THE SHARP C^0-FRAGMENTATION PROPERTY FOR HAMILTONIAN DIFFEOMORPHISMS AND HOMEOMORPHISMS ON SURFACES

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Abstract. In this paper, we present a C^0-fragmentation property for Hamiltonian diffeomorphisms. More precisely, it is known that for a given open covering $\mathcal{U}$ of a compact symplectic surface we can write each C^0-small enough Hamiltonian diffeomorphism as the composition of Hamiltonian diffeomorphisms compactly supported inside the open sets of the covering $\mathcal{U}$. We show that such a decomposition can be done with a Lipschitz estimate on the C^0-norm of the fragments. We also show the same property for the kernel of θ, the mass-flow homomorphism for homeomorphisms. This answers a question from Buhovsky and Seyfaddini.

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1. Introduction and main results

1.1. The C^0-fragmentation property on surfaces.

1.1.1. *The C^0-fragmentation for the group of Hamiltonian diffeomorphisms.* The fragmentation property on a given manifold M allows to decompose diffeomorphisms (or homeomorphisms) of various kinds into a composition of diffeomorphisms supported in small balls. The more refined C^0-fragmentation property provides a control on the size of the fragments. We will be interested in fragmenting elements of the group of Hamiltonian diffeomorphisms on a surface Σ. We denote $\mathrm{Ham}(X, \omega)$ the group of Hamiltonian diffeomorphisms on a symplectic manifold (X, ω). We also denote $\mathrm{Ham}_c(X, \omega)$ the group of compactly supported Hamiltonian diffeomorphisms on (X, ω). We will sometimes drop the symplectic form ω in the notation if it is clear which symplectic form is used.

One motivation for having C^0-fragmentation is the following corollary:

Corollary 1. *Let (Σ, ω) be a closed symplectic surface and d a distance induced by some Riemannian metric. There exists a constant $C > 0$, such that for all ϕ in the group of Hamiltonian* *diffeomorphisms, there exists* $\{\phi_t\}$ *a Hamiltonian isotopy such that* $\phi_0 = Id$, $\phi_1 = \phi$, *that satisfies the following estimate for all* $t \in [0, 1]$*:*

$$\|\phi_t\|_{C^0} \leq C\|\phi\|_{C^0}.$$

The fragmentation property has been introduced in [Thu], [Ban78] and [Ban97] by Thurston and Banyaga in order to study the simplicity and perfectness of the groups of diffeomorphisms preserving a symplectic form or preserving a volume form. Later, in his study of the structure of the group of homeomorphisms preserving a full nonatomic measure, Fathi proved in [Fa80] a similar fragmentation property for measure-preserving homeomorphisms. This led to the proof of the simplicity of the group of compactly supported measure-preserving homeomorphism of the ball of dimension n whenever $n \geq 3$. The case of the dimension 2 turns out to be much harder. Only recently Cristofaro-Gardiner, Humilière and Seyfaddini proved that there exists a normal subgroup of the group of area-preserving homeomorphisms of the disk in [CHS]. In earlier work ([LeR10]), Le Roux showed that the simplicity of the group of area-preserving homeomorphism of the disk is equivalent to the existence of some fragmentation property for homeomorphisms. After Le Roux's work Entov, Polterovich and Py and then Seyfaddini proved quantitative versions of the fragmentation property of C^0-small Hamiltonian diffeomorphisms preserving a volume form in the 2 dimensional case, see [EPP12, Section 1.6.2] and [Sey13, Proposition 3.1]. On the one hand, the goal in [EPP12] was to construct quasi-morphisms on the group of diffeomorphisms, thus understanding better the algebraic properties of this group and on the other hand, Seyfaddini gave a Hölder-type bound on the C^0-norm of the fragments in order to show the C^0-continuity of the Oh-Swharz spectral invariants. In the present paper we adapt their proof to show the sharpest C^0-fragmentation possible, that is a Lipschitz bound on the C^0-norm of the fragments. This is the content of our first theorem.

Theorem 1. *Let* (Σ, ω) *be a closed surface equipped with an area form* ω *and let* d *be a distance induced by some Riemannian metric. Let* $\mathcal{W} = (W_i)_{i=1}^m$ *be a finite open covering of the surface* Σ *by disks. Then there exists a* C^0*-neighborhood* N *of the identity in the group* $\mathrm{Ham}(\Sigma)$ *of Hamiltonian diffeomorphisms such that for each* $\phi \in N$*, we can decompose* ϕ *as:*

$$\phi = \phi_1 \circ \phi_2 \circ \cdots \circ \phi_m,$$

where for all $1 \leq i \leq m$, ϕ_i *belongs to* $\mathrm{Ham}_c(W_i)$*. Moreover, we have the following estimate for all* $1 \leq i \leq m$*,*

$$\|\phi_i\|_{C^0} \leq C\|\phi\|_{C^0},$$

for some constant $C > 0$ *independent of* ϕ*.*

Remark 1.1. *Since any symplectic diffeomorphism supported inside a disk is a Hamiltonian diffeomorphism, the fragmentation property for elements in* $\mathrm{Symp}_0(\Sigma)$ *cannot be true.*

