Recognizing Relating Edges in Graphs without Cycles of Length 6

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Abstract

A graph G is *well-covered* if all maximal independent sets are of the same cardinality. Let $w : V(G) \longrightarrow \mathbb{R}$ be a weight function. Then G is w*-well-covered* if all maximal independent sets are of the same weight. An edge $xy \in E(G)$ is *relating* if there exists an independent set S such that both $S \cup \{x\}$ and $S \cup \{y\}$ are maximal independent sets in the graph. If xy is relating then $w(x) = w(y)$ for every weight function w such that G is w-well-covered. Relating edges play an important role in investigating w-well-covered graphs.

The decision problem whether an edge in a graph is relating is **NP-**complete [4]. We prove that the problem remains **NP-**complete when the input is restricted to graphs without cycles of length 6. This is an unexpected result because recognizing relating edges is known to be polynomially solvable for graphs without cycles of lengths 4 and 6 [20], graphs without cycles of lengths 5 and 6 [22], and graphs without cycles of lengths 6 and 7 [30].

A graph G belongs to the class $\mathbf{W_2}$ if every two pairwise disjoint independent sets in G are included in two pairwise disjoint maximum independent sets [29]. It is known that if G belongs to the class $\mathbf{W_2}$ then it is well-covered. A vertex $v \in V(G)$ is *shedding* if for every independent set $S \subseteq V(G) \setminus N[v]$ there exists $u \in N(v)$ such that $S \cup \{u\}$ is independent [34]. Shedding vertices play an important role in studying the class $\mathbf{W_2}$. Recognizing shedding vertices is co-**NP-**complete, even when the input is restricted to triangle-free graphs [24]. We prove that the problem is co-**NP-**complete for graphs without cycles of length 6.

1 Introduction

1.1 Definitions and Notation

Throughout this paper G is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V(G)$ and edge set $E(G)$. Cycles of k vertices are denoted by C_k. When we say that G does not contain C_k for some $k \geq 3$, we mean that G does not admit subgraphs isomorphic to C_k. It is important to mention that these subgraphs are not necessarily induced. Let $\mathcal{G}(\widehat{C_{i_1}},..,\widehat{C_{i_k}})$ denote the family of all graphs which do not contain C_{i_1},...,C_{i_k}.

Let $S \subseteq V(G)$ be a non-empty set of vertices, and let $i \in \mathbb{N}$. Then

$$N_i(S) = \{v \in V(G) | \ min_{s \in S} \ d(v, s) = i\}, \ N_i[S] = \{v \in V(G) | \ min_{s \in S} \ d(v, s) \leq i\}$$

where $d(x, y)$ is the minimal number of edges required to construct a path between x and y, or infinite if such a path does not exist. If $i \neq j$ then $N_i(S) \cap N_j(S) = \varnothing$. We abbreviate $N_1(S)$ and $N_1[S]$ to $N(S)$ and $N[S]$, respectively. If $S = \{v\}$ for some $v \in V(G)$, then $N_i(\{v\})$, $N_i[\{v\}]$, $N(\{v\})$, and $N[\{v\}]$, are abbreviated to $N_i(v)$, $N_i[v]$, $N(v)$, and $N[v]$, respectively.

Let $T \subseteq V(G)$. Then S *dominates* T if $T \subseteq N[S]$. If $N[S] = V(G)$ then S dominates the whole graph. The induced subgraph of G with vertex set $S \subseteq V(G)$ is $G[S]$, and denote $G \setminus S = G[V(G) \setminus S]$.

1.2 Relating Edges

A set of vertices $S \subseteq V(G)$ is *independent* if for every $x, y \in S$, x and y are not adjacent. Obviously, an empty set is independent. An independent set is called *maximal* if it is not contained in another independent set. An independent set is *maximum* if the graph does not contain an independent set with a higher cardinality. A graph is called *well-covered* if every maximal independent set is maximum. The problem of finding a maximum cardinality independent set in an input graph is **NP**-hard. However, if the input is restricted to well-covered graphs, then a maximum cardinality independent set can be found polynomially using the *greedy algorithm*.

