# THE SYMPLECTIC FORM ASSOCIATED TO A SINGULAR POISSON ALGEBRA 

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#### Abstract

Given an affine Poisson algebra, that is singular one may ask whether there is an associated symplectic form. In the smooth case the answer is obvious: for the symplectic form to exist the Poisson tensor has to be invertible. In the singular case, however, derivations do not form a projective module and it is less clear what non-degenerate means. For a symplectic singularity one may naively ask if there is indeed an analogue of a symplectic form. We examine an example of a symplectic singularity, namely the double cone, and show that here such a symplectic form exists. We use the naive de Rham complex of a Lie-Rinehart algebra. Our analysis of the double cone relies on Gröbner bases calculations.


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## 1. Introduction

Throughout this article $\boldsymbol{k}$ denotes a field of characteristic zero. We consider a polynomial $\boldsymbol{k}$-algebra $P=\boldsymbol{k}\left[x^{1}, \ldots, x^{n}\right]$ with symplectic Poisson bracket $\{$,$\} . Let us write for the associated Poisson tensor$ $\Pi^{i j}=\left\{x^{i}, x^{j}\right\} \in P$. Let $I=\left(f_{1}, \ldots, f_{\ell}\right)$ be a Poisson ideal in $P$, i.e., a multiplicative ideal such that $\{I, P\} \subseteq I$ (for examples see [7]). Its generators $f_{\mu}$ have the property that $\left\{x^{i}, f_{\mu}\right\}=\sum_{\nu=1}^{\ell} Z_{\mu}^{i \nu} f_{\nu}$ for some (in general, non-unique) $Z_{\mu}^{i \nu} \in P$. As the $Z_{\mu}^{i \nu}$ can be interpreted as Christoffel symbols of a connection (see [6]) of the conormal module $I / I^{2}$ we refer to them as the Poissoffel symbols of the Poisson ideal. The quotient $A=P / I$ becomes a Poisson $\boldsymbol{k}$-algebra. We refer to this type of algebra as an affine Poisson algebra.

Many singular affine Poisson algebras that arise in nature have 'symplectic' properties. For example, Poisson algebras associated to symplectic quotients or coadjoint orbits typically have symplectic singularities [1, 8, 4]. However, we do not know of any attempt to construct a symplectic form for such a singular Poisson algebra. In this paper, we propose a general framework for doing so and elaborate an example of such a symplectic form. We hope that the likely explanation of our construction in terms of symplectic singularities can be worked out in the future. We expect complexifications of symplectic reductions of unitary group actions [8] to have symplectic forms.

Our idea is to search for a symplectic form in the naive de Rham complex (see Appendix A) of the LieRinehart algebra $(A, \operatorname{Der}(A))$. The main difficulty is that in the singular case the $A$-module of derivations $\operatorname{Der}(A)=\{X: A \rightarrow A \mid X(a b)=X(a) b+a X(b)\}$ is not projective and not every derivation $X \in \operatorname{Der}(A)$ can be written as an $A$-linear combination of Hamiltonian vector fields $\{a\},, a \in A$. To get our hands on explicit descriptions of $\operatorname{Der}(A)$ in terms of generators and relations we use Gröbner basis calculations.

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## 2. The main idea

As a warm-up and to motivate our approach we recall how to invert a non-degenerate constant Poisson tensor on $\boldsymbol{k}^{n}$. Assume that $\Pi^{i j}=\left\{x^{i}, x^{j}\right\} \in \boldsymbol{k}$ is a non-degenerate Poisson tensor of the Poisson algebra $\left(P=\boldsymbol{k}\left[x^{1}, \ldots, x^{n}\right],\{\},\right)$. Then $\left(\mathrm{d} x^{i}\right)^{\sharp}=\sum_{j} \Pi^{i j} \partial_{j}$ defines the musical isomorphism $\sharp: \Omega_{P \mid \boldsymbol{k}} \rightarrow \operatorname{Der}(P)$ from the $P$-module of Kähler differentials $\Omega_{P \mid \boldsymbol{k}}$ to the $P$-module vector fields $\operatorname{Der}(P)$. Then the symplectic form $\omega=\sum_{i, j} \frac{1}{2} \omega_{i j} \mathrm{~d} x^{i} \wedge \mathrm{~d} x^{j}$ is defined by

$$
\omega\left(\left(\mathrm{d} x^{i}\right)^{\sharp}, \partial_{k}\right)=\sum_{j} \omega\left(\Pi^{i j} \partial_{j}, \partial_{k}\right)=\sum_{j} \Pi^{i j} \omega_{j k}=\delta_{k}^{i}=\partial_{k} x^{i} .
$$

