straightforwardly incorporated into our general framework. We note that an infinite temperature state with quantum memory based on a subsystem code was previously proposed in Ref. [52], and in our framework, constitutes an imTO.

In general, the map from decohered density matrices to TSSCs conveniently allows one to use results regarding the latter to obtain a partial classification of the former. In particular, since Ref. [36] showed that any (non-modular) Abelian anyon theory can be realized by a TSSC, it immediately provides a partial classification of imTOs in terms of Abelian anyon theories.

IV. DECOHERENCE AS GAUGING OUT IN GENERAL UMTCS

While the precise relation between imTOs and TSSCs can only be made in the context of Abelian anyon theories with gappable boundaries, we expect that the general relation between decoherence induced imTO, anyon condensation in a doubled Hilbert space, and "gauging out" anyons should hold more generally. Indeed, gauging out a proper subset of anyons $\hat{\mathcal{A}}$ from a parent UMTC $\mathcal{C}$ is nothing but anyon condensation in a doubled Hilbert space[7] with the resulting deconfined anyons $\mathcal{A}$ given by those that have support purely in the ket or bra space. We will show through examples that gauging out Abelian anyons in an otherwise non-Abelian theory is conceptually straightforward. Similarly, Abelian anyons can also be gauged out from parent chiral UMTCs. This suggests the intriguing possibility of realizing non-modular anyon theories by appropriately gauging out anyons from a UMTC. We schematically describe this below, leaving a complete algebraic decription for future work.

Let us assume that we always begin with a pure state that is the ground state of some local, gapped Hamiltonian in 2D. That is, our parent theory has TO characterized by a UMTC $\mathcal{C}$ with a finite set of anyons $\{a\}$. As is well-established by now, in the doubled Hilbert space this corresponds to the doubled TO $\mathcal{C} \times \bar{\mathcal{C}}$, with anyons labeled by the ordered pair $a\bar{b} = (a_+ b_-)$. Note that the TO in the doubled space is nothing but the Drinfeld centre of $\mathcal{C}$: $\mathcal{Z}(\mathcal{C}) = \mathcal{C} \times \bar{\mathcal{C}}$. The theory in the doubled space is then equivalent to that of a string-net model [53], for

[7] This is distinct from anyon condensation in a *physical* bilayer.

which the input theory is the UMTC $\mathcal{C}$. In such a theory, it is always possible to condense excitations of the form (a_+a_-), which are obviously bosonic [50, 54–56]. In the physical Hilbert space, this corresponds to subjecting the initial pure state to local error channels, which can be written in terms of short string operators for $a \in \mathcal{C}$.

In the doubled Hilbert space, maximal decoherence corresponds to conventional anyon condensation [37], whereby any anyons (r_+s_-) that braid non-trivially with (a_+a_-) are confined and, of the resulting anyons, those that differ only up to fusion by (a_+a_-) are identified. For any non-Abelian anyons that remain deconfined, one must also check their fusion rules: if the the vacuum superselection sector appears more than once, then the non-Abelian anyon splits into other deconfined anyons. From our preceding discussion, we know that the resulting mixed-state TO is encoded in the set of anyons with support only on the ket (or bra) space (the Wilson loops of the remainder are frozen into classical ensembles). These are given by the set $\mathcal{A} = \{r \in \mathcal{C} | \mathcal{B}_\theta(a, r) = 1\}$, of which some may be transparent anyons i.e., the resulting anyon theory may be non-modular or even chiral, both of which we have already encountered.

Thus, we can now define gauging out anyons in the same way as before, but in a more general context: starting with a UMTC $\mathcal{C}$ and a proper subset of anyons $\hat{\mathcal{A}}$ to be gauged out, the resulting anyon theory (the code space of the decohered theory) is given by those anyons in $\mathcal{C}$ which braid trivially with those in $\hat{\mathcal{A}}$. Moreover, if any anyons in $\hat{\mathcal{A}}$ are transparent in $\hat{\mathcal{A}}$, they remain transparent in $\mathcal{A}$, which will generically be a braided fusion category (without the modularity restriction). Formally, given a UMTC $\mathcal{C}$ and a proper subset of objects (anyons) $\hat{\mathcal{A}}$ (i.e., a full subcategory of $\mathcal{C}$), the anyon theory $\mathcal{A}$ that results upon gauging out $\hat{\mathcal{A}}$ is given by the centralizer $C_C(\hat{\mathcal{A}})$ of $\hat{\mathcal{A}}$ in $\mathcal{C}$: $\mathcal{A} \equiv \{x \in \mathcal{C} | \mathcal{B}_\theta(x, y) = 1 \, \forall y \in \hat{\mathcal{A}}\}$, which is a braided fusion category (see Ref. [57]). One could in principle then generate another braided fusion category by gauging out anyons from $\mathcal{A}$ and generate a cascade of imTOs by iteratively gauging out anyons.

