

Consensus formation in quality-sensitive interdependent agent systems

David March Pons,^{1,*} Ezequiel E. Ferrero,^{2,3,4} and M. Carmen Miguel^{2,3}

¹*Departament de Física, Universitat Politècnica de Catalunya, Campus Nord B4, 08034 Barcelona, Spain*

²*Departament de Física de la Matèria Condensada,*

Universitat de Barcelona, Martí i Franquès 1, 08028 Barcelona, Spain.

³*Institute of Complex Systems (UBICS), Universitat de Barcelona, Barcelona, Spain*

⁴*Instituto de Nanociencia y Nanotecnología, CNEA-CONICET, Centro Atómico Bariloche, R8402AGP S. C. de Bariloche, Río Negro, Argentina.*

We perform a comprehensive analysis of a collective decision-making model inspired by honeybee behavior. This model integrates individual exploration for option discovery and social interactions for information sharing, while also considering option qualities. Our assessment of the decision process outcome employs standard consensus metrics and investigates its correlation with convergence time, revealing common trade-offs between speed and accuracy. Furthermore, we investigate fluctuations arising from finite-size effects. In scenarios with strong interdependence, we identify an absorbing phase transition and scrutinize its key features, including critical fluctuations in finite spatial systems. We find that fluctuations play a significant role in the collective decision-making process, particularly when options have similar qualities. Our findings are especially relevant for finite adaptive systems, as they provide insights into navigating decision-making scenarios with similar options more effectively. Finally, we show that this model also fulfills Weber's Law of perception, aligning with previous analyses of collective-decision problems in their adherence to different psychophysical laws.

I. INTRODUCTION

Collective decision making is an emergent, self-organized phenomenon by which a group of agents, each with its own decision mechanisms at the individual level, reaches an agreement on a certain task or topic [1–4]. From humans taking part in elections, social mammal herds, flocking fish, foraging insect colonies or robot swarms, collective decision making is an ubiquitous process across all scales of biological and artificial complexity [1, 2, 5–7]. Thus, addressing this question requires interdisciplinary approaches integrating fields such as sociology, behavioral ecology, biology, physics, computer science and communication studies, among many others [1, 8]. Through empirical observations, field experiments, laboratory studies, and computational modeling, researchers strive to unravel the intricate dynamics of collective decision making and its significance.

Many efforts have been devoted in the modeling of opinion dynamics in order to understand how a collective makes a decision among a set of options. Initial attempts devised models that encompassed simple imitation rules controlling the build up of an agreement between equivalent options, such as the voter model [9, 10] or the majority rule model [11]. These models, though, have been regarded as too unrealistic in their basic formulation, and many more complex models have been proposed in order to capture a richer spectrum of social behaviors, such as partisanship, heterogeneity, personalized information or non linear interactions [12–17].

In the realm of animal behavior, the existence of a remarkable diversity of animal signals has enthralled not

only researchers studying collective motion and consensus formation but also non-experts, as these signals embody some of the most captivating facets of the natural world. Evolving to facilitate and enrich communication, these signals are conspicuous and compelling, even to casual observers. Animal signals prompt a myriad of scientific inquiries probing their functions, the information they convey, their production mechanisms, and the evolutionary processes driving them. These questions encapsulate the essence of comprehending animal communication and its significance in collective decision-making within the natural world.

In recent years, the collective behavior exhibited by social insects has become a focal point of interest within the scientific community. They conceive a paradigmatic example of systems formed by rather simple units exhibiting this kind of collective emergent phenomena. In particular, the nest-site selection process in honeybee swarms, widely studied by biologists and ecologists [18–20], has laid the ground for a great number of collective decision making models inspired in this fascinating process [21–25], and for further exploration of these dynamics in robot swarms [26–29].

In a bee-like house-hunting process, agents can adopt a wider spectrum of behaviors beyond the usual imitation rules. Initially, they must discover available options either through individual exploration or by communicating with peers using different signaling mechanisms. Additionally, they have the ability to abandon options and become uncommitted or neutral. Lastly, they may cross-inhibit other peers with different opinions. Importantly, in honeybee-inspired models, agents are able to estimate the qualities of options, which significantly impacts their subsequent signaling behavior [30]. Specifically, the better they estimate the quality to be, the longer they will

* david.march@upc.edu

advertise this option to their peers. This mechanism allows a positive feedback loop to regulate the group's opinion formation, discarding the worst option in favor of the more suitable ones. Ultimately, a collective agreement is reached through a combination of individual exploration of the environment to identify possible options, estimation of their quality, and information exchange among members of the group.

List, Estholtz and Seeley presented an agent-based model (in the following, the LES model) of the decision making process performed by honeybee swarms [21], further developed analytically later by T. Galla [31], which focuses on these three features mentioned above. The LES model is a simple model that introduces option qualities into the decision process. By balancing individual option exploration and social interactions, it effectively captures decision-making dynamics. The main result highlighted in the original investigation is that when individuals possess sufficiently accurate estimates of the qualities and engage in a high degree of social interaction, the swarm tends to converge towards collective agreement, particularly when faced with various options of differing quality values.

In this paper, we delve deeper into studying the formation of consensus by inspecting the LES model. Our investigation focuses on the interplay between social feedback and individual exploration, which the model balances through a single parameter known as the “interdependence” parameter. The intensity of individual, independent exploration serves a dual purpose. On one hand, it is essential for the group to perceive all available options or to adapt in changing environments [28, 32]. On the other hand, excessive individual exploration can introduce noise that hampers the effectiveness of the aforementioned positive feedback loop. Agents tend to overlook the actual state of their peers and adopt opinions based on the outcome of their own exploration. Consequently, stronger social interaction is required to balance this effect. Recent work highlights the capability of decentralized systems to effectively balance noise with social interactions under various scenarios [33–35].

We approach the problem by considering the mean-field model equations and complementing these results with simulations in fully connected networks and regular lattices in finite dimensions. These simulations help us address finite size effects and convergence times. We conduct an exhaustive parametric exploration of the model, which intriguingly has led us to establish a connection between a specific limit of this model and the well-known contact process. Following recent developments in collective decision-making models inspired by honeybee behavior, we demonstrate how this model may also adhere to certain psychophysical laws that describe organisms' perception when discriminating between various stimuli or options [36].

The paper is organized as follows. In Section II, we provide a brief review of the model formulation and its analytical solution, highlighting results in specific limit-

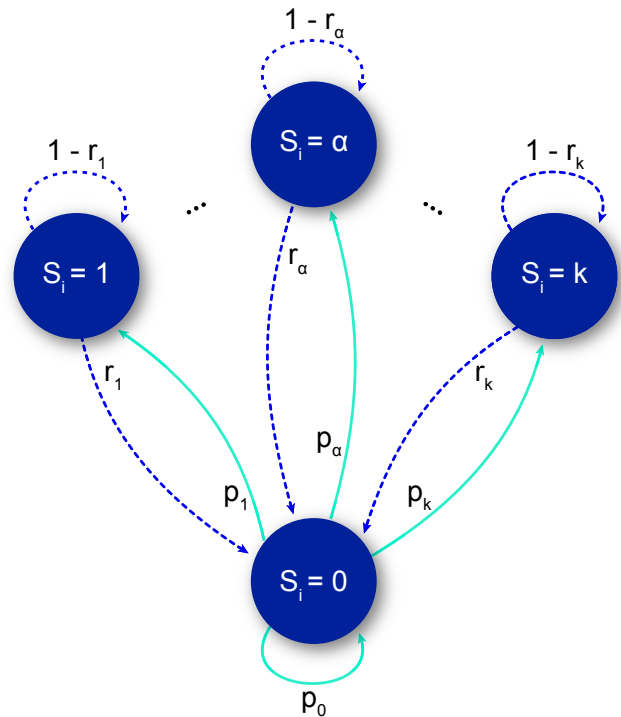


FIG. 1. Schematic representation of the transitions occurring between the commitment states of the LES model. Agents transition from being uncommitted ($s_i = 0$) to being committed to option α ($s_i = \alpha$) with a probability denoted as p_α , where α ranges from 1 to k . If they fail to commit to any of the k options, they remain uncommitted with a probability p_0 . Once an agent is in a committed state, it can transition back to the uncommitted state with a probability r_α . Notice that direct transitions between different committed states are not permitted.

ing scenarios. Section III presents a thorough analysis of the model and its significant findings. In Sections III A and III B, we initiate our investigation by examining the possibility of consensus formation and its characteristic convergence time within a binary decision context. This is followed by an exploration of the significance of finite size effects in Section III C. In Section III D, we investigate the correlation between the equal-quality and equal-discovery binary model and the contact process, along with its characteristic non-equilibrium critical behavior. Furthermore, in Section III E, we discuss the model's alignment with Weber's Psychophysical Law of perception. We close this section by analyzing a multi-site extension of the model that encompasses more options than the typical binary scenario (Section III F). Finally in Section IV, we present our main conclusions.

II. MODELING FULLY-CONNECTED SCOUT-BEE-LIKE SWARMS

List, Elsholtz, and Seeley introduced their honeybee-inspired collective decision-making model in [21]. This

model encapsulates the primary mechanisms through which a swarm of honeybees achieves a collective decision. Honeybees possess the ability to independently discover potential nest site options and evaluate their quality. Subsequently, they communicate their findings to the rest of the swarm. Concurrently, bees can interact with their peers and opt to adopt an opinion already present within the group.

The LES model comprises a swarm of N bees, denoted by $i = 1, \dots, N$, and k potential nest sites, denoted by $\alpha = 1, \dots, k$. Each site is associated with an intrinsic quality, $q_\alpha \geq 0$, and an *a priori* self-discovery probability $\pi_\alpha \geq 0$, which are considered fixed model parameters and do not change over time. Throughout the text, and without loss of generality, we opt to label the sites in ascending order of quality, such that $q_1 \leq \dots \leq q_k$.

The decision process unfolds in discrete time steps. At any given time step, a bee can either be uncommitted, denoted by the state variable $s_i(t) = 0$, or committed to a specific option α , indicated by $s_i(t) = \alpha$, and signaling its preference to its peers. Bees have the flexibility to transition between the uncommitted state and any committed state, but cannot directly switch between different committed states. The system's evolution is governed by a set of transition probabilities $(p_{\alpha,t+1}, r_{\alpha,t+1})$, represented by arrows in Fig. 1, determining commitment and uncommitment events based on the system state at time t for the subsequent time step $t + 1$:

$$\begin{aligned} s_i(t) &= 0 \xrightarrow{p_{\alpha,t+1}} s_i(t+1) = \alpha \\ s_i(t) &= \alpha \xrightarrow{r_{\alpha,t+1}} s_i(t+1) = 0, \end{aligned} \quad (1)$$

The commitment probabilities $\{p_{\alpha,t}\}$ are determined by the following expression:

$$p_{\alpha,t+1} = (1 - \lambda)\pi_\alpha + \lambda f_{\alpha,t}. \quad (2)$$

The first term of this equation represents the likelihood that a scout bee independently discovers site α , while the second term quantifies the probability that the bee commits to state α by following the advice of its peers: $f_{\alpha,t}$ denotes the fraction of agents promoting site α at time t . These terms are weighted by the interdependence parameter, λ . This parameter, ranging between 0 and 1, determines the extent to which bees depend on each other to reach their commitment decision. Vanishing interdependence ($\lambda \rightarrow 0$) signifies that bees will commit to a site solely based on independent exploration. Conversely, high interdependence ($\lambda \rightarrow 1$) means that bees will mostly consider their peers advertisement in order to make a commitment decision. Probabilities must be normalized, so Eq. (2) must satisfy $\sum_{\alpha=0}^k p_{\alpha,t} = 1$. It's important to note that we include the uncommitted state 0 in the normalization condition. Additionally, the normalization condition implies that $\sum_{\alpha=0}^k f_\alpha = 1$, and consequently $\sum_{\alpha=1}^k \pi_\alpha \leq 1$.

The uncommitment transition probabilities $r_{\alpha,t}$ are inherently linked to the qualities of the sites. In the origi-

nal LES model, transitions from committed to uncommitted states occur deterministically after a predetermined commitment time has elapsed. An agent commits to an option and advertises it for a fixed duration. To render the model's equations mathematically tractable, T. Galla [31] replaced this deterministic process by a stochastic process defined by the following rates:

$$r_\alpha = q_0 \left[\frac{\mu}{K} + \frac{1 - \mu}{q_\alpha} \right], \quad \alpha = 1, \dots, k. \quad (3)$$

The parameter μ represents the extent to which bees independently assess the quality of a site. When $\mu = 0$, the duration of a dance relies solely on the quality of the site, indicating that bees independently evaluate the site's quality. Conversely, when $\mu = 1$, the site's quality becomes irrelevant, and bees predominantly advertise an option for a generic period of time, typically related to a new parameter K . This parameter K is uniform for all sites, representing, for example, the maximum quality among all the options. The parameter q_0 ensures that $0 < r_\alpha \leq 1$ and represents the characteristic time scale of the problem. Here, we set $q_0 = 1$, and primarily focus on the case where $\mu = 0$. Consequently, the average duration of the advertisement for site α is $1/r_\alpha$, which is proportional to q_α . The stochastic representation of these transitions still preserves the principal characteristic of the deterministic duration in the original model: the higher the quality of the state, the longer agents will remain committed and advertise for it.

The stochastic problem can be analyzed using a master equation, from which one can derive a set of nonlinear differential equations describing the evolution of the average fraction values $\langle f_{\alpha,t} \rangle$, henceforth referred to as the average dancing frequencies, for each state α . By following the mathematical details outlined in [31], and assuming a fully connected, mean-field-like system, one can arrive at the following equations:

$$\dot{\langle f_{\alpha,t} \rangle} = \langle f_{0,t} \rangle [(1 - \lambda)\pi_\alpha + \lambda \langle f_{\alpha,t} \rangle] - r_\alpha \langle f_{\alpha,t} \rangle \quad (4)$$

where $\langle f_{0,t} \rangle = 1 - \sum_{\alpha=1}^k \langle f_{\alpha,t} \rangle$. Equation (4) can be readily integrated numerically, for example, using the Euler scheme. Furthermore, an expression for the stationary points, denoted as f_α^* , can be obtained by solving a system of k coupled equations, which is derived by setting $\dot{\langle f_{j,t} \rangle} = 0$. By rearranging the resulting equations, one can derive expressions for the stationary values of the population of each site in terms of f_0^* :

$$f_\alpha^* = \left[\frac{r_\alpha}{f_0^*} - \lambda \right]^{-1} (1 - \lambda)\pi_\alpha \quad \alpha = 1, \dots, k. \quad (5)$$

As formulated, the model can be analyzed through agent-based simulations or by numerically integrating Eq. (4). However, equations can be derived to determine the model's stationary points, from which analytical expressions can be obtained under certain specific limits. In the following subsection, we provide an overview of these analytical solutions.

A. Stationary deterministic solutions

Since we can express f_0^* as $f_0^* = 1 - \sum_{\alpha=1}^k f_\alpha^*$, we can sum the k equations given by Eqs. (5) to derive a closed equation for the stationary value f_0^* :

$$(1 - f_0^*) \prod_{\alpha=1}^k (r_\alpha - \lambda f_0^*) = (1 - \lambda) f_0^* \sum_{\alpha=1}^k \pi_\alpha \prod_{\substack{\beta=1, \\ \beta \neq \alpha}}^k (r_\beta - \lambda f_0^*). \quad (6)$$

Eq. (6) can be solved either numerically or rearranged into a $k + 1$ -th degree polynomial. In either approach, we find $k + 1$ roots for f_0 , some of which may lead to nonphysical solutions where $f_0^* > 1$. A linear stability analysis is provided in Appendix A 1. In the specific case of $k = 2$, assuming $q_2 > q_1$ (or $r_2 < r_1$), a single physical and stable solution $f_\alpha^* \forall \alpha$ is obtained. An example of such a solution is depicted in Fig. 2A, with values $r_2 = 1/10$, $r_1 = 1/7$, $\pi_1 = \pi_2 = 0.1$, $\lambda = 0.6$ and $\mu = 0$.