Let $w : V(G) \longrightarrow \mathbb{R}$ be a weight function defined on the vertices of G. For every set $S \subseteq V(G)$, define $w(S) = \Sigma_{s \in S} w(s)$. Then G is *w-well-covered* if all maximal independent sets of G are of the same weight. The set of weight functions w for which G is w-well-covered is a *vector space* [5]. That vector space is denoted $WCW(G)$ [4].

The recognition of well-covered graphs is known to be **co-NP**-complete. This was proved independently in [7] and [27]. The problem remains **co-NP**-complete even when the input is restricted to $K_{1,4}$-free graphs [6], or to circulant graphs [3]. However, the problem is polynomially solvable for $K_{1,3}$-free graphs [31, 32], for graphs with girth at least 5 [11], for graphs that contain neither 4- nor 5-cycles [12], for graphs with a bounded maximal degree [5], and for chordal graphs [26].

Obviously, a graph G is well-covered if and only if $w \equiv 1$ belongs to the vector space $WCW(G)$. Hence, for every family graphs, if recognizing well-covered graphs is **co-NP**-complete, then finding $WCW(G)$ is **co-NP**-hard. On the other hand, if for a specific family of graphs finding $WCW(G)$ can be completed polynomially then also recognizing well-covered graphs is a polynomial task. Polynomial algorithms which find $WCW(G)$ are known for claw-free graphs [21] and for graphs without cycles of lengths 4, 5 and 6 [22].

An edge $xy \in E(G)$ is *relating* if there exists an independent set S such that both $S \cup \{x\}$ and $S \cup \{y\}$ are maximal independent sets in the graph. If xy is relating then $w(x) = w(y)$ for every weight function w such that G is w-well-covered. Relating edges play an important role in investigating w-well-covered graphs.

Problem 1.1 RE
Input: A graph G, and an edge $e \in E(G)$.
Question: Is e relating?

A witness that xy is a relating edge is an independent set S such that both $S \cup \{x\}$ and $S \cup \{y\}$ are maximal independent sets in the graph. The decision problem whether an edge in an input graph is relating is **NP-**complete [4], and it remains **NP-**complete even when the input is restricted to graphs without cycles of lengths 4 and 5 [20] or to bipartite graphs [23]. However, recognizing relating edges can be done in polynomial time if the input is restricted to graphs without cycles of lengths 4 and 6 [20], to graphs without cycles of lengths 5 and 6 [22], and to graphs without cycles of lengths 6 and 7 [30].

1.3 Shedding Vertices

A vertex $v \in V(G)$ is *shedding* if for every independent set $S \subseteq V(G) \setminus N[v]$ there exists $u \in N(v)$ such that $S \cup \{u\}$ is independent. Equivalently, v is shedding if there does not exist an independent set in $V(G) \setminus N[v]$ which dominates $N(v)$ [34]. It is easy to see that v is shedding if and only if there does not exist an independent set in $N_2(v)$ which dominates $N(v)$. Shedding vertices are also called *extendable* [11].

Problem 1.2 SHED
Input: A graph G, and a vertex $v \in V(G)$.
Question: Is v shedding?

If v is not shedding, a *witness* for being not shedding is a an independent set $S \subseteq N_2(v)$ which dominates $N(v)$. It is proved in [24] that recognizing shedding vertices is co-**NP-**complete, even when the input is restricted to triangle-free graphs, but polynomial solvable for claw-free graphs, for graphs without cycles of length 5, and for graphs without cycles of lengths 4 and 6. Theorem 1.3 shows the connection between the RE problem and the SHED problem.

Theorem 1.3 *[24] Let G be a graph without cycles of lengths 4, 5 and 6, and $xy \in E(G)$. Suppose $N(x) \cap N(y) = \varnothing$, $d(x) \geq 2$ and $d(y) \geq 2$. The following assertions are equivalent.*

1. *None of x and y is a shedding vertex.*
2. *xy is a relating edge.*

A graph G belongs to the class $\mathbf{W_2}$ if every two disjoint independent sets in G are included in two disjoint maximum independent sets [29]. All graphs in the class $\mathbf{W_2}$ are well-covered. Recognizing $\mathbf{W_2}$ graphs is co-**NP-**complete [10]. Shedding vertices play an important role in studying the class $\mathbf{W_2}$ due to Theorem 1.4.

