Infinite horizon McKean-Vlasov FBSDEs and applications to mean field control problems

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March 28, 2024

Abstract

In this paper, we study a class of infinite horizon fully coupled McKean-Vlasov forward-backward stochastic differential equations (FBSDEs). We propose a generalized monotonicity condition involving two flexible functions. Under this condition, we establish the well-posedness results for infinite horizon McKean-Vlasov FBSDEs by the method of continuation, including the unique solvability, an estimate of the solution, and the related continuous dependence property of the solution on the coefficients. Based on the solvability result, we study an infinite horizon mean field control problem. Moreover, by choosing appropriate form of the flexible functions, we can eliminate the different phenomenon between the linear-quadratic (LQ) problems on infinite horizon and finite horizon proposed in Wei and Yu (SIAM J. Control Optim. 59: 2594–2623, 2021).

Key words: McKean-Vlasov FBSDE, infinite horizon, monotonicity condition, mean field control.
MSC-classification: 60H10, 93H20, 49N80.

1 Introduction

Mean field control problems have, in the recent years, drawn the attention of the applied mathematics community. Being an extension of the classical optimal control, it has been studied from different angles. The first one is to use the so-called dynamic programming principle (DPP). Compared with the classical control problems, the presence of the distribution of the controlled process in the coefficients brings additional difficulty. One can refer to [3, 20, 21] for related research on establishing DPP for mean field control problems. A second way is based on the Pontryagin's maximum principle. This approach has been successfully developed in many literature, see [8, 6, 1, 9]. Recently, Bayraktar and Zhang [4] studied a mean field control problem on infinite horizon using FBSDE techniques, where the state volatility is a constant.

In this paper, we consider the following infinite horizon mean field control problem: Minimize

$$J(\alpha) := \mathbb{E}\left[\int_0^\infty e^{2Kt} f\left(t, X_t, \mathcal{L}\left(X_t\right), \alpha_t\right) dt\right] \tag{1.1}$$

subject to

$$\begin{cases} dX_t = b\left(t, X_t, \mathcal{L}\left(X_t\right), \alpha_t\right) dt + \sigma(t, X_t, \mathcal{L}\left(X_t\right), \alpha_t) dW_t, \quad t \in [0, \infty), \\ X_0 = \xi, \end{cases} \tag{1.2}$$

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where $K \in \mathbb{R}$ is a constant, W_t is a d-dimensional Brownian motion, $(b, f, \sigma) : [0, \infty) \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \times A \to \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{n\times d}$ are measurable mappings, and $\alpha = (\alpha_t)_{t\geq 0}$ is a progressively measurable process with values in a measurable space $(A, \mathcal{A})$. It can be noticed that we introduce a term e^{2Kt} to "suppress" the growth of the running cost function $f(t, X(t), \mathcal{L}(X_t), \alpha_t)$, thereby obtaining the well-posedness of the cost functional (1.1). In the literature [25, 24, 4], they used the same technique when studying infinite horizon control problems. As indicated in [25], the choice of parameter K depends on the properties of the coefficients. Specifically, if the monotonicity of b with respect to x is sufficiently negative, the parameter K may take a positive real number. Therefore, the appearance of parameter K is friendly with various models and we can always identify the parameter K according to the the intrinsic properties of coefficients and formulate the control problems.

Following the probabilistic approach to finite horizon mean field control problems, we establish an appropriate form of the Pontryagin's maximum principle for the infinite horizon case and then the infinite horizon mean field control problem (1.1)-(1.2) is reduced to solving an infinite horizon McKean-Vlasov FBSDE, which is also called a Hamiltonian system. Motivated by this, we aim to establish the well-posedness of a more general form of infinite horizon coupled McKean-Vlasov FBSDEs. In detail, we consider the following infinite horizon McKean-Vlasov FBSDE:

$$\begin{cases} dX_t = B\left(t, X_t, Y_t, Z_t, \mathcal{L}\left(X_t, Y_t, Z_t\right)\right) dt + \sigma\left(t, X_t, Y_t, Z_t, \mathcal{L}\left(X_t, Y_t, Z_t\right)\right) dW_t, \quad t \in [0, \infty), \\ dY_t = F\left(t, X_t, Y_t, Z_t, \mathcal{L}\left(X_t, Y_t, Z_t\right)\right) dt + Z_t dW_t, \quad t \in [0, \infty), \\ X_0 = \xi, \end{cases} \tag{1.3}$$

where W_t is a d-dimensional Brownian motion, X, Y, Z take values in $\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^{m\times d}$, B, σ, F are progressively measurable functions with appropriate dimensions, ξ is an $\mathcal{F}_0$-measurable square integrable random variable and $\mathcal{L}\left(X_t, Y_t, Z_t\right)$ denote the probability measures induced by (X_t, Y_t, Z_t). We aim to look for solutions (X, Y, Z) to (1.3) in $L^{2,K}_{\mathbb{F}}(0, \infty; \mathbb{R}^{n+m+m\times d})$, where $K \in \mathbb{R}$ and $L^{2,K}_{\mathbb{F}}(0, \infty; \mathbb{R}^n)$ is the Hilbert space of all $\mathbb{R}^n$-valued adapted processes ν_t such that

$$\mathbb{E}\int_0^\infty |e^{Kt}\nu_t|^2 dt < \infty. \tag{1.4}$$

Finite horizon classical FBSDEs were first investigated by Antonelli [2] and a local existence and uniqueness result was obtained. For the global solvability results, there exist four main methods: the method of contraction mapping used by Pardoux and Tang [17], four-step scheme approach introduced by Ma, Protter and Yong [14], continuation method initiated by Hu and Peng [11], Peng and Wu [19] and improved by Yong [29, 30], random decoupling field introduced by Ma et al. [15] and extended by Fromm and Imkeller [10] and Hua and Luo [12]. For more detailed results on finite horizon FBSDEs, one can refer to the monograph by Ma and Yong [16]. The research on finite horizon mean field FBSDEs builds upon these methods and further developments have been made, see [5, 8, 9, 7, 13].

In [18], Peng and Shi, for the first time, investigated fully coupled classical infinite horizon FBSDEs by the method of continuation. Later, Wu [26] studied this problem in some different monotonicity framework. Yin [27, 28] studied the same issue by the method of contraction mapping. Shi and Zhao [23] extended the results [18] to a larger space and studied the connection of infinite horizon FBSDEs with corresponding PDEs. Besides, Yu [31] investigated infinite horizon FBSDEs driven by both Brownian motions and Poisson processes. Bayraktar and Zhang [4] extended the infinite horizon FBSDEs results to a type of infinite horizon McKean-Vlasov FBSDEs where the coefficients B, F depend on $(X, Y, \mathcal{L}(X, Y))$ and the coefficient σ is a constant. Recently, Wei, Xu and Yu [24] studied a kind of infinite horizon linear mean field type FBSDEs with jumps.

In this paper, we establish an existence and uniqueness result and a pair of estimates for the solutions to infinite horizon McKean-Vlasov FBSDE (1.3) with the method of continuation. The key point of the method of continuation lies in proposing suitable motononicity condition. For the finite horizon McKean-Vlasov FBSDEs, Bensoussan et al. [5] extended the results in Peng and Wu [19] to include mean-field terms by proposing the following motononicity condition:

$$\begin{aligned}&\mathbb{E}\left[\langle\Gamma\left(t;\Theta_1,\mathcal{L}(\Theta_1)\right)-\Gamma\left(t;\Theta_2,\mathcal{L}(\Theta_2)\right),\Theta_1-\Theta_2\rangle\right]\\ \leq&-\beta_1\mathbb{E}\left[\left|G(X_1-X_2)\right|^2\right]-\beta_2\left(\mathbb{E}\left[\left|G^\top(Y_1-Y_2)\right|^2\right]+\mathbb{E}\left[\left|G^\top(Z_1-Z_2)\right|^2\right]\right),\end{aligned}$$

for any $\Theta_1=(X_1,Y_1,Z_1),\Theta_2=(X_2,Y_2,Z_2)\in L^2(\mathbb{R}^{n+m+m\times d})$, where $\Gamma=(G^\top F,GB,G\sigma)$, G is a matrix and β_1,β_2 are non-negative constants with $\beta_1+\beta_2>0$. Recently, Reisinge et al. [22] proposed a generalized monotonicity condition by introducing two flexible measurable functions ϕ_1 and ϕ_2:

$$\begin{aligned}&\mathbb{E}\left[\langle\Gamma\left(t;\Theta_1,\mathcal{L}(\Theta_1)\right)-\Gamma\left(t;\Theta_2,\mathcal{L}(\Theta_2)\right),\Theta_1-\Theta_2\rangle\right]\\ &\leq-\beta_1\phi_1\left(X_1,X_2\right)-\beta_2\phi_2\left(t,\Theta_1,\Theta_2,\mathcal{L}(\Theta_1),\mathcal{L}(\Theta_2)\right).\end{aligned}\tag{1.5}$$

For the infinite horizon classical FBSDEs, Wei and Yu [25] proposed the following domination-monotonicity condition:

$$\begin{cases}|g(s,x,y_1,z_1)-g(s,x,y_2,z_2)|\leq\dfrac{1}{\nu}|A(s)(x_1-x_2)|,\\ |F(s,x,y_1,z_1)-F(s,x,y_2,z_2)|\leq\dfrac{1}{\mu}|B(s)(y_1-y_2)+C(s)(z_1-z_2)|,\\ \langle\Gamma(s,\theta_1)-\Gamma(s,\theta_2),\theta_1-\theta_2\rangle+2K\langle x_1-x_2,y_1-y_2\rangle\\ \quad\leq-\nu|A(s)(x_1-x_2)|^2-\mu|B(s)(y_1-y_2)+C(s)(z_1-z_2)|^2,\end{cases}\tag{1.6}$$

for any $\theta_1=(x_1,y_1,z_1),\ \theta_2=(x_2,y_2,z_2)\in\mathbb{R}^{n+n+n\times d}$, where $g=B,\sigma$, and $A(\cdot),B(\cdot),C(\cdot)$ are three bounded matrix-valued stochastic processes and ν,μ are non-negative constants with $\nu+\mu>0$. Motivated by above conditions, we introduce an infinite horizon generalized monotonicity condition involving two flexible functions ϕ_1 and ϕ_2 (see Assumption (H2)(i)). Moreover, different from finite horizon case, two additional monotonicity conditions for the coefficients B and F are proposed (see Assumption (H2)(ii)).

Our work is closely related to [4] and [25]. We provide a comparison with their results and summarize our main innovations as follows:

(i) We study a more general coupled (McKean-Vlasov) FBSDE in comparison with [4] and [25].

(ii) The introduction of the two functions ϕ_1 and ϕ_2 makes our conditions more general and flexible than those considered in [4] and [25], thereby improving application scope of our solvability results. In particular, by selecting appropriate functions ϕ_1 and ϕ_2, our conditions can be reduced to those proposed in [4] and [25] (see Remark 3.2 (ii)).

(iii) As mentioned above, we need to identify the parameter K based on the intrinsic properties of the coefficients, which is an additional consideration compared to finite horizon situation. When studying infinite horizon mean field control problems, under our generalized monotonicity condition, once the mean field control problem is well-defined for parameter K determined by the coefficients, then it is solvable without additional constraint on K. In particular, we can solve the mean field control problem considered in [4] under a rather weaker constraint on K (see Remark 4.7). Moreover, for stochastic LQ control problems on infinite horizon, [25] stated that there exists a phenomenon that whether the cross term coefficient $S(\cdot)$ in the cost functional is equal to zero or not may bring different results to the solvalibility of the LQ problem. With help of our theoretical result of FBSDE, we can eliminate this phenomenon under strictly convex condition (see Remark 4.8).

The rest of the paper is organized as follows. In section 2, we present the necessary notations, concepts and study the well-posedness of the infinite horizon McKean-Vlasov SDEs and McKean-Vlasov BSDEs as a basis of the following study. In section 3, under the infinite horizon generalized monotonicity condition, we obtain the existence, uniqueness and related estimates of the solutions for the infinite horizon McKean-Vlasov FBSDEs with the method of continuation. Finally, in section 4, we investigate an infinite horizon mean field control problem by applying the obtained FBSDE result.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which is defined a d-dimensional Brownian motion W_t, and $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ be the natural filtration of W augmented with an independent σ-algebra $\mathcal{F}_0$. For any given $n, m \in \mathbb{N}$ and $x \in \mathbb{R}^n$, we denote by $\mathbb{I}_n$ the $n\times n$ identity matrix, by $\mathbb{R}^{n\times m}$ the Euclidean space of all $(n \times m)$ real matrices, especially, $\mathbb{R}^n = \mathbb{R}^{n\times 1}$, by 0_n the zero element of $\mathbb{R}^n$ and by δ_x the Dirac measure supported at x. We denote $\langle \cdot, \cdot\rangle$ and $|\cdot|$ to be the respective usual inner product and norm in Euclidean space, and for any $A, B \in \mathbb{R}^{n\times m}$, we define $\langle A, B\rangle \triangleq \operatorname{tr}(A^\top B)$, $|A| = \{\operatorname{tr}(A^\top A)\}^{\frac{1}{2}}$, where the superscript $\top$ denotes the transpose of a vector or matrix. In this paper we use the operator norm of matrices:

$$\|A\| := \sup_{0\neq x\in\mathbb{R}^m} \frac{|Ax|}{|x|}, \quad \text{for any } A \in \mathbb{R}^{n\times m}.$$

Now we introduce some spaces which will be used in our following analysis. For any $t \in [0, \infty)$ and constant $K \in \mathbb{R}$,

- $L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$ is the set of $\mathbb{R}^n$-valued $\mathcal{F}_t$-measurable random variables ξ such that

$$\|\xi\|_{L^2} := \mathbb{E}\left[|\xi|^2\right]^{\frac{1}{2}} < \infty;$$

- $L^{2,K}_{\mathbb{F}}(t, \infty; \mathbb{R}^n)$ is that set of $\mathbb{R}^n$-valued $\mathbb{F}$-progressively measurable processes $f(\cdot)$ such that

$$\|f(\cdot)\|_K := \mathbb{E}\left[\int_t^\infty \left|e^{Ks} f(s)\right|^2 ds\right]^{\frac{1}{2}} < \infty; \tag{2.1}$$

- $L^\infty(t, \infty; \mathbb{R}^{n\times m})$ is the set of all Lebesgue measurable functions $A : [t, \infty) \to \mathbb{R}^{n\times m}$ such that

$$\|A(\cdot)\|_\infty := \operatorname{esssup}_{s\in[t,\infty)} \|A(s)\| < \infty.$$

Clearly, for any $K_1 < K_2$, we have $L^{2,K_2}_{\mathbb{F}}(t, \infty; \mathbb{R}^n) \subset L^{2,K_1}_{\mathbb{F}}(t, \infty; \mathbb{R}^n)$, i.e., the sequence of spaces $\left\{L^{2,K}_{\mathbb{F}}(t, \infty; \mathbb{R}^n)\right\}_{K\in\mathbb{R}}$ is decreasing in K.

In the sequel, we will use the notation $\mathcal{L}(\Theta)$ to denote the law of the random variable Θ. Let $\mathcal{W}_2$ denote 2-Wassertein's distance on $\mathcal{P}_2(\mathbb{R}^n)$ defined by

$$\mathcal{W}_2(\mu_1, \mu_2) \triangleq \inf\left\{\left[\int_{\mathbb{R}^n\times\mathbb{R}^n} |x-y|^2 \pi(dx, dy)\right]^{\frac{1}{2}}, \pi \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n) \text{ with marginals } \mu_1 \text{ and } \mu_2\right\}. \tag{2.2}$$

It is obvious from its definition that

$$\mathcal{W}_2(\mu_1, \mu_2) \leq \mathbb{E}\left[|X_1 - X_2|^2\right]^{\frac{1}{2}},$$

where X_1 and X_2 are n-dimensional random vectors that follow the distributions μ_1 and μ_2 respectively.

For a function defined on space of measures, its Lipschitz continuity and differentiability upon the measure variable μ is understood in the sense of 2-Wassertein distance and L-differentiability, respectively.

Now we briefly introduce the structure of L-derivative for functions defined on space of measures and we refer the readers to [9, Chapter 5] for details. Let $\tilde{\Omega}$ be a Polish space and $\tilde{\mathbb{P}}$ an atomless measure over $\tilde{\Omega}$. The notion of differentiability is based on the lifting of functions $\mathcal{P}_2\left(\mathbb{R}^d\right) \ni \mu \mapsto H(\mu)$ into functions $\tilde{H}$ defined on the Hilbert space $L^2(\tilde{\Omega};\mathbb{R}^d)$ over some probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ by setting $\tilde{H}(\tilde{X}) = H(\tilde{\mathbb{P}}_{\tilde{X}})$ for $\tilde{X} \in L^2(\tilde{\Omega};\mathbb{R}^d)$. Then a function H is said to be differentiable at $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ if there exists a random variable $\tilde{X}_0$ with law μ_0 such that the lifted function $\tilde{H}$ is Fréchet differentiable at $\tilde{X}_0$. Whenever this is the case, the Fréchet derivative of $\tilde{H}$ at $\tilde{X}_0$ can be viewed as an element of $L^2(\tilde{\Omega};\mathbb{R}^d)$, denoted by $D\tilde{H}(\tilde{X}_0)$, by identifying $L^2(\tilde{\Omega};\mathbb{R}^d)$ and its dual. It can be shown that there exists a measurable function : $\partial_\mu H(\mu_0) : \mathbb{R}^d \to \mathbb{R}^d$ such that $\partial_\mu H(\mu_0)(\tilde{X}_0) = D\tilde{H}(\tilde{X}_0)$, $\mathbb{P}$-a.s. Therefore, we define the derivative of H at μ_0 as the measurable function $\partial_\mu H(\mu_0)$, which satisfies

$$H(\mu) = H\left(\mu_0\right) + \tilde{\mathbb{E}}\left[\partial_\mu H\left(\mu_0\right)(\tilde{X}_0)\cdot(\tilde{X}-\tilde{X}_0)\right] + o(\|\tilde{X}-\tilde{X}_0\|_2),$$

where $\mathcal{L}(\tilde{X}) = \mu, \mathcal{L}(\tilde{X}_0) = \mu_0$.

As a basis of the investigation of infinite horizon McKean-Vlasov FBSDEs, we give the well-posedness of infinite horizon McKean-Vlasov SDEs and infinite horizon McKean-Vlasov BSDEs in the rest of this section. The corresponding classical infinite SDEs and BSDEs have been studied in [25] and it can be observed that our conditions can degenerate to their conditions when there are no mean field terms.

2.1 Infinite horizon McKean-Vlasov SDEs

For $t \in [0,\infty)$, consider the following infinite horizon McKean-Vlasov SDE:

$$\begin{cases} dX_s = b\left(s, X_s, \mathcal{L}(X_s)\right) ds + \sigma\left(t, X_s, \mathcal{L}(X_s)\right) dW_s, \quad s \in [t,\infty), \\ X_t = x_t, \end{cases} \tag{2.3}$$

where $b : [t,\infty)\times\Omega\times\mathbb{R}^n\times\mathcal{P}_2(\mathbb{R}^n) \to \mathbb{R}^n$ and $\sigma : [t,\infty)\times\Omega\times\mathbb{R}^n\times\mathcal{P}_2(\mathbb{R}^n) \to \mathbb{R}^{n\times d}$ are measurable functions. We introduce the following assumptions.