1. Limit of fully independent bees

The limit case $\lambda = 0$ can be treated separately, as it permits a simpler analytical solution of Eq. (5) (or, equivalently, Eq. (6)), taking the form:

$$f_0^* = \frac{1}{1 + \sum_{\beta=1}^k \frac{\pi_\beta}{r_\beta}}. \quad (7)$$

$$f_\alpha^* = \frac{\pi_\alpha}{r_\alpha} \frac{1}{1 + \sum_{\beta=1}^k \frac{\pi_\beta}{r_\beta}} \quad 1 \leq \alpha \leq k. \quad (8)$$

2. Limit of fully interdependent bees

In the limit cases where $\pi_\alpha = 0$ for all α , or $\lambda = 1$, where individual sourcing of information has no influence, simpler analytical solutions exist. It should be noted that in dynamics originating from specific initial conditions, such as an all uncommitted population ($f_0(t=0) = 1$), committed populations never have the opportunity to grow in this limit. Beyond this particular case, we can derive the stable analytical fixed point of the model in this limit. The $k + 1$ fixed points solutions of this limit take the form

$$\begin{aligned} f_0^* &= 1, & f_1^* &= \dots = f_k^* = 0 \\ \text{and} & & & \\ f_0^* &= r_\beta / \lambda, & f_\alpha^* &= (1 - r_\beta / \lambda) \delta_{\alpha, \beta}, \end{aligned} \quad (9)$$

where $\delta_{\alpha, \beta}$ is the Kronecker delta, and $1 \leq \alpha, \beta \leq k$. In the case where $\lambda = 1$, the second family of solutions further simplifies to $f_0^* = r_\beta$, $f_\alpha^* = (1 - r_\beta) \delta_{\alpha, \beta}$, $\forall \beta$.

Although all these solutions are physically valid, only two of them (one in the case $\lambda = 1$) are stable. Using linear stability analysis (LSA), we determine that the stability threshold between these two solutions is determined by the highest site quality, denoted as site k . When $\lambda < r_k$, the stable solution is the absorbing state $f_0^* = 1$. Thus, in the absorbing state, the system remains fully uncommitted. Conversely, when $\lambda > r_k$, the stable solution is the one with the best site taking the lead, with $f_\alpha^* = (1 - \frac{r_k}{\lambda}) \delta_{\alpha, k}$, and the remaining proportion of uncommitted population being $f_0^* = \frac{r_k}{\lambda}$. In the special case where we set $\lambda = 1$, the stability threshold vanishes because $\lambda = 1 \geq r_k$, and the only stable solution remaining is $f_0^* = r_k$, $f_\alpha^* = (1 - r_k) \delta_{\alpha, k}$. Details on the LSA analysis are provided in Sec. A 2.

In Fig. 2B,D, we present a summary of the findings for the binary problem with $k = 2$. When $\lambda \geq r_2$, only the fixed point where the best option is imposed remains stable, while the one where the inferior option is imposed acts as a saddle node (stable on the simplex $f_2 = 0$). As λ decreases, these points gradually move upward (reducing either f_1 or f_2) until they merge with the point $f_0 = 1$ at $\lambda = r_\alpha$. In the regime where $\lambda < r_2$, the only remaining stable fixed point is the absorbing state $f_0^* = 1$ fixed point.

3. Limit of equal-quality sites

So far, our analysis has assumed that all sites differ in quality, or at least one site's quality is greater than the others. However, when all qualities are equal, implying $r_\alpha = r$ for $1 \leq \alpha \leq k$, the only parameters that can break the symmetry between the populations of the different states are their discovery probabilities, π_α . If, on top of identical qualities, we also have identical discovery probabilities, $\pi_\alpha = \pi$ for $1 \leq \alpha \leq k$, the system will end up in a deadlock between the k possible nest-site options. In other words, $f_1^* = \dots = f_k^*$, and therefore, no consensus is reached. In this particular limit though, the general solution for f_0^* reduces to a second-degree polynomial, with the physical and stable solution being $f_0^* = (A - \sqrt{A^2 - 4\lambda r}) / (2\lambda)$, where $A = k\pi(1 - \lambda) + \lambda + r$.

In the specific case where we have equal-quality sites and fully interdependent bees ($\pi_\alpha = 0$ for all α), we find two solutions $f_0^* = 1$ and $f_0^* = r/\lambda$ delimited by the stability threshold value $\lambda = r$. In the former solution, all the other frequencies vanish, i.e. $f_\alpha^* = 0$ for $1 \leq \alpha \leq k$, whereas in the latter the actual values of f_α^* remain undetermined (note that a null denominator appears in Eq. (5)). According to LSA, any combination that satisfies $\sum_{\alpha=1}^k f_\alpha = 1 - r/\lambda$ is a potential solution. Consequently, the stationary state reached after numerical integration of Eq. (4) or in numerical simulations of the model with those parameters is highly dependent on the initial conditions and is susceptible to significant finite-size effects (see Secs. III C and III D).

In Figs. 2C and E, we summarize the results once more

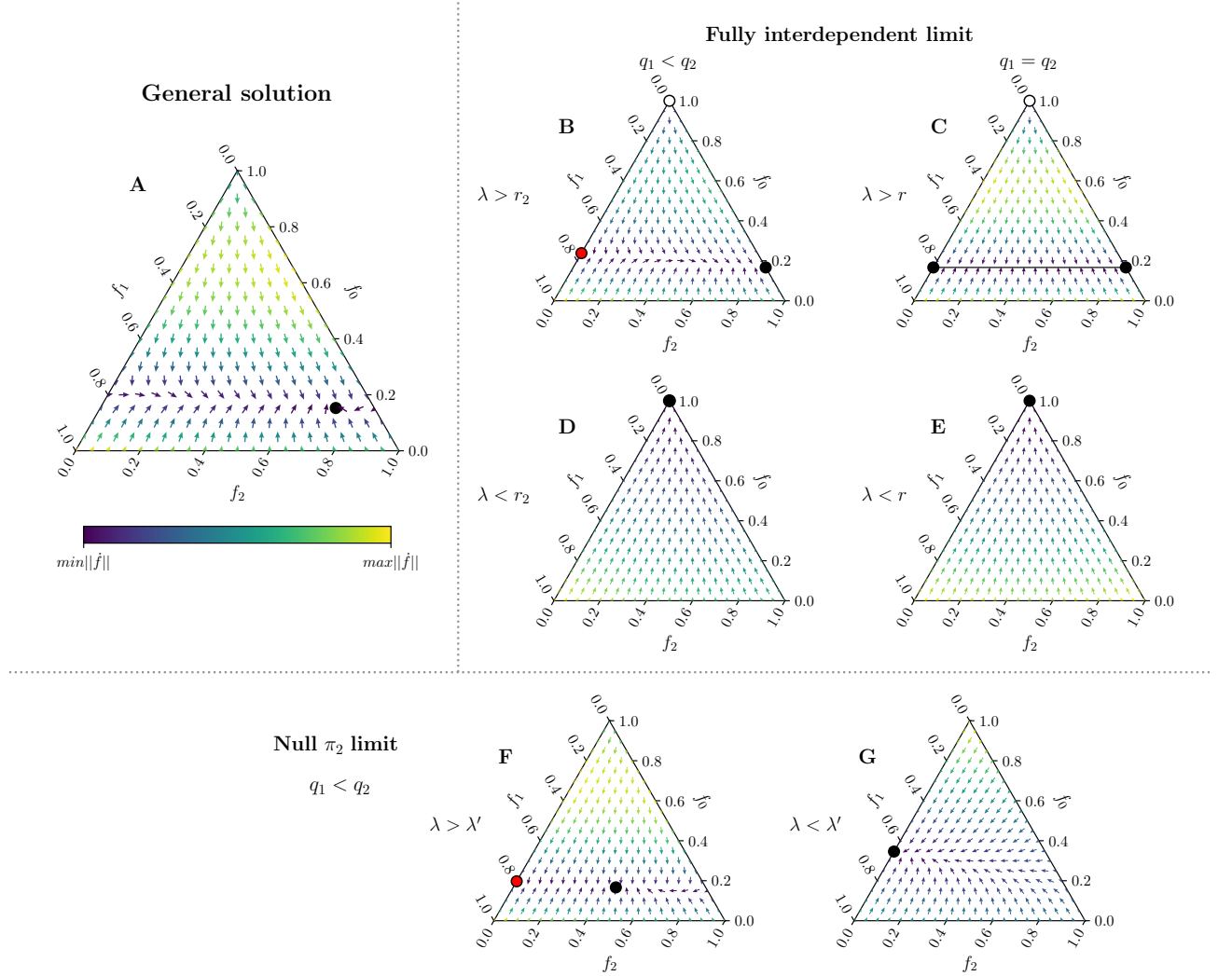


FIG. 2. Flow diagrams depicting the dynamics of collective decision-making in a binary decision scenario. Arrows indicate the direction of the deterministic flow, while the color coding reflects the strength of the flow. Stable, saddle nodes, and unstable fixed points are denoted by black, red, and white circles, respectively. **A**: General solution for $\pi_1 = \pi_2 = 0.1$, $q_1 = 7$, $q_2 = 10$, $\lambda = 0.6$. Panels **B**, **C**, **D**, **E** illustrate solutions in the fully interdependent limit, $\pi_\alpha = 0 \forall \alpha$. In panels B and D, qualities are $q_1 = 7$, $q_2 = 10$, whereas in panels C and E, we have $q_1 = q_2 = 10$. Interdependence is $\lambda = 0.6$ in panels B and C, and $\lambda = 0.05$ in panels D and E. In both cases the absorbing transition appears at $\lambda = r_2$, $r_2 = 0.1$. **F**, **G**: Solutions in the limit of null discovery for the good option, $\pi_2 = 0$, while $\pi_1 = 0.25$. Qualities are $q_1 = 7$, $q_2 = 10$, and $\lambda = 0.6$ (Panel F) or $\lambda = 0.05$ (Panel G). The absorbing transition appears at $\lambda = \lambda' \simeq 0.43$. The analysis assumes perfect independent quality assessment, $\mu = 0$.

for the binary option case in the fully interdependent limit. When $\lambda > r$, all points in the simplex $f_1 + f_2 = 1 - r/\lambda$ (depicted as a black line) are plausible solutions. As λ decreases, these solutions move towards lower values of f_1 or f_2 until they coalesce with the absorbing state $f_0^* = 1$ when $\lambda < r$.

4. Null discovery probability of the best site

The null discovery probability of the best site, $\pi_k = 0$, defines a limit where the best option is never discovered independently. This limit emphasizes the significance of

social influence in decision-making processes, particularly when the inherent quality of options is not apparent to individuals through independent assessment. This specific limit can also be examined analytically. Since $\pi_k = 0$, the differential equation for the best quality site simplifies to a straightforward expression, yielding two stationary solutions: $f_0^* = r_k/\lambda$ or $f_k^* = 0$. With the first solution, one can readily derive the remaining stationary frequencies:

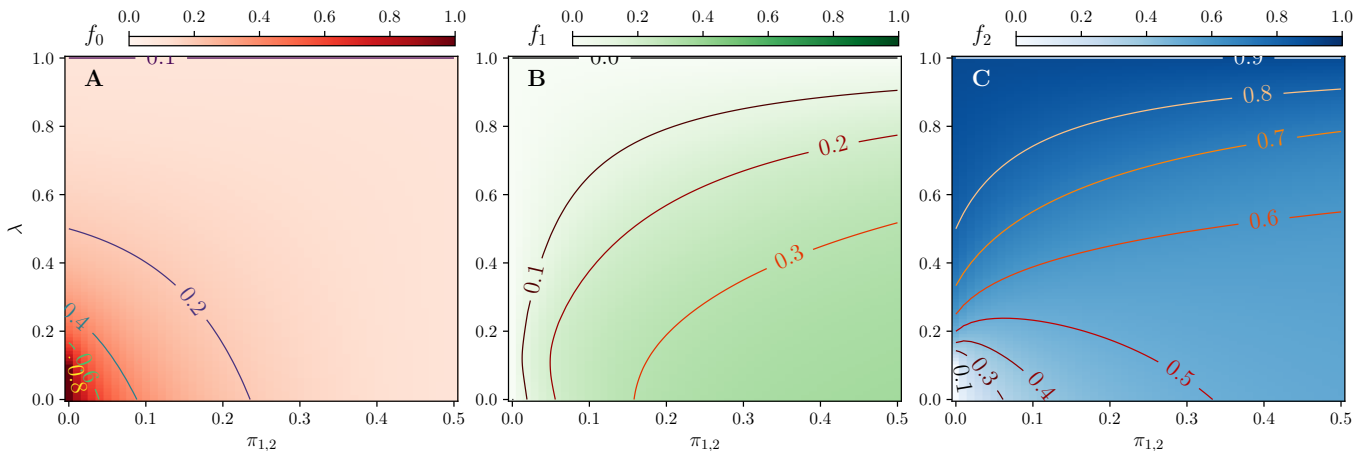


FIG. 3. Color-maps illustrating the stationary values of f_0 , f_1 , and f_2 in the parameter space $(\pi_{1,2}, \lambda)$ are presented in panels **A**, **B**, and **C**, respectively. The site qualities are held constant at $(q_1, q_2) = (7, 10)$. Color gradients are used to indicate the quantitative values of f_α based on the color bars, while continuous lines represent iso-population levels in the parametric space.

$$f_0^* = \frac{r_k}{\lambda}, \quad f_\alpha^* = \frac{\pi_\alpha(1-\lambda)r_k}{\lambda(r_\alpha - r_k)} \quad \alpha = 1, \dots, k-1,$$

$$f_k^* = 1 - \frac{r_k}{\lambda} \left[1 + (1-\lambda) \sum_{\alpha=1}^{k-1} \frac{\pi_\alpha}{r_\alpha - r_k} \right].$$

For the second stationary point $f_k^* = 0$, one arrives at a system of $k-1$ coupled equations for the other committed populations. While it is possible to derive analytical expressions for f_α^* ($\alpha \neq k$) and f_0^* for a small number of sites, it is often more practical to use Eqs. (5) and (6), or to resort to numerical integration of Eq. (4) to obtain the complete solution.

In conclusion, the noteworthy aspect of this particular limit is the transition from a stable state where there is no population for the best option ($f_k^* = 0$) to a state where $f_k^* \neq 0$, as described by Eq. (10). The transition occurs as λ increases, at the specific value

$$\lambda' = \frac{1 + \psi}{r_k^{-1} + \psi} \quad \text{where } \psi = \sum_{\alpha=1}^{k-1} \frac{\pi_\alpha}{r_\alpha - r_k} \quad (10)$$

It should be noted that while conducting a linear stability analysis on these solutions confirms the change in stability of the solution with $f_k^* = 0$ (stable when $\lambda < \lambda'$), the solution provided by Eq. (10) is always stable. However, it becomes nonphysical when $\lambda < \lambda'$, as $f_k^* < 0$. The change in sign occurs precisely at the same value λ' . When the qualities are equal, the regime where $f_k^* \neq 0$ disappears, effectively reducing the system to one with only $k-1$ options, which can be analyzed using the general solution.

