

Exact distance Kneser graphs

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Abstract

For any graph $G = (V, E)$ and positive integer d , the *exact distance- d graph* $G_{=d}$ is the graph with vertex set V , where two vertices are adjacent if and only if the distance between them in G is d . We study the exact distance- d Kneser graphs. For these graphs, we characterize the adjacency of vertices in terms of the cardinality of the intersection between them. We present formulas describing the distance between any pair of vertices and we compute the diameter of these graphs.

Keywords: *exact distance graphs, Kneser graphs, Kneser-like graphs, diameter of graphs.*

1 Introduction and main results

Let G be a connected graph. Given two vertices a and b in G , $dist_G(a, b)$, the *distance* between a and b , is defined as the length of the shortest path in G joining a to b . The diameter of G , denoted $diam(G)$, is defined as the maximum distance between any pair of vertices in G .

The concept of exact distance- d graph, where d is a positive integer, was introduced by Nešetřil and Ossona de Mendez [11] (Section 11.9). Formally, if $G = (V, E)$ is a graph, the exact distance- d graph $G_{=d}$ of G is the graph with vertex set V and where two vertices a and b in $G_{=d}$ are adjacent if and only if $dist_G(a, b) = d$. Note that $G_{=1} = G$.

The main focus in early research of exact distance graphs was on their chromatic number [2, 8, 11, 12]. The exact distance graphs have been much earlier considered for hypercube graphs in the frame of the so-called cube-like graphs [5, 9, 15]. More recently, Brešar et al. [3] considered the structure of exact distance graphs of some graph products. They remarked that the exact distance graphs are not only interesting because of the chromatic number, but also as a general metric graph theory concept.

Let k and r be positive integers and let $[2k + r] = \{1, 2, \dots, 2k + r\}$. Let $[2k + r]^k$ be the set of k -subsets of $[2k + r]$. The Kneser graph $K(2k + r, k)$ is the graph with vertex set $[2k + r]^k$ and where two k -subsets $A, B \in [2k + r]^k$ are joined by an edge if $A \cap B = \emptyset$. Note that $K(5, 2)$ is the well-known Petersen graph. It is easy to show that the Kneser graph $K(2k + r, k)$ is a connected regular graph having $\binom{2k+r}{k}$ vertices of degree $\binom{k+r}{k}$. Theoretical properties of Kneser graphs have been studied in past years (see for example [7, 10, 13, 14] and references therein).

Stahl shows in [13] the following result.

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Lemma 1.1 ([13]). *Let $A, B \in [2k+r]^k$ be two vertices in $K(2k+r, k)$ joined by a path of length $2p$ ($p \geq 0$). Then $|A \cap B| \geq k - rp$.*

The following result is a consequence of Lemma 1.1.

Corollary 1.2. *Let $A, B \in [2k+r]^k$ be two vertices in $K(2k+r, k)$ joined by a path of length $2p+1$ ($p \geq 0$). Then $|A \cap B| \leq rp$.*

Valencia-Pabon and Vera [14] shown the following results.

Proposition 1.3 ([14]). *Let k and r be positive integers, with $k \geq 2$ and $r \geq k - 1$. Then $\text{diam}(K(2k+r, k)) = 2$.*

Lemma 1.4 ([14]). *Let $A, B \in [2k+r]^k$ be two different vertices of the Kneser graph $K(2k+r, k)$, where $1 \leq r < k - 1$. If $|A \cap B| = s$ then*

$$\text{dist}(A, B) = \min \left\{ 2 \left\lceil \frac{k-s}{r} \right\rceil, 2 \left\lceil \frac{s}{r} \right\rceil + 1 \right\}.$$

Theorem 1.5 ([14]). *Let k and r be positive integers. Then $\text{diam}(K(2k+r, k)) = \left\lceil \frac{k-1}{r} \right\rceil + 1$.*

There are some generalizations to Kneser-like graphs in the literature (see for example [1, 4, 6, 7] and references therein). Let $K(n, k, i)$ be the *generalized Kneser graph*, i.e. the graph with vertex set $[n]^k$ and the edges connecting all pairs of vertices with intersection smaller than i . Denote by $J(n, k, i)$ the *generalized Johnson graph*, i.e. the graph with the same vertex set $[n]^k$ and edges connecting all pairs of vertices with intersections exactly i . In particular, in [4] the distance between two vertices of $K(n, k, i)$ was characterized in terms of the size of their intersection, and the diameter of the graph was computed. These results are given in Theorems 1.6 and 1.7.

Theorem 1.6 ([4]). *Let A and B be two vertices in $K(n, k, i)$, where $n = 2k - i + r$ and $0 \leq r < k - 2i - 1$. If $|A \cap B| = s > i$ then*

$$\text{dist}_{K(n, k, i)}(A, B) = \min \left\{ 2 \left\lceil \frac{k-s}{i+r} \right\rceil, 2 \left\lceil \frac{s-i}{i+r} \right\rceil + 1 \right\}.$$

Theorem 1.7 ([4]). *If k and r are positive integers, $i < k$ is a non negative integer and $n = 2k - i + r$ then $\text{diam}(K(n, k, i)) = \left\lceil \frac{k-i-1}{i+r} \right\rceil + 1$.*

Later, in [1], similar results were obtained for $J(n, k, i)$, which are given in Theorems 1.8 and 1.9.

Theorem 1.8 ([1]). *Let A and B be two vertices in $J(n, k, i)$, where $n > k > i$ are non negative integers, $n \geq 2k$ and $(n, k, i) \neq (2k, k, 0)$. If $|A \cap B| = s$ then*

$$\text{dist}_{J(n, k, i)}(A, B) = \begin{cases} 3 & \text{if } s < \min\{i, -n + 3k - 2i\}; \\ \left\lceil \frac{k-s}{k-i} \right\rceil & \text{if } -n + 3k - 2i \leq s < i; \\ \min \left\{ 2 \left\lceil \frac{k-s}{n-2k+2i} \right\rceil, 2 \left\lceil \frac{s-i}{n-2k+2i} \right\rceil + 1 \right\} & \text{if } s \geq i. \end{cases}$$

Theorem 1.9 ([1]). *If $n > k > i$ are non negative integers, $n \geq 2k$ and $(n, k, i) \neq (2k, k, 0)$ then*

$$\text{diam}(J(n, k, i)) = \begin{cases} \left\lceil \frac{k-i-1}{n-2k+2i} \right\rceil + 1 & \text{if } n < 3(k-i) - 1 \text{ or } i = 0; \\ 3 & \text{if } 3(k-i) - 1 \leq n < 3k - 2i \text{ and } i \neq 0; \\ \left\lceil \frac{k}{k-i} \right\rceil & \text{if } n \geq 3k - 2i \text{ and } i \neq 0. \end{cases}$$

In this paper, we are interested in *exact distance- d Kneser graphs*, $K_{=d}(2k+r, k)$, with k and r positive integers. In Section 2 we prove the following theorem, which gives a complete characterization of the adjacency relation on $K_{=d}(2k+r, k)$ in terms of the cardinality of the intersection of its vertices, thus showing that they are another type of generalization of Kneser graphs.

Theorem 1.10. *Let k and r be positive integers with $k \geq 2$ and $1 \leq r < k - 1$. Let $D = \text{diam}(K(2k+r, k))$ and let $1 < d \leq D$ be a fixed integer. Two vertices A and B in $K_{=d}(2k+r, k)$ are adjacent if and only if*

(i) *Case $d = 2p$ (with $p \geq 1$).*

1. $k - rp \leq |A \cap B| \leq k - rp + r - 1$ when $d < D$ or when r divides $k - 1$.
2. $rp - r + 1 \leq |A \cap B| \leq k - rp + r - 1$ if $d = D$ and r does not divide $k - 1$.