Assumption 2.1. *(i) $x_t \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$. For any $x \in \mathbb{R}^n$, $\mu \in \mathcal{P}_2(\mathbb{R}^n)$, the processes $b(\cdot,x,\mu)$ and $\sigma(\cdot,x,\mu)$ are $\mathbb{F}$-progressively measurable. Moreover, there exists a constant $K \in \mathbb{R}$ such that $b(\cdot,0,\delta_{0_n}) \in L^{2,K}_{\mathbb{F}}(t,\infty;\mathbb{R}^n)$ and $\sigma(\cdot,0,\delta_{0_n}) \in L^{2,K}_{\mathbb{F}}\left(t,\infty;\mathbb{R}^{n\times d}\right)$.*
(ii) The functions $b(t,x,\mu)$, $\sigma(t,x,\mu)$ are uniformly Lipschitz in (x,μ), i.e., there exist positive constants l_{bx}, $l_{b\mu}$, $l_{\sigma x}$, $l_{\sigma\mu}$ such that for any $x,x' \in \mathbb{R}^n$, $\mu,\mu' \in \mathcal{P}_2(\mathbb{R}^n)$ and almost all $(s,\omega) \in [t,\infty)\times\Omega$,

$$\begin{aligned} |b(s,x,\mu) - b(s,x',\mu')| &\le l_{bx}|x-x'| + l_{b\mu}\mathcal{W}_2\left(\mu,\mu'\right), \\ |\sigma(s,x,\mu) - \sigma(s,x',\mu')| &\le l_{\sigma x}|x-x'| + l_{\sigma\mu}\mathcal{W}_2\left(\mu,\mu'\right). \end{aligned} \tag{2.4}$$

(iii) There exists a constant $\kappa_x \in \mathbb{R}$ such that for any $\mu \in \mathcal{P}_2(\mathbb{R}^n)$, $x,x' \in \mathbb{R}^n$ and almost all $(s,\omega) \in [t,\infty)\times\Omega$, it holds that

$$\langle x-x', b(s,x,\mu) - b\left(s,x',\mu\right)\rangle \le -\kappa_x\left|x-x'\right|^2. \tag{2.5}$$

It follows from the classical theory of McKean-Vlasov SDEs on finite horizon (see [9]), under Assumption 2.1 (i) (ii), McKean-Vlasov SDE (2.3) admits a unique solution on $[t,\infty)$. Furthermore, similar with the proof of [31, Proposition 2.1]), we can easily get the following result.

Lemma 2.2. *Let Assumption 2.1 (i) (ii) hold. If the solution X to McKean-Vlasov SDE (2.3) belongs to $L^{2,K}_{\mathbb{F}}(t,\infty;\mathbb{R}^n)$, then we have*

$$\lim_{T\to\infty}\mathbb{E}\left[\left|e^{KT}X_T\right|^2\right] = 0. \tag{2.6}$$

Now, we give the main result for McKean-Vlasov SDE (2.3) as follows.

Lemma 2.3. *Let Assumption 2.1 holds. We further assume that* $K < \kappa_x - \frac{(l_{\sigma x}+l_{\sigma\mu})^2}{2} - l_{b\mu}$*. Then the solution* X *to McKean-Vlasov SDE (2.3) belongs to* $L^{2,K}_{\mathbb{F}}(t,\infty;\mathbb{R}^n)$*. Moreover, for any* $\varepsilon>0$*, we have the following estimate:*

$$
\begin{aligned}
&\left(2\kappa_x - 2K - 2l_{b\mu} - (l_{\sigma x}+l_{\sigma\mu})^2 - 3\varepsilon\right)\mathbb{E}\int_t^\infty \left|e^{Ks}X_s\right|^2 ds \\
&\quad\le \mathbb{E}\left\{\left|e^{Kt}x_t\right|^2 + \int_t^\infty \left[\frac{1}{\varepsilon}\left|e^{Ks}b(s,0,\delta_{0_n})\right|^2 + \left(1+\frac{l_{\sigma x}^2+l_{\sigma\mu}^2}{\varepsilon}\right)\left|e^{Ks}\sigma(s,0,\delta_{0_n})\right|^2\right]ds\right\}.
\end{aligned}
\tag{2.7}
$$

Proof. For any $T>t$, applying Itô's formula to $\left|X_s e^{Ks}\right|^2$ on the interval $[t,T]$ yields

$$
\begin{aligned}
&\mathbb{E}\left\{\left|X_T e^{KT}\right|^2 - 2K\int_t^T \left|X_s e^{Ks}\right|^2 ds\right\} \\
&= \mathbb{E}\left\{\left|x_t e^{Kt}\right|^2 + \int_t^T \left[2\langle X_s, b(s,X_s,\mathcal{L}(X_s))\rangle + \left|\sigma\left(s,X_s,\mathcal{L}(X_s)\right)\right|^2\right] e^{2Ks}ds\right\} \\
&\le \mathbb{E}\left\{\left|x_t e^{Kt}\right|^2 + \int_t^T \left[2\left\langle X_s, b(s,X_s,\mathcal{L}(X_s)) - b(s,0,\mathcal{L}(X_s)) + b(s,0,\mathcal{L}(X_s)) - b(s,0,\delta_{0_n})\right\rangle\right.\right. \\
&\qquad \left.\left. + 2|X_s||b(s,0,\delta_{0_n})| + \left(|\sigma(s,X_s,\mathcal{L}(X_s)) - \sigma(s,0,\delta_{0_n})| + |\sigma(s,0,\delta_{0_n})|\right)^2\right] e^{2Ks}ds\right\}.
\end{aligned}
$$

By the monotonicity condition of b and Lipschitz condition of b and σ, we have

$$
\begin{aligned}
&\mathbb{E}\left\{\left|X_T e^{KT}\right|^2 - 2K\int_t^T \left|X_s e^{Ks}\right|^2 ds\right\} \\
&\le \mathbb{E}\left\{\left|x_t e^{Kt}\right|^2 + \int_t^T \left[-2\kappa_x|X_s|^2 + 2l_{b\mu}|X_s|\sqrt{\mathbb{E}[|X_s|^2]} + 2|X_s||b(s,0,\delta_{0_n})|\right.\right. \\
&\qquad \left.\left. + \left(l_{\sigma x}|X_s| + l_{\sigma\mu}\sqrt{\mathbb{E}[|X_s|^2]} + |\sigma(s,0,\delta_{0_n})|\right)^2\right] e^{2Ks}ds\right\}.
\end{aligned}
$$

For any $\varepsilon>0$, with the help of the inequality $2ab \le \varepsilon|a|^2 + (1/\varepsilon)|b|^2$, we derive

$$
\begin{aligned}
&\mathbb{E}\left\{\left|X_T e^{KT}\right|^2 - 2K\int_t^T \left|X_s e^{Ks}\right|^2 ds\right\} \\
&\le \mathbb{E}\left\{\left|x_t e^{Kt}\right|^2 + \int_t^T \left[\left(l_{\sigma x}^2 + l_{\sigma\mu}^2 + 2l_{b\mu} + 2l_{\sigma x}l_{\sigma\mu} + 3\varepsilon - 2\kappa_x\right)|X_s|^2 + \frac{1}{\varepsilon}|b(s,0,\delta_{0_n})|^2\right.\right. \\
&\qquad \left.\left. + \left(1+\frac{l_{\sigma x}^2+l_{\sigma\mu}^2}{\varepsilon}\right)|\sigma(s,0,\delta_{0_n})|^2\right] e^{2Ks}ds\right\}.
\end{aligned}
$$

Let $T\to\infty$, thanks to Lemma 2.2, we have

$$
\begin{aligned}
&\left(2\kappa_x - 2K - 2l_{b\mu} - (l_{\sigma x}+l_{\sigma\mu})^2 - 3\varepsilon\right)\mathbb{E}\int_t^\infty \left|X_s e^{Ks}\right|^2 ds \\
&\quad\le \mathbb{E}\left\{\left|x_t e^{Kt}\right|^2 + \int_t^\infty \left[\frac{1}{\varepsilon}\left|b(s,0,\delta_{0_n})e^{Ks}\right|^2 + \left(1+\frac{l_{\sigma x}^2+l_{\sigma\mu}^2}{\varepsilon}\right)\left|\sigma(s,0,\delta_{0_n})e^{Ks}\right|^2\right]ds\right\}.
\end{aligned}
\tag{2.8}
$$

The condition $K < \kappa - \frac{(l_{\sigma x}+l_{\sigma\mu})^2}{2} - l_{b\mu}$ implies the existence of the number ε such that $3\varepsilon \in \left(0, \kappa - \frac{(l_{\sigma x}+l_{\sigma\mu})^2}{2} - l_{b\mu} - K\right)$. Therefore, the estimate (2.8) and Assumption 2.1 (i) imply that the solution $X \in L^{2,K}_{\mathbb{F}}(t,\infty;\mathbb{R}^n)$. $\square$

2.2 Infinite horizon McKean-Vlasov BSDEs

For $t \in [0,\infty)$, we introduce the following infinite horizon McKean-Vlasov BSDE:

$$dY_s = f(s, Y_s, Z_s, \mathcal{L}(Y_s), \mathcal{L}(Z_s))ds + Z_s dW_s, \quad s \in [t,\infty), \tag{2.9}$$

where $f : [t,\infty)\times\Omega\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathcal{P}_2(\mathbb{R}^m)\times\mathcal{P}_2(\mathbb{R}^{m\times d}) \to \mathbb{R}^m$ is a measurable function and satisfies the following assumption.

Assumption 2.4. *(i) For any $y\in\mathbb{R}^m$, $z\in\mathbb{R}^{m\times d}$, $\mu\in\mathcal{P}_2(\mathbb{R}^m)$, $\nu\in\mathcal{P}_2(\mathbb{R}^{m\times d})$, the process $f(\cdot,y,z,\mu,\nu)$ is $\mathbb{F}$-progressively measurable. Moreover, there exists a constant $K\in\mathbb{R}$ such that $f(\cdot,0,0,\delta_{0_m},\delta_{0_{m\times d}})\in L^{2,K}_{\mathbb{F}}(t,\infty;\mathbb{R}^m)$.*
(ii) $f(s,y,z,\mu,\nu)$ is Lipschitz in (y,z,μ,ν), i.e., there exist positive constants $l_y, l_z, l_{\mu_y}, l_{\mu_z}$ such that for any $y,y'\in\mathbb{R}^m$, $z,z'\in\mathbb{R}^{m\times d}$, $\mu,\mu'\in\mathcal{P}_2(\mathbb{R}^m)$, $\nu,\nu'\in\mathcal{P}_2(\mathbb{R}^{m\times d})$ and almost all $(s,\omega)\in[t,\infty)\times\Omega$,

$$|f(s,y,z,\mu,\nu)-f(s,y',z',\mu',\nu')| \le l_y|y-y'| + l_z|z-z'| + l_{\mu_y}\mathcal{W}_2(\mu,\mu') + l_{\mu_z}\mathcal{W}_2(\nu,\nu'). \tag{2.10}$$

(iii) There exists a constant $\kappa_y\in\mathbb{R}$ such that for any $y,y'\in\mathbb{R}^m$, $z\in\mathbb{R}^{m\times d}$, $\mu\in\mathcal{P}_2(\mathbb{R}^m)$, $\nu\in\mathcal{P}_2(\mathbb{R}^{m\times d})$ and almost all $(s,\omega)\in[t,\infty)\times\Omega$, it holds that

$$\langle y-y', f(s,y,z,\mu,\nu) - f(s,y',z,\mu,\nu)\rangle \ge -\kappa_y |y-y'|^2. \tag{2.11}$$

First, we give the following an a priori estimate.

Lemma 2.5. *Let f be a coefficient satisfying Assumption 2.4 and $K > \kappa_y + l_{\mu_y} + l_z^2 + l_{\mu_z}^2$. Let $(Y,Z)\in L^{2,K}_{\mathbb{F}}\left(t,\infty;\mathbb{R}^{m+m\times d}\right)$ be a solution to McKean-Vlasov BSDE (2.9). Then, for any $\varepsilon>0$, we have*

$$\begin{aligned}\mathbb{E}\bigg\{\left|Y_t e^{Kt}\right|^2 + \int_t^\infty \Big[&\left(2K - 2\kappa_y - 2l_z^2 - 2l_{\mu_z}^2 - 2l_{\mu_y} - 3\varepsilon\right)\left|Y_s e^{Ks}\right|^2 \\ &+ \frac{\varepsilon}{l_z^2 + l_{\mu_z}^2 + \varepsilon}\left|Z_s e^{Ks}\right|^2\Big] ds\bigg\} \le \frac{1}{\varepsilon}\mathbb{E}\int_t^\infty \left|f(s,0,0,\delta_{0_m},\delta_{0_{m\times d}})e^{Ks}\right|^2 ds.\end{aligned} \tag{2.12}$$

Proof. For any $T \in (t,\infty)$, applying Itô's formula to $\left|Y_s e^{Ks}\right|^2$ on the interval $[t,T]$ leads to

$$
\begin{aligned}
&\mathbb{E}\left\{\left|e^{Kt}Y_t\right|^2 + \int_t^T \left[2K\left|e^{Ks}Y_s\right|^2 + \left|e^{Ks}Z_s\right|^2\right] ds\right\} \\
&= \mathbb{E}\Bigg\{ \left|e^{KT}Y_T\right|^2 - 2\int_t^T \Big[\Big\langle Y_s, f(s,Y_s,Z_s,\mathcal{L}(Y_s),\mathcal{L}(Z_s)) - f(s,0,Z_s,\mathcal{L}(Y_s),\mathcal{L}(Z_s))\Big\rangle \\
&\quad + \Big\langle Y_s, f(s,0,Z_s,\mathcal{L}(Y_s),\mathcal{L}(Z_s)) - f(s,0,0,\mathcal{L}(Y_s),\mathcal{L}(Z_s))\Big\rangle \\
&\quad + \Big\langle Y_s, f(s,0,0,\mathcal{L}(Y_s),\mathcal{L}(Z_s)) - f(s,0,0,\delta_{0_m},\mathcal{L}(Z_s))\Big\rangle \\
&\quad + \Big\langle Y_s, f(s,0,0,\delta_{0_m},\mathcal{L}(Z_s)) - f(s,0,0,\delta_{0_m},\delta_{0_{m\times d}})\Big\rangle\Big] e^{2Ks}ds\Bigg\} \\
&\le \mathbb{E}\Bigg\{ \left|e^{KT}Y_T\right|^2 + \int_t^T \Big[2\kappa|Y_s|^2 + 2l_z|Y_s||Z_s| + 2l_{\mu_y}|Y_s|\sqrt{\mathbb{E}[|Y_s|^2]} + 2l_{\mu_z}|Y_s|\sqrt{\mathbb{E}[|Z_s|^2]} \\
&\quad + 2|Y_s||f(s,0,0,\delta_{0_m},\delta_{0_{m\times d}})|\Big] e^{2Ks}ds\Bigg\},
\end{aligned}
\tag{2.13}
$$

where the monotonicity condition and the Lipschitz condition of f are used.
For any $\varepsilon > 0$, by the inequalities

$$
2a|y||z| \le \left(l_z^2 + l_{\mu_z}^2 + \varepsilon\right)|y|^2 + \frac{a^2}{l_z^2 + l_{\mu_z}^2 + \varepsilon}|z|^2, \quad 2|y||f| \le \varepsilon|y|^2 + \frac{1}{\varepsilon}|f|^2,
$$

we deduce that

$$
\begin{aligned}
&\mathbb{E}\left\{\left|Y_t e^{Kt}\right|^2 + \int_t^T \left[\left(2K - 2\kappa_y - 2l_z^2 - 2l_{\mu_z}^2 - 2l_{\mu_y} - 3\varepsilon\right)\left|Y_s e^{Ks}\right|^2 + \frac{\varepsilon}{l_z^2 + l_{\mu_z}^2 + \varepsilon}\left|Z_s e^{Ks}\right|^2\right] ds\right\} \\
&\le \mathbb{E}\left\{\left|Y_T e^{KT}\right|^2 + \int_t^T \frac{1}{\varepsilon}\left|f(s,0,0,\delta_{0_m},\delta_{0_{m\times d}})e^{Ks}\right|^2 ds\right\}.
\end{aligned}
$$

Then, by letting $T \to \infty$ on both sides of the above inequality, with help of Lemma 2.2, we obtain the estimate (2.12). ☐

Similar to the definition of solutions for classical BSDEs on infinite horizon in [23], a pair of processes $(Y,Z) \in L_{\mathbb{F}}^{2,K}(t,\infty;\mathbb{R}^{m+m\times d})$ is called a solution to McKean-Vlasov BSDE (2.9) if and only if, for any $T \in (t,\infty)$, the pair of processes (Y,Z) satisfies

$$
Y_s = Y_T - \int_s^T f(r,Y_r,Z_r,\mathcal{L}(Y_r),\mathcal{L}(Z_r))dr - \int_s^T Z_r dW_r, \quad s \in [t,T]. \tag{2.14}
$$

With the help of above a priori estimate (2.12), the method of Peng and Shi [18, Theorem 4] is still valid to yield the following result.

Lemma 2.6. *Let the coefficient f satisfies Assumption 2.4 and let $K > \kappa_y + l_{\mu_y} + l_z^2 + l_{\mu_z}^2$. Then, infinite horizon McKean-Vlasov BSDE (2.9) admits a unique solution $(Y,Z) \in L_{\mathbb{F}}^{2,K}\left(t,\infty;\mathbb{R}^{m+m\times d}\right)$.*

3 Infinite horizon McKean-Vlasov FBSDEs

In this section, we establish existence and uniqueness of solution to infinite horizon McKean-Vlasov FBSDE

$$
\begin{cases}
dX_t = B\left(t, X_t, Y_t, Z_t, \mathcal{L}\left(X_t, Y_t, Z_t\right)\right) dt + \sigma\left(t, X_t, Y_t, Z_t, \mathcal{L}\left(X_t, Y_t, Z_t\right)\right) dW_t, \quad t \in [0, \infty), \\
dY_t = F\left(t, X_t, Y_t, Z_t, \mathcal{L}\left(X_t, Y_t, Z_t\right)\right) dt + Z_t dW_t, \quad t \in [0, \infty), \\
X_0 = \xi,
\end{cases} \tag{3.1}
$$

where $\xi \in L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)$.

Definition 3.1. *A triple of $\mathbb{F}$-progressively measurable processes $(X, Y, Z) \in L^{2,K}_{\mathbb{F}}\left(0, \infty; \mathbb{R}^{n+m+m\times d}\right)$ is called a solution of FBSDE (3.1), if (3.1) is satisfied in the following sense: for any $T > 0$,*

$$
\begin{aligned}
X_t &= \xi + \int_0^t B(s, X_s, Y_s, Z_s, \mathcal{L}\left(X_s, Y_s, Z_s\right))ds + \int_0^t \sigma(s, X_s, Y_s, Z_s, \mathcal{L}\left(X_s, Y_s, Z_s\right))dW_s, \quad t \in [0, T], \\
Y_t &= Y_T - \int_t^T F(s, X_s, Y_s, Z_s, \mathcal{L}\left(X_s, Y_s, Z_s\right))ds - \int_t^T Z_s dW_s, \quad t \in [0, T].
\end{aligned}
$$

We introduce the following assumptions.