The two regimes are depicted in Figs. 2F and G, focusing on the specific case of $k=2$. As λ approaches λ' from above, $f_2 \rightarrow 0$ and the stable point approaches the unstable solution (Fig. 2F). At $\lambda = \lambda'$, these two points

merge, and in the regime $\lambda < \lambda'$, the only stable solution is where no population is committed to the best-quality site.

III. RESULTS

Going beyond the mean-field approach, previous studies have delved into the stochastic dynamics of this decision-making model through simulations conducted on a regular lattice [31] and on random networks [29]. Additionally, experiments have been conducted using mini-robots as a physical platform [29], albeit with a limited parameter set explored. In the following, we focus on characterizing the model's behavior using a mean-field approach, conducting an exhaustive exploration across a broad parameter range. To quantify the stationary results, we rely on analytical expressions. Simulations conducted on fully connected systems or square lattices yield average results that align perfectly with these analytical solutions. However, to complement our analysis, we take advantage of simulations to measure the time needed to reach the stationary state and to address finite-size effects.

We will first concentrate on the specific case of a binary decision problem, a topic extensively studied in various general opinion dynamics models [10–12, 14, 17, 37–41]. This scenario has also been the focus of other honeybee-inspired models [22–25]. The generalization to larger values of k is discussed later in Section III F.

As detailed in the model description section, we consistently assume that site 2 holds the highest quality, denoted by $q_2 > q_1$. Additionally, we assume that bees independently assess the qualities of the sites, represented by the model parameter $\mu = 0$. This implies that the abandonment rates of the dances depend solely on the qualities of each site, defined as $r_\alpha = q_0/q_\alpha$. Previous research on this model has shown that, under these con-

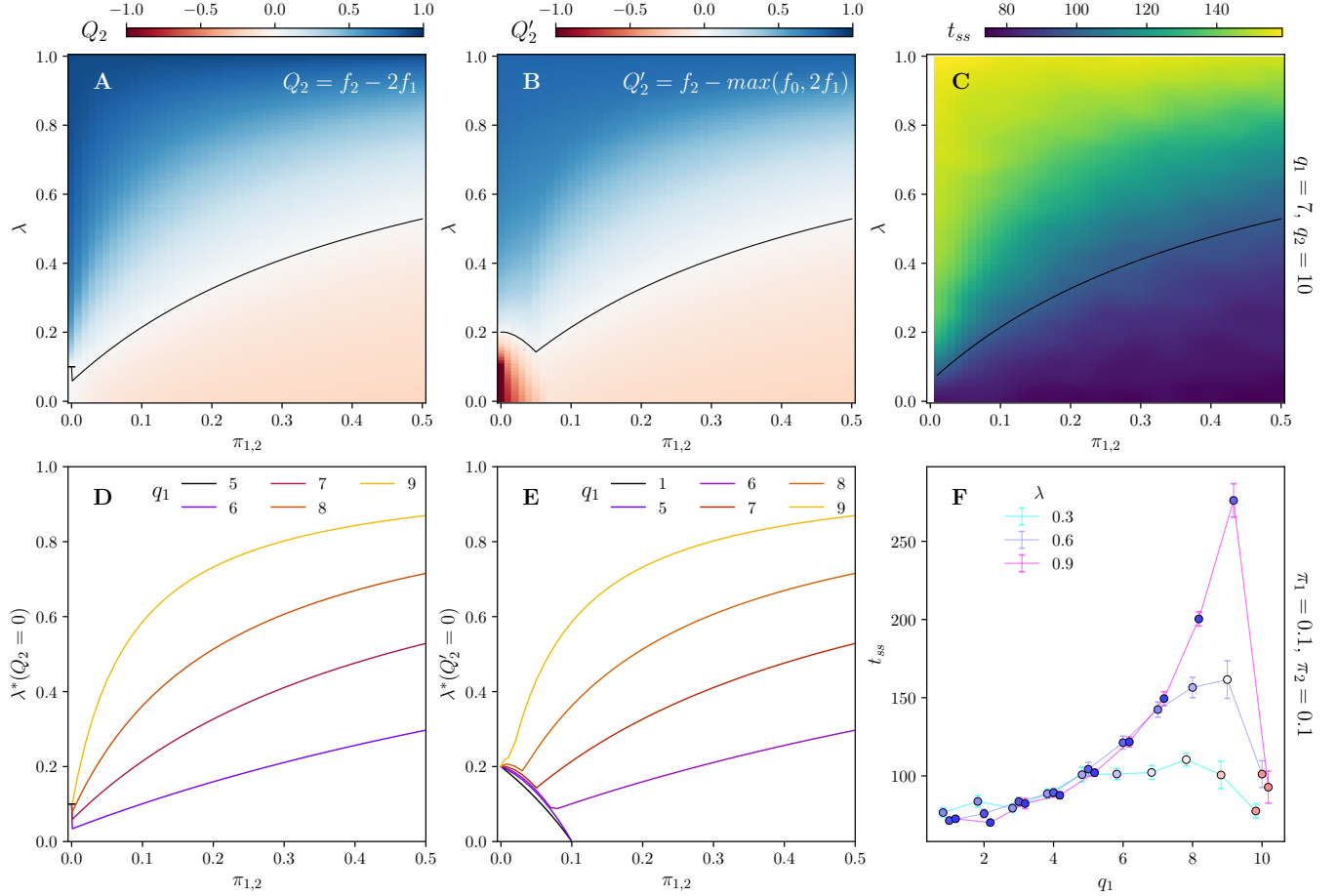


FIG. 4. **A,B:** Color-maps illustrating the consensus parameters Q_2 and Q'_2 , as defined in Eqs. (11) and (12) respectively with $x = 2$, across the parameter space $(\pi_{1,2}, \lambda)$. The black lines represent the crossover points $Q_2 = 0$ or $Q'_2 = 0$. **C:** Time to reach the stationary state, t_{ss} , obtained from simulations with a swarm size of $N = 5000$ bees. The fully interdependent limit ($\pi_\alpha = 0$ or $\lambda = 1$) is not included in the maps (see discussion in Sec.III D). The site quality parameters are held constant at $(q_1, q_2) = (7, 10)$ across all three maps. **D,E:** Crossover lines $\lambda^*(Q_2, Q'_2 = 0)$ as a function of $\pi_{1,2}$. Each curve represents a different value of q_1 , while $q_2 = 10$ is held constant. **F:** Time to reach the stationary state, t_{ss} , obtained from simulations plotted against the quality of the inferior option, q_1 , while maintaining $q_2 = 10$, for three values of the interdependence. Parameters $\pi_1 = \pi_2 = 0.1$ and $N = 5000$ are fixed. Simulations start from uncommitted initial conditions. The markers are color-coded based on the consensus established at the stationary state. They are slightly jittered around the q_1 axis to allow for better visualization of the color code.

ditions, the swarm can identify the best site across a wide range of interdependence values, λ [21, 31]. In this setup, we investigate the interplay between group communication, represented by the model parameter λ , and individual exploration success, captured by π_α . We explore scenarios where sites may vary in their likelihood of being discovered, denoted by $\pi_1 \neq \pi_2$ (asymmetric scenario), or where they share the same discovery probability, denoted by $\pi_1 = \pi_2$ (symmetric scenario).

A. Symmetric discovery scenario

We start by examining the case where the available sites have an equal likelihood of being found, i.e., $\pi_1 = \pi_2 = \pi_{1,2}$, and they only differ in their quality, with

$q_1 < q_2$. We present our findings by exploring the state space defined by $(\pi_{1,2}, \lambda)$ for various combinations of site qualities, while keeping $q_2 = 10$ constant and varying q_1 .

Figure 3A-C represents the particular values of the frequencies f_0 , f_1 and f_2 for a particular choice of the qualities, $(q_1, q_2) = (7, 10)$. In these plots, we observe smooth trends for all these quantities. As the interdependence increases, the frequencies f_0 and f_1 decrease regardless of the value of the discovery probabilities. This decline favors the proportion of bees committed to the best site, f_2 . When bees rely more on the opinions of their peers, the best quality option benefits from the positive feedback generated by longer advertisement times. Conversely, when independent discoveries increase, f_1 increases at the expense of f_0 and f_2 . This shift occurs because as $\pi_{1,2}$ values increase while keeping λ constant, more advertise-

ments are motivated by independent discoveries rather than by opinion sharing. Consequently, information is introduced equivalently for both sites, and the best-quality reinforcement effect of λ is hindered. In the limit where the discovery probabilities are small, the good option is easily imposed by interdependence-mediated discussion. However, when $\pi_{1,2} = 0$, we observe a transition to the absorbing regime, $f_0^* = 1$, at $\lambda = r_2 = 0.1$.

With these results in mind, we can quantify the outcome of the decision process by defining a consensus. This involves setting a threshold condition on the values of the frequencies f_α that is necessary to conclude that the swarm has reached a decision. Such a consensus will be defined with respect to the best option, as our main focus is discerning whether the swarm is able to retrieve the best available option.

The simplest approach is to define a threshold on the value of the population committed to the best option. For instance, one can require that at least half of the total population is committed to the best option in order to consider the system to have made a successful decision. This threshold line is depicted as the contour line $f_2 = 0.5$ in Fig. 3C. We observe that a large portion of the state space $(\pi_{1,2}, \lambda)$ lies above this threshold line. When the interdependence is low, and thus the competition between options is stronger, the system may fail to achieve such a consensus. Furthermore, when the discovery probabilities decrease drastically, the system remains mostly uncommitted, which is insufficient to reach a consensus. Increasing the threshold value for f_2 , such as requiring a two-thirds majority ($f_2 = 2/3$), will shift the crossover point towards higher values of interdependence.

While setting a threshold value on one of the populations provides a straightforward measure of consensus, this method may fail to capture scenarios where the population committed to the best option is substantial, yet faces significant competition from other sites. Additionally, it might overlook situations where one site clearly leads, but the committed population has not yet met the prescribed threshold. To address these concerns, we adopt the consensus definition proposed by LES in their original paper [21]. This approach involves comparing the two largest values of f_α , offering a more comprehensive assessment of the decision-making dynamics. In a general manner, we can define a consensus measure as follows:

$$Q_x = f_2 - x f_1. \quad (11)$$

where x indicates whether we require a simple majority ($x = 1$), a two-thirds majority ($x = 2$) provided that $f_0 = 0$, or any other desired threshold between the two best options. Figure 4A shows the value of Q_2 in the $(\pi_{1,2}, \lambda)$ space for the same choice of qualities as before. The state space is divided between a region of consensus for high λ and region without consensus for small λ , separated by a consensus crossover line, $Q_2(\lambda^*) = 0$. We observe a similar trend as with the f_2 -threshold consensus definition: high values of $\pi_{1,2}$ require high values of

the interdependence in order to retrieve the best option among the *noise*, which is equivalently introduced for both options. On decreasing the discovery probabilities, the range of λ yielding consensus grows, due to the main mechanism under the collective decision being the positive reinforcement on the best quality options through interdependence.

Fig. 4D illustrates the consensus crossover lines for different choices of q_1 , while keeping $q_2 = 10$ fixed. When we increase the quality of the bad option, consequently increasing its advertisement time, the system requires a higher degree of interdependence to favor the higher quality option. This trend would reach a turning point when $q_1 = q_2$. In this scenario, the system reaches a deadlock, where $f_1 = f_2$. Conversely, when q_1 is reduced, the crossover line shifts to lower values of λ^* , potentially leading to a situation where interdependence is not necessary to achieve consensus. For instance, the crossover line for $q_1 = 5$ lies along the x-axis, with $\lambda^* = 0$ for all $\pi_{1,2}$. In this case, the difference in advertisement time is sufficient to maintain a significant difference between f_2 and f_1 , facilitating consensus even in the absence of interdependence. Furthermore, it's notable that the consensus crossover lines terminate abruptly at the point $(\pi_{1,2}, \lambda) = (0.0, 0.1)$. As we demonstrated earlier, in the fully interdependent limit ($\pi_{1,2} = 0$) the system stays fully uncommitted until the threshold $\lambda = r_2 = 0.1$. Beyond this threshold, f_2 begins to take positive values while f_1 remains zero, resulting in trivial consensus.

For $\pi_{1,2} > 0$, we can predict the value of q_1 below which consensus will always be achieved without the need for interdependence. Using the model solution at $\lambda = 0$ and the general definition of consensus, $Q_x = f_2 - x f_1$, we arrive at the following condition:

$$q_1 < \frac{q_2}{x}$$

Consequently, if $x = 2$, consensus is readily achieved if $q_1 < q_2/2$, while a simple majority consensus is always guaranteed if $q_1 < q_2$. Reintroducing the independent quality assessment parameter of the model, μ , modifies the values of these thresholds. We provide a brief discussion of this case in Appendix A 3.

We would also like to acknowledge that the definition of consensus in Eq. (11) may not be very appropriate to describe the low discovery region of the parameter space where $\pi_{1,2} \rightarrow 0$. There, we observe that λ^* decreases, suggesting that a scarce environment with low $\pi_{1,2}$ is the most beneficial situation for the swarm, as it allows for the widest range of λ yielding consensus. However, it's important to note that the range of uncommitted population increases drastically in that region, and the system can even transition to an absorbing state around $\lambda = r_2$. Therefore, this effect should be taken into account in the quorum measure. In the original paper by LES [21], the consensus definition is strengthened by requiring a minimum proportion of agents engaged in the decision process. Here, we propose a slightly different definition to

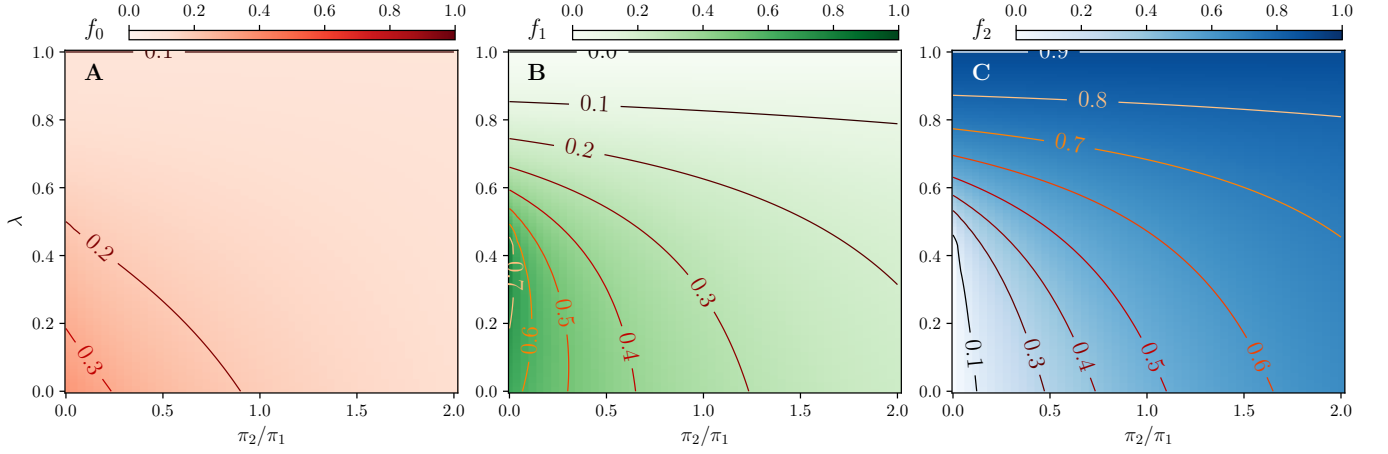


FIG. 5. Color-maps illustrating the stationary values of f_0 , f_1 and f_2 are presented in panels **A**, **B**, and **C**, respectively, for the asymmetric discovery scenario in the parameter space $(\pi_2/\pi_1, \lambda)$. The site site qualities are fixed at $(q_1, q_2) = (7, 10)$, and the discovery probability for site 1 is set to $\pi_1 = 0.25$, while π_2 is changed. Color gradients indicate the quantitative values of f_α according to the color bars, while continuous lines represent iso-population levels in the parametric space.

encompass this effect along with the site competition.

$$Q'_x = f_2 - \max(f_0, x f_1). \quad (12)$$

Consequently, when there is not a sufficient population committed to an option other than the winner, the competition is solely against the undecided population. Unlike when two options compete, where remaining unresolved is not considered a valid outcome of the decision process, this definition only requires that the committed population exceeds the uncommitted to establish a quorum. As shown in Fig. 4B (with $x = 2$), this definition automatically excludes the region of small $\pi_{1,2}$ and λ from consensus. This ensures that we do not conclude there is consensus when a large proportion of the population remains uncommitted, while still capturing the effect of site competition that occurs when the discovery probabilities increase. Note that under this definition, the condition that consensus is granted if $q_1 < q_2/x$ no longer holds (Fig. 4E), at least not for all $\pi_{1,2}$. When the subleading population is uncommitted, we find the threshold value $\pi_2 > r_2$. Consequently, we observe a modified crossover line from $\pi_{1,2} = 0$ up to $\pi_{1,2} = r_2$. Right at the fully interdependent limit ($\pi_{1,2} = 0$), the condition that must be satisfied is $\lambda > 2r_2 = 0.2$.