This can be rewritten in a coordinate free way as follows $\omega\left((\mathrm{d} a)^{\sharp}, X\right)=\sum_{i, j} \frac{\partial a}{\partial x^{i}} \omega\left(\Pi^{i j} \partial_{j}, X^{k} \partial_{k}\right)=X(a)$ for $X \in \operatorname{Der}(P)$ and $a \in P$.

In order to discuss the singular case let us recall the following.
Theorem $2.1([2])$. If $I \subseteq P$ be an ideal in a polynomial $\boldsymbol{k}$-algebra $P$ then $\operatorname{Der}(A) \simeq \operatorname{Der}_{I}(P) / I \operatorname{Der}_{I}(P)$, where $\operatorname{Der}_{I}(P)=\{X \in \operatorname{Der}(P) \mid X(I) \subseteq I\}$.

With this isomorphism understood we define a version of the musical map in the singular case as follows. It is given as the $A$-linear map

$$
\sharp: \Omega_{A \mid \boldsymbol{k}} \rightarrow \operatorname{Der}(A), \quad(\mathrm{d}(a+I))^{\sharp}=\{a,\}+I \operatorname{Der}_{I}(P)
$$

where $\Omega_{A \mid \boldsymbol{k}}$ denotes the $A$-module of Kähler differentials and $a \in P$. Its image is denoted by $\left(\Omega_{A \mid \boldsymbol{k}}\right)^{\sharp}$. As we will see in the next section $\left(\Omega_{A \mid \boldsymbol{k}}\right)^{\sharp}$ may be different from $\operatorname{Der}(A)$. We denote by $\iota^{\mathrm{Ham}}:\left(\Omega_{A \mid \boldsymbol{k}}\right)^{\sharp} \rightarrow \operatorname{Der}(A)$ the inclusion.

Lemma 2.2. The map $\sharp$ is a morphism of Lie-Rinehart algebras and, accordingly, its image $\left(\Omega_{A \mid \boldsymbol{k}}\right)^{\sharp} a$ Lie-Rinehart subalgebra of $(\operatorname{Der}(A), A)$.

Proof. Consider Kähler forms $\left(a_{1}+I\right) \mathrm{d}\left(a_{2}+I\right)$ and $\left(b_{1}+I\right) \mathrm{d}\left(b_{2}+I\right)$ for $a_{1}, a_{2}, b_{1}, b_{2} \in P$. Recall 9 that $\left(\Omega_{A \mid k}, A\right)$ forms a Lie-Rinehart algebra whose bracket is the so-called Koszul bracket:

$$
\begin{aligned}
& {\left[\left(a_{1}+I\right) \mathrm{d}\left(a_{2}+I\right),\left(b_{1}+I\right) \mathrm{d}\left(b_{2}+I\right)\right]} \\
& =\left(a_{1} b_{1}+I\right) \mathrm{d}\left(\left\{a_{2}, b_{2}\right\}+I\right)+\left(a_{1}\left\{a_{2}, b_{1}\right\}+I\right) \mathrm{d}\left(b_{2}+I\right)-\left(b_{1}\left\{b_{2}, a_{1}\right\}+I\right) \mathrm{d}\left(a_{2}+I\right)
\end{aligned}
$$

On the other hand we have the commutator

$$
\left[a_{1}\left\{a_{2},\right\}+I, b_{1}\left\{b_{2},\right\}+I\right]=a_{1} b_{1}\left\{\left\{a_{2}, b_{2}\right\},\right\}+a_{1}\left\{a_{2}, b_{1}\right\}\left\{b_{2},\right\}-b_{1}\left\{b_{2}, a_{1}\right\}\left\{a_{2},\right\}+I
$$