Assumption (H1). *$\xi \in L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)$. Let $(B, F, \sigma) : [0, \infty) \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathcal{P}_2\left(\mathbb{R}^{n+m+m\times d}\right) \to (\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^{m\times d})$ be $\mathbb{F}$- progressively measurable functions satisfying:*
(i)There exists a positive constant l such that for any $x, x' \in \mathbb{R}^n$, $y, y' \in \mathbb{R}^m$, $z, z' \in \mathbb{R}^{m\times d}$, $m, m' \in \mathcal{P}_2\left(\mathbb{R}^{n+m+m\times d}\right)$ and almost all $(t, \omega) \in [0, \infty) \times \Omega$,

$$
\left|(B, F, \sigma)(t, x, y, z, m) - (B, F, \sigma)\left(t, x', y', z', m'\right)\right| \le l\left(|x - x'| + |y - y'| + |z - z'| + \mathcal{W}_2\left(m, m'\right)\right).
$$

(ii) There exists a constant $K \in \mathbb{R}$ such that $(B, F, \sigma)(\cdot, 0, 0, 0, \delta_{0_n}, \delta_{0_m}, \delta_{0_{m\times d}}) \in L^{2,K}_{\mathbb{F}}\left(0, \infty; \mathbb{R}^{n+m+n\times d}\right)$.

Besides Assumption (H1), we introduce the following monotonicity condition to deal with the coupling between the forward equation and the backward equation in (3.1) on infinite horizon.

Assumption (H2). *There exist constants $\kappa_x, \kappa_y \in \mathbb{R}$, $\beta_1, \beta_2 \in [0, \infty)$, $l_\sigma, l_\phi, l_z \in (0, \infty)$, $\gamma \in (0, 1)$, $G \in \mathbb{R}^{m\times n}$ and measurable functions $\phi_1 : [0, \infty) \times L^2\left(\Omega; \mathbb{R}^n\right) \times L^2\left(\Omega; \mathbb{R}^n\right) \to [0, \infty)$, $\phi_2 : [0, \infty) \times L^2\left(\Omega; \mathbb{R}^{n+m+m\times d}\right) \times L^2\left(\Omega; \mathbb{R}^{n+m+m\times d}\right) \times \mathcal{P}_2(\mathbb{R}^{n+m+m\times d}) \times \mathcal{P}_2(\mathbb{R}^{n+m+m\times d}) \to [0, \infty)$ such that for all $t \in [0, \infty)$, $i \in \{1, 2\}$, $\Theta_i := (X_i, Y_i, Z_i) \in L^2\left(\Omega; \mathbb{R}^{n+m+m\times d}\right)$, we have as follows:*
(i) One of the following two monotonicity conditions holds:

$$
\begin{aligned}
&\mathbb{E}\left[\left\langle B\left(t, \Theta_1, \mathcal{L}(\Theta_1)\right) - B\left(t, \Theta_2, \mathcal{L}(\Theta_2)\right), G^\top\left(Y_1 - Y_2\right)\right\rangle\right] \\
&\quad + \mathbb{E}\left[\left\langle \sigma\left(t, \Theta_1, \mathcal{L}(\Theta_1)\right) - \sigma\left(t, \Theta_2, \mathcal{L}(\Theta_2)\right), G^\top\left(Z_1 - Z_2\right)\right\rangle\right] \\
&\quad + \mathbb{E}\left[\left\langle F\left(t, \Theta_1, \mathcal{L}(\Theta_1)\right) - F\left(t, \Theta_2, \mathcal{L}(\Theta_2)\right), G\left(X_1 - X_2\right)\right\rangle\right] \\
&\quad + (\kappa_x + \kappa_y)\mathbb{E}[\langle X_1 - X_2, G^\top(Y_1 - Y_2)\rangle] \\
&\le -\beta_1\phi_1\left(t, X_1, X_2\right) - \beta_2\phi_2\left(t, \Theta_1, \Theta_2, \mathcal{L}(\Theta_1), \mathcal{L}(\Theta_2)\right),
\end{aligned} \tag{3.2}
$$

and

$$
\begin{aligned}
&\mathbb{E}\left[\left\langle B\left(t, \Theta_1, \mathcal{L}(\Theta_1)\right) - B\left(t, \Theta_2, \mathcal{L}(\Theta_2)\right), G^\top\left(Y_1 - Y_2\right)\right\rangle\right] \\
&\quad + \mathbb{E}\left[\left\langle \sigma\left(t, \Theta_1, \mathcal{L}(\Theta_1)\right) - \sigma\left(t, \Theta_2, \mathcal{L}(\Theta_2)\right), G^\top\left(Z_1 - Z_2\right)\right\rangle\right] \\
&\quad + \mathbb{E}\left[\left\langle F\left(t, \Theta_1, \mathcal{L}(\Theta_1)\right) - F\left(t, \Theta_2, \mathcal{L}(\Theta_2)\right), G\left(X_1 - X_2\right)\right\rangle\right] \\
&\quad + (\kappa_x + \kappa_y)\mathbb{E}[\langle X_1 - X_2, G^\top(Y_1 - Y_2)\rangle] \\
&\ge \beta_1\phi_1\left(t, X_1, X_2\right) + \beta_2\phi_2\left(t, \Theta_1, \Theta_2, \mathcal{L}(\Theta_1), \mathcal{L}(\Theta_2)\right).
\end{aligned} \tag{3.3}
$$

(ii) One of the following two cases holds.
Case 1: $\beta_2 > 0$ and for any $t \in [0, \infty)$,

$$
\begin{aligned}
&\mathbb{E}\left[|\sigma(t, \Theta_1, \mathcal{L}(\Theta_1)) - \sigma(t, \Theta_2, \mathcal{L}(\Theta_2))|^2\right] \\
&\quad \leq l_\sigma \mathbb{E}[|X_1 - X_2|^2] + l_\phi \phi_2(t, \Theta_1, \Theta_2, \mathcal{L}(\Theta_1), \mathcal{L}(\Theta_2)), \\
&\mathbb{E}\left[\langle B(t, \Theta_1, \mathcal{L}(\Theta_1)) - B(t, \Theta_2, \mathcal{L}(\Theta_2)), X_1 - X_2\rangle\right] \\
&\quad \leq -\kappa_x \mathbb{E}[|X_1 - X_2|^2] + l_\phi \phi_2(t, \Theta_1, \Theta_2, \mathcal{L}(\Theta_1), \mathcal{L}(\Theta_2)), \\
&\mathbb{E}\left[\langle F(t, \Theta_1, \mathcal{L}(\Theta_1)) - F(t, \Theta_2, \mathcal{L}(\Theta_2)), Y_1 - Y_2\rangle\right] \\
&\quad \geq -(\kappa_y + \frac{l_z}{2})\mathbb{E}\left[|Y_1 - Y_2|^2\right] - \frac{\gamma}{2}\mathbb{E}[|Z_1 - Z_2|^2] \\
&\qquad - l_\phi\left(\mathbb{E}[|X_1 - X_2|^2] + \phi_2(t, \Theta_1, \Theta_2, \mathcal{L}(\Theta_1), \mathcal{L}(\Theta_2))\right).
\end{aligned}
\tag{3.4}
$$

Case 2: $\beta_1 > 0$ and for any $t \in [0, \infty)$,

$$
\begin{aligned}
&\mathbb{E}\left[|\sigma(t, \Theta_1, \mathcal{L}(\Theta_1)) - \sigma(t, \Theta_2, \mathcal{L}(\Theta_2))|^2\right] \\
&\quad \leq l_\sigma \mathbb{E}[|X_1 - X_2|^2] + l_\phi \mathbb{E}\left[|Y_1 - Y_2|^2 + |Z_1 - Z_2|^2\right], \\
&\mathbb{E}\left[\langle B(t, \Theta_1, \mathcal{L}(\Theta_1)) - B(t, \Theta_2, \mathcal{L}(\Theta_2)), X_1 - X_2\rangle\right] \\
&\quad \leq -\kappa_x \mathbb{E}[|X_1 - X_2|^2] + l_\phi \mathbb{E}\left[|Y_1 - Y_2|^2 + |Z_1 - Z_2|^2\right], \\
&\mathbb{E}\left[\langle F(t, \Theta_1, \mathcal{L}(\Theta_1)) - F(t, \Theta_2, \mathcal{L}(\Theta_2)), Y_1 - Y_2\rangle\right] \\
&\quad \geq -(\kappa_y + \frac{l_z}{2})\mathbb{E}\left[|Y_1 - Y_2|^2\right] - \frac{\gamma}{2}\mathbb{E}[|Z_1 - Z_2|^2] - l_\phi \phi_1(t, X_1, X_2).
\end{aligned}
\tag{3.5}
$$

We give the following remark to explain the Assumption (H2) and compare it with the existing literature.

Remark 3.2. *(i) Compared with the finite horizon case in [22], an additional term $(\kappa_x+\kappa_y)\mathbb{E}[\langle X_1 - X_2, G^\top(Y_1 - Y_2)\rangle]$ appears in the monotonicity condition (3.2), which will suit for the later analysis of infinite horizon.*

(ii) Compared with the monotonicity conditions in the literature, we propose a generalized monotonicity condition (3.2) by introducing two functions ϕ_1 and ϕ_2. The introduction of these two functions provides us with more flexibility. Specially, when we choose $n = m = d = 1$, $\beta_1 = 0$, $\beta_2 > 0$, $\phi_2(t, \Theta_1, \Theta_2, \mathcal{L}(\Theta_1), \mathcal{L}(\Theta_2)) = \|(Y_1 - Y_2)\|_{L^2}^2 + \|(X_1 - X_2)\|_{L^2}^2$, the monotonicity condition (3.2) is reduced to the monotonicity condition in [4]. Moreover, our condition can be seen as a generalization of the domination-monotonicity condition proposed in [25] by choosing

$$\phi_1(t, X_1, X_2) = \|A(t)(X_1 - X_2)\|_{L^2}^2,$$

$$\phi_2(t, \Theta_1, \Theta_2, \mathcal{L}(\Theta_1), \mathcal{L}(\Theta_2)) = \|B(t)(Y_1 - Y_2) + C(t)(Z_1 - Z_2)\|_{L^2}^2,$$

where $A(\cdot), B(\cdot), C(\cdot)$ are three bounded matrix-valued stochastic processes. Using a more general function ϕ_2 instead of a linear combination of Y and Z, our solvability results can be easily applied to infinite horizon mean field control problems whose coefficients enjoy specific structural conditions by choosing

$$\phi_2(t, \Theta_1, \Theta_2, \mathcal{L}(\Theta_1), \mathcal{L}(\Theta_2)) = \|\hat{\alpha}(t, X_1, Y_1, Z_1, \mathcal{L}(X_1, Y_1, Z_1)) - \hat{\alpha}(t, X_2, Y_2, Z_2, \mathcal{L}(X_2, Y_2, Z_2))\|_{L^2}^2,$$

where $\hat{\alpha}$ is the optimal control, which will be discussed in detail in section 4.

(iii) The monotonicity conditions of coefficients B and F are necessary assumptions in infinite horizon situations, which is indicated by the analysis of infinite horizon McKean-Vlasov SDEs and

infinite horizon McKean-Vlasov BSDEs in the previous section. From the subsequent analysis, we can observe that the constants κ_x, κ_y will affect the values of the parameter K. Condition (3.4) avoids the influence of the monotonicity of function ϕ_2 on the choice of constants κ_x and κ_y, further the values of parameter K. The benefits of this technique will be demonstrated in investigating mean field control problem (1.1)-(1.2) and LQ control problems (see Remark 4.8).

Now, we give the main result of this section.

Theorem 3.3. *Let Assumptions* (H1) *and* (H2) *hold. Let*

$$\kappa_x-\kappa_y>\max\{l_\sigma,l_z\}\quad \text{and}\quad K=\frac{\kappa_x+\kappa_y}{2}. \tag{3.6}$$

Then, infinite horizon McKean-Vlasov FBSDE (3.1) admits a unique solution $(X,Y,Z)\in L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^{n+m+m\times d})$. Moreover, we have the following estimate:

$$\begin{aligned}&\mathbb{E}\int_0^\infty |e^{Kt}X_t|^2dt+\mathbb{E}\int_0^\infty |e^{Kt}Y_t|^2dt+\mathbb{E}\int_0^\infty |e^{Kt}Z_t|^2dt\\ &\le C\mathbb{E}\int_0^T |(B,F,\sigma)(t,0,0,0,\delta_{0_{n+m+m\times d}})e^{Kt}|^2dt,\end{aligned} \tag{3.7}$$

where $C>0$ is a constant depending on $|G|,\kappa_x,\kappa_y,l_\phi,l_z,l_\sigma,l$ and β_1 or β_2. Furthermore, let $(\bar X,\bar Y,\bar Z)\in L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^{n+m+m\times d})$ be a solution to the FBSDE (3.1) with another set of coefficients $(\bar\xi,\bar B,\bar F,\bar\sigma)$, then we have

$$\begin{aligned}&\mathbb{E}\int_0^\infty |e^{Kt}(X_t-\bar X_t)|^2dt+\mathbb{E}\int_0^\infty |e^{Kt}(Y_t-\bar Y_t)|^2dt+\mathbb{E}\int_0^\infty |e^{Kt}(Z_t-\bar Z_t)|^2dt\\ &\le C\Bigg\{\mathbb{E}[|\xi-\bar\xi|^2]+\mathbb{E}\int_0^\infty |B(t,\bar\Theta_t)-\bar B(t,\bar\Theta_t)e^{Kt}|^2dt+\mathbb{E}\int_0^\infty |F(t,\bar\Theta_t)-\bar F(t,\bar\Theta_t)e^{Kt}|^2dt\\ &\quad+\mathbb{E}\int_0^\infty |\sigma(t,\bar\Theta_t)-\bar\sigma(t,\bar\Theta_t)e^{Kt}|^2dt\Bigg\},\end{aligned} \tag{3.8}$$

where $\bar\Theta_t=(\bar X_t,\bar Y_t,\bar Z_t,\mathcal{L}(\bar X_t,\bar Y_t,\bar Z_t))$ and C is the same constant as in (3.7).

Remark 3.4. *It is easy to verify that (3.6) is equivalent to*

$$\kappa_y+\frac{l_z}{2}<K<\kappa_x-\frac{l_\sigma}{2}\quad \text{and}\quad K=\frac{\kappa_x+\kappa_y}{2}. \tag{3.9}$$

We introduce a family of infinite horizon FBSDEs parameterized by $\lambda\in[0,1]$:

$$\begin{cases} dX_t^\lambda=\Big[\lambda B\left(t,X_t^\lambda,Y_t^\lambda,Z_t^\lambda,\mathcal{L}\left(X_t^\lambda,Y_t^\lambda,Z_t^\lambda\right)\right)-(1-\lambda)\kappa_xX_t^\lambda+\mathcal{I}_t^B\Big]dt\\ \qquad\qquad+\Big[\lambda\sigma\left(t,X_t^\lambda,Y_t^\lambda,Z_t^\lambda,\mathcal{L}\left(X_t^\lambda,Y_t^\lambda,Z_t^\lambda\right)\right)+\mathcal{I}_t^\sigma\Big]dW_t,\\ dY_t^\lambda=\Big[\lambda F\left(t,X_t^\lambda,Y_t^\lambda,Z_t^\lambda,\mathcal{L}\left(X_t^\lambda,Y_t^\lambda,Z_t^\lambda\right)\right)-(1-\lambda)\kappa_yY_t^\lambda+\mathcal{I}_t^F\Big]dt+Z_t^\lambda dW_t,\\ X_0^\lambda=\xi,\end{cases} \tag{3.10}$$

where $\xi\in L^2_{\mathcal{F}_0}(\Omega;\mathbb{R}^n)$ and $(\mathcal{I}^B,\mathcal{I}^F,\mathcal{I}^\sigma)$ are arbitrary processes in $L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^{n+m+n\times d})$. Note that when $\lambda=1$, $\mathcal{I}^B\equiv0$, $\mathcal{I}^F\equiv0$, $\mathcal{I}^\sigma\equiv0$, (3.10) becomes (3.1), and when $\lambda=0$, FBSDE (3.10) is reduced to

$$\begin{cases} dX_t^0=(-\kappa_xX_t^0+\mathcal{I}_t^B)dt+\mathcal{I}_t^\sigma dW_t,\\ dY_t^0=(-\kappa_yY_t^0+\mathcal{I}_t^F)dt+Z_t^0dW_t,\\ X_0^0=\xi.\end{cases} \tag{3.11}$$

It is clear that FBSDE (3.11) is in a decoupled form and we can solve the SDE and BSDE separately. As a direct application of Lemma 2.3 and Lemma 2.6, we have the following result.

Lemma 3.5. *Assume $\kappa_y < K < \kappa_x$, then for any $(\mathcal{I}^B, \mathcal{I}^F, \mathcal{I}^\sigma) \in L^{2,K}_{\mathbb{F}}\left(0, \infty; \mathbb{R}^{n+m+n\times d}\right)$ and $\xi \in L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)$, FBSDE (3.11) admits a unique solution (X^0, Y^0, Z^0) in $L^{2,K}_{\mathbb{F}}\left(0, \infty; \mathbb{R}^{n+m+m\times d}\right)$.*

Next, for any $\lambda_0 \in [0,1]$, we shall establish an a priori estimate for FBSDE (3.10) which plays a key role in the method of continuation.

Lemma 3.6. *Let (ξ, B, F, σ), $(\bar{\xi}, \bar{B}, \bar{F}, \bar{\sigma})$ satisfy Assumptions (H1) and (H2), and $\left(\mathcal{I}^B, \mathcal{I}^F, \mathcal{I}^\sigma\right)$, $(\bar{\mathcal{I}}^B, \bar{\mathcal{I}}^F, \bar{\mathcal{I}}^\sigma) \in L^{2,K}_{\mathbb{F}}\left(0, \infty; \mathbb{R}^{n+m+n\times d}\right)$ and let*

$$\kappa_x - \kappa_y > \max\{l_\sigma, l_z\} \quad \textit{and} \quad K = \frac{\kappa_x + \kappa_y}{2}. \tag{3.12}$$

Suppose $(X, Y, Z), (\bar{X}, \bar{Y}, \bar{Z}) \in L^{2,K}_{\mathbb{F}}(0, \infty; \mathbb{R}^{n+m+m\times d})$ are solutions to FBSDE (3.10) parameterized by $\lambda_0 \in [0,1]$ with $\left(\xi, B, F, \sigma, \mathcal{I}^B, \mathcal{I}^F, \mathcal{I}^\sigma\right)$ and $(\bar{\xi}, \bar{B}, \bar{F}, \bar{\sigma}, \bar{\mathcal{I}}^B, \bar{\mathcal{I}}^F, \bar{\mathcal{I}}^\sigma)$, respectively. Then there exists a constant $C > 0$ only depending on $|G|, \kappa_x, \kappa_y$, l_ϕ, l_z, l_σ and β_1 or β_2, independent of λ_0 such that

$$\begin{aligned}
&\left\|X - \bar{X}\right\|_K^2 + \left\|Y - \bar{Y}\right\|_K^2 + \left\|Z - \bar{Z}\right\|_K^2 \\
\leq\ & C\Big\{\mathbb{E}\left[|\xi - \bar{\xi}|^2\right] + \left\|\lambda_0\left(B(\cdot, \bar{\Theta}) - \bar{B}(\cdot, \bar{\Theta})\right) + (\mathcal{I}^B - \bar{\mathcal{I}}^B)\right\|_K^2 \\
&+ \left\|\lambda_0\left(\sigma(\cdot, \bar{\Theta}) - \bar{\sigma}(\cdot, \bar{\Theta})\right) + (\mathcal{I}^\sigma - \bar{\mathcal{I}}^\sigma)\right\|_K^2 + \left\|\lambda_0\left(F(\cdot, \bar{\Theta}) - \bar{F}(\cdot, \bar{\Theta})\right) + (\mathcal{I}^F - \bar{\mathcal{I}}^F)\right\|_K^2\Big\},
\end{aligned} \tag{3.13}$$

where $\Theta = (X, Y, Z, \mathcal{L}(X, Y, Z))$, $\bar{\Theta} = (\bar{X}, \bar{Y}, \bar{Z}, \mathcal{L}(\bar{X}, \bar{Y}, \bar{Z}))$ and $\|\cdot\|_K^2$ is defined as (2.1).