Up until now, we have focused on the swarm's ability to reach consensus across different combinations of model parameters. However, it's equally important to consider the decision time, whether consensus is achieved or not, to fully understand the effect of these parameters on the decision process. Figure 4C shows the timesteps required for the system to settle at the stationary state, obtained from simulations, across the state space $(\pi_{1,2}, \lambda)$ (details on how t_{ss} is obtained are provided in the Appendix A 4).

The trend observed is similar to that of consensus: interdependence tends to increase the decision time across all values of $\pi_{1,2}$, while the discovery probabilities speed up the process, reducing t_{ss} . This correlation between

the established consensus value and the time to reach the stationary state suggests a speed-accuracy trade-off, commonly observed in biological systems [42] and in models of collective decision making [27, 41]. Achieving the best decision typically takes longer at higher values of λ , while processes are quicker below the consensus crossover line, where a less accurate consensus, such as a simple majority, can be reached much faster. Similarly, increasing the value of the discovery probabilities, and thus the introduction of independent information, accelerates the decision process at the cost of a lower value of Q' , which can be even negative, depending on λ . Only at the limits $\lambda \rightarrow 0$ or $\lambda \rightarrow 1$ $\pi_{1,2}$ has practically no effect: in the former, there's no discussion mediated by interdependence, resulting in a quickly established yet inaccurate stationary state, while in the latter, social feedback dominates the process, with little impact from individual exploration. It's interesting to note the agreement between the stationary times discussed here and the relaxation times obtained from the linear stability analysis eigenvalues, as depicted in Fig. A1.

Finally, we must also acknowledge the influence of the site qualities on the stationary time. Figure 4F illustrates t_{ss} versus q_1 , with fixed q_2 and discovery probabilities, for three values of the interdependence parameter. Each point is color-coded according to the actual value of the achieved consensus at the stationary state. Primarily, we observe that as q_1 increases, the stationary time tends to increase, especially for large λ . This trend persists until the point where the system fails to reach consensus, at which point t_{ss} starts to decrease. While this doesn't necessarily indicate a direct speed-accuracy trade-off (as Q' was already decreasing before t_{ss} started to do so), it's intriguing to note the correlation between t_{ss} and the consensus value. The speed-accuracy trade-off reemerges when examining fixed values of q_1 : increasing λ leads to higher consensus values, but also to longer stationary

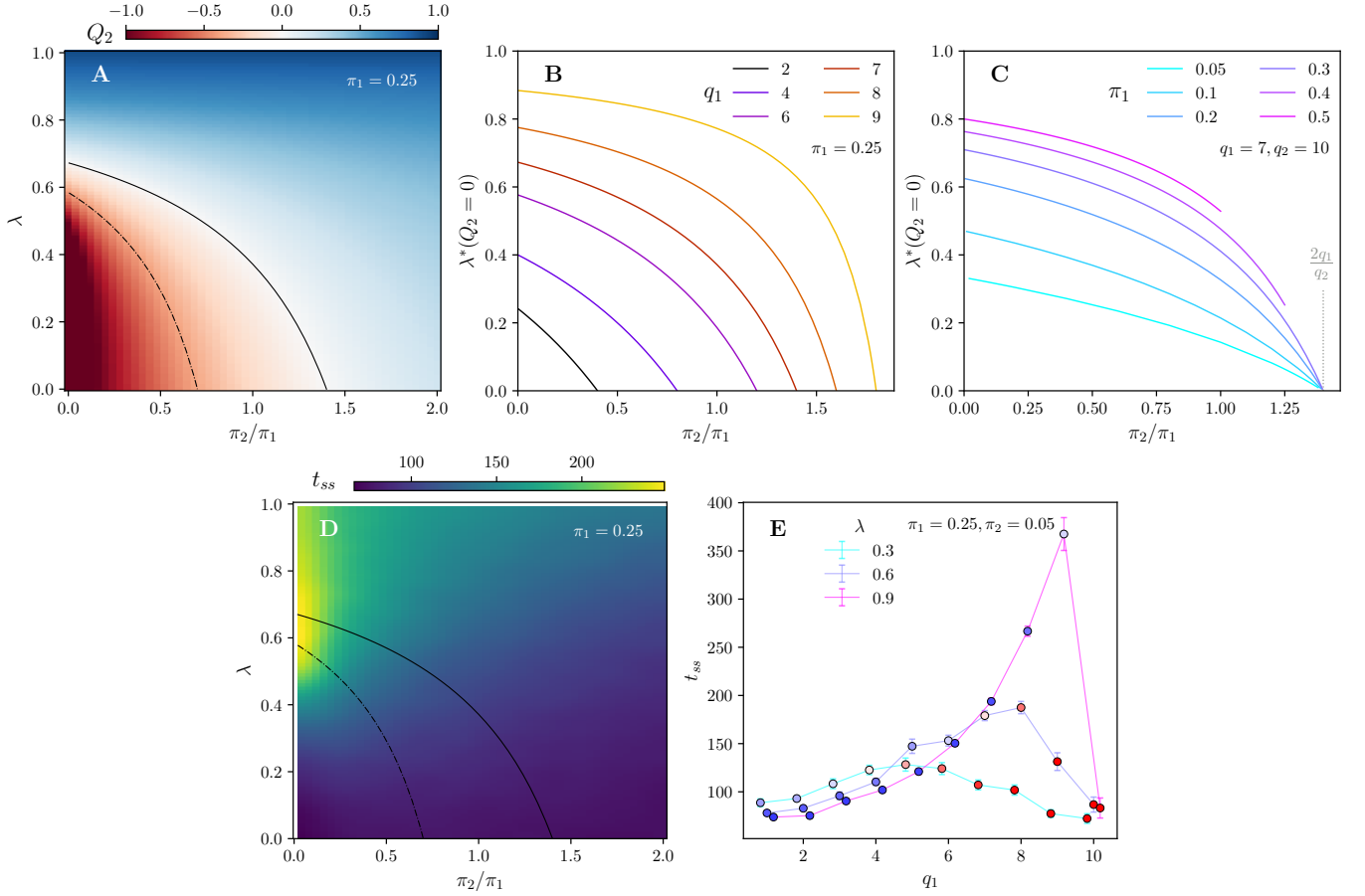


FIG. 6. **A:** Color-map illustrating the stationary value of the consensus parameter Q_2 (as defined from Eq. (11) with $x = 2$) in the parametric space $(\pi_2/\pi_1, \lambda)$. Site qualities are fixed at $(q_1, q_2) = (7, 10)$, and the discovery probability for site 1 is set to $\pi_1 = 0.25$. The solid line indicates $Q_2 = 0$, while the dashed line indicates a simple majority, $Q_1 = 0$. **B:** Crossover lines λ^* above which the consensus Q_2 becomes positive as the self-discovery probability π_2 is varied. Each curve represents a different value of site-1 quality q_1 , while $q_2 = 10$ and $\pi_1 = 0.25$ are held constant. **C:** Crossover lines λ^* above which Q_2 becomes positive as π_2 is varied. In this panel, qualities are held constant, $(q_1, q_2) = (7, 10)$, and we illustrate the effect of different values of π_1 . **D:** Time to reach the stationary state t_{ss} in the state space $(\pi_2/\pi_1, \lambda)$ obtained from simulations. Parameters $\pi_1 = 0.25$, $(q_1, q_2) = (7, 10)$ and $N = 5000$ are fixed. **E:** Time to reach the stationary state, t_{ss} , obtained from simulations plotted against the quality of the inferior option, q_1 , while maintaining $q_2 = 10$, for three values of the interdependence. Parameters $(\pi_1, \pi_2) = (0.25, 0.05)$ and $N = 5000$ are fixed. Simulations start from uncommitted initial conditions. Color markers indicate the consensus value attained at the stationary state. They are slightly jittered around the q_1 axis to allow for better visualization of the color code.

times. This trend holds for moderate values of q_1 ; when the problem difficulty is low ($q_1 \ll q_2$), λ has minimal impact on t_{ss} .

B. Asymmetric discovery scenario

In the following section, we decouple the parameters π_1 and π_2 , allowing one site or the other to be discovered more easily by the swarm. We still maintain $q_1 < q_2$. Consequently, we anticipate that when $\pi_2 > \pi_1$, the group will have no trouble reaching consensus, whereas when $\pi_2 < \pi_1$, interdependence will become crucial for the group to select the best possible option.

Continuing in a similar manner as the previous sec-

tion, we explore the state space for a particular choice of site qualities. With three parameters – π_1 , π_2 and λ –, we maintain π_1 fixed and present our results in the state space $(\pi_2/\pi_1, \lambda)$. Figure 5 illustrates the behavior of f_0 , f_1 and f_2 for site qualities $(q_1, q_2) = (7, 10)$ and $\pi_1 = 0.25$. Once more, we observe a decrease in the proportion of the uncommitted population as either the discovery probability (for site 2) or the interdependence increases. This decrease predominantly benefits the population committed to option 2, as evidenced by the behavior of f_1 and f_2 . As expected, increasing π_2 leads to a growth in f_2 while diminishing the other populations in the system. Similarly, raising the values of the interdependence also results in an increase in the population committed to the better option, accompanied by a

decrease in f_1 . It is noteworthy that even if the discovery probability of the better option is much smaller than that of the other option, sufficiently large values of interdependence enable the swarm to still select the better option. In this regime, only when interdependence is low does the worse option take the lead. In fact, for interdependence below a certain value λ' , the population committed to the better option is exactly null (as indicated by the dash-marker in Fig. 5). Similarly to the symmetric scenario, where we observed an absorbing transition at $\lambda = r_2$ in the fully interdependent limit ($\pi_\alpha = 0$), here, when $\pi_2 = 0$, we observe a transition from $f_2 = 0$ to $f_2 \neq 0$ at $\lambda = \lambda'$. This value of λ' depends on both the qualities and the discovery probability of option 1.

As before, we can quantify the outcome of the decision process in terms of a consensus parameter. Requiring f_2 to reach a certain threshold value produces crossover lines λ^* that decrease with π_2 , as seen in the contour lines on Fig. 5C. This trend holds up to point where consensus is achieved without the need for interdependence, depending on the required threshold value. Beyond this point, the difference in advertising times is enough to impose a majority on the better option, regardless of the particular values of π_1 and π_2 .

Introducing the consensus definition that accounts for the difference in the two most populated sites produces a very similar trend in the crossover lines. In Figure 6A, the value of a two-thirds majority consensus in the voting population (Q_2) across the state space is presented, along with its crossover line (the solid line), as well as the crossover line for a single majority consensus, Q_1 (the dashed line). High interdependence facilitates consensus even in cases of drastic differences in the discovery probabilities, whereas at smaller values of λ , a region where consensus cannot be reached emerges. Below the Q_1 crossover line, the site with the lower quality is imposed, at least, by a simple majority.

Figure 6B illustrates the crossover line $\lambda^*(\pi_2; \pi_1, x = 2)$ for different choices of the bad site quality, q_1 . The region where consensus cannot be reached expands as q_1 increases, pushing the parameters that yield consensus to very high levels of interdependence or a much greater probability to discover the good site. We can determine the value of π_2 from which consensus can be achieved without the need for interdependence. Using the solution at $\lambda = 0$ (Eqs. (7) and (8)), and the consensus definition, this value is given by:

$$\pi_2 \geq x \frac{\pi_1 q_1}{q_2}.$$

As with the symmetric discovery scenario, introducing non-null values of μ will alter the value of this threshold; however, the behavior of the system remains unchanged. Further details are provided in Appendix A 3.

The overall magnitude of the discovery probabilities also influences the consensus crossover. Similar to the observations in the symmetric discovery scenario, smaller magnitudes of the individual π s necessitate smaller val-

ues of λ^* . This pattern persists in the asymmetric scenario as well. Figure 6C illustrates this effect: even with the same ratio π_1/π_2 , if the actual values of π_1 and π_2 are smaller, the consensus crossover line occurs at lower values of interdependence.

It's worth noting that unlike the symmetric discovery probability case studied previously, in this scenario and for the specified parameters, there is not a significant region where the uncommitted population grows noticeably, provided that one of the options has a sufficiently large discovery probability. Hence, there's no need to introduce a modified consensus definition that emphasizes the presence of a large uncommitted group. This feature only becomes relevant if, in the limit $\pi_2 \rightarrow 0$, either $\pi_1 \rightarrow 0$ –which results in the previously discussed scenario–, or $q_1 \rightarrow 0$, implying that option one is effectively expendable.

Finally, we turn our attention to the time required to reach the stationary state, as depicted in Fig. 6D. Once again, we observe the time dilation effect of λ , coupled with an increase in consensus, indicating a speed-accuracy trade-off. However, in contrast, when π_1 is held constant, increasing π_2 does not exhibit the same effect. Increasing the discovery probability for the good option simplifies the decision problem, achieving better consensus in a shorter time. Interestingly, we notice that in the limit $\pi_2 \rightarrow 0$, the stationary time increases abruptly around the consensus crossover. This implies that achieving consensus for the best option in such an unfavorable limit comes at the expense of a significantly prolonged decision process. A similar effect is observed when inspecting the characteristic relaxation times obtained from the solutions' LSA (see Fig. A1). There we observe that this increase in the relaxation time is instead found around the f_2 transition at $\lambda = \lambda'$.

When examining the effect of the bad option quality, we find a similar result to the symmetric discovery scenario. The stationary time increases with increasing q_1 , in parallel with an increase in the value of consensus. However, when the system is unable to achieve consensus, the stationary time begins to decrease.