Definition 2.3. We define the $A$-linear map $\omega^{\text {Ham }}:\left(\Omega_{A \mid \boldsymbol{k}}\right)^{\sharp} \otimes_{A} \operatorname{Der}(A) \rightarrow A$ by

$$
\omega^{\mathrm{Ham}}\left((\mathrm{~d} a+I)^{\sharp}, X+I \operatorname{Der}_{I}(P)\right):=X(a)+I .
$$

## Proposition 2.4.

(1) The form $\omega^{\mathrm{Ham}}$ in Definition 2.3 does not depend on the choice of the representatives $a \in P$, $X \in \operatorname{Der}_{I}(P)$ and is hence well-defined.
(2) With $\delta^{\mathrm{Ham}}: \operatorname{Alt}^{n}\left(\left(\Omega_{A \mid \boldsymbol{k}}\right)^{\sharp}, A\right) \rightarrow \operatorname{Alt}^{n+1}\left(\left(\Omega_{A \mid \boldsymbol{k}}\right)^{\sharp}, A\right)$ the naive de Rham differential of the LieRinehart algebra $\left(\left(\Omega_{A \mid \boldsymbol{k}}\right)^{\sharp}, A\right)$ the restriction of $\omega^{\mathrm{Ham}}$ to $\operatorname{Alt}^{2}\left(\left(\Omega_{A \mid \boldsymbol{k}}\right)^{\sharp}, A\right)$ fulfills $\delta^{\mathrm{Ham}} \omega^{\mathrm{Ham}}=0$.
Proof. As (1) is clear we address (2). Consider $a, b, c \in P$ with $X=\{a\}+,I \operatorname{Der}_{I}(P), Y=\{b\}+,I \operatorname{Der}_{I}(P)$, $Z=\{c\}+,I \operatorname{Der}_{I}(P)$
$\delta^{\operatorname{Ham}} \omega^{\operatorname{Ham}}(X, Y, Z)=X \omega(Y, Z)-Y \omega(X, Z)+Z \omega(X, Y)-\omega([X, Y], Z)+\omega([X, Z], Y)-\omega([Y, Z], X)$, $=\{a,\{c, b\}\}-\{b,\{c, a\}\}+\{c,\{b, a\}\}-\{a,\{c, b\}\}+\{b,\{a, c\}-\{a,\{b, c\}\}\}+I \in I$
by Jacobi's identity.
We are now in the position to formulate the fundamental question of our approach. Does there exist a non-degenerate $\omega \in \operatorname{Alt}^{2}(\operatorname{Der}(A), A)$ such that


Figure 1. The double cone.
(1) $\mathrm{d}^{\mathrm{dR}} \omega=0$ and
(2) $\omega\left(\iota^{\mathrm{Ham}} \otimes \mathrm{id}\right)=\omega^{\mathrm{Ham}} ?$

In this case we say that $\omega$ is a symplectic form on $\operatorname{Spec}(A)$. We will see in the next section that the answer can be affirmative.

## 3. The double cone

Let us check if the program laid out in the Section 2 makes sense for the double cone

$$
A=\boldsymbol{k}\left[x^{1}, x^{2}, x^{3}\right] /\left(x^{1} x^{2}-\left(x^{3}\right)^{2}\right)=: P /(f)
$$

We view $A=\boldsymbol{k}\left[x^{1}, x^{2}, x^{3}\right] /\left(x^{1} x^{2}-\left(x^{3}\right)^{2}\right)$ as the Poisson algebra of invariants of the linear cotangent lifted $\mathbb{Z}_{2}=\mathrm{O}_{2}(\boldsymbol{k})$-action on $\boldsymbol{k}^{2}=T^{*} \boldsymbol{k}$ (see [7]). Here the coordinates $x^{1}, x^{2}$ and $x^{3}$ correspond to the $\mathbb{Z}_{2^{-}}$ invariants $q^{2}, p^{2}$ and $q p$ understood with canonical bracket $\{q, p\}=1$. The variety $\operatorname{Spec}(A)$ has an isolated symplectic singularity at the origin (see [8]).