Proof. The whole proof will be splitted into two cases according to Assumption (H2). Before splitting the proof, we first do some pretreatments for both cases.
First, we can represent the FBSDE (3.10) parameterized by λ_0 in the following form:

$$\begin{aligned}
de^{Kt}X_t &= \left[\lambda_0 e^{Kt} B\left(t, X_t, Y_t, Z_t, \mathcal{L}\left(X_t, Y_t, Z_t\right)\right) + (K - (1-\lambda_0)\kappa_x)e^{Kt}X_t + e^{Kt}\mathcal{I}^B_t\right]dt \\
&\quad + \left[\lambda_0 e^{Kt}\sigma\left(t, X_t, Y_t, Z_t, \mathcal{L}\left(X_t, Y_t, Z_t\right)\right) + e^{Kt}\mathcal{I}^\sigma_t\right]dW_t \\
de^{Kt}Y_t &= \left[\lambda_0 e^{Kt} F\left(t, X_t, Y_t, Z_t, \mathcal{L}\left(X_t, Y_t, Z_t\right)\right) + (K - (1-\lambda_0)\kappa_y)e^{Kt}Y_t + e^{Kt}\mathcal{I}^F_t\right]dt + e^{Kt}Z_t dW_t.
\end{aligned} \tag{3.14}$$

By applying Itô's formula to $\left\langle e^{Kt}(Y_t - \bar{Y}_t), Ge^{Kt}\left(X_t - \bar{X}_t\right)\right\rangle$ on the time interval $[0, T]$, we obtain that

$$\begin{aligned}
&\mathbb{E}\left[\left\langle e^{KT}(Y_T - \bar{Y}_T), Ge^{KT}(X_T - \bar{X}_T)\right\rangle\right] - \mathbb{E}\left[\left\langle Y_0 - \bar{Y}_0, G(\xi - \bar{\xi})\right\rangle\right] \\
=\ & \mathbb{E}\int_0^T \Bigg[\left\langle \lambda_0 e^{Kt}\left(B(t, \Theta_t) - \bar{B}(t, \bar{\Theta}_t)\right) + (K - (1-\lambda_0)\kappa_x)e^{Kt}(X_t - \bar{X}_t) + e^{Kt}(\mathcal{I}^B_t - \bar{\mathcal{I}}^B_t), G^\top e^{Kt}(Y_t - \bar{Y}_t)\right\rangle \\
&+ \left\langle \lambda_0 e^{Kt}\left(F(t, \Theta_t) - \bar{F}(t, \bar{\Theta}_t)\right) + (K - (1-\lambda_0)\kappa_y)e^{Kt}(Y_t - \bar{Y}_t) + e^{Kt}(\mathcal{I}^F_t - \bar{\mathcal{I}}^F_t), Ge^{Kt}(X_t - \bar{X}_t)\right\rangle \\
&+ \left\langle \lambda_0 e^{Kt}\left(\sigma(t, \Theta_t) - \bar{\sigma}(t, \bar{\Theta}_t)\right) + e^{Kt}(\mathcal{I}^\sigma_t - \bar{\mathcal{I}}^\sigma_t), G^\top e^{Kt}(Z_t - \bar{Z}_t)\right\rangle\Bigg]dt.
\end{aligned} \tag{3.15}$$

Then, by adding and subtracting the terms $B\left(t,\bar{\Theta}_t\right)$, $F\left(t,\bar{\Theta}_t\right)$, $\sigma\left(t,\bar{\Theta}_t\right)$, we can deduce that

$$\begin{aligned}
&\mathbb{E}\left[\left\langle e^{KT}(Y_T-\bar{Y}_T), Ge^{KT}(X_T-\bar{X}_T)\right\rangle\right]-\mathbb{E}\left[\left\langle Y_0-\bar{Y}_0, G(\xi-\bar{\xi})\right\rangle\right]\\
=&\,\mathbb{E}\int_0^T\Bigg[\left\langle \lambda_0 e^{Kt}\left(B(t,\Theta_t)-B(t,\bar{\Theta}_t)\right), G^\top e^{Kt}(Y_t-\bar{Y}_t)\right\rangle\\
&\quad+\left\langle \lambda_0 e^{Kt}\left(F(t,\Theta_t)-F(t,\bar{\Theta}_t)\right), Ge^{Kt}(X_t-\bar{X}_t)\right\rangle\\
&\quad+\left\langle \lambda_0 e^{Kt}\left(\sigma(t,\Theta_t)-\sigma(t,\bar{\Theta}_t)\right), G^\top e^{Kt}\left(Z_t-\bar{Z}_t\right)\right\rangle\\
&\quad+\left(2K-(1-\lambda_0)(\kappa_x+\kappa_y)\right)\left\langle e^{Kt}\left(X_t-\bar{X}_t\right), G^\top e^{Kt}\left(Y_t-\bar{Y}_t\right)\right\rangle\\
&\quad+\left\langle \lambda_0 e^{Kt}\left(B(t,\bar{\Theta}_t)-\bar{B}(t,\bar{\Theta}_t)\right)+e^{Kt}\left(\mathcal{I}_t^B-\bar{\mathcal{I}}_t^B\right), G^\top e^{Kt}\left(Y_t-\bar{Y}_t\right)\right\rangle\\
&\quad+\left\langle \lambda_0 e^{Kt}\left(F(t,\bar{\Theta}_t)-\bar{F}(t,\bar{\Theta}_t)\right)+e^{Kt}\left(\mathcal{I}_t^F-\bar{\mathcal{I}}_t^F\right), Ge^{Kt}\left(X_t-\bar{X}_t\right)\right\rangle\\
&\quad+\left\langle \lambda_0 e^{Kt}\left(\sigma(t,\bar{\Theta}_t)-\bar{\sigma}(t,\bar{\Theta}_t)\right)+e^{Kt}(\mathcal{I}_t^\sigma-\bar{\mathcal{I}}_t^\sigma), G^\top e^{Kt}\left(Z_t-\bar{Z}_t\right)\right\rangle\Bigg]dt.
\end{aligned}\tag{3.16}$$

The monotonicity condition (3.2) and $K=(\kappa_x+\kappa_y)/2$ work together to reduce the above equation to

$$\begin{aligned}
&\mathbb{E}\left[\left\langle e^{KT}(Y_T-\bar{Y}_T), Ge^{KT}(X_T-\bar{X}_T)\right\rangle\right]-\mathbb{E}\left[\left\langle Y_0-\bar{Y}_0, G(\xi-\bar{\xi})\right\rangle\right]\\
\le&\,\mathbb{E}\int_0^T\Bigg[\left\langle \lambda_0 e^{Kt}\left(B(t,\bar{\Theta}_t)-\bar{B}(t,\bar{\Theta}_t)\right)+e^{Kt}(\mathcal{I}_t^B-\bar{\mathcal{I}}_t^B), G^\top e^{Kt}(Y_t-\bar{Y}_t)\right\rangle\\
&\quad+\left\langle \lambda_0 e^{Kt}\left(F(t,\bar{\Theta}_t)-\bar{F}(t,\bar{\Theta}_t)\right)+e^{Kt}(\mathcal{I}_t^F-\bar{\mathcal{I}}_t^F), Ge^{Kt}(X_t-\bar{X}_t)\right\rangle\\
&\quad+\left\langle \lambda_0 e^{Kt}\left(\sigma\left(t,\bar{\Theta}_t\right)-\bar{\sigma}\left(t,\bar{\Theta}_t\right)\right)+e^{Kt}(\mathcal{I}_t^\sigma-\bar{\mathcal{I}}_t^\sigma), G^\top e^{Kt}\left(Z_t-\bar{Z}_t\right)\right\rangle\Bigg]dt\\
&-\lambda_0\int_0^T e^{2Kt}\left(\beta_1\phi_1\left(t,X_t,\bar{X}_t\right)+\beta_2\phi_2\left(t,\Theta_t,\bar{\Theta}_t\right)\right)dt.
\end{aligned}\tag{3.17}$$

With help of the inequality $2ab\le\varepsilon a^2+(1/\varepsilon)b^2$ for any $\varepsilon>0$, Lemma 2.2, and letting $T\to\infty$, we have

$$\begin{aligned}
&\lambda_0\int_0^\infty e^{2Kt}\left(\beta_1\phi_1\left(t,X_t,\bar{X}_t\right)+\beta_2\phi_2\left(t,\Theta_t,\bar{\Theta}_t\right)\right)dt\\
\le&\,\varepsilon\left(\left\|Y_0-\bar{Y}_0\right\|_{L^2}^2+\|X-\bar{X}\|_K^2+\|Y-\bar{Y}\|_K^2+\|Z-\bar{Z}\|_K^2\right)\\
&+\frac{|G|^2}{4\varepsilon}\Big\{\mathbb{E}\left[|\xi-\bar{\xi}|^2\right]+\left\|\lambda_0\left(B\left(\cdot,\bar{\Theta}\right)-\bar{B}\left(\cdot,\bar{\Theta}\right)\right)+(\mathcal{I}^B-\bar{\mathcal{I}}^B)\right\|_K^2\\
&+\left\|\lambda_0\left(\sigma\left(\cdot,\bar{\Theta}\right)-\bar{\sigma}\left(\cdot,\bar{\Theta}\right)\right)+(\mathcal{I}^\sigma-\bar{\mathcal{I}}^\sigma)\right\|_K^2+\left\|\lambda_0\left(F\left(\cdot,\bar{\Theta}\right)-\bar{F}\left(\cdot,\bar{\Theta}\right)\right)+(\mathcal{I}^F-\bar{\mathcal{I}}^F)\right\|_K^2\Big\}.
\end{aligned}\tag{3.18}$$

Moreover, under monotonicity condition (3.3), we can still get (3.18) with similar arguments. We have finished the pretreatment work. The next analysis in two cases will be based on (3.18). For simplicity of notations, from now we denote the right-hand side of (3.13) as RHS.

Case 1: $\beta_2>0$. First, applying Itô's formula to $|e^{Kt}(X_t-\bar{X}_t)|^2$ on the time interval $[0,T]$, we have

$$
\begin{aligned}
&\mathbb{E}\left[|e^{KT}(X_T-\bar{X}_T)|^2\right]-\mathbb{E}\left[|\xi-\bar{\xi}|^2\right]\\
&=\mathbb{E}\int_0^T\Bigg[2\left\langle e^{Kt}(X_t-\bar{X}_t),\lambda_0 e^{Kt}\left(B(t,\Theta_t)-B(t,\bar{\Theta}_t)+B(t,\bar{\Theta}_t)-\bar{B}(t,\bar{\Theta}_t)\right)+e^{Kt}(\mathcal{I}_t^B-\bar{\mathcal{I}}_t^B)\right.\\
&\qquad\left.+(K-(1-\lambda_0)\kappa_x)\,e^{Kt}(X_t-\bar{X}_t)\right\rangle+\big|\lambda_0 e^{Kt}\big(\sigma(t,\Theta_t)-\sigma(t,\bar{\Theta}_t)+\sigma(t,\bar{\Theta}_t)-\bar{\sigma}(t,\bar{\Theta}_t)\big)\\
&\qquad+e^{Kt}(\mathcal{I}_t^\sigma-\bar{\mathcal{I}}_t^\sigma)\big|^2\Bigg]dt.
\end{aligned}
\tag{3.19}
$$

Under the condition (3.4) and with the help of the inequality $2ab\le\varepsilon_1 a^2+(1/\varepsilon_1)b^2$ for any $\varepsilon_1>0$, we can deduce that,

$$
\begin{aligned}
&\mathbb{E}\left[|e^{KT}(X_T-\bar{X}_T)|^2\right]-\mathbb{E}\left[|\xi-\bar{\xi}|^2\right]\\
&\le\mathbb{E}\int_0^T\Bigg[(2K-2\kappa_x)e^{2Kt}|X_t-\bar{X}_t|^2+2\lambda_0 l_\phi e^{2Kt}\phi_2(t,\Theta_t,\bar{\Theta}_t)+\varepsilon_1|e^{Kt}(X_t-\bar{X}_t)|^2\\
&\qquad+\frac{1}{\varepsilon_1}|\lambda_0 e^{Kt}\left(B(t,\bar{\Theta}_t)-\bar{B}(t,\bar{\Theta}_t)\right)+e^{Kt}(\mathcal{I}_t^B-\bar{\mathcal{I}}_t^B)|^2+\lambda_0^2\left(l_\sigma|e^{Kt}(X_t-\bar{X}_t)|^2+l_\phi e^{2Kt}\phi_2(t,\Theta_t,\bar{\Theta}_t)\right)\\
&\qquad+|\lambda_0 e^{Kt}\left(\sigma(t,\bar{\Theta}_t)-\bar{\sigma}(t,\bar{\Theta}_t)\right)+e^{Kt}(\mathcal{I}_t^\sigma-\bar{\mathcal{I}}_t^\sigma)|^2+\varepsilon_1\lambda_0^2\left(l_\sigma|e^{Kt}(X_t-\bar{X}_t)|^2+l_\phi e^{2Kt}\phi_2(t,\Theta_t,\bar{\Theta}_t)\right)\\
&\qquad+\frac{1}{\varepsilon_1}|\lambda_0 e^{Kt}\left(\sigma(t,\bar{\Theta}_t)-\bar{\sigma}(t,\bar{\Theta}_t)\right)+e^{Kt}(\mathcal{I}_t^\sigma-\bar{\mathcal{I}}_t^\sigma)|^2\Bigg]dt.
\end{aligned}
\tag{3.20}
$$

Arrange terms, we can obtain that

$$
\begin{aligned}
&\mathbb{E}[|e^{KT}(X_T-\bar{X}_T)|^2]+(2\kappa_x-2K-l_\sigma-(1+l_\sigma)\varepsilon_1)\,\mathbb{E}\int_0^T|e^{Kt}(X_t-\bar{X}_t)|^2dt\\
&\le(2+\varepsilon_1)l_\phi\lambda_0\int_0^T e^{2Kt}\phi_2(t,\Theta_t,\bar{\Theta}_t)dt+\frac{1}{\varepsilon_1}\mathbb{E}\int_0^T e^{2Kt}\left|(\lambda_0\left(B(t,\bar{\Theta}_t)-\bar{B}(t,\bar{\Theta}_t)\right)+\mathcal{I}_t^B-\bar{\mathcal{I}}_t^B\right|^2dt\\
&\quad+\left(\frac{1}{\varepsilon_1}+1\right)\mathbb{E}\int_0^T e^{2Kt}\left|\lambda_0\left(\sigma(\bar{\Theta}_t)-\bar{\sigma}(t,\bar{\Theta}_t)\right)+\mathcal{I}_t^\sigma-\bar{\mathcal{I}}_t^\sigma\right|^2dt+\mathbb{E}[|\xi-\bar{\xi}|^2].
\end{aligned}
\tag{3.21}
$$

The condition $K<\kappa_x-(l_\sigma/2)$ implies the existence of ε_1 such that $(1+l_\sigma)\varepsilon_1\in(0,\kappa_x-K-(l_\sigma/2))$ and we denote

$$
C_1:=\frac{(2+\varepsilon_1)l_\phi}{(2\kappa_x-2K-l_\sigma-(1+l_\phi)\varepsilon_1)\beta_2},\quad C_2:=\frac{(1/\varepsilon_1)+1}{2\kappa_x-2K-l_\sigma-(1+l_\phi)\varepsilon_1}.
\tag{3.22}
$$

When $\beta_2>0$, from (3.18), we obtain,

$$
\begin{aligned}
&\lambda_0\beta_2\int_0^\infty e^{2Kt}\phi_2\left(t,\Theta_t,\bar{\Theta}_t\right)dt\\
&\le\varepsilon\Big(\left\|Y_0-\bar{Y}_0\right\|_{L^2}^2+\|X-\bar{X}\|_K^2+\|Y-\bar{Y}\|_K^2+\|Z-\bar{Z}\|_K^2\Big)+\frac{|G|^2}{4\varepsilon}\mathrm{RHS}.
\end{aligned}
\tag{3.23}
$$

Combine (3.23) and (3.21) and let $T\to\infty$, we have

$$
\begin{aligned}
&\|X-\bar{X}\|_K^2\\
&\le C_1\left\{\varepsilon\Big(\left\|Y_0-\bar{Y}_0\right\|_{L^2}^2+\|X-\bar{X}\|_K^2+\|Y-\bar{Y}\|_K^2+\|Z-\bar{Z}\|_K^2\Big)+\frac{|G|^2}{4\varepsilon}\mathrm{RHS}\right\}+C_2\mathrm{RHS}.
\end{aligned}
\tag{3.24}
$$

Choose ε small enough such that $0<\varepsilon<1/C_1$, and denote

$$\bar{\varepsilon} := \frac{C_1\varepsilon}{1-C_1\varepsilon}, \quad C_3 := \frac{C_1|G|^2}{4\varepsilon(1-C_1\varepsilon)} + \frac{C_2}{1-C_1\varepsilon}. \tag{3.25}$$

Then, (3.24) is reduced to

$$\|X-\bar{X}\|_K^2 \le \bar{\varepsilon}\Big(\big\|Y_0-\bar{Y}_0\big\|_{L^2}^2 + \|Y-\bar{Y}\|_K^2 + \|Z-\bar{Z}\|_K^2 \Big) + C_3\mathrm{RHS}. \tag{3.26}$$