C. Finite Size Effects

The deterministic solution analyzed in the previous subsections aligns perfectly with the results obtained by averaging the stationary state attained in mean-field simulations across all parameter space. However, stochastic simulations enable us to observe significant finite-size fluctuations. An example of such fluctuations for a specific choice of parameters is illustrated in Fig. 7A, where we show the probability density functions (PDFs) of stationary values of the dancing frequencies f_1 and f_2 obtained in stochastic simulations on fully connected systems of various sizes ($N = 50, 250, 1000$). The variance of the PDFs scales as N^{-1} due to the central limit theorem. As a result, when the system size is relatively small,

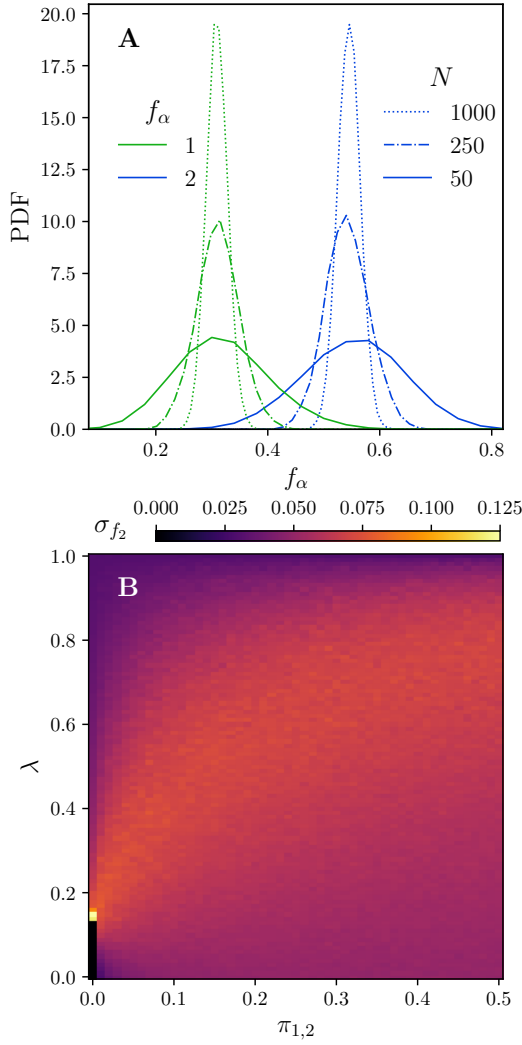


FIG. 7. **A**: Probability density functions (PDFs) of the stationary site frequencies f_1 (green) and f_2 (blue) in simulations with different system sizes N . Other parameters are $\pi_1 = \pi_2 = 0.3$, $q_1 = 7$, $q_2 = 10$ and $\lambda = 0.3$. **B**: Variation of the standard deviation of the PDF for site 2, (σ_{f_2}), across the symmetric discovery parameter space, obtained from simulations with $N = 100$ and $q_1 = 7$, $q_2 = 10$.

there is a notable overlap between the PDFs representing the optimal (site 2) and suboptimal (site 1) options. This indicates that a finite stochastic system may temporarily exhibit at least a simple consensus for the suboptimal (bad quality) option ($f_1 > f_2$). However, as the total population increases, the width of these distributions decreases, enhancing the robustness of the decision-making process. T. Galla further characterized these finite-size effects using the van Kampen expansion on the inverse system size [31].

The interplay between independent exploration and social feedback also affects the magnitude of the fluctuations. In the same state space representation as in the previous sections, Fig. 7B illustrates the standard deviation of f_2 in the symmetric scenario. Excluding some

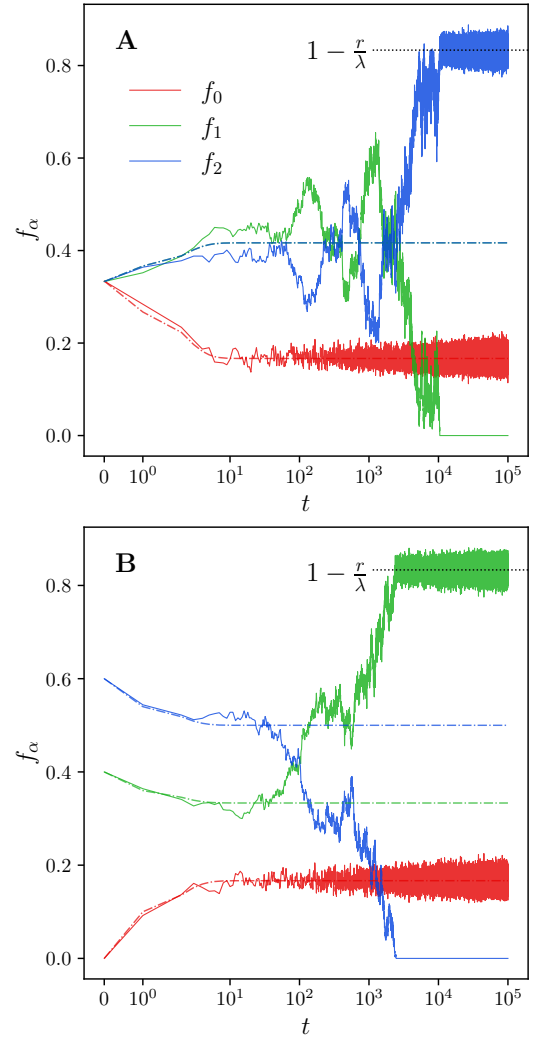


FIG. 8. Temporal evolution of a single experimental realization in the fully interdependent limit ($\pi_\alpha = 0 \forall \alpha$) with equal qualities ($q_\alpha = 10 \forall \alpha$), starting from different initial conditions. **A**: $f_\alpha(t = 0) \simeq (1/3, 1/3, 1/3)$ **B**: $f_\alpha(t = 0) \simeq (0.0, 2/5, 3/5)$. Other parameters are $\lambda = 0.6$, $N = 1000$. The black dotted line indicates the value in which the *winning* population stabilizes, $1 - r/\lambda$, while the other vanishes and $f_0 = r/\lambda$.

extreme values near or at the fully interdependent limits, the fluctuations are most pronounced at intermediate levels of interdependence. Moreover, the region of high fluctuations appears to shift towards greater λ on increasing $\pi_{1,2}$, akin to the consensus crossover lines discussed earlier.

The main rationale behind this observation lies in the intermediate region of the state space, where the competition between the two primary drivers of the system – independence and interdependence – is most pronounced. When high levels of interdependence drive the system, the optimal site is easily favoured by the interplay of longer advertisement and high social feedback. Conversely, when λ is low, the stationary state is primarily

determined by independent discoveries, with minimal or no influence from peer communication. However, in the central region, neither mechanism is dominant enough to outweigh the other, resulting in broader stationary distributions. Despite this broader distribution, consensus may still be established on average.

In the lower-left corner of Fig. 7B, we observe a null variance, followed by a sudden increase at higher values of λ . This corresponds to the region where the system ends up in a fully uncommitted state, occurring when $\pi_{1,2} = 0$ and $\lambda < r_2$ in an infinite system. Although analytically the change in behavior is predicted to occur precisely at $\lambda = r_2$, we note that due to finite-size effects, this behavior extends slightly beyond this threshold. At $\lambda \gtrsim r_2$, fluctuations suddenly increase as most of the simulation results converge around the predicted stable solution, $f_0 = r_2/\lambda$, while, due to finite-size effects, some results still show the system stabilizing at the fully uncommitted state. As the system size increases, the transition becomes sharper around the threshold $\lambda = r_2$.

Lastly, let's consider the scenario where the site qualities are equal (and their discovery probabilities are also equal) in the fully interdependent limit ($\pi_\alpha = 0$). As we've discussed in the analytical solution of the model, detailed in Sec. II A 2, the system may undergo a phase transition between a fully uncommitted absorbing state and an active state with finite fractions of bees advertising the available sites at $\lambda = r$. We further investigate this non-equilibrium phase transition in the following section. Here, we briefly emphasize the influence of finite-size effects under this limit. While the system may temporarily fluctuate around the stationary state predicted by the deterministic equations, finite-size effects ultimately break this tendency, and the system converges towards a consensus for only one of the options, while the other option disappears. In Figs. 8, we illustrate this phenomenon, by representing the temporal evolution of f_0 , f_1 and f_2 for a binary decision problem starting from two different initial conditions. Continuous lines represent the results obtained in stochastic simulations of a finite system consisting of $N = 1000$ bees, while dashed lines represent the integration of deterministic equations Eqs. (4), representing the behavior of an infinite system. We observe that fluctuations introduced by finite-size effects ultimately break the symmetry between the two available options. This can occur even when one option initially appears to be favored.

In real-world applications, whether in ecological systems or synthetic ones like robot swarms, these fluctuations are always expected to be relevant because most of these systems are finite. An important conclusion can be drawn from these results. In ecological systems, the ability to regulate the dissemination of information is crucial for efficient decision-making. For instance, when options have comparable qualities, it may be beneficial for the group to converge quickly towards a consensus to avoid prolonged indecision. In such cases, certain members within the group may play a pivotal role by transmit-

ting stop exploration signals to others [20]. These signals would effectively reduce the parameter π to near zero, signaling to other group members to cease exploration and focus on promoting consensus for one of the options. This adaptive behavior allows the group to navigate decision-making scenarios with similar options more effectively, potentially avoiding resource wastage. However, deterministic modeling approaches, may struggle to capture the dynamics of decision-making processes in real finite systems in similar circumstances.

D. Absorbing phase transition and finite size effects in the fully interdependent limit

The analytical analysis of the model in Section II A 3 reveals that in the fully interdependent limit ($\pi_\alpha = 0$ for all α), the system undergoes a phase transition between two states: an absorbing uncommitted state and a partially-active state. In this limit, the system relies entirely on imitation. Thus, if the imitation parameter λ exceeds a critical value λ_c , imitation alone is sufficient to sustain a steady state with a finite committed population of bees, denoted as $\rho = \sum_{\alpha=1}^{\alpha=k} f_\alpha = 1 - f_0$, at least for the highest quality option. However, below λ_c , no bees actively advertise any site, and the system remains locked in a state where $f_0 = 1$.

Moreover, when qualities are equal for all sites, this problem can be exactly mapped to the well-known contact process, which exhibits the same non-equilibrium critical behavior. In this framework, the parameter $\rho = 1 - f_0$, representing the fraction of agents actively promoting available sites, acts as the order parameter of the transition. Therefore, we anticipate that sufficiently close to the critical threshold λ_c ,

$$\rho \sim (\lambda - \lambda_c)^\beta \quad (13)$$

with β a dimension-dependent critical exponent, which in the mean field approximation is exactly equal to $\beta = 1$.

Indeed, we can rewrite our deterministic differential equations (Eq. (4)) by setting $\pi_\alpha = 0$ and $r_\alpha = r$ for all α :

$$\langle \dot{f}_\alpha \rangle = \lambda \langle f_0 \rangle \langle f_\alpha \rangle - r \langle f_\alpha \rangle \quad \forall \alpha.$$

Summing over all sites, one gets the mean field rate equation for the density of active sites,

$$\frac{d\rho}{dt} = \lambda(1 - \rho)\rho - r\rho. \quad (14)$$

Rescaling by r , to define $\tilde{\lambda} = \lambda/r$ and $\tau = rt$, and rearranging, we arrive at the well-known mean-field equation for the contact process:

$$\frac{d\rho}{d\tau} = (\tilde{\lambda} - 1)\rho - \tilde{\lambda}\rho^2. \quad (15)$$

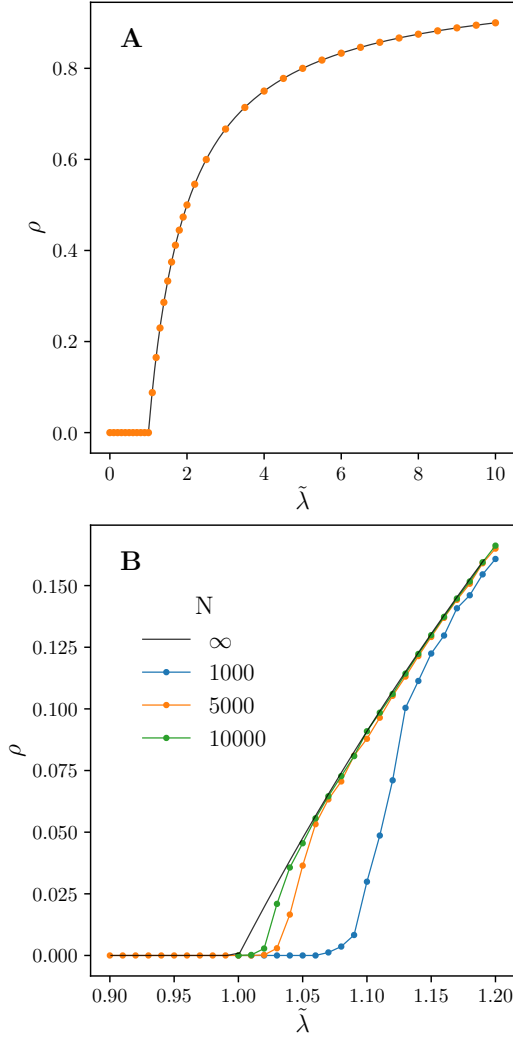


FIG. 9. Absorbing phase transition in the fully interdependent limit ($\pi_\alpha = 0$, $\forall \alpha$) with equal qualities ($q_\alpha = 10$, $\forall \alpha$) for a fully-connected system. **A**: Order parameter $\rho = 1 - f_0$ vs. $\tilde{\lambda} = \lambda/\lambda_c$. **B**: Zoom around the phase transition happening at $\tilde{\lambda}_c = 1$.

This equation has two stationary points: $\rho^* = 0$ and $\rho^* = 1 - \tilde{\lambda}^{-1}$, with the mean-field critical point located at $\tilde{\lambda}_c = 1$. As detailed in Section II A 3, the absorbing state is stable if $\tilde{\lambda} < 1$, while the active state is stable when $\tilde{\lambda} > 1$. Fig. 9 shows ρ vs $\tilde{\lambda}$ for simulations of the fully-connected model alongside the analytical deterministic solution. As anticipated, the analytical solution exhibits the transition at $\tilde{\lambda}_c = 1$. However, due to finite size effects, simulations show deviations from this result, with larger threshold values than expected for an infinite system, $\lambda_c(N) > \tilde{\lambda}_c$.

