QUANTITATIVE STEINITZ THEOREM AND POLARITY

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Abstract. The classical Steinitz theorem asserts that if the origin lies within the interior of the convex hull of a set $S \subset \mathbb{R}^d$, then there are at most $2d$ points in S whose convex hull contains the origin within its interior. Bárány, Katchalski, and Pach established a quantitative version of Steinitz's theorem, showing that for a convex polytope Q in $\mathbb{R}^d$ containing the standard Euclidean unit ball $\mathbf{B}^d$, there exist at most $2d$ vertices of Q whose convex hull Q' satisfies $r\mathbf{B}^d \subset Q'$ with $r \geq d^{-2d}$. Recently, Márton Naszódi and the author derived a polynomial bound on r.

This paper aims to establish a bound on r based on the number of vertices of Q. In other words, we demonstrate an effective method to remove several points from the original set Q without significantly altering the bound on r. Specifically, if the number of vertices of Q scales linearly with the dimension, i.e., cd, then one can select $2d$ vertices such that $r \geq \frac{1}{5cd}$. The proof relies on a polarity trick, which may be of independent interest: we demonstrate the existence of a point p in the interior of a convex polytope $P \subset \mathbb{R}^d$ such that the vertices of the polar polytope $(P-p)^\circ$ sum up to zero.

1. Introduction

The goal of this paper is to establish a quantitative version of the following classical result of E. Steinitz [Ste13].

Proposition 1.1 (Steinitz theorem). *Let the origin belong to the interior of the convex hull of a set $S \subset \mathbb{R}^d$. Then there are at most $2d$ points of S whose convex hull contains the origin in the interior.*

The first quantitative version of this result was obtained in [BKP82], where the authors showed that for a convex polytope Q in $\mathbb{R}^d$ containing the standard Euclidean unit ball $\mathbf{B}^d$, there exist at most $2d$ vertices of Q whose convex hull Q' satisfies $r(d)\mathbf{B}^d \subset Q'$ with $r(d) \geq d^{-2d}$.

With the exception of the planar case $d = 2$ [KMY92, Bra97, BH94], no significant improvement on $r(d)$ has been obtained (see also [DLLHRS17]). Márton Naszódi and the author derived the first polynomial bound $r(d) \geq \frac{1}{6d^2}$ in [IN24], and extended this result to a spherical version [IN23].

The main result of the paper is as follows.

Theorem 1.2. *Let Q be a set of m points of $\mathbb{R}^d$ such that its convex hull* $\operatorname{conv} Q$ *contains the Euclidean unit ball $\mathbf{B}^d$. Then there is $Q' \subset Q$ of size at most $2d$ satisfying* $\operatorname{conv} Q' \supset r\mathbf{B}^d$, *where* $r = \frac{1}{2(m+d)+1}$.

Starting with the breakthrough [BSS14], which led to new results in the area of quantitative combinatorial convexity (see [DLLHRS17], [Nas16], [Bra16], [Bra18], [Bra17], [FVGM22]), one approach to the problems was to initially identify more than $2d$ points (facets, subsets) with desired properties, typically linear in the dimension, and then select the best $2d$ among them. It is worth noting that in some cases, eliminating additional objects poses challenges [DFN21]. The next corollary, which trivially follows from the main result, facilitates this process in the case of the Quantitative Steinitz theorem.

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Corollary 1.3. *Let Q be a set of cd, $c > 1$, points of $\mathbb{R}^d$ such that its convex hull* $\operatorname{conv} Q$ *contains the ball $\lambda\mathbf{B}^d$. Then there are at most $2d$ points of Q whose convex hull Q' satisfies*

$$\frac{\lambda}{5cd}\mathbf{B}^d \subset Q'.$$

As another elementary corollary of the main result, we will get a slightly worse polynomial bound in the Quantitative Steinitz theorem than the quadratic one obtained in [IN24].

Corollary 1.4. *Let Q be a convex polytope in $\mathbb{R}^d$ containing the Euclidean unit ball $\mathbf{B}^d$. Then there are at most $2d$ vertices of Q whose convex hull Q' satisfies*

$$\frac{d^{-\frac{5}{2}}}{7}\mathbf{B}^d \subset Q'.$$

The key observation we will use to prove Theorem 1.2 is the following "polarity trick."
We recall that the *polar* of a set $S \subset \mathbb{R}^d$ is defined by

$$S^\circ = \left\{x \in \mathbb{R}^d : \langle x, s\rangle \le 1 \quad \text{for all} \quad s \in S\right\}.$$

Theorem 1.5. *Let $P \subset \mathbb{R}^d$ be a polytope with non-empty interior. Then there is a point c in its interior such that the sum of vertices of $(P - c)^\circ$ is equal to zero.*

In fact, we will show that for any positive weights, there is a point c from the interior of P such that the sum of vertices of $(P - c)^\circ$ with those weights is zero. We will show that the corresponding point c is a maximizer of a certain functional. Thus, our proof mimics the proof of the existence of the *Santaló point* (see [Gru07, MW98, Leh09, IW21]), which is a point s inside a convex set $K \subset \mathbb{R}^d$ with non-empty interior such that the centroid of $(K - s)^\circ$ is the origin.