Second, we apply Itô's formula to $|e^{Kt}(Y_t-\bar{Y}_t)|^2$ on the time interval $[0,T]$ and we obtain

$$\begin{aligned}
&\mathbb{E}[|e^{KT}(Y_T-\bar{Y}_T)|^2] - \mathbb{E}[|Y_0-\bar{Y}_0|^2] \\
&= \mathbb{E}\int_0^T \Big[2\left\langle e^{Kt}(Y_t-\bar{Y}_t), \lambda_0 e^{Kt}\left(F(t,\Theta_t)-\bar{F}(t,\bar{\Theta}_t)\right) + (K-(1-\lambda_0)\kappa_y)e^{Kt}(Y_t-\bar{Y}_t)\right. \\
&\quad \left. + e^{Kt}(\mathcal{I}_t^F-\bar{\mathcal{I}}_t^F)\right\rangle\Big]dt + \mathbb{E}\int_0^T |e^{Kt}(Z_t-\bar{Z}_t)|^2 dt \\
&= \mathbb{E}\int_0^T \Big[2\Big\langle e^{Kt}(Y_t-\bar{Y}_t), \lambda_0 e^{Kt}\left(F(t,\Theta_t)-F(t,\bar{\Theta}_t)+F(t,\bar{\Theta}_t)-\bar{F}(t,\bar{\Theta}_t)\right) \\
&\quad + (K-(1-\lambda_0)\kappa_y)e^{Kt}(Y_t-\bar{Y}_t) + e^{Kt}(\mathcal{I}_t^F-\bar{\mathcal{I}}_t^F)\Big\rangle\Big]dt + \mathbb{E}\int_0^T |e^{Kt}(Z_t-\bar{Z}_t)|^2 dt.
\end{aligned} \tag{3.27}$$

Under the condition (3.4) and with the inequality, for any $\varepsilon_2>0$, $2ab \ge -\varepsilon_2 a^2 - (1/\varepsilon_2)b^2$, we can deduce that,

$$\begin{aligned}
&\mathbb{E}[|e^{KT}(Y_T-\bar{Y}_T)|^2] - \mathbb{E}[|Y_0-\bar{Y}_0|^2] \\
&\ge -2l_\phi\lambda_0\mathbb{E}\int_0^T e^{2Kt}|X_t-\bar{X}_t|^2dt - 2l_\phi\lambda_0\int_0^T e^{2Kt}\phi_2(t,\Theta_t,\bar{\Theta}_t)dt \\
&\quad + (2K-2\kappa_y-\lambda_0 l_z)\mathbb{E}\int_0^T |e^{Kt}(Y_t-\bar{Y}_t)|^2dt - \lambda_0\gamma\mathbb{E}\int_0^T |e^{Kt}(Z_t-\bar{Z}_t)|^2dt + \mathbb{E}\int_0^T |e^{Kt}(Z_t-\bar{Z}_t)|^2dt \\
&\quad - \varepsilon_2\mathbb{E}\int_0^T |e^{Kt}(Y_t-\bar{Y}_t)|^2dt - \frac{1}{\varepsilon_2}\mathbb{E}\int_0^T \left|\lambda_0 e^{Kt}\left(F(t,\bar{\Theta}_t)-\bar{F}(t,\bar{\Theta}_t)\right) + e^{Kt}(\mathcal{I}_t^F-\bar{\mathcal{I}}_t^F)\right|^2 dt.
\end{aligned} \tag{3.28}$$

Arranging terms, letting $T\to\infty$ and substituting (3.23) into (3.28) yield

$$\begin{aligned}
&\mathbb{E}\left[|Y_0-\bar{Y}_0|^2\right] + (1-\gamma)\mathbb{E}\int_0^\infty |e^{Kt}(Z_t-\bar{Z}_t)|^2dt + (2K-2\kappa_y-l_z-\varepsilon_2)\,\mathbb{E}\int_0^\infty |e^{Kt}(Y_t-\bar{Y}_t)|^2dt \\
&\le 2l_\phi\mathbb{E}\int_0^\infty e^{2Kt}|X_t-\bar{X}_t|^2dt + \frac{2l_\phi\varepsilon}{\beta_2}\Big(\big\|Y_0-\bar{Y}_0\big\|_{L^2}^2 + \|X-\bar{X}\|_K^2 + \|Y-\bar{Y}\|_K^2 + \|Z-\bar{Z}\|_K^2 \Big) \\
&\quad + \frac{l_\phi|G|^2}{2\varepsilon\beta_2}\mathrm{RHS} + \frac{1}{\varepsilon_2}\mathbb{E}\int_0^\infty |\lambda_0 e^{Kt}(F(t,\bar{\Theta}_t)-\bar{F}(t,\bar{\Theta}_t) + (\mathcal{I}_t^F-\bar{\mathcal{I}}_t^F))|^2dt
\end{aligned} \tag{3.29}$$

The condition $K>\kappa_y+(l_z/2)$ implies the existence of the number ε_2 such that $\varepsilon_2\in(0,K-\kappa_y-(l_z/2))$. We can choose ε small enough such that

$$2K-2\kappa_y-l_z-\varepsilon_2-\frac{2l_\phi\varepsilon}{\beta_2}>0, \quad 1-\gamma-\frac{2l_\phi\varepsilon}{\beta_2}>0. \tag{3.30}$$

And we denote

$$C_4 := \min\left\{2K-2\kappa_y-l_z-\varepsilon_2-\frac{2l_\phi\varepsilon}{\beta_2}, 1-\gamma-\frac{2l_\phi\varepsilon}{\beta_2}\right\}. \tag{3.31}$$

Therefore, (3.29) is reduced to

$$\left\|Y_0-\bar{Y}_0\right\|_{L^2}^2+\|Y-\bar{Y}\|_K^2+\|Z-\bar{Z}\|_K^2 \leq C_5\|X-\bar{X}\|_K^2+C_6\mathrm{RHS}, \tag{3.32}$$

where

$$C_5:=\frac{2l_\phi(1+\frac{\varepsilon}{\beta_2})}{C_4}, \quad C_6:=\frac{\frac{l_\phi^2|G|^2}{2\varepsilon\beta_2}+\frac{1}{\varepsilon_2}}{C_4}. \tag{3.33}$$

Choose ε small enough such that $1-\bar{\varepsilon}C_5>0$ and combined with (3.26), we obtain

$$\|X-\bar{X}\|_K^2 \leq \frac{C_6\bar{\varepsilon}+C_3}{1-\bar{\varepsilon}C_5}\mathrm{RHS}. \tag{3.34}$$

Substituting (3.34) into (3.32), we have

$$\left\|X-\bar{X}\right\|_K^2+\left\|Y-\bar{Y}\right\|_K^2+\left\|Z-\bar{Z}\right\|_K^2 \leq C\mathrm{RHS}, \tag{3.35}$$

where

$$C:=\frac{C_6\bar{\varepsilon}+C_3}{1-\bar{\varepsilon}C_5}+\frac{C_5(C_6\bar{\varepsilon}+C_3)}{1-\bar{\varepsilon}C_5}+C_6. \tag{3.36}$$

The desired estimate (3.13) is obtained in case 1.

Case 2: $\beta_1>0$. When $\beta_1>0$, from (3.18), we obtain,

$$\begin{aligned}&\lambda_0\beta_1\int_0^\infty e^{2Kt}\phi_1\left(t,X_1,X_2\right)dt\\&\leq \varepsilon\Big(\left\|Y_0-\bar{Y}_0\right\|_{L^2}^2+\|X-\bar{X}\|_K^2+\|Y-\bar{Y}\|_K^2+\|Z-\bar{Z}\|_K^2\Big)+\frac{|G|^2}{4\varepsilon}\mathrm{RHS}.\end{aligned} \tag{3.37}$$

We first apply Itô's formula to $|e^{Kt}(Y_t-\bar{Y}_t)|^2$ on time interval $[0,T]$. Under condition (3.5), combined with (3.37) and letting $T\to\infty$, (3.27) is reduced to

$$\begin{aligned}&\mathbb{E}\left[|Y_0-\bar{Y}_0|^2\right]+(1-\gamma)\mathbb{E}\int_0^\infty|e^{Kt}(Z_t-\bar{Z}_t)|^2dt+(2K-2\kappa_y-l_z-\varepsilon_2)\,\mathbb{E}\int_0^\infty|e^{Kt}(Y_t-\bar{Y}_t)|^2dt\\&\leq\frac{2l_\phi\varepsilon}{\beta_1}\Big(\left\|Y_0-\bar{Y}_0\right\|_{L^2}^2+\|X-\bar{X}\|_K^2+\|Y-\bar{Y}\|_K^2+\|Z-\bar{Z}\|_K^2\Big)\\&\quad+\frac{l_\phi|G|^2}{2\varepsilon\beta_1}\mathrm{RHS}+\frac{1}{\varepsilon_2}\mathbb{E}\int_0^\infty|\lambda_0e^{Kt}\left(F(t,\bar{\Theta}_t)-\bar{F}(t,\bar{\Theta}_t)\right)+(\mathcal{I}_t^F-\bar{\mathcal{I}}_t^F)|^2dt.\end{aligned} \tag{3.38}$$

We can choose ε small enough such that

$$2K-2\kappa_y-l_z-\varepsilon_2-\frac{2l_\phi\varepsilon}{\beta_1}>0, \quad 1-\gamma-\frac{2l_\phi\varepsilon}{\beta_1}>0, \tag{3.39}$$

and denote

$$C_7:=\min\left\{2K-2\kappa_y-l_z-\varepsilon_2-\frac{2l_\phi\varepsilon}{\beta_1},1-\gamma-\frac{2l_\phi\varepsilon}{\beta_1}\right\}. \tag{3.40}$$

Then, we obtain

$$\left\|Y_0-\bar{Y}_0\right\|_{L^2}^2+\|Y-\bar{Y}\|_K^2+\|Z-\bar{Z}\|_K^2 \leq \tilde{\varepsilon}\|X-\bar{X}\|_K^2+C_8\mathrm{RHS}, \tag{3.41}$$

where

$$\tilde{\varepsilon}:=\frac{2l_\phi\varepsilon}{\beta_1C_7}, \quad C_8:=\frac{\frac{l_\phi^2|G|^2}{2\varepsilon\beta_1}+\frac{1}{\varepsilon_2}}{C_7}. \tag{3.42}$$

Next, we apply Itô's formula to $|e^{Kt}(X_t - \bar{X}_t)|^2$ on time interval $[0, T]$. Under condition (3.5), by similar estimating argument with case 1, we can get

$$\begin{aligned}
&\mathbb{E}[|e^{KT}(X_T - \bar{X}_T)|^2] + (2\kappa_x - 2K - l_\sigma - (1+l_\sigma)\varepsilon_1)\,\mathbb{E}\int_0^T |e^{Kt}(X_t - \bar{X}_t)|^2 dt \\
&\leq \mathbb{E}[|\xi - \bar{\xi}|^2] + (2+\varepsilon_1) l_\phi \int_0^T e^{2Kt}(|Y_t - \bar{Y}_t|^2 + |Z_t - \bar{Z}_t|^2) dt \\
&\quad + \frac{1}{\varepsilon_1}\mathbb{E}\int_0^T e^{2Kt}\left|(\lambda_0\left(B(t,\bar{\Theta}_t) - \bar{B}(t,\bar{\Theta}_t)\right) + \mathcal{I}_t^B - \bar{\mathcal{I}}_t^B\right|^2 dt \\
&\quad + \left(\frac{1}{\varepsilon_1} + 1\right)\mathbb{E}\int_0^T e^{2Kt}\left|\lambda_0\left(\sigma(t,\bar{\Theta}_t) - \bar{\sigma}(t,\bar{\Theta}_t)\right) + \mathcal{I}_t^\sigma - \bar{\mathcal{I}}_t^\sigma\right|^2 dt.
\end{aligned} \tag{3.43}$$

Denote

$$C_9 := \frac{(2+\varepsilon_1) l_\phi}{2\kappa_x - 2K - l_\sigma - (1+l_\phi)\varepsilon_1}. \tag{3.44}$$

Then, letting $T \to \infty$, (3.43) is reduce to

$$\|X - \bar{X}\|_K^2 \leq C_9(\|Y - \bar{Y}\|_K^2 + \|Z - \bar{Z}\|_K^2) + C_2 \text{RHS}. \tag{3.45}$$

Substituting (3.41) into (3.45), we have

$$\|X - \bar{X}\|_K^2 \leq C_9(\tilde{\varepsilon}\|X - \bar{X}\|_K^2 + C_8 \text{RHS}) + C_2 \text{RHS}. \tag{3.46}$$

By choosing ε such that $1 - C_9\tilde{\varepsilon} > 0$, we have

$$\|X - \bar{X}\|_K^2 \leq \frac{C_9 C_8 + C_2}{1 - C_9\tilde{\varepsilon}} \text{RHS}. \tag{3.47}$$

Combined with (3.41), we obtain

$$\left\|X - \bar{X}\right\|_K^2 + \left\|Y - \bar{Y}\right\|_K^2 + \left\|Z - \bar{Z}\right\|_K^2 \leq \tilde{C} \text{RHS}, \tag{3.48}$$

where

$$\tilde{C} := \frac{(1+\tilde{\varepsilon})(C_9 C_8 + C_2)}{1 - C_9\tilde{\varepsilon}} + C_8.$$

The desired estimates (3.13) is obtained in case 2. Therefore, we obtain the estimate (3.13) in two cases. The whole proof is completed. □

Based on the above a priori estimate, we give a continuation lemma.

Lemma 3.7. *Suppose Assumptions* (H1) *and* (H2) *hold and let*

$$\kappa_x - \kappa_y > \max\{l_\sigma, l_z\} \quad \textit{and} \quad K = \frac{\kappa_x + \kappa_y}{2}. \tag{3.49}$$

Then there exists a constant $\delta_0 > 0$ independent of λ_0 such that if for some $\lambda_0 \in [0,1)$ and any $\left(\mathcal{I}^B, \mathcal{I}^F, \mathcal{I}^\sigma\right) \in L_{\mathbb{F}}^{2,K}\left(\mathbb{R}^{n+m+n\times d}\right)$, FBSDE (3.10) admits a unique solution $(X,Y,Z) \in L_{\mathbb{F}}^{2,K}(0,\infty; \mathbb{R}^{n+m+m\times d})$, then the same conclusion is also true for $\lambda = \lambda_0 + \delta$ with $\delta \in [0, \delta_0]$ and $\lambda \leq 1$.

Proof. Let $\delta_0 > 0$ be determined later, and $\delta \in [0, \delta_0]$. For any $\left(\mathcal{I}^B, \mathcal{I}^F, \mathcal{I}^\sigma\right) \in L^{2,K}_{\mathbb{F}}\left(0, \infty; \mathbb{R}^{n+m+n\times d}\right)$ and $\theta = (x, y, z) \in L^{2,K}_{\mathbb{F}}\left(0, \infty; \mathbb{R}^{n+m+m\times d}\right)$, we introduce an infinite horizon McKean-Vlasov FBSDE as follows:

$$\begin{cases} dX_t = \Big[\lambda_0 B\left(t, X_t, Y_t, Z_t, \mathcal{L}(X_t, Y_t, Z_t)\right) - (1-\lambda_0)\kappa_x X_t + \tilde{\mathcal{I}}^B_t\Big] dt \\ \qquad\qquad + \Big[\lambda_0 \sigma\left(t, X_t, Y_t, Z_t, \mathcal{L}\left(X_t, Y_t, Z_t\right)\right) + \tilde{\mathcal{I}}^\sigma_t\Big] dW_t, \\ dY_t = \Big[\lambda_0 F\left(t, X_t, Y_t, Z_t, \mathcal{L}\left(X_t, Y_t, Z_t\right)\right) - (1-\lambda_0)\kappa_y Y_t + \tilde{\mathcal{I}}^F_t\Big] dt + Z_t dW_t, \\ X_0 = \xi, \end{cases} \tag{3.50}$$

where

$$\begin{cases} \tilde{\mathcal{I}}^B_t = \delta\big(B(t, x_t, y_t, z_t, \mathcal{L}(x_t, y_t, z_t)) + \kappa_x x_t\big) + \mathcal{I}^B_t, \\ \tilde{\mathcal{I}}^F_t = \delta\big(F(t, x_t, y_t, z_t, \mathcal{L}(x_t, y_t, z_t)) + \kappa_y y_t\big) + \mathcal{I}^F_t, \\ \tilde{\mathcal{I}}^\sigma_t = \delta\big(\sigma(t, x_t, y_t, z_t, \mathcal{L}(x_t, y_t, z_t)\big) + \mathcal{I}^\sigma_t. \end{cases} \tag{3.51}$$

Since $(x, y, z) \in L^{2,K}_{\mathbb{F}}(0, \infty; \mathbb{R}^{n+m+m\times d})$ and $\left(\mathcal{I}^B, \mathcal{I}^F, \mathcal{I}^\sigma\right) \in L^{2,K}_{\mathbb{F}}\left(\mathbb{R}^{n+m+n\times d}\right)$, by Assumption (H1), it is easy to check that $(\tilde{\mathcal{I}}^B, \tilde{\mathcal{I}}^F, \tilde{\mathcal{I}}^\sigma) \in L^{2,K}_{\mathbb{F}}(0, \infty; \mathbb{R}^{n+m+n\times d})$. Then, by our assumptions, FBSDE (3.50) admits a unique solution $\Theta = (X, Y, Z) \in L^{2,K}_{\mathbb{F}}(0, \infty; \mathbb{R}^{n+m+m\times d})$. Due to the arbitrariness of θ, we can comprehend that FBSDE (3.50) defines a mapping from the Banach space $L^{2,K}_{\mathbb{F}}\left(0, \infty; \mathbb{R}^{n+m+m\times d}\right)$ into itself:

$$\Theta = \mathcal{M}_{\lambda_0+\delta}(\theta).$$

In the following, we shall prove that this mapping is contractive when δ is small enough.

For any $\theta = (x, y, z)$, $\theta' = (x', y', z') \in L^{2,K}_{\mathbb{F}}\left(0, \infty; \mathbb{R}^{n+m+m\times d}\right)$, let $\Theta = (X, Y, Z) = \mathcal{M}_{\lambda_0+\delta}(\theta)$ and $\Theta' = (X', Y', Z') = \mathcal{M}_{\lambda_0+\delta}(\theta')$. By Lemma 3.6, there exists a constant $C > 0$, independent of λ_0 such that

$$\begin{aligned} &\|X - X'\|^2_K + \|Y - Y'\|^2_K + \|Z - Z'\|^2_K \\ \le\ & C\Big\{ \Big\|\delta\Big(B\left(\cdot, \theta, \mathcal{L}(\theta)\right) - B\left(\cdot, \theta', \mathcal{L}(\theta')\right) + \kappa_x (x - x')\Big)\Big\|^2_K \\ &+ \Big\|\delta\Big(\sigma\left(\cdot, \theta, \mathcal{L}(\theta)\right) - \sigma\left(\cdot, \theta', \mathcal{L}(\theta')\right)\Big)\Big\|^2_K \\ &+ \Big\|\delta\Big(F\left(\cdot, \theta, \mathcal{L}(\theta)\right) - F\left(\cdot, \theta', \mathcal{L}(\theta')\right) + \kappa_y (y - y')\Big)\Big\|^2_K \Big\}. \end{aligned} \tag{3.52}$$

Combined the fact that

$$\mathcal{W}_2\left(\mathcal{L}\left(x_t, y_t, z_t\right), \mathcal{L}\left(x'_t, y'_t, z'_t\right)\right) \le \mathbb{E}\Big[|x_t - x'_t|^2\Big]^{\frac{1}{2}} + \mathbb{E}\Big[|y_t - y'_t|^2\Big]^{\frac{1}{2}} + \mathbb{E}\Big[|z_t - z'_t|^2\Big]^{\frac{1}{2}}, \tag{3.53}$$

and the Lipschitz continuity of functions B, F, σ, we have

$$\|X - X'\|^2_K + \|Y - Y'\|^2_K + \|Z - Z'\|^2_K \le \bar{C}\delta^2\left(\|x - x'\|^2_K + \|y - y'\|^2_K + \|z - z'\|^2_K\right), \tag{3.54}$$

where $\bar{C}$ only depending on $|G|, \kappa_x, \kappa_y, l, l_\sigma, l_\phi, l_z, \gamma, \beta_1$ or β_2 and independent of λ_0, which shows we can choose δ_0 independent of λ_0 such that $\bar{C}\delta^2 < 1$ when $\delta \le \delta_0$. Thus $\mathcal{M}_{\lambda_0+\delta}$ is a contraction mapping and the fixed point is the unique solution of infinite horizon McKean-Vlasov FBSDE (3.10) parameterized by $\lambda_0 + \delta$. ∎

Now, we shall establish the well-posedness for infinite horizon FBSDE (3.1) by applying Lemma 3.5 and Lemma 3.7.