The self-discovery probabilities π play the role of an external field, as with non-null π values, an inactive site can spontaneously become active. Similar to the contact process, the introduction of this external field disrupts the absorbing state and consequently the transition

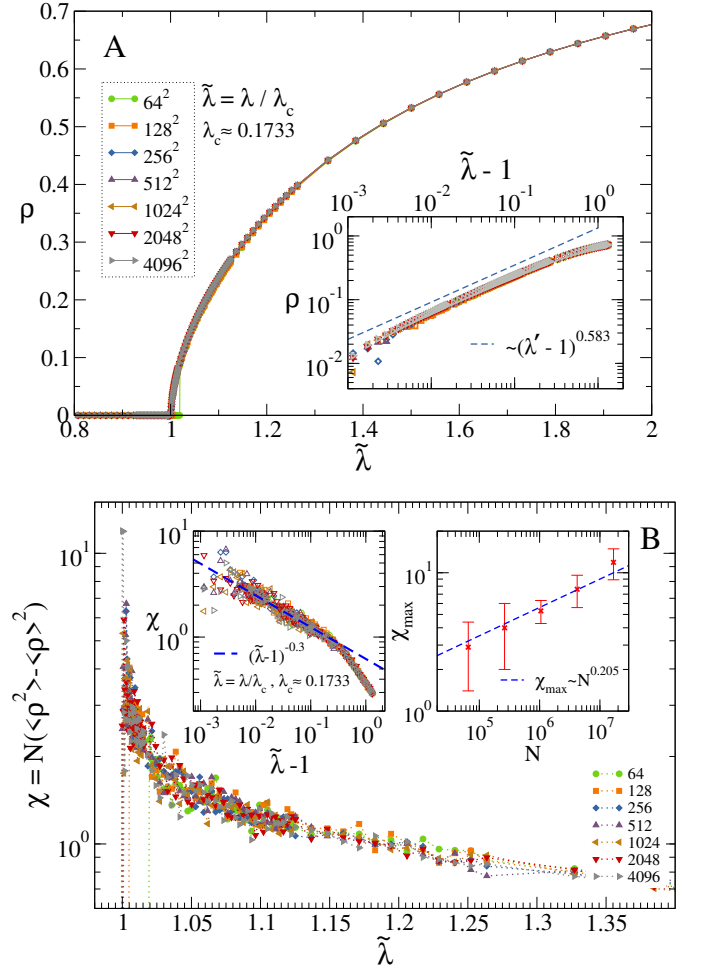


FIG. 10. Phase transition in the fully interdependent limit ($\pi_\alpha = 0$, $\forall \alpha$) with equal qualities ($q_\alpha = 10$, $\forall \alpha$) for stochastic simulations of a binary decision process on a regular square lattice. **A**: Order parameter $\rho = 1 - f_0$ vs. $\tilde{\lambda} = \lambda/\lambda_c$ for different system sizes. The inset shows ρ as a function of $\tilde{\lambda} - 1$. A power-law $\rho \sim (\tilde{\lambda} - 1)^\beta$ is displayed with $\beta = 0.583$ (blue dashed-line) to show consistency with DP exponents. If fitted, $\beta \simeq 0.58 \pm 0.02$, where the error comes from the variability when changing the fitting region and the uncertainty in the determination of λ_c . **B**: Susceptibility $\chi = N(\langle \rho^2 \rangle - \langle \rho \rangle^2)$ vs. $\tilde{\lambda} = \lambda/\lambda_c$ for different system sizes. Insets: (Left) Approaching the critical point from above, the susceptibility scales in a way compatible with $(\lambda/\lambda_c - 1)^{-0.3}$. (Right) Finite-size scaling. At the critical threshold $\lambda_c \simeq 0.1733$, the susceptibility scales in agreement to $\sim N^{0.205}$. Scaling laws show the expected DP critical exponents rather than fitting directly the data.

itself, leading the system away from criticality. However, for sufficiently small values of π , the system still obeys certain scaling laws, as discussed in [43]. We have also confirmed that, for instance, the order parameter in stationary conditions is compatible with the scaling law $\rho \sim \pi^{\beta/\sigma} \sim \pi^{1/2}$, with mean-field exponent values $\beta = 1$ and $\sigma = 2$.

The contact process falls within the directed percola-

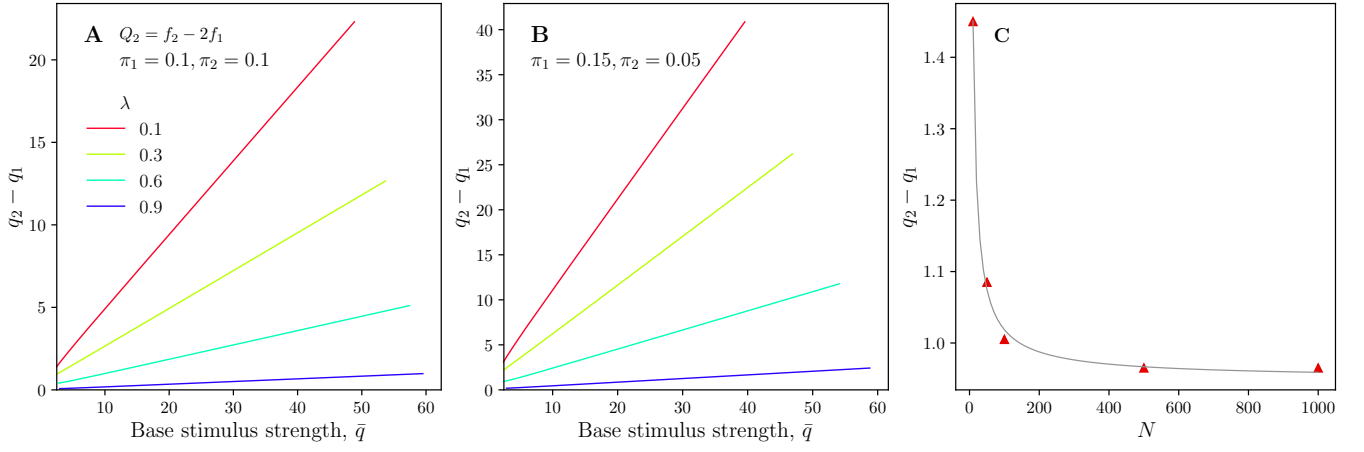


FIG. 11. *Just noticeable quality difference* as a function of the base stimulus strength (**A**, **B**) and the system size (**C**). **A**: Symmetric discovery scenario with $\pi_1 = \pi_2 = 0.1$, **B**: Asymmetric discovery scenario with $\pi_1 = 0.15 > \pi_2 = 0.05$. Results obtained from analysis of the deterministic equations. **C**: Simulations for finite system sizes at a fixed $\lambda = 0.6$ and $\pi_{1,2} = 0.6$. The discernible quality difference decreases with the system size.

tion (DP) universality class. Therefore, beyond mean field, the critical exponents of the uncommitted absorbing phase transition should match those of directed percolation, which depend on the system spatial dimension. For completeness, we also analyze the phase transition on a regular 2-dimensional square lattice with nearest-neighbors interactions. Further details about simulations of the LES model in this geometry are provided in Appendix A 5. In Fig. 10A, we show the density of active sites around the transition for simulations of different system sizes N and $q_1 = q_2 = 10$ (or equivalently $r = 0.1$).

A finite-size scaling analysis enables us to identify the critical interdependence parameter value $\lambda_c \simeq 0.1733$. Beyond this threshold, we note a close correspondence with an order parameter scaling behavior characterized by the exponent $\beta \simeq 0.583$, as expected for DP in two dimensions [43]. Furthermore, as in other phase transitions, one can define a susceptibility for the decision-making problem in this limit as $\chi = N(\langle \rho^2 \rangle - \langle \rho \rangle^2)$, which exhibits a peak at the pseudo-critical threshold $\lambda_c(N)$. For directed percolation, it is expected to scale as $\chi \sim L^{(d\nu_\perp - 2\beta)/\nu_\perp}$ [44, 45], where for $d = 2$, $\beta = 0.583$ and $\nu_\perp = 0.733$. Therefore, $\chi \sim L^{0.409} \sim N^{0.205}$. The scaling properties of this quantity in our simulations on the square lattice are depicted in Fig. 10B. As expected, the susceptibility peaks at $\lambda_c(N)$ for different system sizes N . At the critical threshold $\lambda_c \simeq 0.1733$, it scales in a way compatible with $\chi \sim N^{0.205}$, the expected scaling behavior. Additionally, on approaching the critical point from the active phase, the susceptibility diverges roughly as $\chi \sim (\lambda/\lambda_c - 1)^{-\gamma'}$, with $\gamma' = d\nu_\perp - 2\beta = 0.3$. The measured exponent value is also compatible with a DP exponent in $d = 2$ [45]. Therefore, our results consistently indicate that in the fully interdependent limit, our model reduces to a DP-like process, with a dimension-dependent non-equilibrium critical phase transition un-

dergoing the building of consensus in the swarm.

Non-equilibrium phase transitions are commonly observed in biological systems [46]. These transitions, which occur far from thermodynamic equilibrium, play a crucial role in shaping the collective behavior of living organisms. Here, we find another example which might be relevant for collective decision making processes.

E. Weber's Law of perception

Psychophysics explores how organisms perceive external stimuli, initially concentrating on human perception before broadening its scope to encompass other organisms across varying levels of biological complexity. There's a proposition that a swarm engaged in decision-making tasks can be seen as a superorganism [30], exhibiting cognitive traits akin to those found in vertebrate brains based on neurons. Building on recent research into honeybee-inspired decision-making models [36], we aim to ascertain whether the LES model also reflects such characteristics.

We particularly investigate Weber's Law [47], which posits that the smallest perceptible change between two stimuli (known as the just noticeable difference) is proportional to the intensity of the base stimulus. In our context, the stimuli correspond to the qualities of the two options under consideration in a binary decision problem, and the base stimulus intensity can be defined as the mean quality of the options, denoted as $\bar{q}$. According to Weber's Law, we anticipate observing a linear correlation between the difference in quality and the base stimulus strength ($\bar{q}$), expressed as $q_2 - q_1 = w\bar{q}$, where w represents the Weber fraction.

To evaluate whether a swarm accurately discriminates between two stimuli, achieving consensus is necessary. We assess compliance with Weber's Law by keeping the

quality of one option constant while varying the quality of the other until positive consensus is reached. In our experiment, we fixed the high-quality option while adjusting the inferior-quality option, but similar outcomes were obtained when reversing this setup. Figures 11A and B confirm the linear correlation between the difference in quality and the stimulus strength across different combinations of interdependence and discovery probabilities. This analysis was conducted using the original consensus definition, Eq. (11), although similar results were obtained with alternative definitions discussed earlier.

We tested the model agreement with Weber's Law in the two different discovery scenarios, symmetric ($\pi_1 = \pi_2$, Fig. 11A) and asymmetric ($\pi_1 > \pi_2$, Fig. 11B). While the discovery probabilities influence the actual quality differences discernible for a given level of interdependence, they do not affect the linear relationship with the base stimulus strength. A linear fit of the quality differences to the base quality, $q_2 - q_1 \sim w\bar{q}$, results in $R^2 > 0.99$ (we provide the fit parameters in Table I in the Supplementary Material). All these results suggest a general agreement with the Weber's Law.

Following the analysis in [36], one can also relate the magnitude of finite-size fluctuations within the decision process with the random fluctuations typically observed in a discrimination process. These fluctuations can influence an organism's ability to accurately discriminate between two stimuli [48, 49]. It is widely recognized that collective decisions tend to improve when made by larger groups [50, 51]. As discussed in Sec. III C, fluctuations diminish as the system size increases. Consequently, we anticipate that the system's ability to discriminate between two similar stimuli will improve with the system size, i.e. the just noticeable difference will decrease with N . In agreement with [36], we observe this effect in our model, as illustrated in Fig. 11C, which also exhibits a comparable exponential trend.

In the study by Reina et al. [36], individual commitment transitions (i.e., discovery) are linked to site qualities, unlike in our model. Here, the quality-sensitivity influences system dynamics solely in the uncommitment and recruitment transitions. However, this difference does not appear to impact the model's adherence to Weber's Law. The independent discovery probabilities solely affect the specific value of quality differences that can be discerned. Figures 11 A and B also demonstrate that, for a given stimulus strength, the ability to discriminate decreases with λ . In other words, stronger social interaction enables the system to distinguish between more similar options, but this comes at the expense of a slower decision-making process, as previously noted in Figs. 4 and 6.

Contrary to our findings, Reina et al. in [36] observe that the noticeable quality difference increases (while the decision time decreases) with the signaling ratio, which in their setup reflects the strength of social interactions relative to individual transitions. This disparity underscores the main distinction between the two models: the

presence (or absence, in our case) of negative feedback in social interactions, specifically cross-inhibition. Introducing this mechanism in the decision process leads to quicker decisions, especially in situations involving similarly or equally valued options [24], albeit at the expense of reduced accuracy in the final outcome. These results underscore not only the significance of social interactions but also their nature, as the presence or absence of cross-inhibition yields markedly different outcomes in scenarios initially featuring very similar problems.

F. Multiple sites

Up to now, our attention has been directed towards a binary decision problem, where selecting the best option is imperative. Many collective decision making models have been concerned in generalizing their dynamics to multiple option scenarios [24, 37, 39, 40]. Particularly, previous research on this model, utilizing different parameter setups, examined a specific scenario involving five options. It's worth noting that increasing the number of available options doesn't qualitatively alter the model's behavior. In a general k -site scenario, we still observe the same interplay between independent discovery and social feedback. At high λ values, the system effectively identifies the best site, while at low λ values, it settles into a multi-opinion state where each site is occupied in proportion to its quality and discovery probability. Therefore, in a broader best-of- N scenario, the specific dance frequency values and consensus transitions will rely on the π_α and q_α parameters defining each unique scenario. However, the overall behavioral trend of the system remains consistent across different setups.

In Fig. 12 we represent the consensus parameter together with the dance frequency for the best-site f_k (depicted in the inset) for different number of available sites k . We consider two different symmetric scenarios: one where each site has the same self-discovery probability (independently of the number of sites k), and another where the total sum of self-discovery probabilities remains constant, i.e., $\sum_{\alpha=1}^k \pi_\alpha = 0.6$, and consequently each time a new site is added the overall value of $\pi_\alpha = 0.6/k$ decreases. In both scenarios, site qualities are different and ordered following the sequence $q_1 < q_2 < \dots < q_k$. While there is a slight quantitative change with increasing k , the system's behavioral trend observed in the binary problem remains consistent.

As the number of available options with the same discovery probability increases, the system is exposed to a greater amount of external information. Consequently, slightly higher values of λ are required to achieve the consensus crossover, as evidenced by the rightward shift of the consensus curves in Fig. 12A around $Q = 0$. Contrarily, when the overall level of external information ($\sum \pi_\alpha$) remains constant but the individual discovery probabilities π_α decrease as the number of options increases, as shown in Fig. 12B - the system achieves consensus more

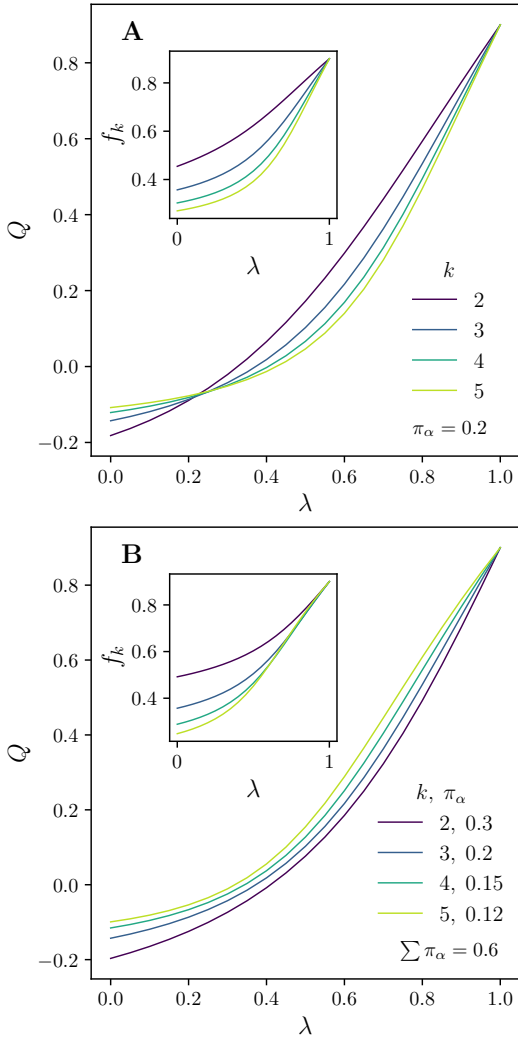


FIG. 12. Consensus parameter Q and dance frequency for the best site f_k (shown in the inset) as a function of the interdependence for different numbers of options, k . The qualities of the sites are defined as follows: $q_k = 10$, $q_{k-1} = 7$, and $q_{k-i} = q_{k-i+1} - 1$ for $2 \leq i \leq k-2$ for options below the top two. **A:** Each site has a self-discovery probability $\pi_\alpha = 0.2$. **B:** Self-discovery probabilities are equal and their sum remains constant, $\sum_{\alpha=1}^k \pi_\alpha = 0.6$, so as the number of sites increases, $\pi_\alpha = 0.6/k$ decreases.

readily as the number of options increases, a result that may appear counter-intuitive compared to other studies on the best-of- N scenario with different models [24]. However, this outcome is a direct consequence of the decision mechanisms embedded in our model: increasing λ strengthens the positive feedback on populations associated with the best quality, leading to the rapid elimination of options with lower qualities from consideration. Consequently, the decision process becomes nearly binary between the top two options. Decreasing the overall values of the discovery probabilities simplifies the discussion process further, facilitating consensus. Supplementary Fig. ?? illustrates all dance frequencies f_α in

the reported scenarios, demonstrating the consistently decreasing trends of the remaining populations in favor of the highest-quality option as λ increases.