(ii) *Case $d = 2p + 1$ (with $p \geq 1$).*

1. $rp - r + 1 \leq |A \cap B| \leq rp$ when $d < D$ or when r divides $k - 1$.
2. $rp - r + 1 \leq |A \cap B| \leq k - rp - 1$ if $d = D$ and r does not divide $k - 1$.

In order to avoid any confusion about the distance between two vertices in $K(2k+r, k)$ and in $K_{=d}(2k+r, k)$, we prefer to denote $\text{dist}_K(A, B)$ instead of $\text{dist}_{K(2k+r, k)}(A, B)$, and $\text{dist}_{(K, d)}(A, B)$ instead of $\text{dist}_{K_{=d}(2k+r, k)}(A, B)$, when the parameters of the Kneser graph and the exact distance- d Kneser graph are clear from the context. In Section 3, we compute the distance between any two vertices in $K_{=d}(2k+r, k)$, and we prove the two following theorems.

Theorem 1.11. *Let A and B be two non adjacent vertices in $K_{=2p}(2k+r, k)$. If $|A \cap B| = s$ then*

$$\text{dist}_{(K, 2p)}(A, B) = \max \left\{ 2, \left\lceil \frac{k-s}{rp} \right\rceil \right\}.$$

Theorem 1.12. *Let A and B be two non adjacent vertices in $K_{=2p+1}(2k+r, k)$. If $|A \cap B| = s$ then*

$$\text{dist}_{(K, 2p+1)}(A, B) = \min \left\{ 1 + 2 \left\lceil \frac{|s-rp|}{2rp+r} \right\rceil, 2 \left\lceil \frac{k-s}{2rp+r} \right\rceil \right\}.$$

Finally, in Section 4, we compute the diameter of graph $K_{=d}(2k+r, k)$, and prove the following theorem.

Theorem 1.13. *Let k and r be positive integers with $k \geq 2$ and $1 \leq r < k - 1$. Let D be the diameter of the Kneser graph $K(2k+r, k)$ and let $1 < d \leq D$. The diameter of the exact distance- d Kneser graph $K_{=d}(2k+r, k)$ is given by*

$$\text{diam}(K_{=d}(2k+r, k)) = \begin{cases} 2 & \begin{array}{l} \text{(i) if } d = D, r \nmid k - 1; \\ \text{(ii) if } d = 2p + 1 = D, r \mid k - 1; \end{array} \\ \left\lceil \frac{k}{rp} \right\rceil & \begin{array}{l} \text{(iii) if } d = 2p < D; \\ \text{(iv) if } d = 2p = D, r \mid k - 1; \end{array} \\ \left\lceil \frac{k+(r/2)}{2rp+r} \right\rceil & \text{(v) if } d = 2p + 1 < D, 2 \mid (k - \frac{r}{2}); \\ 1 + \left\lceil \frac{k+(r/2)-1}{2rp+r} \right\rceil & \text{(vi) if } d = 2p + 1 < D, 2 \nmid (k - \frac{r}{2}). \end{cases}$$

2 Adjacency characterization

The aim of this section is to characterize the adjacency relation of vertices in $K_{=d}(2k+r, k)$ in terms of the cardinality of their intersection.

For the remainder of this article we denote $|A \cap B| = s$, $\text{diam}(K(2k+r, k)) = D$ and assume d to be a fixed integer with $1 \leq d \leq D$.

Notice first that, by Proposition 1.3, $D = 2$ when $r \geq k-1$ and thus, in this case, there are only two different distances: 1 and 2. As $K_{=1}(2k+r, k)$ is isomorphic to $K(2k+r, k)$ and $K_{=2}(2k+r, k)$ is isomorphic to the complement of $K(2k+r, k)$, we have the following remark.

Remark 2.1. *Let k and r be positive integers with $r \geq k-1$. Two vertices A and B in $K_{=2}(2k+r, k)$ are adjacent if and only if $1 \leq s \leq k-1$.*

The following remark is used in several proofs, including the ones in this section.

Remark 2.2. *Let k and r be positive integers with $k \geq 2$ and $1 \leq r < k-1$, and let $k-1 = xr + y$ with x, y integers such that $x > 0$ and $0 \leq y < r$. By Theorem 1.5, $D = \lceil \frac{k-1}{r} \rceil + 1$. Clearly, if r divides $k-1$ then $y = 0$ and $D = x+1$, otherwise $D = x+2$.*

We split the proof of Theorem 1.10 in the following four lemmas.

Lemma 2.3. *Let k and r be positive integers with $k \geq 2$ and $1 \leq r < k-1$, and let $p \geq 1$, such that $2p < D$ or r divides $k-1$. Two vertices A and B in $K_{=2p}(2k+r, k)$ are adjacent if and only if $k-rp \leq s \leq k-rp+r-1$.*

Proof. Trivially, $k-rp \leq k-rp+r-1$. Notice that if $D = x+1$ then $x+1 \geq 2p$ and so $x \geq 2p-1$. Otherwise, $D = x+2$ and $x+2 > 2p$ which also implies that $x \geq 2p-1$.

Assume that $s = k-rp+q$ with $0 \leq q < r$. Now, on the one hand, $2\lceil \frac{k-s}{r} \rceil = 2\lceil \frac{rp-q}{r} \rceil = 2\lceil p - \frac{q}{r} \rceil = 2p$. On the other hand, $2\lceil \frac{s}{r} \rceil + 1 = 2\lceil \frac{k-rp+q}{r} \rceil + 1$. Replacing k by $xr + y + 1$, we have that $2\lceil \frac{s}{r} \rceil + 1 = 2\lceil x - p + \frac{y+1+q}{r} \rceil + 1 \geq 2p+1$, because $x \geq 2p-1$ and $r > y, q \geq 0$. Therefore, by Lemma 1.4, $\text{dist}_K(A, B) = 2p$, and A is adjacent to B in $K_{=2p}(2k+r, k)$.

Conversely, let $\text{dist}_K(A, B) = 2p$. On the one hand, suppose that $s = k-rp-q$ with $q > 0$. As $\text{dist}_K(A, B)$ is even then by Lemma 1.4, it must be equal to $2\lceil \frac{k-s}{r} \rceil = 2\lceil \frac{rp+q}{r} \rceil = 2\lceil p + \frac{q}{r} \rceil \geq 2(p+1)$. On the other hand, if $s = k-rp+q$ with $q \geq r$, then $\text{dist}_K(A, B) = 2\lceil \frac{k-s}{r} \rceil = 2\lceil p - \frac{q}{r} \rceil \leq 2(p-1)$. Either case contradicts the fact that $\text{dist}_K(A, B) = 2p$. Therefore $k-rp \leq s \leq k-rp+r-1$. \square

Lemma 2.4. *Let k and r be positive integers with $k \geq 2$, $1 \leq r < k-1$ such that r does not divide $k-1$, and let $2p = D$ with $p \geq 1$. Two vertices A and B in $K_{=2p}(2k+r, k)$ are adjacent if and only if $rp-r+1 \leq s \leq k-rp+r-1$.*

Proof. Let $k-1 = xr + y$ with $1 \leq y < r$, and notice $x+2 = 2p$. First, $rp-r+1 \leq k-rp+r-1$. In fact, $k-2rp+2r-2 = xr + y - 2rp + 2r - 1 = (2p-2)r + y - 2rp + 2r - 1 = y - 1 \geq 0$. Moreover, the integer interval $[rp-r+1, k-rp+r-1]$ is composed of y integers.