Proof of Theorem 3.3. When $\lambda = 0$, Lemma 3.5 shows that FBSDE (3.10) is uniquely solvable in $L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^{n+m+m\times d})$ for any $\left(\mathcal{I}^B,\mathcal{I}^F,\mathcal{I}^\sigma\right) \in L_{\mathbb{F}}^{2,K}\left(0,\infty;\mathbb{R}^{n+m+n\times d}\right)$. Based on this, Lemma 3.7 further implies that FBSDE (3.10) is uniquely solvable for any $\lambda \in [0,1]$ and any $\left(\mathcal{I}^B,\mathcal{I}^F,\mathcal{I}^\sigma\right) \in L_{\mathbb{F}}^{2,K}\left(0,\infty;\mathbb{R}^{n+m+n\times d}\right)$. Especially, when $\lambda = 1$ and $\left(\mathcal{I}^B,\mathcal{I}^F,\mathcal{I}^\sigma\right) \equiv 0$, (3.10) coincides with the original FBSDE (3.1). Consequently, the unique solvability of (3.1) is obtained.

The estimates (3.8) is followed from (3.13) in Lemma 3.6 by letting $\lambda_0 = 1$, $\left(\mathcal{I}^B,\mathcal{I}^F,\mathcal{I}^\sigma\right) \equiv 0$ and $\left(\bar{\mathcal{I}}^B,\bar{\mathcal{I}}^F,\bar{\mathcal{I}}^\sigma\right) \equiv 0$.

Moreover, when $(\bar{B},\bar{F},\bar{\sigma}) \equiv 0$, $\xi = 0$, it is obvious $(0,0,0)$ is a solution to the corresponding FBSDE (3.1). Then we get the estimate (3.7) from (3.8). ☐

4 Infinite horizon mean field control problems

In this section, we investigate mean field control problems through the solvability results of infinite horizon McKean-Vlasov FBSDEs obtained in previous section. First, in subsection 4.1, we derive the corresponding infinite horizon FBSDE (4.5) by Pontryagin's stochastic maximum principle and solve the control problem given solutions to (4.5). Then in subsection 4.2, we provide sufficient conditions for the existence of solutions to (4.5). Let $A \in \mathbb{R}^m$ $(m \geq 1)$ be a convex control space. Suppose $b : [0,\infty)\times\mathbb{R}^n\times\mathcal{P}_2(\mathbb{R}^n)\times A \to \mathbb{R}^n$, $\sigma : [0,\infty)\times\mathbb{R}^n\times\mathcal{P}_2(\mathbb{R}^n)\times A \to \mathbb{R}^{n\times d}$, $f : [0,\infty)\times\mathbb{R}^n\times\mathcal{P}_2(\mathbb{R}^n)\times A \to \mathbb{R}$ are three measurable functions. We work under the following assumption.

Assumption 4.1. *(i) $b(t,x,\mu,\alpha),\sigma(t,x,\mu,\alpha)$ are Lipschitz in (x,μ,α) and $f(t,x,\mu,\alpha)$ is of at most quadratic growth in (x,μ,a). There exist positive constants $l_{bx}, l_{b\mu}, l_{\sigma x}, l_{\sigma\mu}, l_\alpha$ such that for any $t > 0$, $\alpha,\alpha' \in A$, $x,x' \in \mathbb{R}^n$, $\mu,\mu' \in \mathcal{P}_2(\mathbb{R}^n)$,*

$$
\begin{aligned}
|b(t,x,\mu,\alpha) - b(t,x',\mu',\alpha')| &\leq l_{bx}|x-x'| + l_\alpha|\alpha-\alpha'| + l_{b\mu}\mathcal{W}_2\left(\mu,\mu'\right), \\
|\sigma(t,x,\mu,\alpha) - \sigma(t,x',\mu',\alpha')| &\leq l_{\sigma x}|x-x'| + l_\alpha|\alpha-\alpha'| + l_{\sigma\mu}\mathcal{W}_2\left(\mu,\mu'\right).
\end{aligned}
\tag{4.1}
$$

(ii) There exists a constant $K \in \mathbb{R}$ such that $\int_0^\infty e^{2Kt}\left|f\left(t,0,\delta_{0_n},0\right)\right|dt < +\infty$ and $b\left(\cdot,0,\delta_{0_n},0\right) \in L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^n)$.

(iii) There exists a constant $\kappa > K + \frac{(l_{\sigma x}+l_{\sigma\mu})^2}{2} + l_{b\mu}$ such that for any $t > 0$, $\alpha \in A$, $\mu \in \mathcal{P}_2(\mathbb{R}^n)$, $x,x' \in \mathbb{R}^n$, it holds that

$$
\langle x-x', b(t,x,\mu,\alpha) - b\left(t,x',\mu,\alpha\right)\rangle \leq -\kappa\left|x-x'\right|^2.
$$

Define $\mathcal{A} := L_{\mathbb{F}}^{2,K}(0,\infty;A)$ to be the space of all admissible controls. For any control $\alpha \in \mathcal{A}$, it follows from Lemma 2.3 that under Assumption 4.1, the following controlled McKean-Vlasov SDE (4.3) admits a unique solution $X_t \in L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^n)$.

We consider the following mean field control problem: Minimize

$$
J(\alpha) := \mathbb{E}\left[\int_0^\infty e^{2Kt} f\left(t,X_t,\mathcal{L}\left(X_t\right),\alpha_t\right)dt\right],
\tag{4.2}
$$

over the set $\mathcal{A}$ of admissible control processes, which is finite under Assumption 4.1, subject to the dynamic constraint

$$
\begin{cases}
dX_t = b\left(t,X_t,\mathcal{L}\left(X_t\right),\alpha_t\right)dt + \sigma(t,X_t,\mathcal{L}\left(X_t\right),\alpha_t)dW_t, \\
X_0 = \xi.
\end{cases}
\tag{4.3}
$$

4.1 Pontryagin's stochastic maximum principle

In this subsection, we aim to establish the Pontryagin's stochastic maximum principle for the infinite mean field control problem (4.2)-(4.3). When the volatility σ of the state dynamics is a constant, the corresponding maximum principle was obtained in [4]. Now we extend the maximum principle to a more general state dynamics (4.3).

We define the Hamiltonian $\mathcal{H}$ by

$$\mathcal{H}(t,x,\mu,y,z,\alpha) := b(t,x,\mu,\alpha)\cdot y + \sigma(t,x,\mu,\alpha)\cdot z + f(t,x,\mu,\alpha) + 2Kx\cdot y, \tag{4.4}$$

for $t\in[0,\infty),\ \alpha\in A, x,y\in\mathbb{R}^n,\ z\in\mathbb{R}^{n\times d},\ \mu\in\mathcal{P}_2(\mathbb{R}^n)$, where the dot notation stands for the inner product in Euclidean space. Moreover, the Hamiltonian $\mathcal{H}$, which is assumed to be differentiable in (x,α,μ), is said to be convex in (x,μ,α) if for any $t\in[0,\infty),\ y\in\mathbb{R}^n,\ z\in\mathbb{R}^{n\times d}$, (x,α,μ), $(x',\alpha',\mu')\in\mathbb{R}^n\times A\times\mathcal{P}_2(\mathbb{R}^n)$, we have

$$\begin{aligned}\mathcal{H}\left(t,x',\mu',\alpha',y,z\right) \geq{}& \mathcal{H}(t,x,\mu,\alpha,y,z) + \partial_x\mathcal{H}(t,x,\mu,\alpha,y,z)\cdot(x'-x)\\ &+ \partial_\alpha\mathcal{H}(t,x,\mu,\alpha,y,z)\cdot(\alpha'-\alpha) + \tilde{\mathbb{E}}\left[\partial_\mu\mathcal{H}(t,x,\mu,\alpha,y,z)(\tilde{X})\cdot(\tilde{X}'-\tilde{X})\right],\end{aligned}$$

where $\tilde{X}',\tilde{X}$ are square integrable random variables defined on $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ and have distributions μ',μ respectively.

We assume the existence of a function $(t,x,y,z,\mu)\to\hat{\alpha}(t,x,y,z,\mu)\in A$, which is Lipschitz-continuous with respect to (x,y,z,μ), uniformly in $t\in[0,\infty)$ such that:

$$\hat{\alpha}(t,x,y,z,\mu) = \underset{\alpha\in A}{\operatorname{argmin}}\mathcal{H}(t,x,\mu,y,z,\alpha), \quad t\in[0,\infty),\ x,y\in\mathbb{R}^n,\ z\in\mathbb{R}^{n\times d},\ \mu\in\mathcal{P}_2(\mathbb{R}^n).$$

The existence of such function was proven in the next subsection 4.2 under specific assumptions on the drift b and the running cost function f. Then, we introduce the following infinite horizon McKean-Vlasov FBSDE:

$$\begin{cases} dX_t = b\left(t,X_t,\mathcal{L}(X_t),\hat{\alpha}(t,X_t,Y_t,Z_t,\mathcal{L}(X_t))\right)dt + \sigma(t,X_t,\mathcal{L}(X_t),\hat{\alpha}(t,X_t,Y_t,Z_t,\mathcal{L}(X_t)))dW_t,\\ dY_t = -\partial_x\mathcal{H}\left(t,X_t,\mathcal{L}(X_t),Y_t,Z_t,\hat{\alpha}(t,X_t,Y_t,Z_t,\mathcal{L}(X_t))\right)dt + Z_t dW_t\\ \qquad - \tilde{\mathbb{E}}\left[\partial_\mu\mathcal{H}\left(t,\tilde{X}_t,\mathcal{L}(X_t),\tilde{Y}_t,\tilde{Z}_t,\hat{\alpha}(t,\tilde{X}_t,\tilde{Y}_t,\tilde{Z}_t,\mathcal{L}(X_t))\right)(X_t)\right]dt,\\ X_0=\xi, \end{cases} \tag{4.5}$$

where $(\tilde{X},\tilde{Y},\tilde{Z})$ is an independent copy of (X,Y,Z) defined on the space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$, which is an independent copy of $(\Omega,\mathcal{F},\mathbb{P})$ and $\tilde{\mathbb{E}}$ denotes the expectation on $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$.

Proposition 4.2. *Let (b,f) be differentiable in (x,μ,α), Assumption 4.1 holds and $\mathcal{H}$ be convex in (x,μ,α), $\hat{\alpha}(\cdot,0,0,0,\delta_{0_n})\in L^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^n)$ and $\hat{\alpha}(t,x,y,z,\mu)$ is Lipschitz in (x,y,z,μ). Moreover, suppose infinite horizon McKean-Vlasov FBSDE (4.5) admits a unique solution $(X,Y,Z)\in L^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^{n+n+n\times d})$. Then we have that $J(\hat{\alpha})=\min_\alpha J(\alpha)$.*

Proof. Since we assume infinite horizon McKean-Vlasov FBSDE (4.5) admits a unique solution (X,Y,Z), let us denote $\theta^\wedge_t := (X_t,Y_t,Z_t)$, $\tilde{\theta}^\wedge_t := (\tilde{X}_t,\tilde{Y}_t,\tilde{Z}_t)$, $\Theta^\wedge_t := (\theta^\wedge_t,\mathcal{L}(X_t),\hat{\alpha}(t,\theta^\wedge_t,\mathcal{L}(X_t)))$ and $\tilde{\Theta}^\wedge_t := (\tilde{\theta}^\wedge_t,\mathcal{L}(X_t),\hat{\alpha}(t,\tilde{\theta}^\wedge_t,\mathcal{L}(X_t)))$, where $(\tilde{X}_t,\tilde{Y}_t,\tilde{Z}_t)$ is an independent copy of (X_t,Y_t,Z_t). For an arbitrary admissible control α' and its associated process X', we have that

$$
\begin{aligned}
J(\hat{\alpha}) - J(\alpha') = & \mathbb{E}\left[\int_0^\infty e^{2Kt}\left(\mathcal{H}\left(t, X_t, \mathcal{L}\left(X_t\right), \hat{\alpha}_t, Y_t, Z_t\right) - \mathcal{H}\left(t, X_t', \mathcal{L}\left(X_t'\right), \alpha_t', Y_t, Z_t\right)\right) dt\right] \\
& - \mathbb{E}\left[\int_0^\infty e^{2Kt}\left(b\left(t, X_t, \mathcal{L}\left(X_t\right), \hat{\alpha}_t\right) - b\left(t, X_t', \mathcal{L}\left(X_t'\right), \alpha_t'\right)\right) \cdot Y_t dt\right] \\
& - \mathbb{E}\left[\int_0^\infty e^{2Kt}\left(\sigma\left(t, X_t, \mathcal{L}\left(X_t\right), \hat{\alpha}_t\right) - \sigma\left(t, X_t', \mathcal{L}\left(X_t'\right), \alpha_t'\right)\right) \cdot Z_t dt\right] \\
& - 2K\mathbb{E}\left[\int_0^\infty e^{2Kt}\left(X_t - X_t'\right) \cdot Y_t dt\right].
\end{aligned} \tag{4.6}
$$

It can be easily seen from Lemma 2.2, that there exists a sequence of $T_i \to \infty$ such that

$$
\mathbb{E}\left[e^{2KT_i}\left(X_{T_i} - X_{T_i}'\right) \cdot Y_{T_i}\right] \to 0.
$$

Applying Itô's formula to $e^{2Kt}\left(X_t - X_t'\right) \cdot Y_t$ on time interval $[0, T_i]$, and letting $T_i \to \infty$, we obtain that

$$
\begin{aligned}
& \mathbb{E}\left[\int_0^\infty e^{2Kt}\left(X_t - X_t'\right) \cdot \left(\partial_x \mathcal{H}\left(t, \Theta_t^\wedge\right) + \tilde{\mathbb{E}}\left[\partial_\mu \mathcal{H}(\tilde{\Theta}_t^\wedge)\left(X_t\right)\right]\right) dt\right] \\
= & \mathbb{E}\left[\int_0^\infty e^{2Kt}\left(2K\left(X_t - X_t'\right) + b\left(t, X_t, \mathcal{L}\left(X_t\right), \hat{\alpha}_t\right) - b\left(t, X_t', \mathcal{L}\left(X_t'\right), \alpha_t'\right)\right) \cdot Y_t dt\right] \\
& + \mathbb{E}\left[\int_0^\infty e^{2Kt}\left(\sigma\left(t, X_t, \mathcal{L}\left(X_t\right), \hat{\alpha}_t\right) - \sigma\left(t, X_t', \mathcal{L}\left(X_t'\right), \alpha_t'\right)\right) \cdot Z_t dt\right].
\end{aligned} \tag{4.7}
$$

According to the convexity of $\mathcal{H}$ and the fact that $\hat{\alpha}_t = \operatorname{argmin}_{\alpha \in A} \mathcal{H}\left(t, X_t, \mathcal{L}\left(X_t\right), \alpha, Y_t, Z_t\right)$, it holds that

$$
\begin{aligned}
& \mathcal{H}\left(t, X_t', \mathcal{L}\left(X_t'\right), \alpha_t', Y_t, Z_t\right) - \mathcal{H}\left(t, X_t, \mathcal{L}\left(X_t\right), \hat{\alpha}_t, Y_t, Z_t\right) \\
\geq & \left(X_t' - X_t\right) \cdot \partial_x \mathcal{H}\left(t, \Theta_t^\wedge\right) + \tilde{\mathbb{E}}\left[\partial_\mu \mathcal{H}\left(t, \Theta_t^\wedge\right)\left(\tilde{X}_t\right) \cdot \left(\tilde{X}_t' - \tilde{X}_t\right)\right] \\
& + \left(\alpha_t' - \hat{\alpha}_t\right) \cdot \partial_\alpha \mathcal{H}\left(t, \Theta_t^\wedge\right) \\
\geq & \left(X_t' - X_t\right) \cdot \partial_x \mathcal{H}\left(t, \Theta_t^\wedge\right) + \tilde{\mathbb{E}}\left[\partial_\mu \mathcal{H}\left(t, \Theta_t^\wedge\right)\left(\tilde{X}_t\right) \cdot \left(\tilde{X}_t' - \tilde{X}_t\right)\right].
\end{aligned} \tag{4.8}
$$

Using Fubini's theorem and the fact $\tilde{\Theta}_t^\wedge$ is an independent copy of $\Theta_t^\wedge$, we have that

$$
\mathbb{E}\left[\left(X_t' - X_t\right) \cdot \tilde{\mathbb{E}}\left[\partial_\mu \mathcal{H}(\tilde{\Theta}_t^\wedge)\left(X_t\right)\right]\right] = \mathbb{E}\left[\tilde{\mathbb{E}}\left[\partial_\mu \mathcal{H}\left(t, \Theta_t^\wedge\right)\left(\tilde{X}_t\right) \cdot \left(\tilde{X}_t' - \tilde{X}_t\right)\right]\right]. \tag{4.9}
$$

Combined with (4.6), (4.7), (4.8) and (4.9), we conclude that

$$
\begin{aligned}
J(\hat{\alpha}) - J(\alpha') = & \mathbb{E}\left[\int_0^\infty e^{2Kt}\left(\mathcal{H}\left(t, X_t, \mathcal{L}\left(X_t\right), \hat{\alpha}_t, Y_t, Z_t\right) - \mathcal{H}\left(t, X_t', \mathcal{L}\left(X_t'\right), \alpha_t', Y_t, Z_t\right)\right) dt\right] \\
& - \mathbb{E}\left[\int_0^\infty e^{2Kt}\left(X_t - X_t'\right) \cdot \left(\partial_x \mathcal{H}\left(t, \Theta_t^\wedge\right) + \tilde{\mathbb{E}}\left[\partial_\mu \mathcal{H}\left(\tilde{\Theta}_t^\wedge\right)\left(X_t\right)\right]\right) dt\right] \\
= & \mathbb{E}\left[\int_0^\infty e^{2Kt}\left(\mathcal{H}\left(t, X_t, \mathcal{L}\left(X_t\right), \hat{\alpha}_t, Y_t, Z_t\right) - \mathcal{H}\left(t, X_t', \mathcal{L}\left(X_t'\right), \alpha_t', Y_t, Z_t\right)\right) dt\right] \\
& - \mathbb{E}\left[\int_0^\infty e^{2Kt}\left(\left(X_t - X_t'\right) \cdot \partial_x \mathcal{H}\left(t, \Theta_t^\wedge\right) + \tilde{\mathbb{E}}\left[\partial_\mu \mathcal{H}\left(t, \Theta_t^\wedge\right)\left(\tilde{X}_t\right) \cdot \left(\tilde{X}_t - \tilde{X}_t'\right)\right]\right) dt\right] \\
\leq & 0.
\end{aligned} \tag{4.10}
$$

□

4.2 Solvability of mean field control problems

In this subsection, we give sufficient conditions on the given data for the existence and uniqueness of solutions to infinite horizon FBSDE (4.5). As it is most often the case in applications of the maximum principle, we choose $A = \mathbb{R}^m$ and $\mathcal{A} := L^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^m)$ to be the space of all admissible controls. We consider a linear model for the forward dynamics of the state.