We believe that this picture for obtaining braided fusion categories from parent UMTCs falls squarely within the general class of *mixed TQFTs* proposed by Zini and Wang in Ref. [58], but where the input to the parent 2+1D Turaev-Viro (TV) type TQFT is always modular. In our context, this restriction is physically motivated since we take as input the ground state of a local Hamiltonian (so the anyon theory is a UMTC) and then subject it to *local* noise. In fact, note that the doubled semion example we previously considered is one where the imTO we obtained belongs to the class of mixed TQFTs considered in Ref. [58]. Specifically, in that case the input was the doubled semion UMTC and the output imTO was the chiral semion UMTC, where we can view the latter as the Reshetikhin-Turaev (RT) TQFT of the former: this supports our claim that braided fusion categories provide a partial classification of imTOs. More generally, if the parent theory is some doubled Chern-Simons theory (which have gapped boundaries to vacuum), one might expect that local error channels will lead to the underlying chiral Chern-Simons TO in the decohered mixed-state—we investigate this in a forthcoming work. For now, we consider some simple examples that go beyond Abelian anyon theories to show the generality of our framework.

A. $\mathbb{Z}_2^{(1)}$ TSSC from Chiral Ising UMTC

The chiral Ising anyon theory consists of the anyons $\{1, \sigma, \psi\}$ which satisfy the fusion algebra $\psi \times \psi = 1$, $\sigma \times \psi = \psi \times \sigma = \sigma$, and $\sigma \times \sigma = 1 + \psi$. Here, ψ is a fermion and σ is an Ising anyon, whose non-integer quantum dimension $\sqrt{2}$ reflects its non-Abelian nature. The topological spin (self-statistics) of the theory are $\theta(\psi) = -1$ and $\theta(\sigma) = e^{i\pi/8}$ from which, combined with the fusion rules, one can derive the non-trivial braiding between ψ and σ: $\mathcal{B}_\theta(\sigma, \psi) = -1$.

A physical Hamiltonian that supports a phase with chiral Ising TO is furnished by the Kitaev honeycomb model [46]. Recently, Ref. [27, 59] considered the honeycomb model in the presence of local ψ errors, which are generated by a local error channel. We will not dwell on details of the lattice model but directly reproduce the results using our general framework. Specifically, we wish to gauge out the ψ fermion: since σ braids non-trivially with ψ, the resulting anyon theory describing the decohered state is simply given by $\{1, \psi\}$, which does not encode any quantum memory but still yields a classical memory and retains a well-defined fermionic excitation. This is in precise agreement with the numerical results of Refs. [27, 59], which also verified the stability of this intrinsically mixed-state TO, which we expect on general grounds from the unbroken strong anomalous 1-form symmetry of $\rho_\mathcal{E}$.

We may also consider the corresponding analysis in the doubled Hilbert space. Here, we are condensing $(\psi_+\psi_-)$ in the doubled Ising Chern-Simons theory $\mathcal{C} \times \bar{\mathcal{C}}$. It is well-known that the condensed phase has the following deconfined excitations: $\{1, \psi_+, \psi_-, \sigma_+\sigma_-\}$ where $\sigma_+\sigma_-$ splits since the vacuum sector appears twice in its fusion rules: $\sigma_+\sigma_- \times \sigma_+\sigma_- = 1 + \psi_+ + \psi_- + \psi_+\psi_-$ (where $\psi_+\psi_- \sim 1$) which is identical to the fusion $(e+m) \times (e+m)$ in the $\mathbb{Z}_2$ Toric code. Thus, the TO in the doubled Hilbert space is a $\mathbb{Z}_2$ gauge theory, but back in the physical Hilbert space, this corresponds to the freezing of σ loops into a classical ensemble while ψ remains a well-defined excitation.