The author would like to know the answer to the following conjecture related to Theorem 1.2

Conjecture 1.6. *Let Q be a set of $2d + 1$ points of $\mathbb{R}^d$ such that* $\operatorname{conv} Q \supset \mathbf{B}^d$*. Then there is $Q' \subset Q$ of size at most $2d$ satisfying* $\operatorname{conv} Q' \supset c\mathbf{B}^d$ *for some universal constant c.*

The rest of the paper is organized as follows: In the next Section, we will explain the ideas behind the proof of the Quantitative Steinitz theorem obtained in [IN24] that can be traced back to [IN22] and [AHAK22]; we will try to show why Theorem 1.5 comes naturally as a development of those ideas. In Section 3, we will prove a more general version of Theorem 1.5. Finally, in Section 4 we derive Theorem 1.2 and its corollary.

2. Useful lemmas

We begin by elucidating the proof ideas of the Quantitative Steinitz theorem as outlined in [IN24]. The central strategy revolves around the careful application of polarity, employed twice in succession.

We started with a "Steinitz-type picture", wherein we considered a set $Q \subset \mathbb{R}^d$ whose convex hull contains the unit ball $\mathbf{B}^d$. Subsequently, we transitioned to an equivalent "Helly-type picture" by examining the polar set Q° of Q. This transformation allows us to reformulate the original problem into an equivalent Helly-type statement, justifying the name. Now comes a trick: we chose a point c "deep" in Q° and considered $(Q^\circ - c)^\circ$. So to say, this maneuver returns us to a "Steinitz-type picture" albeit a modified one, as we have altered our original set. In essence, by manipulating the center of polarity, we achieve a more structurally organized convex polytope. We dub the resultant configuration following the second polarity transformation as "Atlantis." For a new set within "Atlantis", we derived the desired polynomial bound utilizing a result from [AHAK22]. Finally, we demonstrated that reverting to the original "Steinitz-type picture" does not significantly degrade our bound.

The crux of the proof lies in selecting the appropriate center c of polarity during the second step. Notably, Theorem 1.5 offers a methodology for choosing an alternative point, which holds intrinsic interest in itself.

Now, we are going to formalize a few statements.

For a natural number n, $[n]$ denotes the set $\{1,\ldots,n\}$; $\mathbf{B}^d$ denotes the standard Euclidean unit ball in $\mathbb{R}^d$; $\langle p,x\rangle$ denotes the inner product of p and x. We use $(a)_+$ to denote $\max\{a,0\}$.

We start with an open problem. In relation to volumetric Helly-type results, the author is interested in the following conjecture: Macbeath [Mac52, Lemma 7.1] showed that for a compact convex set $K\subset\mathbb{R}^d$ with non-empty interior, the function $f(x)=\mathrm{vol}_d\, K\cap(-K+2x)$ attains its maximum in a unique point of the interior of K (here vol_d denotes the d-dimensional volume on $\mathbb{R}^d$, as usual). Let us call this point the *Macbeath point* of K.

Conjecture 2.1. *The Macbeath point p of a compact convex set $K\subset\mathbb{R}^d$ with non-empty interior satisfies the inclusion*

$$K-p\subset -d(K-p).$$

We formulate now the above-mentioned result from [AHAK22].

Lemma 2.2. *Let L be a bounded subset of $\mathbb{R}^d$ linearly spanning the whole space, let $S=\mathrm{conv}\{0,v_1,\ldots,v_d\}$ be the maximal volume simplex among all simplices with d vertices from L and one vertex at the origin. We use P to denote the Minkowski sum of segments $[-v_i,v_i]$, $i\in[d]$. Then the following inclusions hold:*

(1) $L\subset P$.
(2) $P\subset -2dS+(v_1+\cdots+v_d)$.