Assume first that $s = k-rp+r-1-q$ with $0 \leq q \leq y-1$. Then we have to prove that $2\lceil \frac{k-s}{r} \rceil \leq 2\lceil \frac{s}{r} \rceil + 1$. Notice that, $2\lceil \frac{k-s}{r} \rceil = 2\lceil p-1 + \frac{q+1}{r} \rceil = 2p$. Moreover, $2\lceil \frac{s}{r} \rceil + 1 = 2\lceil \frac{k-rp+r-1-q}{r} \rceil + 1 = 2\lceil \frac{xr+y-rp+r-q}{r} \rceil + 1 = 2\lceil \frac{(2p-2)r+y-rp+r-q}{r} \rceil + 1 = 2\lceil p-1 + \frac{y-q}{r} \rceil + 1 = 2p+1$ because $0 < \frac{y-q}{r} < 1$. Therefore, by Lemma 1.4, $\text{dist}_K(A, B) = 2p$, and A is adjacent to B in $K_{=2p}(2k+r, k)$.

Conversely, let $\text{dist}_K(A, B) = 2p$. On the one hand assume that $s = rp-r+1-q$ with $q \geq 1$. In this case, $2\lceil \frac{s}{r} \rceil + 1 < 2p$. In fact, $2\lceil \frac{s}{r} \rceil + 1 = 2\lceil \frac{rp-r+1-q}{r} \rceil + 1 = 2\lceil p-1 + \frac{1-q}{r} \rceil + 1 \leq$

$2(p-1)+1=2p-1$. On the other hand, assume that $s=k-rp+r-1+q$ with $q \geq 1$. Then $\text{dist}_K(A,B)=2\lceil\frac{k-s}{r}\rceil=2\lceil\frac{rp-r+1-q}{r}\rceil=2\lceil p-1+\frac{1-q}{r}\rceil \leq 2(p-1)$. Either case contradicts the fact that $\text{dist}_K(A,B)=2p$. Therefore $rp-r+1 \leq s \leq k-rp+r-1$. \square

Lemma 2.5. *Let k and r be positive integers with $k \geq 2$ and $1 \leq r < k-1$, and let $p \geq 1$, such that $2p+1 < D$ or r divides $k-1$. Two vertices A and B in $K_{=2p+1}(2k+r,k)$ are adjacent if and only if $rp-r+1 \leq s \leq rp$.*

Proof. Trivially, $rp-r+1 \leq rp$, also by Remark 2.2, we have that $x \geq 2p$. Assume that $s=rp-r+1+q$ with $0 \leq q < r$. On the one hand, $2\lceil\frac{k-s}{r}\rceil=2\lceil\frac{xr+y-rp+r-q}{r}\rceil=2\lceil x-p+1+\frac{y-q}{r}\rceil \geq 2(p+1)$ because $-1 < \frac{y-q}{r} < 1$. On the other hand, $2\lceil\frac{s}{r}\rceil+1=2\lceil\frac{rp-r+1+q}{r}\rceil+1=2\lceil p-1+\frac{1+q}{r}\rceil+1=2p+1$. Therefore, by Lemma 1.4, $\text{dist}_K(A,B)=2p+1$, and A is adjacent to B in $K_{=2p+1}(2k+r,k)$.

Conversely, let $\text{dist}_K(A,B)=2p+1$. On the one hand, suppose that $s=rp-r+1-q$ with $q > 0$. As $\text{dist}_K(A,B)$ is odd then by Lemma 1.4, it must be equal to $2\lceil\frac{s}{r}\rceil+1=2\lceil\frac{rp-r+1-q}{r}\rceil+1=2\lceil p-1+\frac{1-q}{r}\rceil+1 \leq 2(p-1)+1$. On the other hand, suppose that $s=rp+q$ with $q > 0$. Thus, $\text{dist}_K(A,B)=2\lceil\frac{rp+q}{r}\rceil+1=2\lceil p+\frac{q}{r}\rceil+1 \geq 2(p+1)+1$. Either case contradicts the fact that $\text{dist}_K(A,B)=2p+1$. Therefore $rp-r+1 \leq s \leq rp$. \square

Lemma 2.6. *Let k and r be positive integers with $k \geq 2$, $1 \leq r < k-1$ such that r does not divide $k-1$, and let $2p+1=D$ with $p \geq 1$. Two vertices A and B in $K_{=2p+1}(2k+r,k)$ are adjacent if and only if $rp-r+1 \leq s \leq k-rp-1$.*

Proof. Let $k-1=xr+y$ with $1 \leq y < r$ and notice $x+2=2p+1$. First, $rp-r+1 \leq k-rp-1$. In fact, $k-2rp+r-2=xr+y-2rp+r-1=(2p-1)r+y-2rp+r-1=y-1 \geq 0$. Moreover, the integer interval $[rp-r+1, k-rp-1]$ is composed of y integers.

Assume first that $s=rp-r+1+q$ with $0 \leq q \leq y-1$. Then we must prove that $2\lceil\frac{k-s}{r}\rceil \geq 2\lceil\frac{s}{r}\rceil+1$. Notice that, $2\lceil\frac{k-s}{r}\rceil=2\lceil\frac{xr+y-rp+r-q}{r}\rceil=2\lceil x-p+1+\frac{y-q}{r}\rceil=2\lceil p+\frac{y-q}{r}\rceil=2(p+1)$, because $0 < \frac{y-q}{r} < 1$. Moreover, $2\lceil\frac{s}{r}\rceil+1=2\lceil\frac{rp-r+1+q}{r}\rceil+1=2\lceil p-1+\frac{q+1}{r}\rceil+1=2p+1$ because $0 < \frac{q+1}{r} < 1$. Therefore, by Lemma 1.4, $\text{dist}_K(A,B)=2p+1$, and A is adjacent to B in $K_{=2p+1}(2k+r,k)$.

Conversely, let $\text{dist}_K(A,B)=2p+1$. On the one hand assume that $s=rp-r+1-q$ with $q \geq 1$. As $\text{dist}_K(A,B)$ is odd then by Lemma 1.4, $\text{dist}_K(A,B)=2\lceil\frac{s}{r}\rceil+1=2\lceil\frac{rp-r+1-q}{r}\rceil+1=2\lceil p-1+\frac{1-q}{r}\rceil+1=2(p-1)+1$ because $-1 < \frac{1-q}{r} \leq 0$. On the other hand, assume that $s=k-rp-1+q$ with $q \geq 1$. Then $2\lceil\frac{k-s}{r}\rceil < 2p+1$. In fact, $2\lceil\frac{k-s}{r}\rceil=2\lceil\frac{rp+1-q}{r}\rceil=2\lceil p+\frac{1-q}{r}\rceil=2p$ because $-1 < \frac{1-q}{r} \leq 0$. Either case contradicts the fact that $\text{dist}_K(A,B)=2p+1$. Therefore $rp-r+1 \leq s \leq k-rp-1$. \square

As a direct consequence of Theorem 1.10 we have the following result for $r=1$.

Corollary 2.7. *Let k be positive integer with $k \geq 2$ and $1 < d \leq k$ be a fixed integer. Two vertices A and B in $K_{=d}(2k+1,k)$ are adjacent if and only if*

$$s = \begin{cases} k - \frac{d}{2} & \text{if } d \text{ is even;} \\ \frac{d-1}{2} & \text{if } d \text{ is odd.} \end{cases}$$

3 Computing the distance function

In this section, we compute the distance function of two non adjacent vertices in the exact distance- d Kneser graph $K_{=d}(2k+r,k)$, where k, r and d are positive integers, with $r < k-1$. We assume

$d \geq 2$, as the exact distance-1 Kneser graph is exactly the original Kneser graph whose distance function was computed in [14]. By Remark 2.1, we obtain the following.

Remark 3.1. *Let k and r be positive integers with $r \geq k - 1$. If A and B are two non-adjacent vertices in $K_{=2}(2k + r, k)$ then $\text{dist}_{(K,2)}(A, B) = 2$.*

Let A and B be two vertices in $K_{=d}(2k + r, k)$. We denote by C the set $A \cap B$ and we denote by Z the set $[2k + r] \setminus C$. Notice that $|C| = s$ and $|Z| = r + s$. Thus, in this section, we use the following notation for sets A, B, C and Z : $A = \{c_1, \dots, c_s, a_1, \dots, a_{k-s}\}$, $B = \{c_1, \dots, c_s, b_1, \dots, b_{k-s}\}$, $C = \{c_1, \dots, c_s\}$, and $Z = \{z_1, \dots, z_{r+s}\}$.