Theorem 1.4 *[18] For every well-covered graph G having no isolated vertices, the following assertions are equivalent:*

1. *G is in the class $\mathbf{W_2}$.*

2. *All vertices of G are shedding.*

1.4 SAT

Let $\mathcal{X} = \{x_1, ..., x_n\}$ be a set of 0-1 variables. We define the set of *literals* $L_{\mathcal{X}}$ over $\mathcal{X}$ by $L_{\mathcal{X}} = \{x_i, \overline{x_i} : i = 1, ..., n\}$, where $\overline{x} = 1 - x$ is the *negation* of x. A *truth assignment* to $\mathcal{X}$ is a mapping $t : \mathcal{X} \longrightarrow \{0, 1\}$ that assigns a value $t(x_i) \in \{0, 1\}$ to each variable $x_i \in \mathcal{X}$. We extend t to $L_{\mathcal{X}}$ by putting $t(\overline{x_i}) = 1 - t(x_i)$. A literal $l \in L_{\mathcal{X}}$ is true under t if $t(l) = 1$. A *clause* over $\mathcal{X}$ is a conjunction of some literals of $L_{\mathcal{X}}$, such that for every variable $x \in \mathcal{X}$, the clause contains at most one literal out of x and its negation. Let $\mathcal{C} = \{c_1, ..., c_m\}$ be a set of clauses over $\mathcal{X}$. A truth assignment t to $\mathcal{X}$ *satisfies* a clause $c_j \in \mathcal{C}$ if c_j contains at least one true literal under t. The number of times that a variable *appears* in $\mathcal{C}$ is the number of clauses that include the variable or its negation.

Problem 1.5 SAT
Input: A set of variables $\mathcal{X} = \{x_1, ..., x_n\}$, and a set of clauses $\mathcal{C} = \{c_1, ..., c_m\}$ over $\mathcal{X}$.
Question: Is there a truth assignment to $\mathcal{X}$ which satisfies all clauses of $\mathcal{C}$?

The SAT problem is well-known to be **NP-**complete [13]. The number of times that a variable *appears* in an instance of SAT is the number of clauses which contain the variable or its negation. The SAT problem was learned thoroughly in recent years. The complexity statuses of many restricted cases of the problem were found.

Horn SAT is a restricted case of the SAT problem where every clause contains at most one unnegeted literal. This problem is known to be polynomial solvable [17]. MONOTONE SAT is the SAT problem in the restricted case that every clause contains either negated literals or unnegated literals, but not both. MONOTONE SAT is **NP-**complete [15].

Let $k \geq 2$. The k-SAT problem is a restricted case of the SAT problem where each clause contains exactly k different literals. For every $k \geq 3$, the k-SAT problem is well-known to be **NP-**complete [13], while the 2-SAT problem is polynomial solvable [9]. In the MAX 2-SAT problem the input is a set of clauses of size 2 and a positive integer, k. It should be decided whether there exists a truth assignment which satisfies at least k clauses. This problem was proved to be NP-complete [14]. Moreover, even if every variable appears 3 times, the MAX 2-SAT problem is NP-complete [16].

MONOTONE 3-SAT is NP-complete when each variable appears exactly 2 times negated and 2 times unnegated [8]. On the other hand, 3-SAT is always satisfiable, and therefore polynomial, when each variable appears at most 3 times [33]. However, allowing clauses of size 2 and 3, with each variable appearing 3 times, is NP-complete [25].

The 1-in-3 SAT is another restricted case of the SAT problem, denoted X3SAT. In X3SAT every clause contains 3 literals. It should be decided whether there exists a truth assignment which satisfies exactly one literal in every clause. X3SAT is NP-complete even when all its variables occuring unnegated [28].

1.5 Main Results

In Section 2 a new restricted version of the SAT problem, called 23SAT, is defined. It is proved that 23SAT is **NP-**complete.

In Section 3 we prove that the SHED problem is co-**NP**-complete when its input is restricted to graphs without cycles of length 6. The proof is based on a polynomial reduction from the complement of the 23SAT problem.

In Section 4 it is proved that the RE problem is **NP**-complete for graphs without cycles of length 6. The proof is based on a polynomial reduction from the complement of the SHED problem.