B. Non-modular imTO from Doubled Ising UMTC

Building on the previous example, let us now consider a pure state which belongs to the ground state manifold of the doubled Ising string-net [53]. The anyons in this theory are $\{1, \psi, \sigma\} \times \{1, \bar{\psi}, \bar{\sigma}\}$ with fusion rules that can be inferred from those of the chiral Ising UMTC. Now, suppose we wish to consider $\psi\bar{\psi}$ errors: these are induced by local short-string operators that are explicitly provided in e.g. Ref. [56]. As above, we will not delve into details of the specific lattice Hamiltonian or the short-string operators here as we can directly infer the imTO of the decohered density matrix.

Maximal decoherence of the $\psi\bar{\psi}$ errors is equivalent to gauging out this bosonic anyon. As before, only those excitations that braid trivially with $\psi\bar{\psi}$ remain as deconfined anyons in the resulting decohered state. Thus, the resulting mixed-state TO is given by the set $\mathcal{A} = \{1, \psi, \bar{\psi}, \sigma\bar{\sigma}, \psi\bar{\psi}\}$. Notably, this is distinct from typical anyon condensation of $\psi\bar{\psi}$ in the doubled Ising string-net, where $\psi\bar{\psi}$ disappears into the condensate, ψ and $\bar{\psi}$ are identified, and $\sigma\bar{\sigma}$ splits into Abelian anyons. Decohering $\psi\bar{\psi}$ instead results in a *non-modular* imTO, characterized by the anyons $\mathcal{A}$, amongst which $\psi\bar{\psi}$ is transparent [8]. We can infer the presence of a quantum memory in the logical subsystem of the decohered code space from the presence of non-trivial braiding between the remaining opaque anyons in $\mathcal{A}$.

C. Non-modular imTOs from Doubled $SU(2)_k$ UMTC

As a final example, we can consider doubled $\mathrm{SU}(2)_k$ string-net models, whose lattice models and short-string operators are given in Ref. [54]. Anyons in this theory are labeled by pairs (j_1, j_2) where $j = 0, 1/2, 1, \ldots, k/2$. Let us now subject a ground state of this model to local errors that incoherently proliferate the anyon $(k/2, k/2)$ (which is a boson). In order to read off the resulting imTO in the decohered density matrix, after $(k/2, k/2)$ has been gauged out, we require the braiding relations of this theory. In particular, the braiding between an anyon (j_1, j_2) and $(k/2, k/2)$ is given by $\mathcal{B}_\theta((j_1, j_2), (k_1, k_2)) = (-1)^{2(j_1+j_2)}$. Thus, the resulting imTO is characterized by the anyon theory $\mathcal{A} = \{(j_1, j_2) | j_1 + j_2 \in \mathbb{Z}\}$ with $j_1, j_2 = 0, 1/2, 1, \ldots, k/2$. Of these, $(k/2, k/2)$ is a transparent boson, which is sufficient to conclude that the decohered theory is a *non-modular* anyon theory.

Thus, we can obtain a large family of imTOs by exposing the ground states of string-net models to local error channels, where the decohered code space generically retains logical information i.e., it is a decoherence-free subspace. The presence of non-trivial logical information (or a quantum memory) is encoded in the Wilson-loop algebra (equivalently, the S-matrix of $\mathcal{A}$). Since we have shown that the resulting imTOs can host transparent anyons, we obtain a partial classification of imTO in terms of braided fusion categories.

[8] For the cognoscenti, we note that maximally decohering $\psi\bar{\psi}$ in the doubled Ising string-net results in precisely the same anyon content as in each layer of the Ising cage-net [56]; this suggests that "p-string" condensation [60] may have an interpretation in terms of gauging out certain anyons.