Sketch of the proof. Clearly, the volume of S is strictly positive. The simplex S can be represented as

$$S=\left\{x\in\mathbb{R}^d:\ x=\alpha_1v_1+\ldots+\alpha_dv_d \quad \text{for}\ \ \alpha_i\ge 0 \text{ and } \sum_{i=1}^d\alpha_i\le 1\right\}. \tag{1}$$

It is easy to see that P is a paralletope that can be represented as

$$P=\{x\in\mathbb{R}^d:\ x=\beta_1v_1+\ldots+\beta_dv_d \quad \text{for } \beta_i\in[-1,1]\}. \tag{2}$$

Since S is chosen maximally, equation (2) shows that for any vertex v of L, $v\in P$. By convexity,

$$L\subset P.$$

Let $S'=-2dS+(v_1+\ldots+v_d)$. By (1),

$$S'=\left\{x\in\mathbb{R}^d:\ x=\gamma_1v_1+\ldots+\gamma_dv_d \quad \text{for } \gamma_i\le 1 \text{ and } \sum_{i=1}^d\gamma_i\ge -d\right\},$$

which, together with (2), yields $P\subseteq S'$, completing the proof. □

Now we want to show that the whole way from "Steinitz-type picture" to "Atlantis" and back does not cost much in terms of the bound on the radius.

Let P be a polytope in $\mathbb{R}^d$ with a non-empty interior. It is well known that for any point c of the interior of P, there is a one-to-one correspondence between the facets of P and the vertices $(P-c)^\circ$. For two points c_1 and c_2 of the interior of P, we will say that a vertex of $(P-c_1)^\circ$ and a vertex of $(P-c_2)^\circ$ are *polar corresponding* if they correspond to the same facet of P.

Lemma 2.3 (Vertex correspondence)**.** *Let $P\subset\mathbb{R}^d$ be a polytope containing the origin and a point c in its interior. Denote $Q=P^\circ$ and $L=(P-c)^\circ$. Then v is a vertex of Q if and only if $\frac{v}{1-\langle c,v\rangle}$ is a vertex of L. Moreover, the vertex v of Q and the vertex $\frac{v}{1-\langle c,v\rangle}$ are polar corresponding.*

Proof. A point v is a vertex of Q if and only if the half-space $H_v=\{x\in\mathbb{R}^d:\ \langle x,v\rangle\le 1\}$ supports P by a facet. The latter is true if and only if H_v-c supports $P-c$ in a facet. On the other hand, since c is in the interior of P, $\langle c,v\rangle<1$, and thus,

$$H_v-c=\{x\in\mathbb{R}^d:\ \langle x,v\rangle\le 1\}-c=\{y\in\mathbb{R}^d:\ \langle y,v\rangle\le 1-\langle c,v\rangle\}=$$

$$\{y \in \mathbb{R}^d : \left\langle y, \frac{v}{1-\langle c, v\rangle} \right\rangle \leq 1\}.$$

Consequently, $\frac{v}{1-\langle c,v\rangle}$ is a vertex of L if and only if v is a vertex of Q. $\square$

Lemma 2.4 (Atlantis: There and back again)**.** *Let $P \subset \mathbf{B}^d \subset \mathbb{R}^d$ be a polytope containing the origin and a point c in its interior. Denote $K_1 = P^\circ$ and $K_2 = (P-c)^\circ$. If some vertices $w_1, \ldots, w_k$ of K_2 satisfy the inclusion* $\operatorname{conv}\{w_1, \ldots, w_k\} \supset \lambda \mathbf{B}^d$ *for some positive λ, then their polar corresponding vertices $v_1, \ldots, v_k$ of K_1 satisfy* $\operatorname{conv}\{v_1, \ldots, v_k\} \supset \frac{\lambda}{1+\lambda}\mathbf{B}^d$.

Proof. By Lemma 2.3, the vertices $v_1, \ldots, v_k$ of K_1 polar corresponding to $w_1, \ldots, w_k$ satisfy $w_1 = \frac{v_1}{1-\langle v_1, c\rangle}, \ldots, w_k = \frac{v_k}{1-\langle v_k, c\rangle}$. Next, $(\operatorname{conv}\{w_1, \ldots, w_k\})^\circ \subset \frac{1}{\lambda}\mathbf{B}^d$ and

$$(\operatorname{conv}\{w_1, \ldots, w_k\})^\circ = \bigcap_{i\in[k]} \{y \in \mathbb{R}^d : \langle y, w_i\rangle \leq 1\} = \bigcap_{i\in[k]} \left\{y \in \mathbb{R}^d : \frac{\langle y, v_i\rangle}{1-\langle v_i, c\rangle} \leq 1\right\} =$$

$$\bigcap_{i\in[k]} \{y \in \mathbb{R}^d : \langle y, v_i\rangle \leq 1 - \langle v_i, c\rangle\} = \bigcap_{i\in[k]} \{y \in \mathbb{R}^d : \langle y + c, v_i\rangle \leq 1\} =$$

$$\bigcap_{i\in[k]} \left(\{x \in \mathbb{R}^d : \langle x, v_i\rangle \leq 1\} - c\right) = -c + \bigcap_{i\in[k]} \{x \in \mathbb{R}^d : \langle x, v_i\rangle \leq 1\}.$$