2 23SAT

Let $I = (\mathcal{X}, \mathcal{C})$ be an instance of SAT. A *major literal* is a literal in $L_{\mathcal{X}}$ that belongs to at least two clauses of $\mathcal{C}$, while a *minor literal* belongs to only one clause. The following problem is a restricted case of the SAT problem.

Problem 2.1 23SAT
Input: A set of variables $\mathcal{X} = \{x_1, ..., x_n\}$, and a set of clauses $\mathcal{C} = \{c_1, ..., c_m\}$ over $\mathcal{X}$ such that every clause contains 2 or 3 literals, and every clause contains at most 1 major literal.
Question: Is there a truth assignment to $\mathcal{X}$ which satisfies all clauses of $\mathcal{C}$?

Theorem 2.2 *23SAT is **NP**-complete.*

Proof. Clearly, the problem is **NP**. We prove **NP**-completeness by a polynomial reduction from SAT. Let $I_1 = (\mathcal{X} = \{x_1, ..., x_n\}, \mathcal{C} = \{c_1, ..., c_m\})$ be an instance of SAT. For every $1 \leq j \leq m$, denote $c_j = \{l_{j,1}, \ldots, l_{j,k(j)}\}$, $k(j) \geq 2$, and define

$$f(c_j) = \{\{l_{j,1}, y_{j,1}\}, \{\overline{y_{j,1}}, l_{j,2}, y_{j,2}\}, \{\overline{y_{j,2}}, l_{j,3}, y_{j,3}\}, \ldots, \{\overline{y_{j,k(j)-1}}, l_{j,k(j)}\}\}$$

where $y_{j,1}, \ldots, y_{j,k(j)-1}$ are new variables. Clearly, $y_{j,1}, \overline{y_{j,1}}, \ldots, y_{j,k(j)-1}, \overline{y_{j,k(j)-1}}$ are minor literals. Let

$$\mathcal{X}' = \mathcal{X} \cup \{y_{j,r} : 1 \leq j \leq m, 1 \leq r \leq k(j) - 1\}, \quad \mathcal{C}' = \bigcup_{1 \leq j \leq m} f(c_j)$$

Then $I_2 = (\mathcal{X}', \mathcal{C}')$ is an instance of 23SAT, since the size of every clause is 2 or 3, and every clause contains at most one major literal. It remains to prove that I_1 and I_2 are qeuivalent.

If I_1 is positive then there exists a truth assignment $t_1 : \mathcal{X} \longrightarrow \{0, 1\}$ which satisfies $\mathcal{C}$. For every $1 \leq j \leq m$ the fact that t_1 satisfies c_j implies that there exists $1 \leq r(j) \leq k(j) - 1$ such that $t_2(l_{j,r(j)}) = 1$. Let $t_2 : \mathcal{X}' \longrightarrow \{0, 1\}$ be the following extraction of t_1. Define $t_2(y_{j,r'}) = 1$ for every $1 \leq r' < r(j)$ and $t_2(y_{j,r'}) = 0$ for every $r(j) \leq r' \leq k(j) - 1$. Clearly, t_2 satisfies all clauses of $f(c_j)$ for every $1 \leq j \leq m$. Consequently, I_2 is positive.

Conversely, if I_2 is positive then there exists a truth assignment $t_2 : \mathcal{X}' \longrightarrow \{0, 1\}$ which satisfies $\mathcal{C}'$. Hence, t_2 satisfies $f(c_j)$ for every $1 \leq j \leq m$. However, $f(c_j)$ contains $k(j)$ clauses, and $k(j) - 1$ variables form $\mathcal{X}' \setminus \mathcal{X}$. Therefore, there exists $1 \leq r(j) \leq k(j) - 1$ such that $t_2(l_{j,r(j)}) = 1$. Define $t_1 : \mathcal{X} \longrightarrow \{0, 1\}$ by $t_1(x_i) = t_2(x_i)$ for every $1 \leq i \leq n$. Then t_1 satisfies c_j because $t_1(l_{j,r(j)}) = 1$ for every $1 \leq j \leq m$. Consequently, t_1 satisfies $\mathcal{C}$, and I_1 is positive. ■

Let $I = (\mathcal{X}, \mathcal{C})$ be an instance of 23SAT. A *bad pair* in I is a set of 2 clauses $\{c_1, c_2\} \subseteq \mathcal{C}$ such that there exist 2 literals, l_1 and l_2, for which $\{l_1, l_2\} \subseteq c_1$ and $\{\overline{l_1}, \overline{l_2}\} \subseteq c_2$.