V. HIGHER-FORM SYMMETRY AND NON-UNITARY EXFOLIATION OF WALKER-WANG MODELS

We now place our results in a broader context by characterizing imTO states via their *higher-form* symmetry structure [61, 62], which we have already alluded to above in specific examples, and by relating them to anomalous surface states of 3D pure state TO. First, we recall that q-form symmetries are generated by operators acting on a closed, codimension $q-1$ manifold of spacetime. In the $2+1$-dimensional case, 1-form symmetries thus are both generated by, and act on, one-dimensional loop-like objects. Indeed, in a 2D TO, the Wilson loops associated with (Abelian) anyons may be understood as being generators of 1-form symmetries. In this language, the non-trivial ground state degeneracy—and hence the non-trivial code space—of a TO on the torus is often understood in terms of the spontaneous breaking of these 1-form symmetries. For instance, the Toric code possesses a $\mathbb{Z}_2^e \times \mathbb{Z}_2^m$ 1-form symmetry, generated by the e and m Wilson loops. Like conventional symmetries, 1-form symmetries can be gauged which, in the context of 2D TO, amounts to condensing the corresponding anyon [9]. Thus, a 1-form symmetry is anomalous if the corresponding anyon has non-trivial self-statistics (i.e., is not bosonic). In the $\mathbb{Z}_2$ Toric code, the $\mathbb{Z}_2^e$ and $\mathbb{Z}_2^m$ 1-form symmetries are hence not individually anomalous, as we may gauge either to obtain a trivial state; correspondingly, we may condense either of these anyons. Instead, the 1-form symmetries for e and m have a mixed anomaly, reflecting the non-trivial braiding between e and m, and that we cannot condense f.

In order to extend this analysis to mixed-state order, we must distinguish between *strong* and *weak* symmetries of a density matrix [23, 63, 64]. Given a unitary representation U_g of a symmetry g in some symmetry group G, we say that the density matrix ρ is strongly symmetric under G if for every $g \in G$, $U_g \rho = \rho U_g^\dagger = \rho$. In the doubled space picture, this constraint translates to $U_{g+} |\rho\rangle\rangle = U_{g-}^* |\rho\rangle\rangle = |\rho\rangle\rangle$. Conversely, ρ is weakly symmetric if we only have $U_g \rho U_g^\dagger = \rho$ or, equivalently, $U_{g+} U_{g-}^* |\rho\rangle\rangle = |\rho\rangle\rangle$.

Let us focus on the Toric code first for concreteness. The initial pure state TO density matrix trivially has a *strong* $\mathbb{Z}_2^e \times \mathbb{Z}_2^m$ 1-form symmetry. Working in the doubled space picture, we then see that the $\mathbb{Z}_2^{(0)}$ imTO resulting from e-decoherence still has a strong $\mathbb{Z}_2^e$ 1-form symmetry generated by Wilson loops associated to the deconfined, quantum excitation e_+, but only a *weak* $\mathbb{Z}_2^m$ 1-form symmetry, generated by $m_+ m_-$. Indeed, in each of the examples we have studied, we see that decoherence

[9] The relation between gauging 1-form symmetries and anyon condensation only holds for Abelian anyons; more generally, condensing non-Abelian anyons can be understood in terms of gauging *non-invertible* 1-form symmetries.

has a non-trivial effect on the underlying strong 1-form symmetries of the parent TO. We can thus rephrase our results for generic Abelian TOs in the language of 1-form symmetry: when gauging out an anyon a via decoherence, the resulting imTO density matrix retains *strong* 1-form symmetries for those symmetries generated by anyons which braid trivially with a, while the remaining 1-form symmetries are reduced to *weak* symmetries. In other words, only those 1-form symmetries which do not have a mixed anomaly with the 1-form symmetry generated by Wilson loops of a remain as strong 1-form symmetries, while the remainder are reduced to weak symmetries. As noted previously, in the TSSC framework, the strong 1-form symmetries of the decohered state are manifest in Eq. (6), where the stabilizers may be viewed as closed Wilson loops for the deconfined anyons. This observation further reinforces the idea that the logical subsystem forms a decoherence free subspace under local noise that incoherently proliferates certain anyons, affecting only degrees of freedom in the gauge subsystem.