3.1 Case $d = 2p$

The proof of Theorem 1.11 follows from Lemmas 3.2, 3.3, 3.4, 3.6 and Corollary 3.5 given below.

Lemma 3.2. *Let $2p = D$ and r does not divide $k - 1$. If A and B are two vertices in $K_{=2p}(2k + r, k)$ which are not adjacent, then $\text{dist}_{(K,2p)}(A, B) = 2$.*

Proof. As vertices A and B are not adjacent then $\text{dist}_{(K,2p)}(A, B) \geq 2$. Moreover, by Theorem 1.10 (Case (i).2), $s \leq rp - r$ or $s \geq k - rp + r$.

- (i) Let $0 \leq s \leq rp - r$. As $rp - r + 1 \leq k$, $rp \geq r - 1$ and $k - rp + r - 1 - s \geq 0$, we have the following vertex X in $K_{=2p}(2k + r, k)$:

$$X = \{c_1, \dots, c_s, a_1, \dots, a_{rp-r+1-s}, b_1, \dots, b_{k-rp+r-1-s}, z_1, \dots, z_s\}.$$

Clearly, $|A \cap X| = rp - r + 1$ and $|B \cap X| = k - rp + r - 1$. Thus, by Theorem 1.10 (Case (i).2), X is adjacent to A and to B in $K_{=2p}(2k + r, k)$. Therefore, $\text{dist}_{(K,2p)}(A, B) \leq 2$.

- (ii) Let $k - rp + r \leq s \leq k - 1$. As $rp - r + 1 \leq k - rp + r - 1$ and $|Z| = s + r > s \geq k - rp + r \geq rp - r + 2 > rp - r + 1$, we have the following vertex X in $K_{=2p}(2k + r, k)$:

$$X = \{c_1, \dots, c_{k-rp+r-1}, z_1, \dots, z_{rp-r+1}\}.$$

Clearly, $|A \cap X| = |B \cap X| = k - rp + r - 1$. Thus, by Theorem 1.10 (Case (i).2), X is adjacent to A and to B in $K_{=2p}(2k + r, k)$. Therefore, $\text{dist}_{(K,2p)}(A, B) \leq 2$. \square

Lemma 3.3. *Let $2p < D$ or r divides $k - 1$. Let A and B be two vertices in $K_{=2p}(2k + r, k)$ which are not adjacent. If $s \geq k - rp + r$ then $\text{dist}_{(K,2p)}(A, B) = 2$. Otherwise, if $0 \leq s < k - rp$ then $\text{dist}_{(K,2p)}(A, B) \leq \lceil \frac{k-s}{rp} \rceil$.*

Proof. By Theorem 1.10 (Case (i).1), two vertices A and B are adjacent if and only if $k - rp \leq s \leq k - rp + r - 1$. If $s \geq k - rp + r$ then as in the proof of Lemma 3.2 (Case (ii)), $\text{dist}_{(K,2p)}(A, B) = 2$. Thus, assume that $0 \leq s \leq k - rp - 1$. Let $k - rp - s = trp + q$, with $t \geq 0$ and $0 \leq q < rp$. We split the proof in two cases, either $t = 0$, or $t > 0$.

- $t = 0$. In this case, $k - rp - s < rp$, which implies $k - s < 2rp$ and therefore, $2rp + s - k > 0$. In order to prove that $2rp + s - k \leq r + s$, it suffices to prove that $k \geq 2rp - r$. Recall that $k - 1 = xr + y$ with $x \geq 1$ and $0 \leq y \leq r - 1$.
 - Assume r divides $k - 1$. In this case $D = x + 1$. So, if $x + 1 \geq 2p$ then $k \geq (2p - 1)r + 1 > 2pr - r$.

- Assume r does not divide $k - 1$. In this case, $D = x + 2$. By theorem hypothesis, $x + 2 > 2p$ and so, $k \geq (2p - 1)r + y + 1 > 2pr - r$ because $y + 1 \geq 2$.

Now we have the following vertex X in $K_{=2p}(2k + r, k)$:

$$X = \{c_1, \dots, c_s, a_1, \dots, a_{k-rp-s}, b_1, \dots, b_{k-rp-s}, z_1, \dots, z_{2rp+s-k}\}.$$

Clearly, $|X \cap A| = |X \cap B| = k - rp$ and thus, X is adjacent to both A and B .

- $t > 0$. Let $A = X_0$ and for $1 \leq i \leq t$, consider the following vertices in $K_{=2p}(2k + r, k)$:

$$X_i = \{c_1, \dots, c_s, a_1, \dots, a_{k-irp-s}, b_1, \dots, b_{irp}\}.$$

Notice that $|X_i \cap X_{i+1}| = s + (irp) + (k - (i + 1)rp - s) = k - rp$ and thus X_i is adjacent to X_{i+1} for $0 \leq i < t$. Moreover, X_i is not adjacent to X_j , for $j > i + 1$ or $j < i - 1$ because $|X_i \cap X_j| < k - rp$. Now, we consider two cases:

- rp divides $k - rp - s$.

In this case, $X_t = \{c_1, \dots, c_s, a_1, \dots, a_{rp}, b_1, \dots, b_{k-rp-s}\}$. Thus, $|X_t \cap B| = k - rp$ which implies that X_t and B are adjacent. Hence, $dist_{(K, 2p)}(A, B) \leq t + 1$.

- rp does not divide $k - rp - s$.

In this case, $X_t = \{c_1, \dots, c_s, a_1, \dots, a_{trp+q}, b_1, \dots, b_{k-rp-s-q}\}$. Thus, $|X_t \cap B| = k - rp - q < k - rp$ because $q > 0$, which implies that X_t and B are not adjacent.

Let's prove that $k - s - trp + rp - q \leq k - s$. As $t, q > 0$ then $rp \leq trp + q$ and so $k - s + rp - (trp + q) \leq k - s$. Now we have the following vertex:

$$X_{t+1} = \{c_1, \dots, c_s, a_1, \dots, a_q, a_{k-s-trp+1}, \dots, a_{k-s-trp+rp-q}, b_1, \dots, b_{trp+q}\}.$$

Notice that $|X_t \cap X_{t+1}| = s + trp + q = k - rp = |X_{t+1} \cap B|$ and thus, X_{t+1} is adjacent to both X_t and B . Finally, for this case, we obtain $dist_{(K, 2p)}(A, B) \leq t + 2$.

Therefore, $dist_{(K, 2p)}(A, B) \leq \lceil \frac{k-rp-s}{rp} \rceil + 1 = \lceil \frac{k-s}{rp} \rceil$. □

In order to obtain a lower bound for the distance function when d is even and, $d < D$ or r divides $k - 1$, we need the following results.

Lemma 3.4. *Let $d < D$ or r divides $k - 1$. Let A and B be two vertices in $K_{=2p}(2k + r, k)$. If there is a path of length ℓ from A to B , then $s \geq k - \ell rp$.*

Proof. The proof follows by induction. For the base case, if there is a path of length 1, then $s \geq k - rp$ by Theorem 1.10.