Theorem 2.3 *There exists a polynomial algorithm which receives as its input an instance of 23SAT, and finds an equivalent instance of 23SAT without bad pairs.*

Proof. The following algorithm receives as its input an instance I_1 of 23SAT with bad pairs. The algorithm finds an equivalent instance, I_2, of 23SAT such that the number of bad pairs in I_2 is smaller than the number of bad pairs in I_1. By invoking the algorithm repeatedly, one can find an instance of 23SAT which is equivalent to the original one, and does not contain bad pairs.

Denote $I_1 = (\mathcal{X}, \mathcal{C})$. There exist clauses $c_1, c_2 \in \mathcal{C}$, and literals, l_1 and l_2, such that $\{l_1, l_2\} \subseteq c_1$ and $\{\overline{l_1}, \overline{l_2}\} \subseteq c_2$. By definition of 23SAT, every clause contains at most one major literal. Assume without loss of generality that l_1 is a minor literal. At least one of $\overline{l_2}$ and $\overline{l_1}$ is a minor literal.

If $\overline{l_2}$ is a minor literal then construct a truth assignment t for I_1 by assigning $t(l_1) = t(\overline{l_2}) = 0$. This assigment satisfies both c_1 and c_2. Moreover, if I_1 contains other clauses with $\overline{l_1}$ and l_2, these clauses are satisfied, too. Let I_2 be the instance of 23SAT obtained from I_1 by omitting all clauses which contain $\overline{l_1}$ or l_2.

We show that I_1 and I_2 are equivalent. If there exists a truth assignment t_2 that satisfies I_2, then assign $t(x) = t_2(x)$ for every variable x of I_2. Clearly, t satisfies I_1. On the other hand, since every clause of I_2 belongs also to I_1, if there does not exist a truth assigment that satisfies I_2, then there does not exist a truth assignment which satisfies I_1. Therefore, I_1 and I_2 are equivalent, and the number of bad pairs in I_2 is smaller than the number of bad pairs in I_1.

Suppose $\overline{l_1}$ is a minor literal. Assigning $l_1 = \overline{l_2}$ satisfies both c_1 and c_2, and does not affect the other clauses of I_1. Hence, let I_2 be the instance of 23SAT obtained from I_1 by omitting c_1 and c_2. Clearly, I_2 is equivalent to I_1, and the number of bad pairs in I_2 is smaller than the number of bad pairs in I_1. ■

3 Shedding Vertices

Let $I = (\mathcal{X} = \{x_1, ..., x_n\}, \mathcal{C} = \{c_1, ..., c_m\})$ be an instance of SAT. Define G_I to be the following graph.

$$V(G_I) = \{v\} \cup \{w_j : 1 \leq j \leq m\} \cup \{u_i, u_i' : 1 \leq i \leq n\}$$

$$E(G_I) = \{vw_j : 1 \leq j \leq m\} \cup \{w_j u_i : x_i \in c_j\} \cup \{w_j u_i' : \overline{x_i} \in c_j\} \cup \{u_i u_i' : 1 \leq i \leq n\}$$

Note that the subgraph induced by $N_2(v)$ is a *matching*, i.e. a disjoint union of copies of K_2. Every maximal independent set of $N_2(v)$ contains exactly one of u_i and u_i', for every $1 \leq i \leq n$.

Lemma 3.1 *An instance I of SAT is satisfiable if and only if there exists in G_I an independent set $S \subseteq N_2(v)$ which dominates $N(v)$.*

Proof. Suppose that there exists an independent set S of $N_2(v)$ which dominates $N(v)$. Define a truth assignment $t : \mathcal{X} \longrightarrow \{0,1\}$ as follows. For every $1 \leq i \leq n$, $t(x_i) = 1$ if and only if $u_i \in S$. Otherwise, $t(x_i) = 0$. The fact that S dominates all vertices of $N(v)$ implies that all clauses of $\mathcal{C}$ are satisfied by t.