The 1-form symmetry structure of imTOs provides a useful language with which to characterize the stability of these states to local noise as well as their utility as memories [10]. Regarding the former, the characterization of the decohered mixed-state in terms of strong 1-form symmetries immediately implies that these fixed point density matrices belong to an *imTO phase*, where each state in that phase shares the same 1-form symmetry structure and these states can be connected via strong 1-form symmetry preserving finite-depth LPQCs. Similar to gapped ground states of local Hamiltonians, we also expect stability up to some finite noise threshold under general *local* noise channels since LPQCs satisfy an area-law (i.e., satisfy a Lieb-Robinson bound) and cannot generate arbitrarily long-range correlations that destroy the state at infinitesimally small noise rates [65] (note however that a notion of a "gap" remains lacking in this context). Each strong 1-form symmetry, being equivalent to the presence of an anyon in the theory, moreover implies the existence of non-local operators commuting with the stabilizer group—the corresponding anyon Wilson loop along the non-contractible cycles of the torus.

We may then employ the anomalies of the 1-form symmetries to characterize the structure of the code space. Specifically, if two 1-form symmetries have a mixed anomaly, they give rise to a pair of logical operators and hence a quantum memory, as in the $\mathbb{Z}_4^{(1)}$ TSSC. If a 1-form symmetry has no mixed anomalies but has a $\mathbb{Z}_N$ anomaly with $N > 2$, its corresponding Wilson loops along the two cycles of the torus also yield logical operators and a quantum memory, as in the chiral semion TSSC. Finally, if a 1-form symmetry has no mixed anomalies and at most a $\mathbb{Z}_2$ anomaly (i.e. it is either a boson or fermion), its corresponding non-contractible Wilson loops only yield non-local stabilizers, thus yielding a classical memory. In passing we note that while our focus has been on the implications of strong symmetries, excitations charged under weak 1-form symmetries (hence these are confined), correspond to "fluxes" in the language of TSSCs [31, 36]. Higher form symmetries thus provide a convenient language with which to characterize the TSSC structure of imTOs.

There is also a striking analogy between imTO and anomalous surface states of certain pure state 3D TOs, which suggests potential generalizations of our scheme to other intrinsically mixed-states. As we have emphasized throughout, imTO generally supports chiral and non-modular TO in a purely 2D system. In the context of local gapped Hamiltonians, such states naturally arise at the 2D surfaces of 3D topological orders, specifically those realized in the Walker-Wang (WW) models [38]. These are 3D exactly solvable lattice models which, given a potentially non-modular TO $\mathcal{C}$, realize $\mathcal{C}$ as its surface theory. In particular, if we consider a slab geometry with open boundary conditions in, say, the z direction, one obtains $\mathcal{C}$ on the top surface and $\overline{\mathcal{C}}$ on the bottom surface. If $\mathcal{C}$ is modular, then the bulk has trivial topological order. Conversely, if $\mathcal{C}$ is non-modular, then the bulk is topologically ordered and supports both point-like and loop-like excitations, generated at the ends of Wilson lines and edges of Wilson surfaces, respectively, which braid non-trivially with each other. These loop-like excitations can be absorbed by the surfaces. Importantly, the transparent anyons in $\mathcal{C}$ also correspond to deconfined point-like excitations in the bulk, and so can freely move from the top $\mathcal{C}$ surface, into the bulk, and onto the bottom $\overline{\mathcal{C}}$ surface. Additionally, a "tube-like" Wilson surface stretching between a loop on the top surface and a loop on the bottom surface serves as a symmetry of the ground state, as the loop-like excitations are condensed on the surfaces.

Remarkably, this structure *exactly* parallels that of the vectorized density matrix for an imTO in the doubled Hilbert space, with the ket and bra spaces identified with the top and bottom surfaces of a WW model. Much like the surface states of WW models, the deconfined excitations with support solely on the ket (or bra) space can realize non-modular or chiral TO. The aforementioned weak 1-form symmetries (which act simultaneously on the ket and bra spaces) mirror the effect of the tube-like Wilson surfaces in the WW model, when they terminate on the top and bottom surfaces. Moreover, in the doubled Hilbert space representation of the imTO, the transparent anyons can move freely between the ket and bra spaces, just as the transparent anyons in the WW model can move between the top and bottom surfaces. Indeed, in the doubled Hilbert space, if we condense (a_+a_-), all anyons of the form $(a_+^m 1_-)$ (for integer m) are transparent and are equivalent to anyons of the form $(1_+a_-^m)$ via fusion with the condensate. Thus, at least at the level of analogy, decoherence induced imTO provides a physical means of realizing the anomalous surface states of 3D pure state TO as realized by WW models. This lends fur-