Remark 1.2. *The result proved by Seyfaddini in* [Sey13, Proposition 3.1] *is, for a very specific covering given by the handle decomposition of the surface, the estimate:*

$$\|\phi_i\|_{C^0} \leq C \, \|\phi\|_{C^0}^{2^{1-N}},$$

where $N = 2g + 2$ *depends on the genus* g *of the surface.*

The fragmentation property stated in Theorem 1 relies on Lemma 1.3, which is an improvement of the extension lemma in [Sey13, Section 3.4.2].

In the following lemma and for the rest of the paper we will denote by $\mathbb{A}_y$ the annulus $S^1 \times [-y, y]$ for y a positive real number and we will also denote $\mathbb{A}_{y,y'} := S^1 \times ([-y', -y] \cup [y, y'])$ if $y' > y$. We recall that for a subset C of a manifold M,the notation $Op(C)$ means any open neighborhood of C in M.

Lemma 1.3 (Area-preserving extension lemma for the annulus)**.** *Consider the annulus* $\mathbb{A}_2$ *with a symplectic form* ω*. Let* $\mathcal{E}$ *be the set of smooth area-preserving embeddings* $\phi : Op(\mathbb{A}_1) \to \mathbb{A}_2$ *which are homotopic to the inclusion and such that for some* $y \in (-1, 1)$*, and hence for all, the signed area in* $\mathbb{A}_2$ *bounded by* $S^1 \times y$ *and* $\phi(S^1 \times y)$ *is zero.*

Then there exists $\delta, D, C > 0$*, such that for all* $\phi \in \mathcal{E}$ *with* $\|\phi\|_{C^0} \leq \delta$*, there exists* $\psi \in Ham(\mathbb{A}_2)$ *such that* $\psi|_{\mathbb{A}_{1-D\|\phi\|_{C^0}}} = \phi|_{\mathbb{A}_{1-D\|\phi\|_{C^0}}}$ *and*

$$\|\psi\|_{C^0} \leq C\|\phi\|_{C^0}.$$

Moreover, if for some arc $I \subset S^1$ *we have that* $\phi = Id$ *outside a quadrilateral* $I \times [-1,1]$ *and* $\phi(I \times [-1,1]) \subset I \times [-2,2]$*, then the extension* ψ *can be chosen to be the identity outside of* $I \times [-2,2]$*.*

Remark 1.4. *In* [Sey13]*, Seyfaddini expended the method developed by Entov, Polterovich and Py to show the bound*

$$\|\psi\|_{C^0} \leq C\left(\|\phi\|_{C^0}\right)^{\frac{1}{2}}$$

for some constant $C > 0$*. The method employed then cannot give a better estimate than the one he obtained.*

1.1.2. *The* C^0*-fragmentation for the kernel of the mass-flow homomorphism.* As in [Fa80], it is sensible to ask whether the C^0-fragmentation holds also for homeomorphisms preserving a "good" measure. It is necessary then to restrict our study to a continuous analogue of Hamiltonian diffeomorphisms, that is the group $\mathrm{Ham}(\Sigma)$ is replaced by the group $\mathrm{Ker}(\theta)$. Where θ denotes the mass-flow homomorphism as defined in [Sch57]. Under this assumption the fragmentation property still holds. We need to define the notion of Oxtoby-Ulam measure: an *Oxtoby-Ulam measure* μ on a compact manifold M is a measure that is nonatomic, of full support and is zero on the boundary.

Theorem 2. *Let* (Σ, μ) *be a closed surface equipped with an Oxtoby-Ulam measure* μ *and* d *be a distance induced by a Riemannian metric. Let* $\mathcal{W} = (W_i)_{i=1}^m$ *be a finite open covering of the surface by disks. Then there exists a* C^0*-neighborhood* N *of the identity in the group* $\mathrm{Ker}(\theta)$ *such that for each* $\phi \in N$*, we can decompose* ϕ *as:*

$$\phi = \phi_1 \circ \phi_2 \circ \cdots \circ \phi_m,$$

where for all $1 \leq i \leq m$*,* ϕ_i *belongs to* $\mathrm{Homeo}_c(W_i, \mu)$*, the group of compactly supported homeomorphisms preserving the area in* W_i*. Moreover, we have the following estimate for all* $1 \leq i \leq m$*,*

$$\|\phi_i\|_{C^0} \leq C\|\phi\|_{C^0},$$

for some constant $C > 0$ *independent of* ϕ*.*

In the same spirit as for Corollary 1, we show that for all homeomorphisms ϕ in $\mathrm{Ker}(\theta)$ there exists a C^0-small isotopy in $\mathrm{Ker}(\theta)$ between Id and ϕ.