On the other hand, assume that there exists a truth assignment $t : \mathcal{X} \longrightarrow \{0, 1\}$ which satisfies $\mathcal{C}$. Define $S = \{u_i : t(x_i) = 1\} \cup \{u'_i : t(x_i) = 0\}$. Clearly, $S \subseteq N_2(v)$. The fact that for every $1 \leq i \leq n$ the set S contains exactly one of u_i and u'_i implies that S is independent. The fact that t satisfies all clauses of $\mathcal{C}$ implies that S dominates all vertices of $N(v)$. $\blacksquare$

Lemma 3.2 *Let $I = (\mathcal{X}, \mathcal{C})$ be an instance of 23SAT without bad pairs. Then $G_I \in \mathcal{G}(\widehat{C}_6)$.*

Proof. Assume on the contrary that v belongs to a cycle of length 6. Then there exist vertices w_1, w_2, w_3 in $N(v)$ and z_1, z_2 in $N_2(v)$ such that $(v, w_1, z_1, w_2, z_2, w_3)$ is a cycle of length 6. Let l_1 and l_2 be the literals which represent z_1 and z_2, respectively. Let c_1, c_2 and c_3 be the clauses which represent w_1, w_2 and w_3, respectively. Then l_1 is a major literal since it belongs to both c_1 and c_2. Similarly, l_2 is a major literal, since it belongs to both c_2 and c_3. Hence c_2 contains two major literals, which is a contradiction. Therefore, v is not a part of a cycle of length 6.

Assume on the contrary that $G_I \setminus \{v\}$ contains a cycle C of length 6. Since $N(v)$ is independent, $|C \cap N(v)| \leq 3$. The fact that every connected component of $N_2(v)$ is K_2 implies that $|C \cap N(v)| \geq 2$.

If $|C \cap N(v)| = 3$ and $|C \cap N_2(v)| = 3$ then the vertices in $C \cap N_2(v)$ represent major literals, and the vertices of $C \cap N(v)$ represent clauses that each of them contains at least 2 major literals, which is a contradiction. Consequently, $|C \cap N(v)| = 2$ and $|C \cap N_2(v)| = 4$. That means that either $C = (u_1, u'_1, w_1, u_2, u'_2, w_2)$ or $C = (u_1, u'_1, w_1, u'_2, u_2, w_2)$ for u_1, u'_1, u_2, u'_2 in $N_2(v)$ and w_1, w_2 in $N(v)$. Therefore, there exist clauses c_1 and c_2 and literals l_1 and l_2 such that $\{l_1, l_2\} \subseteq c_1$ and $\{\overline{l_1}, \overline{l_2}\} \subseteq c_2$. Hence, I contains a bad pair, which is a contradiction. $G_I \in \mathcal{G}(\widehat{C}_6)$. $\blacksquare$

Theorem 3.3 *Recognizing shedding vertices is **co-NP**-complete, even for graphs without cycles of length 6.*

Proof. It should be proved that the complement problem is **NP**-complete. An instance of the problem is $I = (G, v)$, where G is a graph without cycles of length 6, and $v \in V(G)$. The instance is positive if and only if there exists an independent set in $N_2(v)$ which dominates $N(v)$.

The problem is obviously **NP**. We prove **NP**-completeness by a polynomial reduction from 23SAT. Let I_1 be an instance of 23SAT. By Theorem 2.3, it is possible to find in polynomial time an equivalent instance I_2 of 23SAT without bad pairs. Let $I_3 = (G_{I_2}, v)$ be an instance of the complement of the SHED problem. By Lemma 3.2, $G_{I_2} \in \mathcal{G}(\widehat{C}_6)$. By Lemma 3.1, I_2 is satisfiable if and only if there exists an independent set in $N_2(v)$ which dominates $N(v)$. $\blacksquare$

4 Relating Edges

Theorem 4.1 is the main result of this section.

Theorem 4.1 *Recognizing relating edges is **NP**-complete for graphs without cycles of length 6.*

Proof. Clearly, the problem is **NP**. We prove **NP**-completeness by a polynomial reduction from the complement of the SHED problem for graphs without cycles of length 6. Let $I_1 =$

(G, x) be an instance of the comlement of SHED such that $G \in \mathcal{G}(\widehat{C}_6)$. Then I_1 is positive if and only if there exists an independent set in $N_2(x)$ which dominates $N(x)$. Define a new graph G' as follows. $V(G') = V(G) \cup \{y\}$, when y is a new vertex, and $E(G') = E(G) \cup \{xy\}$. The fact that $G \in \mathcal{G}(\widehat{C}_6)$ implies $G' \in \mathcal{G}(\widehat{C}_6)$. Let $I_2 = (G', xy)$ be an instance of the RE problem. It remains to prove that I_1 and I_2 are equivalent.