[10] Here, we are specifically referring to stability under *finite-depth* local noise channels.

ther credence to our claim that a partial classification of imTOs should be provided by braided tensor categories, as WW models are classified by the same. In this sense, one may view decoherence as a means of "exfoliating" surface states of a 3D TO into a purely 2D mixed-state using only a finite-depth LPQCs. One may imagine applying this non-unitary exfoliation procedure to isolate anomalous surface states of other exotic pure states, an avenue we intend to pursue in future work.

VI. DISCUSSION

In this work, we have proposed a framework for classifying a large family of intrinsically mixed-state topological orders, obtained via local decoherence of parent pure state topological order. We demonstrated that local decoherence, previously shown to correspond to anyon condensation in the vectorized density matrix obtained via the Choi-Jamiołkowski isomorphism, in fact provides a physical mechanism for the gauging out of anyons. As a consequence, for parent Abelian topological order, the resulting imTO is naturally characterized as a topological subsystem code and thus classified in terms of (degenerate) braided tensor category theory. Hence, 2D pure state TOs provide resource states, under *local decoherence*, for the preparation of non-modular and even chiral states. We also illustrated that this procedure naturally extends to non-Abelian states, though the resulting imTOs are no longer identified as TSSCs. Finally, we characterized the family of imTOs under consideration by their strong and weak 1-form symmetries, and demonstrated that they correspond to the anomalous surface states of 3D pure state topological orders, to wit, Walker-Wang models. This provides a natural interpretation of decoherence as a means of non-unitarily exfoliating surface states of topological states in one higher dimension, a perspective which may find use in generating other classes of intrinsically mixed phases of matter. Our general framework provides many exciting avenues for further exploration, some of which we address in forthcoming work.

A pressing issue is to characterize our family of imTOs via their entanglement structure. While the entanglement entropy has previously been studied in mixed-state TO [51, 66], a more natural probe of entanglement in mixed-states is provided by the entanglement negativity which, unlike the entanglement entropy, is a good measure of quantum correlations in a mixed state [67–71]. In pure state TO, the negativity has been shown to receive universal contributions which are sensitive to the modular data of the TO (namely, the total quantum dimension) [12, 72–77]. This topological entanglement negativity (TEN) has also been shown to be sensitive to the breakdown of TO in thermal states [12, 78], which the entanglement entropy does not accurately reflect. Since we have shown that imTO is generally characterized by *non-modular* anyon theories, it is an intriguing question as to what universal data the TEN captures in these states. In one specific instance, Ref. [27] distinguished between the $\mathbb{Z}_2^{(0)}$ and $\mathbb{Z}_2^{(1)}$ imTOs (obtained via decoherence of the $\mathbb{Z}_2$ Toric code) by the respective absence and presence of topological contributions to the negativity. Recalling that these two states correspond to quantum condensates of bosonic and fermionic loops, it is tempting to conjecture that the TEN remains sensitive to the spins of the underlying deconfined anyon excitations. In a forthcoming work, we address more comprehensively the connection between TEN and the braided tensor category structure of imTO. It would likewise be interesting to understand novel decoherence induced negativity transitions [15, 27, 79, 80] that may result from (competition between) the different decoherence channels discussed in this work.

While the entanglement negativity is a good measure of bipartite entanglement, it has recently been understood that pure state TO can be more finely characterized by its *tripartite* entanglement structure [81–88]. Specifically, it has been argued that chiral TO supports tripartite entanglement beyond that of the Greenberger-Horne-Zeilinger (GHZ) type [81, 82, 88]. As we have shown, decoherence of the double semion state can induce an imTO characterized by the chiral semion TSSC. Intriguingly, this suggests that decoherence has transmuted one form of many-body tripartite entanglement (i.e. GHZ-like entanglement) into another. It is conceivable that non-unitary processes may stabilize patterns of multipartite entanglement in many-body systems which do not arise naturally in the ground states of local gapped Hamiltonians. Understanding in more detail the multipartite entanglement of imTOs and the transmutation between different classes of entanglement via non-unitary processes promises to be a fruitful direction for further research. In a similar vein, Ref. [89] recently argued that a state that is strongly symmetric with respect to an anomalous 0-form symmetry in 2D must be 4-partite non-separable. It is an intriguing question whether similar constraints exist for systems respecting an anomalous strong 1-form symmetry.