Assume there is a path of length ℓ from A to B , and let X be the vertex adjacent to B in such path. As there is a path of length $(\ell - 1)$ from A to X , by the inductive hypothesis, we have $|A \cap X| \geq k - (\ell - 1)rp$. On the other hand, as B and X are adjacent, we have $|X \cap B| \geq k - rp$. Then

$$\begin{aligned} s &= |A \cap B| \geq |A \cap B \cap X| \\ &\geq |A \cap X| - |X \cap \overline{B}| \\ &= |A \cap X| - |X| + |X \cap B| \\ &\geq k - (\ell - 1)rp - k + k - rp \\ &= k - \ell rp. \end{aligned}$$

Thus, the result follows by induction. □

Corollary 3.5. *Let $d < D$ or r divides $k - 1$. If A and B are two vertices in $K_{=d}(2k + r, k)$ then $dist_{(K,2p)}(A, B) \geq \frac{k-s}{rp}$.*

Proof. There is a path of length $dist_{(K,2p)}(A, B)$ from A to B . Thus, applying Lemma 3.4 with $\ell = dist_{(K,2p)}(A, B)$ yields,

$$|A \cap B| \geq k - dist_{(K,2p)}(A, B)rp.$$

Solving for $dist_{(K,2p)}(A, B)$ we get

$$dist_{(K,2p)}(A, B) \geq \frac{k - |A \cap B|}{rp}.$$

Thus, the result follows. \square

Lemma 3.6. *Let $d < D$ or r divides $k - 1$. Let A and B be two vertices in $K_{=d}(2k + r, k)$ which are not adjacent. If $0 \leq s < k - rp$ then $dist_{(K,2p)}(A, B) = \lceil \frac{k-s}{rp} \rceil$.*

Proof. The result follows from Lemma 3.3 and Corollary 3.5. \square

To complete the proof of Theorem 1.11, we must show that $\lceil \frac{k-|A \cap B|}{rp} \rceil \leq 2$ whenever $dist_{(K,2p)}(A, B) = 2$. This is split in two cases. Namely, the case when $|A \cap B| \geq k - rp + r$ and the case when $d = D$. First, when $|A \cap B| \geq k - rp + r$, we have

$$\begin{aligned} \frac{k - |A \cap B|}{rp} &\leq \frac{rp - r}{rp} \\ &\leq 1. \end{aligned}$$

Thus, $\lceil \frac{k-|A \cap B|}{rp} \rceil \leq 1 < 2$.

When $d = D$ we have

$$p = \frac{D}{2} = \frac{1}{2} \left(\left\lceil \frac{k-1}{r} \right\rceil + 1 \right).$$

Thus,

$$\begin{aligned} \frac{k - |A \cap B|}{rp} &= 2 \left(\frac{k - |A \cap B|}{r \lceil \frac{k-1}{r} \rceil + r} \right) \\ &\leq 2 \left(\frac{k}{r \left(\frac{k-1}{r} \right) + r} \right) \\ &\leq 2 \left(\frac{k}{k} \right) \\ &= 2. \end{aligned}$$

Similarly, this implies $\lceil \frac{k-|A \cap B|}{rp} \rceil \leq 2$.

3.2 Case $d = 2p + 1$

The proof of Theorem 1.12 follows from Lemmas 3.7, 3.8, 3.11 and 3.12, and Corollary 3.10 given below.

Lemma 3.7. *Let A and B be two vertices in $K_{=2p+1}(2k+r, k)$ which are not adjacent. If $2p+1 = D$ and r does not divide $k - 1$, then $dist_{(K,2p+1)}(A, B) = 2$.*

Proof. As vertices A and B are not adjacent, $\text{dist}_{(K,2p+1)}(A, B) \geq 2$. Moreover, by Theorem 1.10 (Case (ii).2), $|A \cap B| \leq rp - r$ or $|A \cap B| \geq k - rp$. As r does not divide $k - 1$ then $k - 1 = xr + y$, with $0 < y < r$ and $D = x + 2$. So, in this case, $x + 2 = 2p + 1$ which implies that $x = 2p - 1$ and therefore, $k = 2rp - r + y + 1$.

- (i) Case $0 \leq |A \cap B| \leq rp - r$. As $r, y > 0$, $s \leq rp - r$, and $k - rp - 1 - s \geq rp - r + y - s \geq y$, the quantities $rp - r + 1 - s$ and $k - rp - 1 - s$ are both positive. Thus, we can consider the vertex X in $K_{=2p+1}(2k + r, k)$:

$$X = \{a_1, \dots, a_{rp-r+1-s}, b_1, \dots, b_{k-rp-1-s}, z_1, \dots, z_{s+r}\} \cup C.$$

Clearly, $|A \cap X| = rp - r + 1$ and $|B \cap X| = k - rp - 1$. Hence, by Theorem 1.10 (Case (ii).2), X is adjacent to A and to B in $K_{=2p+1}(2k + r, k)$. Therefore, $\text{dist}_{(K,2p+1)}(A, B) \leq 2$.

- (ii) Case $k - rp \leq |A \cap B| \leq k - 1$. As $k = 2rp - r + y + 1$, we have $0 < k - rp - 1 < s$ and $s + r > rp + 1$. Thus, we can consider the following vertex X in $K_{=2p+1}(2k + r, k)$:

$$X = \{c_1, \dots, c_{k-rp-1}, z_1, \dots, z_{rp+1}\}.$$

Clearly, $|A \cap X| = |B \cap X| = k - rp - 1$. Hence, by Theorem 1.10 (Case (ii).2), X is adjacent to A and to B in $K_{=2p+1}(2k + r, k)$. Therefore, $\text{dist}_{(K,2p+1)}(A, B) \leq 2$. \square

Lemma 3.8. *Let A and B be two vertices in $K_{=2p+1}(2k + r, k)$ which are not adjacent. If $2p + 1 = D$ and r divides $k - 1$, then $\text{dist}_{(K,2p+1)}(A, B) = 2$.*

Proof. As vertices A and B are not adjacent then $\text{dist}_{(K,2p+1)}(A, B) \geq 2$. Moreover, by Theorem 1.10 (Case (ii).1), $s \leq rp - r$ or $s \geq rp + 1$. Recall that in this case, $D = x + 1$ and thus $k = 2rp + 1$.

- (i) Case $0 \leq s \leq rp - r$. This case is similar to the Case (i) in Lemma 3.7.
- (ii) Case $rp + 1 \leq s \leq k - 1$. Notice that $k - rp = rp + 1 < s + r$. Thus, we can consider the following vertex X in $K_{=2p+1}(2k + r, k)$:

$$X = \{c_1, \dots, c_{rp}, z_1, \dots, z_{k-rp}\}.$$

Clearly, $|A \cap X| = |B \cap X| = rp$. Hence, by Theorem 1.10 (Case (ii).1), X is adjacent to A and to B in $K_{=2p+1}(2k + r, k)$. Therefore, $\text{dist}_{(K,2p+1)}(A, B) \leq 2$. \square

Lemma 3.9. *Let $d < D$ and let A and B be two vertices in $K_{=d}(2k + r, k)$. If $s \leq rp - r$, then*

$$\text{dist}_{(K,2p+1)}(A, B) = \begin{cases} 2 & \text{if } s \geq k - 2rp - r; \\ 3 & \text{if } s < k - 2rp - r. \end{cases}$$

Proof. (i) Case $s \geq k - 2rp - r$. As $s \leq rp - r$, by Theorem 1.10, A and B are not adjacent. Thus, for this case it is enough to show that $\text{dist}_{(K,2p+1)}(A, B) \leq 2$. Notice that the set

$$X = \{a_1, \dots, a_{rp}, b_1, \dots, b_{rp}, z_1, \dots, z_{k-2rp}\}$$

is a vertex in $K_{=2p+1}(2k + r, k)$ because $k > 2rp$ when $d < D$. Clearly, $|A \cap X| = |X \cap B| = rp$ and, by Theorem 1.10 (Case (ii).1), X is adjacent to A and to B in $K_{=2p+1}(2k + r + k)$. Therefore, $\text{dist}_{(K,2p+1)}(A, B) = 2$.