Other scenarios, such as a generally asymmetric discovery scenario (where $\pi_1 > \pi_2 > \dots > \pi_k$) or equivalent inferior options ($q_1 = \dots = q_{k-1}$), exhibit similar trends to the discussed symmetric discovery scenario. Naturally, the specific values of f_k or the consensus may vary, as it becomes more challenging for the swarm to achieve consensus under these conditions. However, the interplay between interdependence, site qualities, and discovery probabilities remains unchanged. In Supplementary Fig. ?? we present the results obtained in these scenarios to emphasize the qualitative similarities discussed earlier.

IV. CONCLUSIONS

We have performed an analytical study of an agent-based model inspired by the collective behavior of honeybees [21], following the mathematical framework outlined in [31]. This model incorporates key features such as the balance between independent exploration and social interdependence, as well as the sensitivity of agents to option quality. Using analytical reductions of this agent-based model, we extensively analyzed its behavior in the space of relevant parameters. Our goal was to investigate whether the system achieves consensus for the best available option when balancing information acquisition and social interactions. Furthermore, we supplemented this analysis with simulations to report the time required for the system to converge to its stationary state.

We have explored two scenarios: one where options are equally likely to be discovered, and another where the worst quality option (among two) is more likely to be discovered. In both cases, when the system prioritizes interdependence, the established consensus is stronger. This highlights the role of social interactions in achieving a higher accuracy in the final decision, consistent with similar honeybee-inspired models [23–25]. Interdependence plays a critical role, especially when the system deals with high values of the self-discovery probabilities or when the superior option faces a disadvantage in terms of discovery likelihood. In such scenarios, interdependence emerges as a crucial *noise*-reducing mechanism, allowing the system to maintain a high degree of coherence even when individuals are more prone to random changes in their state.

However, escalating social feedback diminishes the extent of individual exploration, leading to a prolonged time required to reach the stationary state. This observation underscores a trade-off between speed and accuracy, indicating that a moderate level of social interactions is most beneficial when a system seeks to balance consensus accuracy and convergence time [52]. This finding aligns with previous research investigating the emergence of a speed-accuracy trade-off in opinion dynamics models [27, 41]. In those studies, interactions governed by a majority rule revealed a trade-off between decision accu-

racy and the size of the interaction group.

When confronted with a decision problem between two equal-quality options, the deterministic solution of the LES model predicts a system unable to break the symmetry and achieve consensus for one option. In such situations, the introduction of more complex higher-order interactions, such as cross-inhibition, becomes necessary. This need for higher-order interactions has been observed in both on-field honeybee experiments [20], computational models [23, 25, 53], and robot swarm experiments [26, 33, 53, 54]. However, finite-size fluctuations can readily break the symmetry between the options if the system is driven around the so-called "fully interdependent" limit, where individual exploration behavior plays a minimal role. This finding suggests a simple adaptive behavior of the model: reducing the strength of the self-discovery probabilities as the decision process progresses, especially in scenarios where options are advertised similarly. By doing so, the system can break the symmetry even between identical options. However, it's important to note that while this mechanism allows for simple symmetry breaking, it makes the system less flexible when it needs to adapt in changing environments [28, 32].

We have discovered an interesting correspondence between the behavior of the model at the fully interdependent limit and the well-known contact process [43]. In this limit, where the discovery probabilities are negligible, the dynamics are solely governed by interactions. By adjusting the interdependence, which represents the strength of these interactions, the system undergoes a non-equilibrium phase transition. Below a critical value $\tilde{\lambda}_c$, the system settles into an inactive, absorbing state—referred to as the uncommitted state in opinion dynamics terminology. Conversely, above the critical value, the system can maintain a non-negligible proportion of active or committed population for either of the two options, particularly when they are of equal quality. We have derived the model equations, which exactly map to the mean-field contact process equations. In addition, making use of $d = 2$ lattice simulations, we have explored the critical behavior and some of the critical exponents surrounding the phase transition. Our findings confirm that the critical exponents of the order parameter and susceptibility consistently align with those reported for the contact process in $d = 2$, which falls within the directed percolation universality class.

We have further evaluated the model's compliance with Weber's Law of psychological perception, which suggests that the smallest noticeable change between two stimuli is proportional to the intensity of the base stimulus [47]. This relation was first studied in individual organisms, but recently efforts have been devoted to study swarms faced with perception tasks as *superorganisms*, capable of obeying the same laws [30, 36]. This assessment was conducted in both symmetric and asymmetric discovery scenarios. While the discovery probabilities do impact the discernible quality differences for a given level of in-

terdependence, they do not alter the linear relationship with the base stimulus strength. Thus, our findings indicate a broad concordance with Weber's Law. In this context, we have also confirmed that the system's capability to distinguish between two similar stimuli enhances with an increase in system size.

While our primary focus has been on a binary decision problem, we have also evaluated the model's robustness when expanding the number of available options. Unlike other studies [24], we observe that increasing the number of sites consistently improves the swarm performance, or accuracy, when making decisions among non-equivalent options. As discussed in [24], the main distinction between our approaches lies in the absence or inclusion of negative social feedback, such as cross-inhibition between populations representing different options. Consequently, the effect of increasing social interactions, or signaling, has varying impacts on the system dynamics.

In conclusion, our study of a simple honeybee-inspired collective decision-making model reveals the intricate interplay between individual exploration and social interactions in shaping consensus and decision outcomes. We have demonstrated how the balance of these factors influences the system's ability to discriminate between options, and to achieve strong enough consensus. The role of finite-size or critical fluctuations becomes particularly relevant in decision-making processes of adaptive systems when available alternatives are very similar. Indeed, these fluctuations play a crucial role in breaking deadlocks or abandoning fully uncommitted states, thus avoiding potentially dangerous situations in ecological systems. Our findings contribute to a deeper understanding of collective behavior in biological systems and provide insights that may inform the design and optimization of decision-making algorithms in artificial systems.

ACKNOWLEDGMENTS

We acknowledge financial support from the Spanish MCIN/AEI/10.13039/501100011033, through projects PID2019-106290GB-C21, PID2019-106290GB-C22, PID2022-137505NB-C21 and PID2022137505NB-C22. D.M. acknowledges support from the fellowship FPI-UPC2022, granted by Universitat Politècnica de Catalunya. E.E.F. acknowledges support from the Maria Zambrano program of the Spanish Ministry of Universities through the University of Barcelona and PIP 2021-2023 CONICET Project N° 0757.

Appendix A: Appendix

1. Linear Stability Analysis on the general solution

Studying the effects of perturbations on the general deterministic solution up to linear order leads to a square-

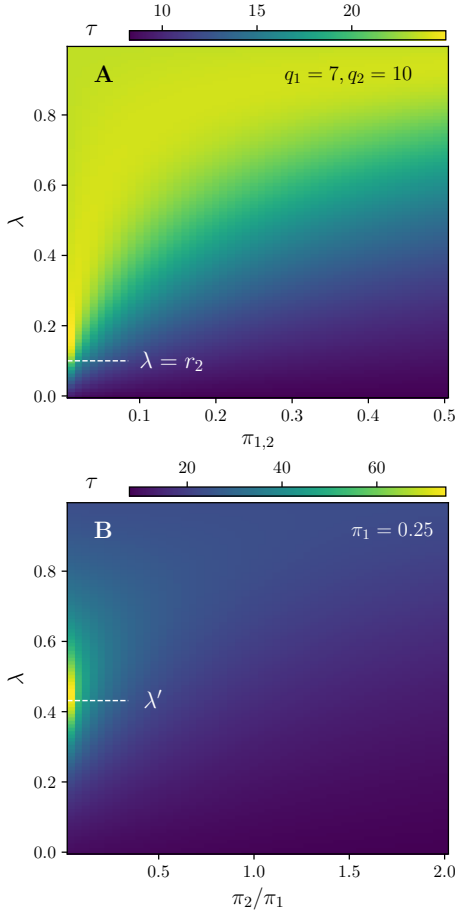


FIG. A1. Characteristic relaxation times τ of linear perturbations obtained from the LSA eigenvalues. **A**: Symmetric discovery scenario. **B**: Asymmetric discovery scenario with $\pi_1 = 0.25$. In both scenarios, the qualities are $(q_1, q_2) = (7, 10)$. White dashed lines indicate the transitions occurring at the fully interdependent limit $\pi_\alpha = 0$ (null- π_k limit) for the symmetric (asymmetric) scenario. For the exact value of $\lambda' \simeq 0.43$, refer to Eq. (10).

k matrix with coefficients:

$$\begin{aligned} a_{\alpha\alpha} &= \lambda(f_0^* - f_\alpha^*) - (1 - \lambda)\pi_\alpha - r_\alpha \\ a_{\alpha\beta} &= -(1 - \lambda)\pi_\alpha - \lambda f_\alpha^* \end{aligned}$$

where $\alpha = 1, \dots, k$. The stability of any solution can be verified by obtaining the matrix eigenvalues numerically, once the actual solution $(f_0^*, f_1^*, \dots, f_k^*)$ is known.

The eigenvalues obtained from the numerical analysis of this matrix can be used to compute the relaxation time of a perturbation in the stationary state, denoted as τ . This represents the characteristic time for the system to return to the state $(f_0^*, f_1^*, \dots, f_k^*)$ after a small perturbation f'_α is introduced in any value of f_α^* (note that these perturbations must satisfy that $\sum_{\alpha=0}^k f'_\alpha = 0$). Specifically, the relaxation time is determined by the smallest eigenvalue, $\tau^{-1} = \min(\theta_\alpha)$. In Fig. A1, we illustrate the characteristic relaxation times for a fixed set of qualities

in both the symmetric and asymmetric discovery scenarios.

We observe that in the limit of small π , near the transition occurring at the fully interdependent limit (marked with a white-dashed line at the appropriate value of λ in both scenarios), the relaxation times increase sharply. This suggests the onset of a critical slowing down of the system dynamics around these points. Finally, it's worth noting the similarity between these relaxation times and the stationary times discussed in the main text (see sections III A and III B, and Figs. 4C and 6D). The same mechanisms that govern the convergence time to reach the stationary state also play a role when the system is relaxing from a small perturbation. However, in simulations of the asymmetric discovery scenario, the increase in time when $\pi_2 \rightarrow 0$ occurs at slightly larger values of λ , above the transition at λ' , and while it slightly decreases with increasing λ , it does not return to the level prior to the transition, unlike the relaxation time of linear perturbations.

2. Linear Stability Analysis on the Fully Interdependent limit

In the fully interdependent limit ($\pi = 0$), we encounter $k + 1$ feasible fixed point solutions (see Eqs. (9)). To ascertain their stability, we conduct a linear stability analysis. Our analysis reveals that the best-site winning solution f_k^* yields all negative eigenvalues, indicating its stability when $\lambda > r_k$. Specifically, the resulting eigenvalues are as follows:

$$\begin{aligned} \theta_m &= r_k - r_m < 0 \quad m = 1, \dots, k-1 \\ \theta_k &= r_k - \lambda < 0 \quad \text{if } \lambda > r_k \end{aligned}$$

If $\lambda < r_k$, this fixed point becomes unstable, and the stable solution is the absorbing fixed point $f_0^* = 1$. The eigenvalues for this stable solution are of the form: $\theta_\alpha = \lambda - r_\alpha < 0$, confirming the change of stability between solutions.

Alternative fixed points that correspond to any other $\alpha \neq k$ less-quality winning site have, at least, $k - \alpha$ positive eigenvalues, indicating that these solutions are actually saddle points. For instance, the eigenvalues for the winning option $\alpha = k - 1$ are:

$$\begin{aligned} \theta_m &= r_{k-1} - r_m < 0 \quad m = 1, \dots, k-2 \\ \theta_{k-1} &= r_{k-1} - \lambda \\ \theta_k &= r_{k-1} - r_k > 0 \end{aligned}$$

3. Imperfect quality assessment: The case of non-null parameter μ

As we explained in the main text, the parameter μ in the original LES model governs the bees' quality assessment behavior. When $\mu \rightarrow 0$, bees independently assess

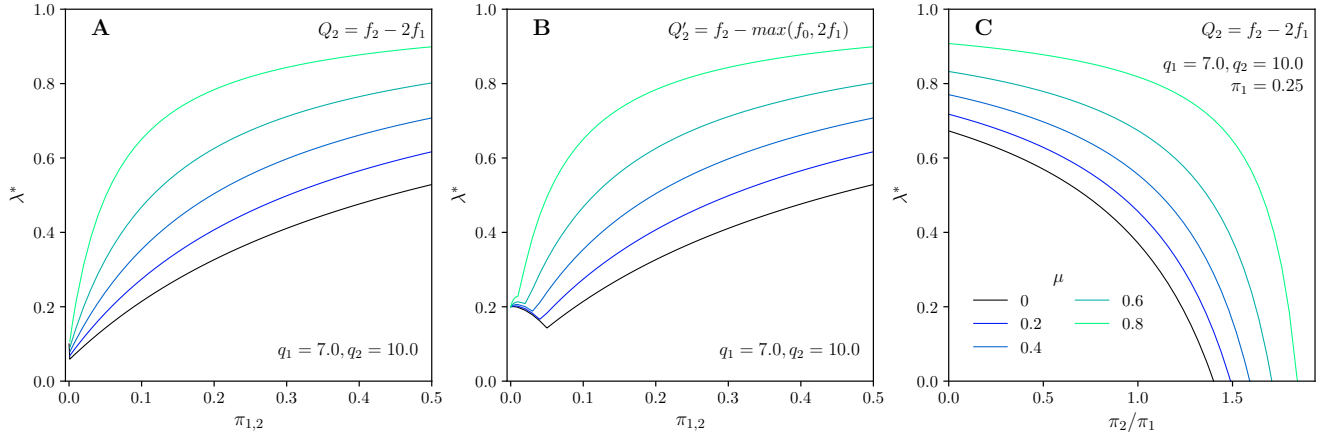


FIG. A2. Consensus crossover lines for different values of the independent quality assessment parameter μ . **A**: Consensus Q_2 in the symmetric discovery scenario. **B**: Consensus Q'_2 in the symmetric discovery scenario. **C**: Consensus Q_2 in the asymmetric discovery scenario with $\pi_1 = 0.25$ and varying π_2 . The quality parameters are $q_1 = 7$, $q_2 = 10$.

the quality of sites, and thus, the duration of the advertisement period strongly correlates with the actual site quality. Conversely, when $\mu \rightarrow 1$, bees do not individually assess site qualities but rather imitate their peers without regard to the options' qualities. Consequently, the advertisement duration is influenced by a generic parameter K that is the same for all sites, as indicated in Eq. (3). Although we have primarily considered $\mu = 0$ in the text to explore the interplay between other parameters such as interdependence, discovery probabilities, and qualities, here we provide a brief overview of how the results change when nonzero values of the independent quality assessment parameter, μ , are considered. For simplicity and in accordance with the prescription by List et al. in the original formulation of the model [21], we set K equal to the maximum quality among all sites, i.e., $K = q_k$.