In the spirit of fleshing out the structure of the family of imTOs we have obtained, an important avenue for further development is a more thorough classification of non-Abelian imTO. While we have demonstrated that the process of gauging out via decoherence extends naturally to the non-Abelian case, we do not yet have a comprehensive understanding of the algebraic structure of the resulting imTO, although we have provided compelling evidence that the appropriate mathematical framework is that of braided fusion categories. To that end, it would be prudent to understand more fully the connections with the mixed TQFTs proposed by Zini and Wang [58]. In particular, it remains to be understood whether the class of mixed TQFTs proposed in that work can be realized in a physical setting i.e., by exposing some parent state to *local* noise. On a related note, it would be interesting to understand whether there exists a non-equilibrium, continuum field theory formulation for describing generic imTO, most likely in the language of the Schwinger-Keldysh path integral.

The general framework we have developed also has potential exciting applications beyond the context of 2D mixed-state topological order. An obvious extension is to incorporate the ground states of 3D local gapped Hamiltonians into our framework and study the resulting decohered mixed-states. Our picture of decoherence in d-dimensions as non-unitary exfoliation of anomalous surface states of $d+1$-dimensional systems suggests a route towards realizing anomalous 3D topological orders [90–92] via local noise channels, where these states generically host transparent loop-like excitations (in analogy with the transparent anyons in our imTOs). We also expect that 3D fracton orders can be prepared by subjecting a 3D stack of 2D TO layers to an appropriate noise channel. Secondly, it is natural to consider the possibility of replacing correlated decoherence with correlated *disorder*; as in the context of intrinsically average SPTs stabilized by disorder [18, 24], one may expect intrinsically average TO, the classification of which would likely be similar to, but distinct from, that of imTO.

We conclude by commenting on practical implications of our work for imTO in open quantum systems. Currently, preparing states with chiral TO requires sequential quantum circuits [35] and, although unproven, it is widely believed that no finite-depth quantum circuit can disentangle such states from the surface of a 3D WW model; for instance, Ref. [34] proved that either there exists a commuting projector Hamiltonian for the 2D chiral semion TO (which Ref. [1] argued should not exist) or that the circuit that disentangles this TO from the surface of a 3D WW is not finite-depth. On the other hand, single shot measurement and feedback protocols for preparing ground states with Abelian TO have recently been proposed [47]. Our results thus open the door towards the dissipative preparation of chiral TOs using finite-depth LPQCs: given a parent TO that can be prepared using a single-shot measurement and feedback circuit, we have shown that appropriately engineering a locally correlated noise channel can lead to chiral imTO. Surprisingly, since the doubled state in our construction can always be represented as a fixed point projected entangled pair state (PEPS) with finite bond dimension as we have a topological stabilizer code in the doubled space, our work also indicates the existence of a fixed point projected entangled pair operator (PEPO) representation for density matrices exhibiting chiral topological order (with finite bond dimension). This is an intriguing implication, as it is widely believed—though not proven—that there do not exist exact PEPS representations for (interacting) chiral topological pure states.

More generally, we can imagine beginning from a topologically ordered pure state that can be efficiently prepared using existing protocols. Exposing such a state to noise channels will generically decrease its encoded logical information (as in each of our examples), such that the resulting decohered state represents a genuinely distinct phase of matter [13, 19]. Heuristically, this is clear since no quasi-local recovery map can reconstruct the logical information stored in the parent state. We can then imagine a cascade of descendant TOs that be prepared from a parent state by carefully selecting error channels that gauge out anyons in a prescribed manner. This suggests a classification of mixed-state phases of matter in terms of the complexity of their code space, whereby no state in the sequence can recover the information of a precursor via LPQCs. Understanding the appropriate equivalence relation on the space of mixed-states is a question we intend to address in a future work.

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