Corollary 2. *Let* (Σ, ω) *be a closed symplectic surface and* d *be a distance induced by some Riemannian metric. There exists a constant* $C > 0$*, such that for all* ϕ *in* $\mathrm{Ker}(\theta)$*, there exists an isotopy* $\{\phi_t\}$ *of area-preserving homeomorphisms in* $\mathrm{Ker}(\theta)$ *such that* $\phi_0 = Id$*,* $\phi_1 = \phi$*, that satisfies the following estimate for all* $t \in [0,1]$*:*

$$\sup_t \|\phi_t\|_{C^0} \leq C\|\phi\|_{C^0}.$$

We will prove Theorem 2 by using an analog of Lemma 1.3 for homeomorphisms.

Lemma 1.5. *Consider* $\mathbb{A}_2$ *with an Oxtoby-Ulam measure* μ*. Let* $\mathcal{E}'$ *be the set of continuous measure-preserving embeddings* $\phi : Op(\mathbb{A}_1) \to \mathbb{A}_2$ *which are homotopic to the inclusion and such that for some* $y \in (-1,1)$*, and hence for all, the signed area in* $\mathbb{A}_2$ *bounded by* $S^1 \times y$ *and* $\phi(S^1 \times y)$ *is zero.*

Then there exists $\delta, D, C > 0$*, such that for all* $\phi \in \mathcal{E}'$ *with* $\|\phi\|_{C^0} \leq \delta$*, there exists* $\psi \in \ker(\theta)$ *such that* $\psi|_{\mathbb{A}_{1-D\|\phi\|_{C^0}}} = \phi|_{\mathbb{A}_{1-D\|\phi\|_{C^0}}}$ *and*

$$\|\psi\|_{C^0} \leq C\|\phi\|_{C^0}.$$

Moreover, if for some arc $I \subset S^1$ *we have that* $\phi = Id$ *outside a quadrilateral* $I \times [-1,1]$ *and* $\phi(I \times [-1,1]) \subset I \times [-2,2]$*, then the extension* ψ *can be chosen to be the identity outside of* $I \times [-2,2]$*.*

1.2. **Organization of the paper.** We review briefly the content of each section. In Section 3 we prove Corollary 1 and 2 assuming the we know the C^0-fragmentation property (Theorem 1 and Theorem 2). In Section 4 we show that the area-preserving extension lemmas yield the C^0-fragmentation property for diffeomorphisms and for homeomorphisms. In Section 5 we prove the area-preserving extension lemmas and in Section 6 we prove an extension lemma that we use in Section 5.

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2. Some definitions and notations

In this section we go through some definitions and notations that will be used later on and describe geometrical interpretations of the flux homomorphism and the mass-flow homomorphism.

Let (M, ω) be a connected symplectic manifold. If M is closed, let $\mathrm{Symp}(M, \omega)$ be the set of smooth diffeomorphisms of M that preserve ω. We then let $\mathrm{Symp}_0(M, \omega)$ be the connected component of Id in $\mathrm{Symp}(M, \omega)$, i.e. $\phi \in \mathrm{Symp}_0(M, \omega)$ if and only if there exists a smooth family of symplectic diffeomorphisms $(\phi_t)_{t\in[0,1]}$ such that $\phi_0 = Id$ and $\phi_1 = \phi$.

A *Hamiltonian* H is a smooth function $H : [0, 1] \times M \to \mathbb{R}$ compactly supported in $[0, 1] \times M$. A Hamiltonian H induces the *Hamiltonian flow*

$$\phi_H^t : M \to M, \quad (0 \leq t \leq 1),$$

by integrating the unique time-dependent vector field X_H satisfying $\iota_{X_H}\omega = dH_t$, the isotopy ϕ_H^t is a symplectic isotopy. A *Hamiltonian diffeomorphism* is a diffeomorphism obtained as the time-1 map of a Hamiltonian flow. We will denote $\mathrm{Ham}(M, \omega) \subset \mathrm{Symp}_0(M, \omega)$ the set of such diffeomorphisms. We will eliminate the symplectic form ω from the above notation when no confusion is possible. We recall that if H and G are two Hamiltonians with flows ϕ_H^t and ϕ_G^t, then the Hamiltonian $H\#G(t, x) = H(t, x) + G(t, (\phi_H^t)^{-1}(x))$ generates the Hamiltonian flow $\phi_H^t \circ \phi_G^t$.