Suppose I_1 is positive. The graph G contains an independent set $S \subseteq N_2(x)$ which dominates $N(x)$. Let S^* be a maximal independent set of $G \setminus \{x\}$ which contains S. In the graph G' the sets $S^* \cup \{x\}$ and $S^* \cup \{y\}$ are maximal independent sets. Therefore, S^* is a witness that xy is relating, and I_2 is positive.

Conversely, if I_2 is positive then there exists an independent set $S \subseteq V(G')$ such that $S \cup \{x\}$ and $S \cup \{y\}$ are maximal independent sets of G'. Obviously, $S \cap N_2(x)$ is an independent set of $N_2(x)$ which dominates $N(x)$, and I_1 is positive. ■

5 Conclusions

We proved that the RE problem is **NP**-complete for graphs without cycles of length 6. This result is suprising and unexpected since the RE problem is polynomially solvable for graphs without cycles of lengths 6 and 4 [20], for graphs without cycles of lengths 6 and 5 [22], and for graphs without cycles of lengths 6 and 7 [30]. Each of the algorithms presented in the three above-mentioned papers works as follows. It finds an independent set S_x in $N_2(x) \cap N_3(y)$ which dominates $N(x) \cap N_2(y)$. Then it finds similarly an independent set S_y in $N_2(y) \cap N_3(x)$ which dominates $N(y) \cap N_2(x)$. Since the graph does not contain cycles of length 6, there are no edges between S_x and S_y. Hence, if both S_x and S_y exist then $S = S_x \cup S_y$ is an independent set which is a witness that xy is relating, and the algorithm terminates announcing that the edge is relating. On the other hand, if at least one of S_x and S_y does not exist then the instance of the problem is negative. First, we conjectured that an algorithm for graphs without cycles of length 6 would work according to the same principle, and thus generalize the last three results. However, we found out that the existence of such an algorithm would imply that **P=NP**.

Theorem 1.3 shows a connection between the RE and SHED problems. In each of these problems it should be decided whether there exists an independent set in a subset of $V(G)$ which dominates another subset of $V(G)$. Hence, for many families of graphs either RE is **NP**-complete and SHED is co-**NP**-complete, or both problems are polynomially solvable. That holds also for graphs without cycles of length 6, the family of graphs which is studied in this paper. However, some families are exceptional. For example, concerning graphs without cycles of lengths 4 and 5, RE is **NP**-complete [20], but SHED is a polynomially solvable [24].

References

[1] A. Ahadi, A. Dehghan, *(2/2/3)-SAT problem and its applications in dominating set problems*, Discrete Mathematics & Theoretical Computer Science **21** (2019) #9.

[2] E. Boros, V. Gurvich, M. Milanič, *On CIS circulants*, Discrete Mathematics **318** (2014) 78-95.

[3] J. I. Brown, R. Hoshino, *Well-covered circulant graphs*, Discrete Mathematics **311** (2011) 244-251.

[4] J. I. Brown, R. J. Nowakowski, I. E. Zverovich, *The structure of well-covered graphs with no cycles of length* 4, Discrete Mathematics **307** (2007) 2235-2245.

[5] Y. Caro, N. Ellingham, G. F. Ramey, *Local structure when all maximal independent sets have equal weight*, SIAM Journal on Discrete Mathematics **11** (1998) 644-654.

[6] Y. Caro, A. Sebő, M. Tarsi, *Recognizing greedy structures*, Journal of Algorithms **20** (1996) 137-156.

[7] V. Chvatal, P. J. Slater, *A note on well-covered graphs*, Quo vadis, Graph Theory Ann Discr Math 55, North Holland, Amsterdam, 1993, 179-182.