The deterministic solutions described in Section II A are expressed as functions of the rates, r_α , and therefore remain valid for any value of μ . However, the results presented in the subsequent analysis of the model, particularly in the binary decision problem concerning consensus and stationary time, were obtained with $\mu = 0$. After introducing $\mu \neq 0$, achieving the same levels of consensus would simply require increased values of interdependence compared to the case of $\mu = 0$. Figure A2 illustrates how the consensus crossover lines change, shifting towards greater λ^* , when varying μ for both symmetric and asymmetric discovery scenarios.

The condition to achieve consensus without interdependence, i.e. by simply relying on the different advertisement times for each option, $q_1 < q_2/x$, is now modified as follows. For the symmetric discovery scenario, either fixing (q_2, μ) or (q_1, q_2) , we obtain:

$$\lambda = 0: \quad q_1 \leq q_2 \frac{1 - \mu}{x - \mu} \quad \text{or} \quad \mu \leq \frac{q_2 - 2q_1}{q_2 - q_1}. \quad (\text{A1})$$

For the asymmetric discovery probabilities, fixing (π_1, q_1, q_2, μ) or (π_1, π_2, q_1, q_2) , respectively, we obtain:

$$\lambda = 0: \quad \pi_2 \geq \frac{\pi_1 x}{\mu + \frac{q_2}{q_1}(1 - \mu)} \quad \text{or} \quad \mu \leq \frac{q_2 - q_1 \frac{\pi_1 x}{\pi_2}}{q_2 - q_1}. \quad (\text{A2})$$

4. Criterion for measuring convergence times to the stationary state

The criterion for measuring the time when the system reaches the stationary state is as follows: We monitor the time evolution of each population fraction $f_\alpha(t)$, including the uncommitted state, and calculate the average of f_α over a time window of size w , i.e., over the time interval $[t, t + w]$. We consider a population to have reached the stationary state if the absolute difference between this average value and the value of f_α at the subsequent time step is below a certain threshold κ . In other words, if:

$$|f_2(t + w + 1) - \langle f_2 \rangle_{t, w+t}| < \kappa, \quad (\text{A3})$$

we identify the stationary time for this particular population as $t_{ss} = t + w + 1$. The longest time among all populations will be considered the effective simulation's stationary time. For our simulations in the binary decision problem, we have used a block size of $w = 50$ and a threshold $\kappa = 5 \times 10^{-4}$. Other values within the intervals $w \in [25, 100]$ or $\kappa \in [5 \times 10^{-4}, 5 \times 10^{-3}]$ produce qualitatively similar results.

5. Lattice simulations in $d = 2$

We additionally implement a spatially distributed version of the model described in Sec II. In the spatially

distributed version of the model, each scout bee occupies a specific location on a square lattice with periodic boundary conditions. The lattice has a lateral length of L , resulting in a total of $N = L \times L$ nodes. Each scout bee interacts only with its nearest neighbors on the lattice.

The state s_i of each bee i now corresponds to a specific location $\mathbf{r} = (x, y)$ on the lattice, denoted as $s_i(t) \equiv s(\mathbf{r}, t) \equiv s(x, y; t)$. The fractions $f_{\alpha, t}$ of bees promoting site α at time t , which enter into the equation defining the system dynamics (Eq. 2), are computed as follows:

$$f_{\alpha}(\mathbf{r}, t) = \frac{1}{4} \sum_{\langle n, n', \mathbf{r}' \rangle} \delta(\alpha, s(\mathbf{r}', t)), \quad (\text{A4})$$

where δ is the Kronecker's delta, the sum runs over the four nearest neighbors of $\mathbf{r}$ on the square lattice, and the index $\alpha = 1, \dots, k$ indicates the possible committed states. In addition to the spatial dependence of f_{α} , the state update rules at each site, as described by Eq. (1), remain the same. Each time step involves updating (or attempting to update) the states of every bee or node in the lattice. We implement this process using a massively parallel programming approach and ensure that the evolution of interdependent nodes on each time step is serialized by using a checkerboard lattice decomposition.

The spatially distributed model, similar to [31], is not designed to replicate realistic spatial behavior of bees but rather to analyze the dynamic competition of states within a spatio-temporal framework. As discussed in the main text, the general stationary results of this spatial model align perfectly with mean field (i.e., fully-connected) simulations. Analyses such as the time to reach the stationary state, as depicted in Fig. 4F or Fig. 6E, also yield qualitatively similar results, exhibiting

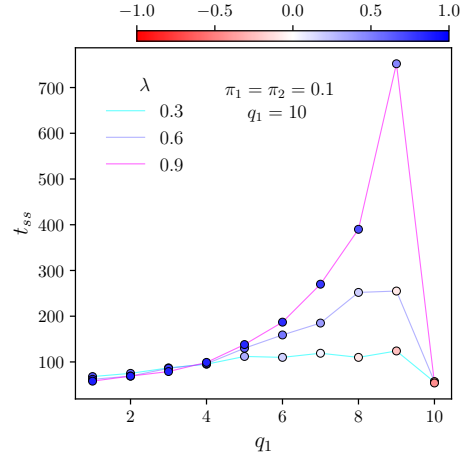


FIG. A3. Time to reach the stationary state, t_{ss} , plotted as a function of the bad-quality option, q_1 , while maintaining $q_2 = 10$ in lattice simulations. Lattice size is $L = 1024$. As in the main text, three values of the interdependence parameter have been considered, namely $\lambda = 0.3, 0.6, 0.9$, with discovery probabilities $\pi_1 = \pi_2 = 0.1$.

the same trends discussed in the main text - see Fig. A3.

On the other hand, two-dimensional lattice simulations enable the study of critical behavior and some critical exponents around the absorbing phase transition occurring in equal-quality, and almost fully-interdependent, scenarios. In this limit, lattice simulations provide valuable insights into the universality class of the phase transition and help classify the critical behavior of the system beyond the mean-field approximation, as these simulations incorporate spatial fluctuations.

-
- [1] J. R. Dyer, A. Johansson, D. Helbing, I. D. Couzin, and J. Krause, *Philosophical Transactions of the Royal Society B: Biological Sciences* **364**, 781 (2009).
 - [2] J. Krause and G. Ruxton, *Living in Groups* (2002).
 - [3] A. Baronchelli, *Royal Society Open Science* **5**, 172189 (2018), publisher: Royal Society.
 - [4] T. Bose, A. Reina, and J. A. Marshall, *Current Opinion in Behavioral Sciences* **16**, 30 (2017).
 - [5] D. J. T. Sumpter, *Collective Animal Behavior* (Princeton University Press, 2010).
 - [6] J. Smith and D. Harper, *Animal Signals* (2003).
 - [7] N. E. Leonard, A. Bizyaeva, and A. Franci, *Annual Review of Control, Robotics, and Autonomous Systems* **7**, null (2024).
 - [8] T. Sasaki and S. C. Pratt, *Annual Review of Entomology* **63**, 259 (2018).
 - [9] R. A. Holley and T. M. Liggett, *The Annals of Probability* **3**, 643 (1975).
 - [10] C. Castellano, S. Fortunato, and V. Loreto, *Rev. Mod. Phys.* **81**, 591 (2009), publisher: American Physical Society.
 - [11] Galam, S., *Eur. Phys. J. B* **25**, 403 (2002).
 - [12] S. Galam, *International Journal of Modern Physics C* **19**, 409 (2008).
 - [13] C. Castellano, M. A. Muñoz, and R. Pastor-Satorras, *Phys. Rev. E* **80**, 041129 (2009).
 - [14] S. Redner, *Comptes Rendus Physique* **20**, 275 (2019).
 - [15] G. De Marzo, A. Zaccaria, and C. Castellano, *Physical Review Research* **2**, 043117 (2020), publisher: American Physical Society.
 - [16] A. Bizyaeva, A. Franci, and N. E. Leonard, *IEEE Transactions on Automatic Control* **68**, 1415 (2023), conference Name: IEEE Transactions on Automatic Control.
 - [17] L. S. Ramirez, F. Vazquez, M. San Miguel, and T. Galla, *Phys. Rev. E* **109**, 034307 (2024).
 - [18] N. F. Britton, N. R. Franks, S. C. Pratt, and T. D. Seeley, *Proceedings of the Royal Society of London. Series B: Biological Sciences* **269**, 1383 (2002).
 - [19] T. D. Seeley and S. C. Buhrman, *Behavioral Ecology and Sociobiology* **49**, 416 (2001).
 - [20] T. D. Seeley, P. K. Visscher, T. Schlegel, P. M. Hogan, N. R. Franks, and J. A. R. Marshall, *Science* **335**, 108

- (2012).
- [21] C. List, C. Elsholtz, and T. D. Seeley, *Philosophical Transactions of the Royal Society B: Biological Sciences* **364**, 755 (2008).
 - [22] G. Valentini, H. Hamann, and M. Dorigo, in *Proceedings of the 2014 International Conference on Autonomous Agents and Multi-Agent Systems*, AAMAS '14 (International Foundation for Autonomous Agents and Multiagent Systems, Richland, SC, 2014) p. 45–52.
 - [23] A. Reina, G. Valentini, C. Fernandez-Oto, M. Dorigo, and V. Trianni, *PLoS one* **10**, e0140950 (2015).
 - [24] A. Reina, J. A. R. Marshall, V. Trianni, and T. Bose, *Phys. Rev. E* **95**, 052411 (2017), publisher: American Physical Society.
 - [25] R. Gray, A. Franci, V. Srivastava, and N. E. Leonard, *IEEE Transactions on Control of Network Systems* **5**, 793 (2018), conference Name: IEEE Transactions on Control of Network Systems.
 - [26] A. Reina, T. Bose, V. Trianni, and J. A. R. Marshall, in *Distributed Autonomous Robotic Systems: The 13th International Symposium*, edited by R. Groß, A. Kolling, S. Berman, E. Frazzoli, A. Martinoli, F. Matsuno, and M. Gauci (Springer International Publishing, Cham, 2018) pp. 461–473.
 - [27] G. Valentini, E. Ferrante, H. Hamann, and M. Dorigo, *Autonomous Agents and Multi-Agent Systems* **30**, 553 (2016).
 - [28] M. S. Talamali, A. Saha, J. A. R. Marshall, and A. Reina, *Science Robotics* **6**, eabf1416 (2021), publisher: American Association for the Advancement of Science.
 - [29] D. March, J. Múgica, E. E. Ferrero, and M. C. Miguel, Honeybee-like collective decision making in a kilobot swarm (2023), [arXiv:2310.15592 \[cond-mat.dis-nn\]](https://arxiv.org/abs/2310.15592).
 - [30] K. Passino, T. Seeley, and P. Visscher, *Behavioral Ecology and Sociobiology* **62**, 401 (2008).
 - [31] T. Galla, *Journal of Theoretical Biology* **262**, 186 (2010).
 - [32] T. Aust, M. S. Talamali, M. Dorigo, H. Hamann, and A. Reina, in *Swarm Intelligence*, edited by M. Dorigo, H. Hamann, M. López-Ibáñez, J. García-Nieto, A. Engelbrecht, C. Pinciroli, V. Strobil, and C. Camacho-Villalón (Springer International Publishing, Cham, 2022) pp. 234–247.
 - [33] R. Zakir, M. Dorigo, and A. Reina, in *Swarm Intelligence: 13th International Conference, ANTS 2022, Málaga, Spain, November 2–4, 2022, Proceedings* (Springer-Verlag, Berlin, Heidelberg, 2022) pp. 209–221.
 - [34] Y. Khaluf, I. Rausch, and P. Simoens, in *Swarm Intelligence*, edited by M. Dorigo, M. Birattari, C. Blum, A. L. Christensen, A. Reina, and V. Trianni (Springer International Publishing, Cham, 2018) pp. 252–263.
 - [35] I. Rausch, A. Reina, P. Simoens, and Y. Khaluf, *Swarm Intell* **13**, 321 (2019).
 - [36] A. Reina, T. Bose, V. Trianni, and J. A. R. Marshall, *Scientific Reports* **8**, 4387 (2018), number: 1 Publisher: Nature Publishing Group.
 - [37] M. Starnini, A. Baronchelli, and R. Pastor-Satorras, *Journal of Statistical Mechanics: Theory and Experiment* **2012**, P10027 (2012).
 - [38] G. Valentini, E. Ferrante, and M. Dorigo, *Frontiers in Robotics and AI* **4** (2017).
 - [39] F. Vazquez, E. S. Loscar, and G. Baglietto, *Phys. Rev. E* **100**, 042301 (2019).
 - [40] L. Ramirez, M. San Miguel, and T. Galla, *Phys. Rev. E* **106**, 054307 (2022).
 - [41] A. Reina, T. Njougouo, E. Tuci, and T. Carletti, Studying speed-accuracy trade-offs in best-of-n collective decision-making through heterogeneous mean-field modelling (2023), [arXiv:2310.13694 \[physics\]](https://arxiv.org/abs/2310.13694).
 - [42] L. Chittka, P. Skorupski, and N. E. Raine, *Trends in Ecology & Evolution* **24**, 400–407 (2009).
 - [43] H. Hinrichsen, *Advances in Physics* **49**, 815 (2000).
 - [44] I. Jensen and R. Dickman, *Phys. Rev. E* **48**, 1710 (1993).
 - [45] S. Lübeck and R. D. Willmann, *Journal of Statistical Physics* **115**, 1231 (2004).
 - [46] M. A. Muñoz, *Rev. Mod. Phys.* **90**, 031001 (2018).
 - [47] G. T. Fechner, in *Readings in the history of psychology*, Century psychology series (Appleton-Century-Crofts, East Norwalk, CT, US, 1948) pp. 206–213.
 - [48] L. L. Thurstone, *The American Journal of Psychology* **38**, 368 (1927), publisher: University of Illinois Press.
 - [49] S. S. Stevens, *Psychological Review* **64**, 153 (1957), place: US Publisher: American Psychological Association.
 - [50] D. Austen-Smith and J. S. Banks, *American Political Science Review* **90**, 34–45 (1996).
 - [51] J. Marshall, G. Brown, and A. Radford, *Trends in Ecology & Evolution* **32** (2017).
 - [52] K. M. Passino and T. D. Seeley, *Behavioral Ecology and Sociobiology* **59**, 427 (2006).
 - [53] A. Reina, R. Zakir, G. De Masi, and E. Ferrante, *Communications Physics* **6**, 1 (2023), publisher: Nature Publishing Group.
 - [54] M. S. Talamali, J. A. R. Marshall, T. Bose, and A. Reina, in *2019 International Conference on Robotics and Automation (ICRA)* (IEEE Press, Montreal, QC, Canada, 2019) pp. 9652–9659.