By the assumption of the lemma, $c \in P \subset \mathbf{B}^d$. Hence, $\bigcap_{i\in[k]} \{x \in \mathbb{R}^d : \langle x, v_i\rangle \leq 1\} \subset \left(\frac{1}{\lambda}+1\right)\mathbf{B}^d$.

Consequently, $\operatorname{conv}\{v_1, \ldots, v_k\} = \left(\bigcap_{i\in[k]} \{x \in \mathbb{R}^d : \langle x, v_i\rangle \leq 1\}\right)^\circ \supset \frac{\lambda}{1+\lambda}\mathbf{B}^d$. The lemma is proven. $\square$

We note that Lemma 2.4 allows to go both ways from "Steinitz-type" picture to "Atlantis" and back. For example, if $K_2 \supset \mathbf{B}^d$, then $K_1 \supset \frac{1}{2}\mathbf{B}^d$.

3. Polarity trick

In this section, we prove the following result, which implies Theorem 1.5.

Theorem 3.1. *Let $P \subset \mathbb{R}^d$ be a polytope with non-empty interior with facets $F_1, \ldots, F_n$. For any positive weights $\alpha_1, \ldots, \alpha_n$, there is a point c in the interior of P such that*

$$\sum_{i\in[n]} \alpha_i w_i = 0,$$

where w_i is the vertex of $(P-c)^\circ$ corresponding to the facet F_i of P.

The key observation is the following:

Lemma 3.2. *Let $P \subset \mathbb{R}^d$ be a polytope containing the origin in its interior defined by the linear inequalites $\{x \in \mathbb{R}^d : \langle x, v\rangle \leq 1$ for all $v \in Q\}$, for some finite $Q \subset \mathbb{R}^d$. Denote*

$$F(x) = \prod_{v\in Q} (1 - \langle x, v\rangle)_+^{\beta_v},$$

where all $\beta_v > 0$ for all $v \in Q$. Then F attains its maximum at a unique point c of the interior of P satisfying the identity

$$\sum_{v\in Q} \frac{\beta_v v}{1-\langle c, v\rangle} = 0.$$

Proof. Clearly, the function F vanishes outside of the interior of P. The function F is smooth, and the identity $F(x) = \prod\limits_{v\in Q} (1 - \langle x, v\rangle)^{\beta_v}$ holds for any x in the interior of P. By compactness, F attains its maximum at a point c of the interior of P. The function $\ln F$ is strictly convex on its support, which implies the uniqueness of c.

Fix a unit vector u. For any positive β and $b \neq 0$,

$$\frac{\mathrm{d}(at+b)^{\beta}}{\mathrm{d}t}\bigg|_{t=0} = \beta a \, \frac{(at+b)^{\beta}}{(at+b)}\bigg|_{t=0}.$$

Thus,

$$-\frac{\mathrm{d}F(c+tu)}{\mathrm{d}t}\bigg|_{t=0} = -\left(\prod_{v\in Q}(1-\langle c, v\rangle - t\langle u, v\rangle)^{\beta_v}\right)'\Bigg|_{t=0} =$$

$$\sum_{v\in Q}\beta_v\langle u, v\rangle \frac{\prod\limits_{v\in Q}(1-\langle c, v\rangle - t\langle u, v\rangle)^{\beta_v}}{1-\langle c, v\rangle - t\langle u, v\rangle}\Bigg|_{t=0} = \sum_{v\in Q}\beta_v\langle u, v\rangle\frac{F(c)}{1-\langle c, v\rangle} = F(c)\left\langle u, \sum_{v\in Q}\frac{\beta_v v}{1-\langle c, v\rangle}\right\rangle$$