(ii) Case $s < k - 2rp - r$. The proof of this case is split in two parts. First we show that $\text{dist}_{(K, 2p+1)}(A, B) \leq 3$. As $rp - r + 1 - s > 0$ because $s \leq rp - r$ we can consider the following vertices X_1, X_2 in $K_{=2p+1}(2k + r, k)$:

$$X_1 = \{c_1, \dots, c_s, a_1, \dots, a_{rp-r+1-s}, b_1, \dots, b_{k-rp+r-1}\},$$

$$X_2 = \{c_1, \dots, c_s, a_{rp-r+1-s+1}, \dots, a_{k-s}, b_1, \dots, b_{rp-r+1-s}\}.$$

Notice that $|X_1 \cap X_2| = |A \cap X_1| = |X_2 \cap B| = rp - r + 1$. Thus, by Theorem 1.10, X_1 is adjacent to A and to X_2 , and B is adjacent to X_2 . Hence, $\text{dist}_{(K, 2p+1)}(A, B) \leq 3$.

To complete the proof, we need to show that no vertex Y can be adjacent to both A and B . As $|A \cap B| < k - r - 2rp$, we have $|Z| = r + s < k - 2rp$. Thus, if $|Y| = k$, then $|Y \cap (A \cup B)| \geq |Y| - |Z| \geq 2rp + 1$. This implies that $\max\{|Y \cap A|, |Y \cap B|\} \geq rp + 1$. Hence, by Theorem 1.10, Y cannot be simultaneously adjacent to A and to B . Therefore, $\text{dist}_{(K, 2p+1)}(A, B) \geq 3$. \square

Corollary 3.10. *Let A and B be two vertices in $K_{=d}(2k + r, k)$. If $d = D$ or $s \leq rp - r$, then*

$$\text{dist}_{(K, 2p+1)}(A, B) = \min \left\{ 1 + 2 \left\lceil \frac{|s - rp|}{2rp + r} \right\rceil, 2 \left\lceil \frac{k - s}{2rp + r} \right\rceil \right\}.$$

Proof. Let $m = \min \left\{ 1 + 2 \left\lceil \frac{|s - rp|}{2rp + r} \right\rceil, 2 \left\lceil \frac{k - s}{2rp + r} \right\rceil \right\}$. By Lemma 3.9, we need to show that

$$m = \begin{cases} 2 & (i) \text{ if } d = D, \\ & (ii) \text{ if } d < D, k - 2rp - r \leq s \leq rp - r; \\ 3 & \text{if } d < D, s \leq rp - r, \text{ and } s < k - 2rp - r. \end{cases}$$

Notice that $s \notin \{rp, k\}$, as A and B are not adjacent and $A \neq B$. This implies that

$$1 + 2 \left\lceil \frac{|s - rp|}{2rp + r} \right\rceil \geq 3,$$

and that

$$2 \left\lceil \frac{k - s}{2rp + r} \right\rceil \geq 2.$$

Thus, in order to prove that $m = 2$, it suffices to show that $\left\lceil \frac{k - s}{2rp + r} \right\rceil \leq 1$.

When $d = D$, we have $2p + 1 = \left\lceil \frac{k-1}{r} \right\rceil + 1$. Then

$$p = \frac{1}{2} \left\lceil \frac{k-1}{r} \right\rceil \geq \frac{k-1}{2r}.$$

Thus,

$$\begin{aligned} \left\lceil \frac{k - s}{2rp + r} \right\rceil &\leq \left\lceil \frac{k}{k - 1 + r} \right\rceil \\ &= 1. \end{aligned}$$

When $d < D$ and $k - 2rp - r \leq s \leq rp - r$,

$$\begin{aligned} \left\lceil \frac{k-s}{2rp+r} \right\rceil &\leq \left\lceil \frac{k-k+2rp+r}{2rp+r} \right\rceil \\ &=1. \end{aligned}$$

Thus, in these cases $m = 2$.

Assume now that $d < D$, $s < k - 2rp - r$ and $s \leq rp - r$. Notice that

$$\begin{aligned} \frac{k-s}{2rp+r} &\geq \frac{k-k+2rp+r+1}{2rp+r} \\ &=1 + \frac{1}{2rp+r}. \end{aligned}$$

Thus,

$$\begin{aligned} 2 \left\lceil \frac{k-s}{2rp+r} \right\rceil &\geq 2 \left\lceil 1 + \frac{1}{2rp+r} \right\rceil \\ &=2 \cdot 2 \\ &=4. \end{aligned}$$

But, as $s \leq rp - r$,

$$\begin{aligned} 1 + 2 \left\lceil \frac{|s-rp|}{2rp+r} \right\rceil &=1 + 2 \left\lceil \frac{rp-s}{2rp+r} \right\rceil \\ &\leq 1 + 2 \left\lceil \frac{rp}{2rp+r} \right\rceil \\ &\leq 1 + 2 \\ &=3. \end{aligned}$$

Therefore, in this case, $m = 3$. □

For the remaining cases, we first show the existence of a path of a certain length between A and B to give a lower bound for $|A \cap B|$. Then, we use this bound to find the actual distance.

Lemma 3.11. *Let $d < D$. If there is a path of length 2ℓ from A to B , then $|A \cap B| \geq k - 2\ell r p - r\ell$. If there is a path of length $2\ell + 1$ from A to B , then $|A \cap B| \leq (2\ell + 1)rp + r\ell$.*

Proof. The proof follows by induction. For the base case, when $\ell = 0$, if there is a path of length $1 = 2 \cdot 0 + 1$, then $|A \cap B| \leq (2\ell + 1)rp + r\ell$ by Theorem 1.10.

Assume there is a path of length 2ℓ from A to B , and let X be the vertex adjacent to B in such path. As there is a path of length $2(\ell - 1) + 1$ from A to X , by the inductive hypothesis, we have

$$\begin{aligned} |A \cap X| &\leq [2(\ell - 1) + 1]rp + r(\ell - 1) \\ &=(2\ell - 1)rp + r(\ell - 1). \end{aligned}$$

Thus, we have an upper bound for the size of the complement of $A \cup X$,

$$\begin{aligned} |\overline{A \cup X}| &=2k + r - |A| - |X| + |A \cap X| \\ &\leq 2k + r - k - k + (2\ell - 1)rp + r(\ell - 1) \\ &=(2\ell - 1)rp + r\ell. \end{aligned}$$

On the other hand, as B and X are adjacent, by Theorem 1.10 we have

$$rp - r + 1 \leq |B \cap X| \leq rp.$$

Hence,

$$\begin{aligned} |B \cap \bar{A}| &= |B \cap (\overline{A \cup X})| + |B \cap \bar{A} \cap X| \\ &\leq |\overline{A \cup X}| + |B \cap X| \\ &\leq (2\ell - 1)rp + r\ell + rp \\ &= 2\ell rp + r\ell. \end{aligned}$$

Thus,

$$\begin{aligned} |B \cap A| &= k - |B \cap \bar{A}| \\ &\geq k - 2\ell rp - r\ell. \end{aligned}$$

Which proves the even case.