Proposition 3.1. Let $I=(f) \subset P$ be a principal ideal and $A=P / I$. Consider the intersection $J:=$ $\mathrm{Jac}_{f} \cap(f)$ of the Jacobian ideal $\mathrm{Jac}_{f}=\left(\frac{\partial f}{\partial x^{1}}, \ldots, \frac{\partial f}{\partial x^{n}}\right) \subset P$ with $(f)$. Since $J$ is in $\mathrm{Jac}_{f}$ we can write any $\xi \in J$ as $\sum_{i} X^{i} \frac{\partial f}{\partial x^{i}}$ with $X^{i} \in P$. The choice of each $X^{i}$ is unique up to the $P$-module $\operatorname{Syz}\left(\operatorname{Jac}_{f}\right)$ of first syzygies of $\mathrm{Jac}_{f}$. Any such $\xi$ gives rise to an $X^{\xi}:=\sum_{i} X^{i} \frac{\partial}{\partial x^{i}} \in \operatorname{Der}_{I}(P)$. We write

$$
\hat{f}: J \rightarrow \operatorname{Der}_{I}(P), \quad \xi \mapsto X^{\xi}
$$

for the corresponding morphism of P-modules. Define for each $\left(Y^{1}, \ldots, Y^{n}\right) \in \operatorname{Syz}^{\left(\mathrm{Jac}_{f}\right)}$ the derivation $Y=\sum_{i} Y^{i} \frac{\partial}{\partial x^{i}}$ so that $Y(f)=0$. Write for the corresponding P-linear map

$$
\tilde{f}: \operatorname{Syz}\left(\operatorname{Jac}_{f}\right) \rightarrow \operatorname{Der}(P), \quad\left(Y^{1}, \ldots, Y^{n}\right) \mapsto Y
$$

Then $\operatorname{Der}_{I}(P)=\operatorname{im}(\hat{f})+\operatorname{im}(\tilde{f})$.
Proof. For any $\xi \in J$ we have by construction $X^{\xi}(f) \in(f)$ and, conversely any such vector field can be obtained this way. The ambiguities in the choice of $X^{\xi}$ are dealt with by including all first syzygies of $\mathrm{Jac}_{f}$ into the list of generators of $\operatorname{Der}_{I}(P)$, by interpreting them as vector fields.

The table of Poisson brackets is

| $\{\}$, | $x^{1}$ | $x^{2}$ | $x^{3}$ |
| :---: | :---: | :---: | :---: |
| $x^{1}$ | 0 | $4 x^{3}$ | $2 x^{1}$ |
| $x^{2}$ | $-4 x^{3}$ | 0 | $-2 x^{2}$ |
| $x^{3}$ | $-2 x^{1}$ | $2 x^{2}$ | 0 |

and the generators of $\left(\Omega_{A \mid \boldsymbol{k}}\right)^{\sharp}$ as an $A$-module are given by the $I \operatorname{Der}_{I}(P)$ classes of

$$
\begin{equation*}
\left\{x^{1},\right\}=4 x^{3} \frac{\partial}{\partial x^{2}}+2 x^{1} \frac{\partial}{\partial x^{3}},\left\{x^{2},\right\}=-4 x^{3} \frac{\partial}{\partial x^{1}}-2 x^{2} \frac{\partial}{\partial x^{3}},\left\{x^{3},\right\}=2 x^{2} \frac{\partial}{\partial x^{2}}-2 x^{1} \frac{\partial}{\partial x^{1}} \tag{3.1}
\end{equation*}
$$

It turns out that the Poissoffel symbols vanish, i.e., the polynomial $f=x^{1} x^{2}-\left(x^{3}\right)^{2}$ is actually a Casimir. Moreover, $\operatorname{im}(\tilde{f})=\left(\Omega_{A \mid \boldsymbol{k}}\right)^{\sharp}$. We have $\operatorname{Jac}_{f} \cap(f)=(f)$ since $2 f=2 x^{1} \partial f / \partial x^{1}+x^{3} \partial f / \partial x^{3}$. So the list (3.1) has to be amended by

$$
\begin{equation*}
Z:=2 x^{1} \frac{\partial}{\partial x^{1}}+x^{3} \frac{\partial}{\partial x^{3}} \tag{3.2}
\end{equation*}
$$