[8] J. Döcker, *Monotone 3-Sat-(2, 2) is NP-complete*, (2019) arXiv:1912.08032

[9] S. Even, A. Itai, A. Shamir, *On the complexity of timetable and multicommodity flow problems*, SIAM Journal on Computing **5** (1976) 691-703.

[10] C. Feghali, M. Marin, *Three remarks on* $\mathbf{W_2}$ *graphs*, Theoretical Computer Science, **990** (2024) 114403 https://doi.org/10.1016/j.tcs.2024.114403.

[11] A. Finbow, B. Hartnell, R. Nowakowski, *A characterization of well-covered graphs of girth 5 or greater*, Journal of Combinatorial Theory Ser. B. **57** (1993) 44-68.

[12] A. Finbow, B. Hartnell, R. Nowakowski *A characterization of well-covered graphs that contain neither 4- nor 5-cycles*, Journal of Graph Theory **18** (1994) 713-721.

[13] M. R. Garey, D. S. Johnson, *Computers and Intractability; A Guide to the Theory of NP-Completeness*, W. H. Freeman & Co. New York, NY, USA, 1990.

[14] M. R. Garey, D. S. Johnson, L. Stockmeyer, *Some simplified NP-complete graph problems*, Theoretical Computer Science **1** (1976) 237-267.

[15] E. M. Gold, *Complexity of automaton identification from given data*, Information and Control **37** (1978) 302–320.

[16] M. Halaby, *On the computational complexity of MaxSAT*, Electronic Colloquium on Computational Complexity (2016) #34.

[17] A. Horn, *On sentences which are true of direct unions of algebras*, Journal of Symbolic Logic, **16** (1951) 14-21.

[18] V. E. Levit, E. Mandrescu, $\mathbf{W_2}$*-graphs and shedding vertices*, Electronic Notes in Discrete Mathematics **61** (2017) 797-803.

[19] V. E. Levit, D. Tankus *Weighted well-covered graphs without* C_4*,* C_5*,* C_6*,* C_7, Discrete Applied Mathematics **159** (2011) 354-359.

[20] V. E. Levit, D. Tankus, *On relating edges in graphs without cycles of length 4*, Journal of Discrete Algorithms **26** (2014) 28-33.

[21] V. E. Levit, D. Tankus, *Weighted well-covered claw-free graphs*, Discrete Mathematics **338** (2015) 99-106.

[22] V. E. Levit, D. Tankus, *Well-covered graphs without cycles of lengths 4, 5 and 6*, Discrete Applied Mathematics **186** (2015) 158-167.

[23] V. E. Levit, D. Tankus, *Complexity results for generating subgraphs*, Algorithmica **80** (2018) 2384-2399.

[24] V. E. Levit, D. Tankus, *Recognizing W_2 graphs*, Graphs and Combinatorics (2024) accepted.

[25] C. H. Papadimitriou, *Combinatorial Complexity*, Pearson, 1993.

[26] E. Prisner, J. Topp and P. D. Vestergaard, *Well-covered simplicial, chordal and circular arc graphs*, Journal of Graph Theory **21** (1996) 113-119.

[27] R. S. Sankaranarayana, L. K. Stewart, *Complexity results for well-covered graphs*, Networks **22** (1992) 247-262.

[28] T. J. Schaefer, *The complexity of satisfiability problems*, Proc. 10th Ann. ACM Symp. on Theory of Computing (1978) 216-226.

[29] J. W. Staples, *On some subclasses of well-covered graphs*, Ph.D. Thesis, 1975, Vanderbilt University.

[30] D. Tankus, *Recognizing generating subgraphs in graphs without cycles of lengths 6 and 7*, Discrete Applied Mathematics **283** (2020) 189-198.

[31] D. Tankus, M. Tarsi, *Well-covered claw-free graphs*, Journal of Combinatorial Theory Ser. B. **66** (1996) 293-302.

[32] D. Tankus, M. Tarsi, *The structure of well-covered graphs and the complexity of their recognition problems*, Journal of Combinatorial Theory Ser. B. **69** (1997) 230-233.

[33] C. A. Tovey, *A simplified NP-complete satisfiability problem*, Discrete Applied Mathematics **8** (1984) 85-89.

[34] R. Woodroofe, *Vertex decomposable graphs and obstructions to shellability*, Proceedings of the American Mathematical Society **137** (2009), 3235-3246.