For the odd case, assume there is a path of length $2\ell + 1$ from A to B , and let X be the vertex adjacent to B in such path. As there is a path of length 2ℓ from A to X , by the inductive hypothesis, we have

$$|A \cap X| \geq k - 2\ell rp - r\ell.$$

Thus, we have an upper bound for the size of $A \cap \bar{X}$,

$$\begin{aligned} |A \cap \bar{X}| &\leq k - (k - 2\ell rp - r\ell) \\ &= 2\ell rp + r\ell. \end{aligned}$$

On the other hand, as B and X are adjacent, by Theorem 1.10 we have

$$rp - r + 1 \leq |B \cap X| \leq rp.$$

Hence

$$\begin{aligned} |A \cap B| &= |A \cap B \cap \bar{X}| + |A \cap B \cap X| \\ &\leq |A \cap \bar{X}| + |B \cap X| \\ &\leq 2\ell rp + r\ell + rp \\ &= (2\ell + 1)rp + r\ell. \end{aligned}$$

Therefore, the result follows by induction. □

By Lemma 3.11, if $s \geq rp + 1$, then $\text{dist}_{(K, 2p+1)}(A, B)$ is given by

$$\min(\{2\ell + 1 \mid (2\ell + 1)rp + r\ell \geq s\} \cup \{2\ell \mid k - 2\ell rp - r\ell \leq s\}).$$

Assume $d = \min\{\ell \mid (2\ell + 1)rp + r\ell \geq s\}$. Then

$$(2(d - 1) + 1)rp + r(d - 1) < s \leq (2d + 1)rp + rd.$$

Which yields

$$\frac{s - rp}{2rp + r} \leq d < \frac{s + rp + r}{2rp + r}.$$

But, as d is an integer,

$$\begin{aligned} \left\lceil \frac{s - rp}{2rp + r} \right\rceil &\leq d < \left\lceil \frac{s + rp + r}{2rp + r} \right\rceil \\ \left\lceil \frac{s - rp}{2rp + r} \right\rceil &\leq d < \left\lceil \frac{s - rp + 2rp + r}{2rp + r} \right\rceil \\ \left\lceil \frac{s - rp}{2rp + r} \right\rceil &\leq d < \left\lceil \frac{s - rp}{2rp + r} + 1 \right\rceil \\ \left\lceil \frac{s - rp}{2rp + r} \right\rceil &\leq d < \left\lceil \frac{s - rp}{2rp + r} \right\rceil + 1. \end{aligned}$$

Which implies, $d = \left\lceil \frac{s - rp}{2rp + r} \right\rceil$, and $2d + 1 = 2 \left\lceil \frac{s - rp}{2rp + r} \right\rceil + 1$.

Similarly, if $d = \min\{\ell \mid k - 2\ell rp - r\ell \leq s\}$, we have

$$k - 2d rp - rd \leq s < k - 2(d - 1)rp - r(d - 1),$$

which, solving for d , implies

$$\frac{k - s}{2rp + r} \leq d < \frac{k - s}{2rp + r} + 1.$$

Thus, $d = \left\lceil \frac{k - s}{2rp + r} \right\rceil$, and $2d = 2 \left\lceil \frac{k - s}{2rp + r} \right\rceil$. Hence, the distance between A and B is

$$\text{dist}_{(K, 2p+1)}(A, B) = \min \left\{ 1 + 2 \left\lceil \frac{s - rp}{2rp + r} \right\rceil, 2 \left\lceil \frac{k - s}{2rp + r} \right\rceil \right\}.$$

Therefore, we obtain the following result.

Lemma 3.12. *Let $d < D$ and let A and B be two vertices in $K_{=d}(2k + r, k)$. Let $s \geq rp + 1$. Then*

$$\text{dist}_{(K, 2p+1)}(A, B) = \min \left\{ 1 + 2 \left\lceil \frac{s - rp}{2rp + r} \right\rceil, 2 \left\lceil \frac{k - s}{2rp + r} \right\rceil \right\}.$$

Proof. The result follows from the preceding discussion. \square

4 Computing the diameter

Notice first that, by Remark 3.1, we have the following result.

Remark 4.1. *If k and r are positive integers with $k \geq 2$ and $r \geq k - 1$, then, $\text{diam}(K_{=2}(2k + r, k)) = 2$.*

We can now proceed with the proof of Theorem 1.13.

Theorem 1.13. *Let k and r be positive integers with $k \geq 2$ and $1 \leq r < k - 1$. Let D be the diameter of the Kneser graph $K(2k + r, k)$ and let $1 < d \leq D$. The diameter of the exact distance- d Kneser graph $K_{=d}(2k + r, k)$ is given by*

$$\text{diam}(K_{=d}(2k + r, k)) = \begin{cases} 2 & \begin{array}{l} \text{(i) if } d = D, r \nmid k - 1; \\ \text{(ii) if } d = 2p + 1 = D, r \mid k - 1; \end{array} \\ \left\lceil \frac{k}{rp} \right\rceil & \begin{array}{l} \text{(iii) if } d = 2p < D; \\ \text{(iv) if } d = 2p = D, r \mid k - 1; \end{array} \\ \left\lceil \frac{k+(r/2)}{2rp+r} \right\rceil & \text{(v) if } d = 2p + 1 < D, 2 \mid k - (r/2); \\ 1 + \left\lceil \frac{k+(r/2)-1}{2rp+r} \right\rceil & \text{(vi) if } d = 2p + 1 < D, 2 \nmid k - (r/2). \end{cases}$$

Proof. Cases (i) and (ii) follow from Lemma 3.2, 3.7, and 3.8. Cases (iii) and (iv) follow from Lemma 3.6 by having $A \cap B = \emptyset$.

For case (v), assume $d = 2p + 1 < D$. By Lemma 3.12 we have

$$\text{diam}(K_{=d}(2k + r, k)) = \max_s \min \left\{ 1 + 2 \left\lceil \frac{s - rp}{2rp + r} \right\rceil, 2 \left\lceil \frac{k - s}{2rp + r} \right\rceil \right\}.$$

Let A and B be two vertices in $K_{=d}(2k + r, k)$, and notice

$$\text{dist}_{(K,d)}(A, B) \leq 1 + 2 \left\lceil \frac{s - rp}{2rp + r} \right\rceil < 3 + 2 \left(\frac{s - rp}{2rp + r} \right)$$

and

$$\text{dist}_{(K,d)}(A, B) \leq 2 \left\lceil \frac{k - s}{2rp + r} \right\rceil < 2 + 2 \left(\frac{k - s}{2rp + r} \right).$$

We find the value of s that maximizes the distance by doing

$$\begin{aligned} 3 + 2 \left(\frac{s - rp}{2rp + r} \right) &= 2 + 2 \left(\frac{k - s}{2rp + r} \right) \\ 2s &= k + rp - \frac{2rp + r}{2} \\ s &= \frac{k - (r/2)}{2}. \end{aligned}$$

In the case when $\frac{k - (r/2)}{2}$ is an integer, it is a valid value to take for s . In such a case $\text{diam}(K_{=d}(2k + r, k))$ is given by

$$\min \left\{ 1 + 2 \left\lceil \frac{(k - (r/2))/2 - rp}{2rp + r} \right\rceil, 2 \left\lceil \frac{k - (k - (r/2))/2}{2rp + r} \right\rceil \right\}.$$

Letting $k = (4rp + 2r)a + b$, with $0 \leq b < 4rp + 2r$

$$\begin{aligned} 2 \left\lceil \frac{k - (k - (r/2))/2}{2rp + r} \right\rceil &= 2 \left\lceil \frac{(k + (r/2))/2}{2rp + r} \right\rceil \\ &= 2a + 2 \left\lceil \frac{b + (r/2)}{4rp + 2r} \right\rceil \\ &= \begin{cases} 2a + 2 & \text{if } b \leq 4rp + r + (r/2) \\ 2a + 4 & \text{if } b > 4rp + r + (r/2), \end{cases} \end{aligned}$$

and

$$\begin{aligned} 1 + 2 \left\lceil \frac{(k - (r/2))/2 - rp}{2rp + r} \right\rceil &= 1 + 2 \left\lceil \frac{k - (r/2) - 2rp}{4rp + 2r} \right\rceil \\ &= 1 + 2(a - 1) + 2 \left\lceil \frac{4rp + 2r + b - (r/2) - 2rp}{4rp + 2r} \right\rceil \\ &= 1 + 2(a - 1) + 2 \left\lceil \frac{2rp + b + (3r/2)}{4rp + 2r} \right\rceil \\ &= \begin{cases} 2a + 1 & \text{if } b \leq 2rp + (r/2) \\ 2a + 3 & \text{if } b > 2rp + (r/2). \end{cases} \end{aligned}$$