Assumption 4.3. *(i) The drift b and the volatility σ are linear in μ, x and α. They read*

$$
\begin{aligned}
b(t,x,\mu,\alpha) &= b_0(t) + b_1(t)x + b_2(t)\bar{\mu} + b_3(t)\alpha,\\
\sigma(t,x,\mu,\alpha) &= \sigma_0(t) + \sigma_1(t)x + \sigma_2(t)\bar{\mu} + \sigma_3(t)\alpha,
\end{aligned}
$$

for some bounded measurable deterministic functions b_0, b_1, b_2, b_3 with values in $\mathbb{R}^n, \mathbb{R}^{n\times n}, \mathbb{R}^{n\times n}$ and $\mathbb{R}^{n\times m}$ and $\sigma_0, \sigma_1, \sigma_2$, σ_3 with values in $\mathbb{R}^{n\times d}$, $\mathbb{R}^{(n\times d)\times n}$, $\mathbb{R}^{(n\times d)\times n}$ and $\mathbb{R}^{(n\times d)\times m}$(the parentheses around $n\times d$ indicating that $\sigma_i(t)u_i$ is seen as an element of $\mathbb{R}^{n\times d}$ whenever $u_i \in \mathbb{R}^n$, with $i = 1, 2$, or $u_i \in \mathbb{R}^m$, with $i = 3$), $b_0(\cdot) \in L^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^n)$, $\sigma_0(\cdot) \in L^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^{n\times d})$ and we use the notation $\bar{\mu} = \int x d\mu(x)$ for the mean of a measure μ.
(ii) f is differentiable with respect to (x,μ,α) and $|f(\cdot,0,\delta_{0_n},0)| \in L^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R})$. Moreover, the derivatives satisfy that $\partial_x f(\cdot,0,\delta_{0_n},0), \partial_\mu f(\cdot,0,0,\delta_{0_n})(0) \in L^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^n)$ and $\partial_\alpha f(\cdot,0,\delta_{0_n},0) \in L^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^m)$.
(iii) There exists a positive constant $\tilde{L}$ such that for all $t \geq 0$, the functions $\partial_x f, \partial_\alpha f$ are $\tilde{L}$-Lipschitz continuous with respect to (x,α,μ). Moreover, there exists a version of $\partial_\mu f(t,x',\mu,\alpha)(\cdot)$ such that $\partial_\mu f(t,x',\mu,\alpha)(x)$ is $\tilde{L}$-Lipschitz in (x',μ,α,x) [1], the Lipschitz property in the variable μ being understood in the sense of the 2-Wassertein distance.
(iv) The function f is convex with respect to (x,μ,α) for $t \geq 0$ in such a way that, for some $\lambda > 0$,

$$
\begin{aligned}
&f(t,x',\mu',\alpha') - f(t,x,\mu,\alpha) - \partial_{(x,\alpha)} f(t,x,\mu,\alpha)\cdot(x'-x,\alpha'-\alpha)\\
&\quad - \tilde{\mathbb{E}}\left[\partial_\mu f(t,x,\mu,\alpha)(\tilde{X})\cdot(\tilde{X}'-\tilde{X})\right] \geq \lambda\left|\alpha'-\alpha\right|^2,
\end{aligned}
$$

whenever $\tilde{X}, \tilde{X}'$ with distributions μ and μ', respectively.

Now, we introduce some notations about bounded measurable functions valued in $\mathbb{R}^{n\times n}$. It is obvious that for any $t > 0$, the matrix $(b_1(t) + b_1(t)^\top)$ is symmetrical. Let $\lambda_{\max}(b_1(t)+b_1(t)^\top)$ denote the largest eigenvalue of $(b_1(t)+b_1(t)^\top)$. Due to the boundedness of $b_1(t)$, it is clear that

$$
\frac{1}{2}\sup_{t\in[0,\infty)} \lambda_{\max}(b_1(t)+b_1(t)^\top) < +\infty. \tag{4.11}
$$

Then, we have

$$
\begin{aligned}
\langle b_1(t)x, x\rangle &= \frac{1}{2}\left\langle (b_1(t)+b_1(t)^\top)x, x\right\rangle \leq \frac{1}{2}\lambda_{\max}(b_1(t)+b_1(t)^\top)|x|^2\\
&\leq \frac{1}{2}\sup_{t\in[0,\infty)} \lambda_{\max}(b_1(t)+b_1(t)^\top)|x|^2,
\end{aligned} \tag{4.12}
$$

for any $t > 0$ and $x \in \mathbb{R}^n$.
Then in this linear setting, to ensure the control problem well-defined, we can choose κ in Assumption 4.1 (iii) as

$$
\kappa = -\frac{1}{2}\sup_{t\in[0,\infty)} \lambda_{\max}(b_1(t)+b_1(t)^\top), \tag{4.13}
$$

[1]For the Lipschitz property of $(t,x',\mu,\alpha,x) \mapsto \partial_\mu f(t,x',\mu,\alpha)(x)$, [9, Lemma 5.41] provides a simple criterion.

and the corresponding relationship of κ and K is reduced to

$$K < -\frac{1}{2}\sup_{t\in[0,\infty)} \lambda_{\max}(b_1(t)+b_1(t)^\top) - \|b_2(\cdot)\|_\infty - \frac{(\|\sigma_1(\cdot)\|_\infty + \|\sigma_2(\cdot)\|_\infty)^2}{2}. \tag{4.14}$$

Since the drift and the volatility are linear, the Hamiltonian defined as (4.4) takes the following particular form:

$$\begin{aligned} H(t,x,\mu,y,z,\alpha) = &[b_0(t)+b_1(t)x+b_2(t)\bar{\mu}+b_3(t)\alpha]\cdot y \\ &+ [\sigma_0(t)+\sigma_1(t)x+\sigma_2(t)\bar{\mu}+\sigma_3(t)\alpha]\cdot z + f(t,x,\mu,\alpha) + 2Kx\cdot y, \end{aligned} \tag{4.15}$$

for $t\in[0,\infty),\ \alpha\in\mathbb{R}^m,\ x,y\in\mathbb{R}^n,\ z\in\mathbb{R}^{n\times d},\ \mu\in\mathcal{P}_2(\mathbb{R}^n)$.

First, using similar argument in [9, Lemma 3.3, Lemma 6.18], we can easily obtain the following result about the minimization of the Hamiltonian (4.15).

Lemma 4.4. *Let Assumption 4.3 holds. Then for any $(t,x,\mu,y,z,\alpha)\in[0,\infty)\times\mathbb{R}^n\times\mathcal{P}_2(\mathbb{R}^n)\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\times\mathbb{R}^m$, there exists a unique minimizer $\hat{\alpha}(t,x,\mu,y,z)$ of Hamiltonian H. Moreover, the function $[0,\infty)\times\mathbb{R}^n\times\mathcal{P}_2(\mathbb{R}^n)\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\ni(t,x,\mu,y,z)\mapsto\hat{\alpha}(t,x,\mu,y,z)\in\mathbb{R}^m$ is measurable, locally bounded and Lipschitz continuous with respect to (x,μ,y,z), uniformly in $t\in[0,\infty)$, the Lipschitz constant depending only upon λ, $\|b_3(\cdot)\|_\infty$, $\|\sigma_3(\cdot)\|_\infty$ and the Lipschitz constant $\tilde{L}$ of $\partial_\alpha f$ in (x,μ). In fact, an explicit upper bound for $\hat{\alpha}$ reads:*

$$\begin{aligned} &\forall(t,x,\mu,y,z)\in[0,\infty)\times\mathbb{R}^n\times\mathcal{P}_2(\mathbb{R}^n)\times\mathbb{R}^n\times\mathbb{R}^{n\times d}, \\ &\qquad |\hat{\alpha}(t,x,\mu,y,z)| \le \lambda^{-1}(|\partial_\alpha f(t,x,\mu,\beta)| + |b_3(t)||y| + |\sigma_3(t)||z|) + |\beta_t|, \end{aligned} \tag{4.16}$$

where β_t is any admissible control in $\mathcal{A}$, and then $\hat{\alpha}(\cdot,0,\delta_{0_n},0,0)\in L^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^m)$.

Then the infinite horizon McKean-Vlasov FBSDE corresponding to (4.5) reads

$$\begin{cases} dX_t = [b_0(t)+b_1(t)X_t+b_2(t)\mathbb{E}[X_t]+b_3(t)\hat{\alpha}(t,X_t,\mathcal{L}(X_t),Y_t,Z_t)]\,dt \\ \qquad\quad + [\sigma_0(t)+\sigma_1(t)X_t+\sigma_2(t)\mathbb{E}[X_t]+\sigma_3(t)\hat{\alpha}(t,X_t,\mathcal{L}(X_t),Y_t,Z_t)]\,dW_t, \\ dY_t = -[\partial_x f(t,X_t,\mathcal{L}(X_t),\hat{\alpha}(t,X_t,\mathcal{L}(X_t),Y_t,Z_t)) + b_1(t)Y_t + 2KY_t + \sigma_1(t)Z_t]\,dt + Z_t dW_t \\ \qquad\quad - \left\{\tilde{\mathbb{E}}\left[\partial_\mu f(t,\tilde{X}_t,\mathcal{L}(X_t),\hat{\alpha}(t,\tilde{X}_t,\mathcal{L}(X_t),\tilde{Y}_t,\tilde{Z}_t))(X_t)\right] + b_2(t)\mathbb{E}[Y_t] + \sigma_2(t)\mathbb{E}[Z_t]\right\}dt. \end{cases} \tag{4.17}$$

Next, based on Lemma 4.4, we can follow the arguments in the proof of [4, Lemma 3.2] to prove that the following function is Lipschitz:

$$\begin{aligned} \Psi(t,x,m) &:= \tilde{\mathbb{E}}\left[\partial_\mu f(t,\tilde{X}_t,\mathcal{L}(X_t),\hat{\alpha}(t,\tilde{X}_t,\mathcal{L}(X_t),\tilde{Y}_t,\tilde{Z}_t))(X_t)\right] \\ &= \int_{x',y',z'} \partial_\mu f(t,x',\mu,\hat{\alpha}(t,x',\mu,y',z'))(x)dm(x',y',z'), \end{aligned} \tag{4.18}$$

where $m\in\mathcal{P}_2(\mathbb{R}^{n+n+n\times d})$ and μ is the first marginal of m. To avoid repetition, the detailed proof of the following lemma is omitted.

Lemma 4.5. *Under Assumption 4.3, for any $t>0$, $x,\bar{x}\in\mathbb{R}^n, m,\bar{m}\in\mathcal{P}_2\left(\mathbb{R}^{n+n+n\times d}\right)$, it holds that*

$$|\Psi(t,x,m)-\Psi(t,\bar{x},\bar{m})| \le C_\Psi\mathcal{W}_2(m,\bar{m}) + \tilde{L}|x-\bar{x}|, \tag{4.19}$$

where C_Ψ depending only upon the Lipschitz constant of $\hat{\alpha}$ in (x,μ,y,z), the Lipschitz constant $\tilde{L}$ of $\partial_\mu f$ in (x',μ,α,x).

Now we give the main result of this section. It is worth noting that the condition (4.20) for parameter K is exactly the requirement (4.14) to ensure the infinite horizon mean field control problems well-defined.

Theorem 4.6. *Suppose Assumption 4.3 holds. Let*

$$K < -\frac{1}{2}\sup_{t\in[0,\infty)} \lambda_{\max}(b_1(t)+b_1(t)^\top) - \|b_2(\cdot)\|_\infty - \frac{(\|\sigma_1(\cdot)\|_\infty + \|\sigma_2(\cdot)\|_\infty)^2}{2}. \tag{4.20}$$

Then infinite horizon FBSDE (4.17) admits a unique solution $(X,Y,Z) \in L^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^{n+n+n\times d})$ and $\hat{\alpha}(t,X_t,\mathcal{L}(X_t),Y_t,Z_t)$, $t\in[0,\infty)$ is the optimal control of the infinite horizon mean field control problem (4.2)-(4.3).

Proof. Under Assumption 4.3, by Lemma 4.4, $\hat{\alpha}_t$ is Lipschitz in (x,μ,y,z) and $\hat{\alpha}(\cdot,0,0,0,\delta_{0_n}) \in L^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^m)$. The linearity of (b,σ) and the convexity of f in Assumption 4.3 imply that the Hamiltonian H is convex in (x,μ,α). Indeed, for all $(t,y,z)\in[0,T]\in\mathbb{R}^n\times\mathbb{R}^{n\times d}$, (x,μ,α), $(x',\mu',\alpha')\in \mathbb{R}^n\times\mathcal{P}_2(\mathbb{R}^n)\times\mathbb{R}^m$, we have

$$\begin{aligned}
& H(t,x',\alpha',\mu',y,z) - H(t,x,a,\mu,y,z) - \left\langle \partial_{(x,\alpha)}H(t,x,\alpha,\mu,y,z),(x'-x,\alpha'-\alpha)\right\rangle \\
&\quad - \tilde{\mathbb{E}}\left[\langle \partial_\mu H(t,x,\alpha,\mu,y,z)(\tilde{X}),\tilde{X}'-\tilde{X}\rangle\right] \\
&= b_2(\bar{\mu}'-\bar{\mu})\cdot y + \sigma_2(\bar{\mu}'-\bar{\mu})\cdot z + f(t,x',\alpha',\mu') - f(t,x,\alpha,\mu) \\
&\quad - \tilde{\mathbb{E}}\left[\langle b_2 y + \sigma_2 z + \partial_\mu f(t,x,\alpha,\mu)(\tilde{X}),\tilde{X}'-\tilde{X}\rangle\right] \\
&= f(t,x',\alpha',\mu') - f(t,x,\alpha,\mu) - \tilde{\mathbb{E}}\left[\langle \partial_\mu f(t,x,\alpha,\mu)(\tilde{X}),\tilde{X}'-\tilde{X}\rangle\right] \\
&\geq \lambda\left|\alpha'-\alpha\right|^2,
\end{aligned} \tag{4.21}$$

whenever $\tilde{X},\tilde{X}'\in L^2\left(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}};\mathbb{R}\right)$ with distributions μ and μ' and we use the notation $\bar{\mu},\bar{\mu}'$ for the mean of a measure μ, μ', respectively.

Then, from Proposition 4.2, it remains to prove the well-posedness of FBSDE (4.17) and we will apply Theorem 3.3 to obtain the solvability results. By Assumption 4.3, Lemma 4.4 and Lemma 4.5, it is easy to show Assumption (H1) holds. Now it remains to verify Assumption (H2) and condition (3.6) to obtain the desired conclusion.
For simplicity of notation, we denote $\Theta_1 := (X_1,Y_1,Z_1)$, $\Theta_2 := (X_2,Y_2,Z_2)\in L^2(\Omega;\mathbb{R}^{n+n+n\times d})$ and

$$\alpha_i = \hat{\alpha}(t,X_i,\mathcal{L}(X_i),Y_i,Z_i), \quad \tilde{\alpha}_i = \hat{\alpha}(t,\tilde{X}_i,\mathcal{L}(X_i),\tilde{Y}_i,\tilde{Z}_i), \quad i=1,2. \tag{4.22}$$

First, we show the monotonicity condition (3.2) holds. It is clear that the Hamiltonian system (4.17) is a special case of FBSDE (3.1) with

$$\begin{aligned}
B(t,\Theta,\mathcal{L}(\Theta)) &= b_0(t) + b_1(t)X + b_2(t)\mathbb{E}[X] + b_3(t)\hat{\alpha}(t,X,\mathcal{L}(X),Y,Z) \\
\sigma(t,\Theta,\mathcal{L}(\Theta)) &= \sigma_0(t) + \sigma_1(t)X + \sigma_2(t)\mathbb{E}[X] + \sigma_3(t)\hat{\alpha}(t,X,\mathcal{L}(X),Y,Z) \\
F(t,\Theta,\mathcal{L}(\Theta)) &= -\Big\{\partial_x f(t,X,\mathcal{L}(X),\hat{\alpha}(t,X,\mathcal{L}(X),Y,Z)) + b_1(t)Y + 2KY + \sigma_1(t)Z \\
&\qquad + \tilde{\mathbb{E}}\left[\partial_\mu f(t,\tilde{X},\mathcal{L}(X),\hat{\alpha}(t,\tilde{X},\mathcal{L}(X),\tilde{Y},\tilde{Z}))(X)\right] + b_2(t)\mathbb{E}[Y] + \sigma_2(t)\mathbb{E}[Z]\Big\}.
\end{aligned} \tag{4.23}$$

Let κ_x,κ_y be determined later, $K=\frac{\kappa_x+\kappa_y}{2}$, $G=\mathbb{I}_n$ and choose the measurable function ϕ_2 appearing in Assumption (H2) to be

$$\phi_2(t,\Theta_1,\Theta_2,\mathcal{L}(\Theta_1),\mathcal{L}(\Theta_2)) := \mathbb{E}\left[|\hat{\alpha}(t,X_1,\mathcal{L}(X_1),Y_1,Z_1) - \hat{\alpha}(t,X_2,\mathcal{L}(X_2),Y_2,Z_2)|^2\right].$$

Denote

$$\bar{H}(t,x,\mu,y,z,\alpha) = [b_0(t)+b_1(t)x+b_2(t)\bar{\mu}+b_3(t)\alpha]\cdot y + [\sigma_0(t)+\sigma_1(t)x+\sigma_2(t)\bar{\mu}+\sigma_3(t)\alpha]\cdot z \\ + f(t,x,\mu,\alpha). \tag{4.24}$$