Since $F(c) > 0$ and u was chosen arbitrary, we conclude that

$$\sum_{v\in Q}\frac{\beta_v v}{1-\langle c, v\rangle} = 0,$$

completing the proof of the lemma. □

Proof of Theorem 3.1 . Returning to our theorem, we shift P in such a way that it contains the origin in its interior. Denote $Q = P^{\circ}$ and let v_i is the vertex of Q corresponding to the facet F_i. Applying Lemma 3.2 for $F(x) = \prod\limits_{i\in[m]}(1-\langle x, v_i\rangle)_+^{\alpha_i}$, we get

$$\sum_{i\in[m]}\frac{\alpha_i v_i}{1-\langle c, v_i\rangle} = 0.$$

By Lemma 2.3, the sum is equal to $\sum\limits_{i\in[m]}\alpha_i w_i$, w_i is the vertex of $(P-c)^{\circ}$ corresponding to the facet F_i of P. The proof of Theorem 3.1 is complete. □

4. Proof of the main result and its corollary

The following consequence of the Carathéodory lemma comes in handy. The proof can be found in [Bár21, Theorem 2.3].

Lemma 4.1. *Assume a point p belongs to the convex hull of a set $Q \subset \mathbb{R}^d$. Then there are $v_1, \dots, v_d$ (some of them might coincide) of Q satisfying $p \in \operatorname{conv}\{0, v_1, \dots, v_d\}$.*

Proof of Theorem 1.2. Set $P = Q^{\circ}$. By Theorem 1.5, there is a point c in the interior of P such that the vertices of $(P-c)^{\circ}$ sum up to zero. Denote $L = (P-c)^{\circ}$.

Using Lemma 2.4 with $K_2 = Q$ and $K_1 = L$, one sees that $\frac{\mathbf{B}^d}{2} \subset L$.

Consider $S = \operatorname{conv}\{0, w_1, \dots, w_d\}$ the maximal volume simplex among all simplices with d vertices from L and one vertex at the origin. Then the sum of all other vertices of L is equal to $-(w_1 + \dots + w_d)$. And thus the centroid p of all others is equal to $-\frac{(w_1+\dots+w_d)}{m-d}$. Thus, by Lemma 4.1, there are vertices $w_{d+1}, \dots, w_{2d}$ such that p belongs to $\operatorname{conv}\{w_{d+1}, \dots, w_{2d}, 0\}$. Thus, the convex hull of $\{w_1, \dots, w_{2d}\}$ contains the simplex $\operatorname{conv}\{0, w_1, \dots, w_d\}$.

By Lemma 2.2,

$$\frac{\mathbf{B}^d}{2} \subset L \subset -2d\operatorname{conv}\{0, w_1, \dots, w_d\} + (w_1 + \dots + w_d) \subset -2d\operatorname{conv}\{w_1, \dots, w_{2d}\} - p(m-d) \subset$$

$$-2d\operatorname{conv}\{w_1, \dots, w_{2d}\} - (m-d)\operatorname{conv}\{w_1, \dots, w_{2d}\} = -(m+d)\operatorname{conv}\{w_1, \dots, w_{2d}\}.$$

Thus, $\frac{\mathbf{B}^d}{2(m+d)} \subset \operatorname{conv}\{w_1, \ldots, w_{2d}\}$, and by Lemma 2.4 with $K_2 = L$ and $K_1 = Q$,, one sees that the corresponding vertices $v_1, \ldots, v_{2d}$ of Q satisfy $\frac{\mathbf{B}^d}{2(m+d)+1} \subset \operatorname{conv}\{v_1, \ldots, v_{2d}\}$. □

Proof of Corollary 1.4. The first step is to reduce the number of points to a quadratic in d. It is easy to find $2d^2$ points of Q such that their convex hall contains $\frac{\mathbf{B}^d}{\sqrt{d}}$. Take arbitrary standard cross-polytope inscribed in the unit ball $\mathbf{B}^d$, say the convex hull of vectors of the standard basis $\{e_1, \ldots, e_d\}$ of $\mathbb{R}^d$ and their opposites $\{-e_1, \ldots, -e_d\}$. By Lemma 4.1, for each point $p \in \{\pm e_1, \ldots, e_d\}$, there are d points, say $v_1, \ldots, v_d$, of Q with the property $p \in \operatorname{conv}\{0, v_1, \ldots, v_d\}$. The convex hull of the union of such d-tuples of points for all $p \in \{\pm e_1, \ldots, \pm e_d\}$, contains the cross-polytope and hence contains the ball $\frac{\mathbf{B}^d}{\sqrt{d}}$.

Now, it suffices to apply Theorem 1.2 to go from $2d^2$ points to $2d$ points whose convex hull contains the ball $\frac{1}{2(2d^2+d)+1} \cdot \frac{\mathbf{B}^d}{\sqrt{d}} \supset \frac{d^{-\frac{5}{2}}}{7}\mathbf{B}^d$. □

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