Thus, when $\frac{k - (r/2)}{2}$ is an integer, we have

$$\text{diam}(K_{=d}(2k + r, k)) = 2a + \begin{cases} 1 & \text{if } b \leq 2rp + \lfloor (r/2) \rfloor \\ 2 & \text{if } 2rp + \lfloor (r/2) \rfloor < b \leq 4rp + r + \lfloor (r/2) \rfloor \\ 3 & \text{if } 4rp + r + \lfloor (r/2) \rfloor < b. \end{cases}$$

Notice that in this case

$$\begin{aligned} \left\lceil \frac{k + (r/2)}{2rp + r} \right\rceil &= \left\lceil \frac{2a(2rp + r) + b + (r/2)}{2rp + r} \right\rceil \\ &= 2a + \left\lceil \frac{b + (r/2)}{2rp + r} \right\rceil \\ &= 2a + \begin{cases} 1 & \text{if } b \leq 2rp + \lfloor (r/2) \rfloor \\ 2 & \text{if } 2rp + \lfloor (r/2) \rfloor < b \leq 4rp + r + \lfloor (r/2) \rfloor \\ 3 & \text{if } 4rp + r + \lfloor (r/2) \rfloor < b. \end{cases} \end{aligned}$$

Thus, when $2p + 1 = d < D$ and $\frac{k - (r/2)}{2}$ is an integer,

$$\text{diam}(K_{=d}(2k + r, k)) = \left\lceil \frac{k + (r/2)}{2rp + r} \right\rceil.$$

For case (vi), when $\frac{k - (r/2)}{2}$ is not an integer, notice that $1 + 2 \left\lceil \frac{s - rp}{2rp + r} \right\rceil$ and $2 \left\lceil \frac{k - s}{2rp + r} \right\rceil$ are a non-decreasing and a non-increasing function of s , respectively. Thus, for $1 + 2 \left\lceil \frac{s - rp}{2rp + r} \right\rceil$ we consider the value $s = \frac{k - (r/2) - 1}{2}$ and for $2 \left\lceil \frac{k - s}{2rp + r} \right\rceil$ the value $s = \frac{k - (r/2) + 1}{2}$. As the diameter is the maximum

of the distances, in this case $diam(K_{=d}(2k+r, k))$ is given by

$$\max \left\{ 1 + 2 \left\lceil \frac{(k - (r/2) - 1)/2 - rp}{2rp + r} \right\rceil, 2 \left\lceil \frac{k - (k - (r/2) + 1)/2}{2rp + r} \right\rceil \right\}.$$

Letting $k = (4rp + 2r)a + 2b + \xi$, with $0 \leq 2b + \xi < 4rp + 2r$ and $0 \leq \xi \leq 1$ and $r = 4q + 2t + \zeta$, with $0 \leq t, \zeta \leq 1$ and $q = \lfloor \frac{r}{4} \rfloor$, we have

$$\begin{aligned} \left\lceil \frac{(k - (r/2) - 1)/2 - rp}{2rp + r} \right\rceil &= \left\lceil \frac{(2rp + r)a + b - q - rp + (\xi - t - (\zeta/2) - 1)/2}{2rp + r} \right\rceil \\ &= (a - 1) + \left\lceil \frac{b + 2rp + 4q + 2t + \zeta - q - rp + (\xi - t - (\zeta/2) - 1)/2}{2rp + r} \right\rceil \\ &= (a - 1) + \left\lceil \frac{b + rp + 3q + 2t + \zeta + (\xi - t - (\zeta/2) - 1)/2}{2rp + r} \right\rceil. \end{aligned}$$

Thus,

$$\begin{aligned} 1 + 2 \left\lceil \frac{(k - (r/2) - 1)/2 - rp}{2rp + r} \right\rceil &= 1 + 2(a - 1) + 2 \left\lceil \frac{b + rp + 3q + 2t + \zeta + (\xi - t - (\zeta/2) - 1)/2}{2rp + r} \right\rceil \\ &= \begin{cases} 2a + 1 & \text{if } b \leq rp + q - (\xi - t - (\zeta/2) - 1)/2 \\ 2a + 3 & \text{if } b > rp + q - (\xi - t - (\zeta/2) - 1)/2 \end{cases} \end{aligned}$$

and

$$\begin{aligned} 2 \left\lceil \frac{k - (k - (r/2) + 1)/2}{2rp + r} \right\rceil &= 2 \left\lceil \frac{(4rp + 2r)a + 2b + \xi - ((4rp + 2r)a + 2b + \xi - 2q - t - (\zeta/2) + 1)/2}{2rp + r} \right\rceil \\ &= 2 \left\lceil \frac{(2rp + r)a + b + \xi + q - (\xi - t - (\zeta/2) + 1)/2}{2rp + r} \right\rceil \\ &= 2a + 2 \left\lceil \frac{b + q + (\xi + t + (\zeta/2) - 1)/2}{2rp + r} \right\rceil \\ &= \begin{cases} 2a + 2 & \text{if } b \leq 2rp + 3q + 2t + \zeta - (\xi + t + (\zeta/2) - 1)/2 \\ 2a + 4 & \text{if } b > 2rp + 3q + 2t + \zeta - (\xi + t + (\zeta/2) - 1)/2. \end{cases} \end{aligned}$$

Thus,

$$diam(K_{=d}(2k+r, k)) = \begin{cases} 2a + 2 & \text{if } b \leq rp + q - \frac{\xi}{2} + \frac{t}{2} + \frac{\zeta}{4} + \frac{1}{2} \\ 2a + 3 & \text{if } rp + q - \frac{\xi}{2} + \frac{t}{2} + \frac{\zeta}{4} + \frac{1}{2} < b \leq 2rp + 3q + 2t - \frac{\xi}{2} - \frac{t}{2} + \frac{3\zeta}{4} + \frac{1}{2} \\ 2a + 4 & \text{if } b > 2rp + 3q + 2t - \frac{\xi}{2} - \frac{t}{2} + \frac{3\zeta}{4} + \frac{1}{2}. \end{cases}$$

But notice that

$$\begin{aligned}
1 + \left\lceil \frac{k + (r/2) - 1}{2rp + r} \right\rceil &= 1 + \left\lceil \frac{(2rp + r)2a + 2b + \xi + 2q + t + (\zeta/2) - 1}{2rp + r} \right\rceil \\
&= 1 + 2a + \left\lceil \frac{2b + \xi + 2q + t + (\zeta/2) - 1}{2rp + r} \right\rceil \\
&= 2a + \begin{cases} 2 & \text{if } 2b \leq 2rp + 2q + t + (\zeta/2) - \xi + 1 \\ 3 & \text{if } 2rp + 2q + t + (\zeta/2) - \xi + 1 < 2b \leq 4rp + 6q + 3t + 3(\zeta/2) - \xi + 1 \\ 4 & \text{if } 2b > 4rp + 6q + 3t + 3(\zeta/2) - \xi + 1 \end{cases} \\
&= 2a + \begin{cases} 2 & \text{if } b \leq rp + q - \frac{\xi}{2} + \frac{t}{2} + \frac{\zeta}{4} + \frac{1}{2} \\ 3 & \text{if } rp + q - \frac{\xi}{2} + \frac{t}{2} + \frac{\zeta}{4} + \frac{1}{2} < b \\ & \text{and } b \leq 2rp + 3q + 2t - \frac{\xi}{2} - \frac{t}{2} + \frac{3\zeta}{4} + \frac{1}{2} \\ 4 & \text{if } b > 2rp + 3q + 2t - \frac{\xi}{2} - \frac{t}{2} + \frac{3\zeta}{4} + \frac{1}{2}. \end{cases}
\end{aligned}$$

Thus, in this case,

$$diam(K_{=d}(2k + r, k)) = 1 + \left\lceil \frac{k + (r/2) - 1}{2rp + r} \right\rceil.$$

□

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