It is obvious that $H(t,x,\mu,y,z,\alpha) = \bar{H}(t,x,\mu,y,z,\alpha) + 2Kx\cdot y$ and it has the same convex property as (4.21). From the definition of B and the linearity of $\bar{H}$ in (y,z), we can deduce that

$$\begin{aligned} &\mathbb{E}\Big[\left\langle B\left(t,\Theta_1,\mathcal{L}(\Theta_1)\right) - B\left(t,\Theta_2,\mathcal{L}(\Theta_2)\right), Y_1 - Y_2\right\rangle + \left\langle \sigma\left(t,\Theta_1,\mathcal{L}(\Theta_1)\right) - \sigma\left(t,\Theta_2,\mathcal{L}(\Theta_2)\right), Z_1 - Z_2\right\rangle\Big] \\ &= \mathbb{E}\Big[\bar{H}\left(t,X_1,\alpha_1,\mathcal{L}(X_1),Y_1,Z_1\right) - \bar{H}\left(t,X_1,\alpha_1,\mathcal{L}(X_1),Y_2,Z_2\right)\Big] \\ &\quad - \mathbb{E}\Big[\bar{H}\left(t,X_2,\alpha_2,\mathcal{L}(X_2),Y_1,Z_1\right) - \bar{H}\left(t,X_2,\alpha_2,\mathcal{L}(X_2),Y_2,Z_2\right)\Big]. \end{aligned} \tag{4.25}$$

Moreover, by the definition of F, we can obtain that

$$\begin{aligned} &\mathbb{E}[\langle F\left(t,\Theta_1,\mathcal{L}(\Theta_1)\right) - F\left(t,\Theta_2,\mathcal{L}(\Theta_2)\right), X_1 - X_2\rangle] \\ &= -\mathbb{E}\left[\left\langle \partial_x\bar{H}\left(t,X_1,\alpha_1,\mathcal{L}(X_1),Y_1,Z_1\right), X_1 - X_2\right\rangle\right] \\ &\quad - \mathbb{E}\Big[\tilde{\mathbb{E}}[\langle \partial_\mu\bar{H}\left(t,X_1,\alpha_1,\mathcal{L}(X_1),Y_1,Z_1\right)(\tilde{X}_1), \tilde{X}_1 - \tilde{X}_2\rangle]\Big] \\ &\quad + \mathbb{E}\left[\left\langle \partial_x\bar{H}\left(t,X_2,\alpha_2,\mathcal{L}(X_2),Y_2,Z_2\right), X_1 - X_2\right\rangle\right] \\ &\quad + \mathbb{E}\Big[\tilde{\mathbb{E}}[\langle \partial_\mu\bar{H}\left(t,X_2,\alpha_2,\mathcal{L}(X_2),Y_2,Z_2\right)(\tilde{X}_2), \tilde{X}_1 - \tilde{X}_2\rangle]\Big] \\ &\quad - 2K\mathbb{E}\left[\langle X_1 - X_2, Y_1 - Y_2\rangle\right], \end{aligned} \tag{4.26}$$

where we have also applied Fubini's theorem and the fact that $\mathcal{L}\left(X_i,Y_i,Z_i,\alpha_i\right) = \mathcal{L}(\tilde{X}_i,\tilde{Y}_i,\tilde{Z}_i,\tilde{\alpha}_i)$ for $i = 1,2$. Therefore, we can conclude that

$$\begin{aligned} &\mathbb{E}\left[\langle B\left(t,\Theta_1,\mathcal{L}(\Theta_1)\right) - B\left(t,\Theta_2,\mathcal{L}(\Theta_2)\right), Y_1 - Y_2\rangle\right] \\ &\quad + \mathbb{E}\left[\langle \sigma\left(t,\Theta_1,\mathcal{L}(\Theta_1)\right) - \sigma\left(t,\Theta_2,\mathcal{L}(\Theta_2)\right), Z_1 - Z_2\rangle\right] \\ &\quad + \mathbb{E}\left[\langle F\left(t,\Theta_1,\mathcal{L}(\Theta_1)\right) - F\left(t,\Theta_2,\mathcal{L}(\Theta_2)\right), X_1 - X_2\rangle\right] \\ &\quad + 2K\mathbb{E}[\langle X_1 - X_2, Y_1 - Y_2\rangle] \\ &= \mathbb{E}\Bigg[\bar{H}\left(t,X_1,\alpha_1,\mathcal{L}(X_1),Y_1,Z_1\right) - \bar{H}\left(t,X_2,\alpha_2,\mathcal{L}(X_2),Y_1,Z_1\right) \\ &\quad - \left\langle \partial_x\bar{H}\left(t,X_1,\alpha_1,\mathcal{L}(X_1),Y_1,Z_1\right), X_1 - X_2\right\rangle \\ &\quad - \tilde{\mathbb{E}}\left[\langle \partial_\mu\bar{H}\left(t,X_1,\alpha_1,\mathcal{L}(X_1),Y_1,Z_1\right)(\tilde{X}_1), \tilde{X}_1 - \tilde{X}_2\rangle\right]\Bigg] \\ &\quad - \mathbb{E}\Bigg[\bar{H}\left(t,X_1,\alpha_1,\mathcal{L}(X_1),Y_2,Z_2\right) - \bar{H}\left(t,X_2,\alpha_2,\mathcal{L}(X_2),Y_2,Z_2\right) \\ &\quad - \left\langle \partial_x\bar{H}\left(t,X_2,\alpha_2,\mathcal{L}(X_2),Y_2,Z_2\right), X_1 - X_2\right\rangle \\ &\quad - \tilde{\mathbb{E}}\left[\langle \partial_\mu\bar{H}\left(t,X_2,\alpha_2,\mathcal{L}(X_2),Y_2,Z_2\right)(\tilde{X}_2), \tilde{X}_1 - \tilde{X}_2\rangle\right]\Bigg] \\ &\leq -2\lambda\mathbb{E}\left[|\alpha_1 - \alpha_2|^2\right] \\ &\leq -2\lambda\phi_2(t,\Theta_1,\Theta_2,\mathcal{L}(\Theta_1),\mathcal{L}(\Theta_2)), \end{aligned}$$

which corresponds to $\beta_2 > 0$ in monotonicity condition (3.2). Then, we check the condition of case 1 in Assumption (H2)(ii).
For the coefficient function σ, for any $\varepsilon_\sigma > 0$, we have

$$
\begin{aligned}
&\mathbb{E}\left[|\sigma(t,\Theta_1,\mathcal{L}(\Theta_1)) - \sigma(t,\Theta_2,\mathcal{L}(\Theta_2))|^2\right] \\
&= \mathbb{E}\left[\left|\sigma_1(t)(X_1 - X_2) + \sigma_2(t)(\mathbb{E}[X_1] - \mathbb{E}[X_2]) + \sigma_3(t)(\alpha_1 - \alpha_2)\right|^2\right] \\
&\leq \Big((\|\sigma_1(\cdot)\|_\infty + \|\sigma_2(\cdot)\|_\infty)^2 + \varepsilon_\sigma\Big)\mathbb{E}[|X_1 - X_2|^2] \\
&\quad + \left(\|\sigma_3(\cdot)\|_\infty^2 + \frac{1}{4\varepsilon_\sigma}\|\sigma_3(\cdot)\|_\infty^2(\|\sigma_1(\cdot)\|_\infty^2 + \|\sigma_2(\cdot)\|_\infty^2)\right)\phi_2(t,\Theta_1,\Theta_2,\mathcal{L}(\Theta_1),\mathcal{L}(\Theta_2)), \\
&= l_\sigma\mathbb{E}[|X_1 - X_2|^2] + \left(\|\sigma_3(\cdot)\|_\infty^2 + \frac{1}{4\varepsilon_\sigma}\|\sigma_3(\cdot)\|_\infty^2(\|\sigma_1(\cdot)\|_\infty^2 + \|\sigma_2(\cdot)\|_\infty^2)\right)\phi_2(t,\Theta_1,\Theta_2,\mathcal{L}(\Theta_1),\mathcal{L}(\Theta_2)),
\end{aligned}
\tag{4.27}
$$

where

$$
l_\sigma = (\|\sigma_1(\cdot)\|_\infty + \|\sigma_2(\cdot)\|_\infty)^2 + \varepsilon_\sigma.
$$

As for coefficient function B, we have

$$
\begin{aligned}
&\mathbb{E}\left[\langle B(t,\Theta_1,\mathcal{L}(\Theta_1)) - B(t,\Theta_2,\mathcal{L}(\Theta_2)), X_1 - X_2\rangle\right] \\
&= \mathbb{E}[\langle b_1(t)(X_1 - X_2) + b_2(t)(\mathbb{E}[X_1] - \mathbb{E}[X_2]) + b_3(t)(\alpha_1 - \alpha_2), X_1 - X_2\rangle] \\
&\leq \left(\frac{1}{2}\sup_{t\in[0,\infty)}\lambda_{\max}(b_1(t) + b_1(t)^\top) + \|b_2(\cdot)\|_\infty + \varepsilon_1\right)\mathbb{E}[|X_1 - X_2|^2] \\
&\quad + \frac{\|b_3(\cdot)\|_\infty}{4\varepsilon_1}\phi_2(t,\Theta_1,\Theta_2,\mathcal{L}(\Theta_1),\mathcal{L}(\Theta_2)) \\
&\leq -\kappa_x\mathbb{E}[|X_1 - X_2|^2] + \frac{\|b_3(\cdot)\|_\infty}{4\varepsilon_1}\phi_2(t,\Theta_1,\Theta_2,\mathcal{L}(\Theta_1),\mathcal{L}(\Theta_2)),
\end{aligned}
\tag{4.28}
$$

where the κ_x satisfying

$$
\kappa_x < -\frac{1}{2}\sup_{t\in[0,\infty)}\lambda_{\max}(b_1(t) + b_1(t)^\top) - \|b_2(\cdot)\|_\infty,
\tag{4.29}
$$

and the condition (4.29) ensures the existence of $\varepsilon_1 > 0$ such that last inequality of (4.28) holds.
Moreover, by the Lipschitz property of $\partial_x f$ and $\partial_\mu f$, for any $\varepsilon_z > 0$, we have

$$
\begin{aligned}
&\mathbb{E}\left[\langle F(t,\Theta_1,\mathcal{L}(\Theta_1)) - F(t,\Theta_2,\mathcal{L}(\Theta_2)), Y_1 - Y_2\rangle\right] \\
&= \mathbb{E}\Bigg[\Big\langle \partial_x f(t,X_2,\mathcal{L}(X_2),\alpha_2) - \partial_x f(t,X_1,\mathcal{L}(X_1),\alpha_1) + b_1(t)(Y_2 - Y_1) + b_2(t)(\mathbb{E}[Y_2] - \mathbb{E}[Y_1]) \\
&\quad + 2K(Y_2 - Y_1) + \tilde{\mathbb{E}}\big[\partial_\mu f(t,\tilde{X}_2,\mathcal{L}(X_2),\tilde{\alpha}_2)(X_2)\big] - \tilde{\mathbb{E}}[\partial_\mu f(t,\tilde{X}_1,\mathcal{L}(X_1),\tilde{\alpha}_1)(X_1)] \\
&\quad + \sigma_1(t)(Z_1 - Z_2) + \sigma_2(t)(\mathbb{E}[Z_1] - \mathbb{E}[Z_2]), Y_1 - Y_2\Big\rangle\Bigg] \\
&\geq -\left(\frac{1}{2}\sup_{t\in[0,\infty)}\lambda_{\max}(b_1(t) + b_1(t)^\top) + \|b_2(\cdot)\|_\infty + 2K + 7\varepsilon_2\right)\mathbb{E}[|Y_1 - Y_2|^2] \\
&\quad - \frac{(\|\sigma_1(\cdot)\|_\infty + \|\sigma_2(\cdot)\|_\infty)^2 + \varepsilon_z}{2}\mathbb{E}[|Y_1 - Y_2|^2] - \frac{(\|\sigma_1(\cdot)\|_\infty + \|\sigma_2(\cdot)\|_\infty)^2}{2\Big((\|\sigma_1(\cdot)\|_\infty + \|\sigma_2(\cdot)\|_\infty)^2 + \varepsilon_z\Big)}\mathbb{E}\left[|Z_1 - Z_2|^2\right]
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \frac{5\tilde{L}^2}{4\varepsilon_2}\mathbb{E}[|X_1 - X_2|^2] - \frac{2\tilde{L}^2}{4\varepsilon_2}\phi_2(t,\Theta_1,\Theta_2,\mathcal{L}(\Theta_1),\mathcal{L}(\Theta_2)) \\
&\geq -\left(\kappa_y + \frac{l_z}{2}\right)\mathbb{E}[|Y_1 - Y_2|^2] - \frac{\gamma}{2}\mathbb{E}\left[|Z_1 - Z_2|^2\right] - \frac{5\tilde{L}^2}{4\varepsilon_2}\mathbb{E}[|X_1 - X_2|^2] \\
&\quad - \frac{2\tilde{L}^2}{4\varepsilon_2}\phi_2(t,\Theta_1,\Theta_2,\mathcal{L}(\Theta_1),\mathcal{L}(\Theta_2)),
\end{aligned}
\tag{4.30}
$$

where

$$
l_z = (\|\sigma_1(\cdot)\|_\infty + \|\sigma_2(\cdot)\|_\infty)^2 + \varepsilon_z, \quad \gamma = \frac{(\|\sigma_1(\cdot)\|_\infty + \|\sigma_2(\cdot)\|_\infty)^2}{\left((\|\sigma_1(\cdot)\|_\infty + \|\sigma_2(\cdot)\|_\infty)^2 + \varepsilon_z\right)}
$$

satisfying $0 < \gamma < 1$ and κ_y satisfying

$$
\kappa_y > \frac{1}{2}\sup_{t\in[0,\infty)}\lambda_{\max}(b_1(t) + b_1(t)^\top) + \|b_2(\cdot)\|_\infty + 2K, \tag{4.31}
$$

and the condition (4.31) ensures the existence of $\varepsilon_2 > 0$ such that last inequality of (4.30) holds. In fact, for any K satisfying condition (4.20), we can find $\delta > 0$ such that

$$
K = -\frac{1}{2}\sup_{t\in[0,\infty)}\lambda_{\max}(b_1(t) + b_1(t)^\top) - \|b_2(\cdot)\|_\infty - \frac{(\|\sigma_1(\cdot)\|_\infty + \|\sigma_2(\cdot)\|_\infty)^2}{2} - \delta.
$$

Let

$$
\kappa_x = -\frac{1}{2}\sup_{t\in[0,\infty)}\lambda_{\max}(b_1(t) + b_1(t)^\top) - \|b_2(\cdot)\|_\infty - \frac{\delta}{2},
$$

and

$$
\kappa_y = \frac{1}{2}\sup_{t\in[0,\infty)}\lambda_{\max}(b_1(t) + b_1(t)^\top) + \|b_2(\cdot)\|_\infty + 2K + \frac{\delta}{2},
$$

which satisfy (4.29) and (4.31), and we have

$$
K = \frac{\kappa_x + \kappa_y}{2},
$$

and

$$
\begin{aligned}
\kappa_x - \kappa_y &= -\sup_{t\in[0,\infty)}\lambda_{\max}(b_1(t) + b_1(t)^\top) - 2\|b_2(\cdot)\|_\infty - 2K - \delta \\
&= -\sup_{t\in[0,\infty)}\lambda_{\max}(b_1(t) + b_1(t)^\top) - 2\|b_2(\cdot)\|_\infty - \delta \\
&\quad - 2\left(-\frac{1}{2}\sup_{t\in[0,\infty)}\lambda_{\max}(b_1(t) + b_1(t)^\top) - \|b_2(\cdot)\|_\infty - \frac{(\|\sigma_1(\cdot)\|_\infty + \|\sigma_2(\cdot)\|_\infty)^2}{2} - \delta\right) \\
&= (\|\sigma_1(\cdot)\|_\infty + \|\sigma_2(\cdot)\|_\infty)^2 + \delta \\
&> (\|\sigma_1(\cdot)\|_\infty + \|\sigma_2(\cdot)\|_\infty)^2 + \delta_0,
\end{aligned}
\tag{4.32}
$$

for any $0 < \delta_0 < \delta$. Therefore, letting $\varepsilon_\sigma = \varepsilon_z = \delta_0$ in (4.27) and (4.30), we have shown that if the condition (4.20) is satisfied, we can choose κ_x, κ_y satisfying (4.29) and (4.31) respectively, such that

$$
\kappa_x - \kappa_y > \max\{l_\sigma, l_z\} \quad \text{and} \quad K = \frac{\kappa_x + \kappa_y}{2}. \tag{4.33}
$$

The proof is finished. $\square$

At the end of this section, let us recall the infinite horizon control problems studied in [4] and [25] and provide some remarks to make a comparison.

Remark 4.7. *Bayraktar and Zhang [4] considered an infinite horizon mean field control problem as follows.*
Minimize the problem

$$J(\alpha) := \mathbb{E}\left[\int_0^\infty e^{2Kt} f\left(t, X_t, \mathcal{L}\left(X_t\right), \alpha_t\right) dt\right], \tag{4.34}$$

subject to

$$\begin{cases} dX_t = b\left(t, X_t, \mathcal{L}\left(X_t\right), \alpha_t\right) dt + \sigma dW_t, \\ X_0 = \xi. \end{cases} \tag{4.35}$$

When solving the Hamiltonian system, they let $b(t,x,\mu,\alpha) = b_0(t) + b_1(t)x + \bar{b}_1(t)\bar{\mu} + b_2(t)\alpha$ *and proposed the following condition*

$$\left|\bar{b}_1(t)\right| \leq l \quad \text{ and } -\max_t b_1(t) \geq l + K, \tag{4.36}$$

which can be easily verified to correspond to our condition (4.20) on parameter K. Not only did they need (4.36), they also needed to supplement another constraint for K to solve the mean field control problem (4.34)-(4.35) (see [4, Theorem 3.1]). Thus, compared with [4], we weaken the conditions on the values of parameter K and extend the results to a larger space.

Remark 4.8. *Wei and Yu [25] studied an infinite horizon LQ optimal control problems as follows. Minimize:*

$$J(\alpha) = \frac{1}{2}\mathbb{E}\int_0^\infty e^{2Ks}[\langle Q(s)x(s), x(s)\rangle + 2\langle S(s)x(s), \alpha(s)\rangle + \langle R(s)\alpha(s), \alpha(s)\rangle]ds, \tag{4.37}$$

subject to

$$\begin{cases} dx(s) = [A(s)x(s) + B(s)\alpha(s)]ds + \left[C(s)x(s) + D(s)\alpha(s)\right] dW_s, \quad s \in [0,\infty), \\ x(0) = x. \end{cases} \tag{4.38}$$

They pointed out that if $S(\cdot)$ is a nonzero matrix, the values of parameter K will be adjusted accordingly. It is obvious that the optimization problem (4.2) covers LQ problem (4.37)-(4.38). Thus it follows from Theorem 4.6 that if there exists a constant $\lambda > 0$ such that

$$Q(\cdot) - S(\cdot)^\top R(\cdot)^{-1} S(\cdot) - \lambda I \geq 0,$$

and

$$K < -\frac{1}{2}\sup_{t\in[0,\infty)} \lambda_{\max}(A(t) + A(t)^\top) - \frac{\|C(\cdot)\|_\infty^2}{2},$$

the Hamiltonian system corresponding to the LQ problem (4.37)-(4.38) admits a unique solution $(X,Y,Z) \in L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^{n+n+n\times d})$. Therefore, we can obtain the same solvability results for the LQ problem (4.37)-(4.38) whether the cross term coefficient $S(\cdot)$ is equal to zero or not. The root causing this different phenomenon is that the values of $S(\cdot)$ will directly affect the monotonicity condition proposed in [25], but it will not affect our condition since it only appears in optimal control and we do not need to consider its monotonicity under our condition.

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