to get the generators of $\operatorname{Der}_{I}(P)$ as a $P$-module. Note that $2 x^{2} \frac{\partial}{\partial x^{2}}+x^{3} \frac{\partial}{\partial x^{3}}=Z-\left\{x^{3},\right\}$. According to Macaulay2, there is a short exact sequence of $A$-modules

$$
0 \leftarrow \operatorname{Der}(A) \stackrel{\left[\begin{array}{cccc}
0 & -4 x^{3} & -2 x^{1} & 2 x^{1}  \tag{3.3}\\
4 x^{3} & 0 & 2 x^{2} & 0 \\
2 x^{1} & -2 x^{2} & 0 & x^{3}
\end{array}\right]}{\longleftarrow} A^{4} \stackrel{\left[\begin{array}{c}
x^{2} \\
x^{1} \\
-2 x^{3} \\
0
\end{array}\right]}{\longleftarrow} A \leftarrow 0
$$

so that $\operatorname{pd}(\operatorname{Der}(A))=1$.
Since $\operatorname{dim}\left(\operatorname{coker}\left(\iota^{\mathrm{Ham}}\right)\right)=1$ the form $\omega \in \operatorname{Alt}^{2}(\operatorname{Der}(A), A)$ is already defined by Definition 2.3. It remains to check that $\omega$ is non-degenerate and that $\mathrm{d}^{\mathrm{dR}} \omega=0$. To this end let us evaluate

$$
\left(\mathrm{d}^{\mathrm{dR}} \omega\right)(X, Y, Z)=X \omega(Y, Z)-Y \omega(X, Z)+Z \omega(X, Y)-\omega([X, Y], Z)+\omega([X, Z], Y)-\omega([Y, Z], X)
$$

where $X, Y$ are distinct $\left\{x^{i},\right\}$ with $i=1,2,3$. In fact, we have

$$
\begin{aligned}
& \left(\mathrm{d}^{\mathrm{dR}} \omega\right)\left(\left\{x^{1},\right\},\left\{x^{2},\right\}, Z\right) \\
& =\left\{x^{1}, \omega\left(\left\{x^{2},\right\}, Z\right)\right\}-\left\{x^{2}, \omega\left(\left\{x^{1},\right\}, Z\right)\right\}+Z \omega\left(\left\{x^{1},\right\},\left\{x^{2},\right\}\right) \\
& \quad-\omega\left(\left[\left\{x^{1},\right\},\left\{x^{2},\right\}\right], Z\right)+\omega\left(\left[\left\{x^{1},\right\}, Z\right],\left\{x^{2},\right\}\right)-\omega\left(\left[\left\{x^{2},\right\}, Z\right],\left\{x^{1},\right\}\right) \\
& =\left\{x^{1}, Z\left(x^{2}\right)\right\}-\left\{x^{2}, Z\left(x^{1}\right)\right\}+Z\left(\left\{x^{2}, x^{1}\right\}\right) \\
& \quad-\omega\left(\left\{\left\{x^{1}, x^{2}\right\},\right\}, Z\right)+\omega\left(\left\{x^{1},\right\},\left\{x^{2},\right\}\right)-\omega\left(\left\{x^{2},\right\},\left\{x^{1},\right\}\right) \\
& =0-2\left\{x^{2}, x^{1}\right\}+Z\left(4 x^{3}\right)-\omega\left(\left\{4 x^{3},\right\}, Z\right)+4 x^{3}+4 x^{3} \\
& =-8 x^{3}-4 x^{3}+4 x^{3}+4 x^{3}+4 x^{3}=0 .
\end{aligned}
$$

All expressions above are to be understood modulo $I \operatorname{Der}_{I}(P)$ and $I$, respectively. To unclutter the notation we did not annotate these expressions and continue with this habit later on. We used the auxiliary evaluations:

$$
\begin{aligned}
{\left[\left\{x^{1},\right\},\left\{x^{2},\right\}\right] } & =\left\{x^{1},\left\{x^{2},\right\}\right\}-\left\{x^{2},\left\{x^{1},\right\}\right\}=\left\{\left\{x^{1}, x^{2}\right\},\right\}, Z\left(x^{2}\right)=0, \quad Z\left(x^{1}\right)=2 x^{1} \\
{\left[\left\{x^{1},\right\}, Z\right] } & =\left[4 x^{3} \frac{\partial}{\partial x^{2}}+2 x^{1} \frac{\partial}{\partial x^{3}}, 2 x^{1} \frac{\partial}{\partial x^{1}}+x^{3} \frac{\partial}{\partial x^{3}}\right]=-2 x^{1} \frac{\partial}{\partial x^{3}}-4 x^{3} \frac{\partial}{\partial x^{2}}=-\left\{x^{1},\right\} \\
{\left[\left\{x^{2},\right\}, Z\right] } & =\left[4 x^{3} \frac{\partial}{\partial x^{2}}+2 x^{1} \frac{\partial}{\partial x^{3}}, 2 x^{1} \frac{\partial}{\partial x^{1}}+x^{3} \frac{\partial}{\partial x^{3}}\right]=-4 x^{3} \frac{\partial}{\partial x^{1}}-2 x^{2} \frac{\partial}{\partial x^{3}}=\left\{x^{2},\right\}
\end{aligned}
$$

Next we determine

$$
\begin{aligned}
& \left(\mathrm{d}^{\mathrm{dR}} \omega\right)\left(\left\{x^{1},\right\},\left\{x^{3},\right\}, Z\right) \\
& =\left\{x^{1}, \omega\left(\left\{x^{3},\right\}, Z\right)\right\}-\left\{x^{3}, \omega\left(\left\{x^{1},\right\}, Z\right)\right\}+Z \omega\left(\left\{x^{1},\right\},\left\{x^{3},\right\}\right) \\
& \quad-\omega\left(\left[\left\{x^{1},\right\},\left\{x^{3},\right\}\right], Z\right)+\omega\left(\left[\left\{x^{1},\right\}, Z\right],\left\{x^{3},\right\}\right)-\omega\left(\left[\left\{x^{3},\right\}, Z\right],\left\{x^{1},\right\}\right) \\
& =\left\{x^{1}, Z\left(x^{3}\right)\right\}-\left\{x^{3}, Z\left(x^{1}\right)\right\}+Z\left(\left\{x^{3}, x^{1}\right\}\right)-\omega\left(\left\{\left\{x^{1}, x^{3}\right\},\right\}, Z\right)+\omega\left(-\left\{x^{1},\right\},\left\{x^{3},\right\}\right)-0 \\
& =\left\{x^{1}, x^{3}\right\}-2\left\{x^{3}, x^{1}\right\}-Z\left(2 x^{1}\right)-Z\left(2 x^{1}\right)+\left\{x^{1}, x^{3}\right\}=4\left\{x^{1}, x^{3}\right\}-8 x^{1}=0,
\end{aligned}
$$

where we have used

$$
\begin{aligned}
{\left[\left\{x^{3},\right\}, Z\right] } & =\left[2 x^{2} \frac{\partial}{\partial x^{2}}-2 x^{1} \frac{\partial}{\partial x^{1}}, 2 x^{1} \frac{\partial}{\partial x^{1}}+x^{3} \frac{\partial}{\partial x^{3}}\right]=0, \quad Z\left(x^{3}\right)=x^{3}, \\
{\left[\left\{x^{1},\right\},\left\{x^{3},\right\}\right] } & =\left\{x^{1},\left\{x^{3},\right\}\right\}-\left\{x^{3},\left\{x^{1},\right\}\right\}=\left\{\left\{x^{1}, x^{3}\right\},\right\} .
\end{aligned}
$$

Finally, we calculate

$$
\begin{aligned}
& \left(\mathrm{d}^{\mathrm{dR}} \omega\right)\left(\left\{x^{2},\right\},\left\{x^{3},\right\}, Z\right) \\
& =\left\{x^{2}, \omega\left(\left\{x^{3},\right\}, Z\right)\right\}-\left\{x^{3}, \omega\left(\left\{x^{2},\right\}, Z\right)\right\}+Z \omega\left(\left\{x^{2},\right\},\left\{x^{3},\right\}\right) \\
& \quad-\omega\left(\left[\left\{x^{2},\right\},\left\{x^{3},\right\}\right], Z\right)+\omega\left(\left[\left\{x^{2},\right\}, Z\right],\left\{x^{3},\right\}\right)-\omega\left(\left[\left\{x^{3},\right\}, Z\right],\left\{x^{2},\right\}\right) \\
& =\left\{x^{2}, Z\left(x^{3}\right)\right\}-\left\{x^{3}, Z\left(x^{2}\right)\right\}+Z\left(\left\{x^{2}, x^{3}\right\}\right)-\omega\left(\left\{\left\{x^{2}, x^{3}\right\},\right\}, Z\right)+\omega\left(\left\{x^{2},\right\},\left\{x^{3},\right\}\right) \\
& =\left\{x^{2}, x^{3}\right\}-0+0+\omega\left(\left\{2 x^{2},\right\}, Z\right)+\left\{x^{3}, x^{2}\right\}=0,
\end{aligned}
$$

proving that $\mathrm{d}^{\mathrm{dR}} \omega=0$.
To check non-degeneracy we used Macaulay2 [5] to calculate the kernel of the matrix

| $\omega()$, | $\left\{x^{1},\right\}$ | $\left\{x^{2},\right\}$ | $\left\{x^{3},\right\}$ | $Z$ |
| :---: | :---: | :---: | :---: | :---: |
| $\left\{x^{1},\right\}$ | 0 | $-4 x^{3}$ | $-2 x^{1}$ | $2 x^{1}$ |
| $\left\{x^{2},\right\}$ | $4 x^{3}$ | 0 | $2 x^{2}$ | 0 |
| $\left\{x^{3},\right\}$ | $2 x^{1}$ | $-2 x^{2}$ | 0 | $x^{3}$ |
| $Z$ | $-2 x^{1}$ | 0 | $-x^{3}$ | 0 |

over $A=P /(f)$. By the exact sequence (3.3) the transposed gradient vector of $f,\left[\begin{array}{lll}x^{2} & x^{1} & -2 x^{3}\end{array} 0^{\top}\right.$, generates the kernel when the matrix is interpreted as a 2 -form on the free module $A^{4}$. This means that the 2 -form $\omega$ is well-defined on $\operatorname{Der}(A)$ and nondegenerate. We have no explanation for the fact that its determinant is $(4 f)^{2}$.

We have proven that $\omega$ is a symplectic form on the double cone.

## 4. Conclusion and outlook

We proposed a general framework of how to make sense of a symplectic form for a singular affine Poisson variety and showed that it is not void by exhibiting a symplectic form on the double cone. The setup is not at all restricted to the hypersurface case. Proposition 3.1 can be easily generalized for affine algebras. It then can happen that $\operatorname{dim} \operatorname{Der}(A) /\left(\Omega_{A \mid \boldsymbol{k}^{\sharp}}\right)>1$. If one is attempting to construct the symplectic form the simplest assumption to try is to suppose that the generators of $\operatorname{Der}_{I}(A)$ not belonging to $\Omega_{A \mid \boldsymbol{k}^{\sharp}}$ are isotropic. Then the calculations checking closedness of $\omega$ appear to be straight forward. Yet the catch is that those generators are unique up to $\Omega_{A \mid k^{\sharp}}{ }^{\sharp}$, which in turn is typically not isotropic. It should be said that the concrete data of an affine Poisson algebra are typically bulky, if available at all. A more systematic empirical study must rely on computer implementations to be practically feasible. Of course, a conceptual way to prove the existence of the symplectic form is desirable.

It should be also said that algebraic geometry is not the proper setting for symplectic geometry since Hamiltonian flows do not respect polynomial observables. The appropriate framework for singular symplectic geometry appears to be a Poisson differential space $\left(X, \mathcal{C}^{\infty}(X),\{\},\right)$ in the sense of [3], or variations thereof such as, e.g., [13, 10. This is because symplectic reductions by compact group actions and gauge theoretic moduli spaces are to be described in this language. The notion of the module of differentials $\mathfrak{D}(X)$ for a differential space has been developed in 11 and the idea of using the naive de Rham complexes of the Lie-Rinehart algebras $\left(\mathcal{C}^{\infty}(X), \mathfrak{D}(X)^{\sharp}\right)$ and $\left(\mathcal{C}^{\infty}(X), \operatorname{Der}\left(\mathcal{C}^{\infty}(X)\right)\right.$ goes through without difficulty. One faces however the problem that it is not so clear how to gain explicit descriptions of the $\mathcal{C}^{\infty}(X)$-module $\operatorname{Der}\left(\mathcal{C}^{\infty}(X)\right)$, since the ideal theory of $\mathcal{C}^{\infty}\left(\mathbb{R}^{n}\right)$ is more subtle. As the naive de Rham complex is natural it is expected to be straight forward to show that the singular symplectic form restricts to the symplectic forms on the symplectic strata (compare [13]).

## Appendix A. Lie-Rinehart algebras and the naive de Rham complex

A Lie-Rinehart algebra $(L, A)$ (see [12]) is a commutative $\boldsymbol{k}$-algebra $A$ and an $A$-module $L$ such that
(1) $L$ is a $\boldsymbol{k}$-Lie algebra,
(2) $L$ acts on $A$ by derivations via $L \otimes_{\boldsymbol{k}} A \rightarrow A, X \otimes a \mapsto X(a)$,
(3) $[X, a Y]=X(a) Y+a[X, Y]$ for all $X, Y \in L$ and $a \in A$,
(4) $(a X)(b)=a X(b)$ for all $X \in L$ and $a, b \in A$.

The $A$-module of $n$-cochains of the naive de Rham complex of the Lie-Rinehart algebra $(L, A)$ is given by the space $\operatorname{Alt}^{n}(L, A)$ of alternating $A$-linear forms of arity $n$ with values in $A$. The differential d: $\operatorname{Alt}^{n}(L, A) \rightarrow$ $\mathrm{Alt}^{n+1}(L, A)$ of the naive de Rham complex is given by the Koszul formula
$(\mathrm{d} \omega)\left(X_{0}, X_{1}, \ldots, X_{n}\right)=\sum_{i=0}^{n}(-1)^{i} X_{i}\left(\omega\left(X_{0}, \ldots, \widehat{X}_{i}, \ldots, X_{n}\right)\right)+\sum_{i<j}^{n}(-1)^{i+j}\left(X_{0}, \ldots, \widehat{X}_{i}, \ldots, \widehat{X}_{j}, \ldots, X_{n}\right)$,
where $X_{0}, X_{1}, \ldots, X_{n} \in L$ and $\operatorname{Alt}^{n}(L, A)$. The ${ }^{\wedge}$ indicates omission of the corresponding term. It is wellknown that $\left(\operatorname{Alt}^{n}(L, A), \mathrm{d}\right)$ forms a differential graded $\boldsymbol{k}$-algebra with respect to the product $\cup: \operatorname{Alt}^{p}(L, A) \times$ $\operatorname{Alt}^{q}(L, A) \rightarrow \operatorname{Alt}^{p+q}(L, A)$

$$
\omega \cup \eta\left(X_{1}, X_{2}, \ldots, X_{p+q}\right)=\sum_{\sigma \in \operatorname{Sh}_{p, q}}(-1)^{\sigma} \omega\left(X_{\sigma(1)}, \ldots, X_{\sigma(q)}\right) \eta\left(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)}\right)
$$

where $\mathrm{Sh}_{p, q}$ denotes the set of $p, q$-shuffle permutations. If $L=\operatorname{Der}(A)$ we use the notation $\mathrm{d}^{\mathrm{dR}}:=\mathrm{d}$. Rinehart [12] has shown that if $L$ is a projective $A$-module then the $n$th cohomology of $\left(\mathrm{Alt}^{n}(L, A), \mathrm{d}\right)$ computes $\operatorname{Ext}_{U(L, A)}^{n}(A, A)$, where $U(L, A)$ denotes Rinehart's universal enveloping algebra of $(L, A)$. If $L$ is not projective there is no such result to be expected in general. For this reason the complex is called naive.

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