We will now give an alternative definition of $\mathrm{Ham}(M)$ using the Flux, we will need this definition later on, for a proof of the properties stated below we refer the reader to [McS98, Chapter 10]. We define first the homomorphism

$$\widetilde{\mathrm{Flux}} : \widetilde{\mathrm{Symp}}_0(M, \omega) \to H^1(M, \mathbb{R}),$$

where $\widetilde{\mathrm{Symp}}_0(M, \omega)$ is the covering space of $\mathrm{Symp}_0(M, \omega)$ in the following way. Let $\{\phi_t\}$ be a symplectic isotopy from Id to ϕ_1 and let X_t be the time-dependent vector field defined via the relation

$$\frac{d}{dt}\phi_t = X_t \circ \phi_t.$$

Since $\{\phi_t\}$ is a symplectic isotopy, the 1-form $\iota_{X_t}\omega$ is a closed form. We then define:

$$\widetilde{\mathrm{Flux}}(\phi_t) := \int_0^1 [\iota_{X_t}\omega]\, dt \in H^1(M, \mathbb{R}).$$

This 1-form depends only on the choice of the homotopy class of the isotopy ϕ_t with fixed endpoints so $\widetilde{\mathrm{Flux}}$ is well-defined. Also one can see by the natural identification of $H^1(M, \mathbb{R})$ and $\mathrm{Hom}(\pi_1(M), \mathbb{R})$ that $\widetilde{\mathrm{Flux}}(\{\phi_t\})$ acts on $\pi_1(M)$, and roughly speaking this action describes how much "mass" goes through a loop γ in M during the isotopy. If $\{\phi_t\}$ is a Hamiltonian isotopy, then the 1-form $\iota_{X_t}\omega$ is exact and thus $\widetilde{\mathrm{Flux}}(\phi_t) = 0$, moreover one can show that if ϕ_t is such that $\widetilde{\mathrm{Flux}}(\phi_t) = 0$ then ϕ is a Hamiltonian diffeomorphism. A proof of this non-trivial fact can again be found in [McS98, Chapter 10].

We would like to define a reciprocal statement to define Hamiltonian diffeomorphisms and not having to take care of the choice of the isotopy $\{\phi_t\}$, this is why we want to define the Flux on $\mathrm{Symp}_0(M, \omega)$ in such a way that $\mathrm{Ker}(\mathrm{Flux}) = \mathrm{Ham}(\Sigma)$. In order to define the Flux we will take the quotient by the space of loops based at Id inside the group $\mathrm{Symp}_0(M,\omega)$. If we denote by Γ the image under $\widetilde{\mathrm{Flux}}$ of the loops based at Id in $\mathrm{Symp}_0(M,\omega)$, then the homomorphism $\widetilde{\mathrm{Flux}}$ descends to $\mathrm{Flux} : \mathrm{Symp}_0(M) \to H^1(M,\mathbb{R})/\Gamma$. Then we have our reciprocal statement, a symplectomorphism ϕ is in $\mathrm{Ham}(M)$ if and only if for any isotopy $\{\phi_t\}$ with $\phi_0 = Id$ and $\phi_1 = \phi$, we have $\mathrm{Flux}(\{\phi_t\}) = 0$.

On a symplectic manifold (M,ω) endowed with a distance d induced by a Riemannian metric, the C^0-*norm* (or *uniform norm*) is defined by

$$\|\phi\|_{C^0} := \max_x d\left(Id, \phi(x)\right).$$

Similarly, given a symplectic isotopy $\{\phi_t\}$, we define its C^0-*norm* by

$$\|\{\phi_t\}\|_{C^0}^{\text{path}} := \max_{x,t} d(Id, \phi_t(x)) = \sup_t \|\phi_t\|_{C^0}.$$

This norm induces what is called the C^0-*topology*. We denote

$$\overline{\mathrm{Ham}}(\Sigma,\omega) \subset \mathrm{Homeo}(\Sigma,\omega)$$

the closure for the C^0-norm of the group of Hamiltonian diffeomorphisms. Note that every diffeomorphism of $\overline{\mathrm{Ham}}(\Sigma,\omega)$ is in $\mathrm{Ham}(\Sigma,\omega)$.

In this paragraph we focus on the case of a closed surface (Σ,μ) equipped with a measure μ. We define $\mathrm{Homeo}(\Sigma,\mu)$ as the set of homeomorphisms of Σ that preserve the measure μ and $\mathrm{Homeo}_0(\Sigma,\mu)$ the identity component of $\mathrm{Homeo}(\Sigma,\mu)$. We will describe first the *mass-flow homomorphism* for homeomorphisms in the case of surfaces as it was introduced by Schwartzman in [Sch57]. For the definition of the mass-flow homomorphism on general compact metric spaces we refer to [Fa80]. Let $\widetilde{\mathrm{Homeo}_0}(\Sigma,\mu)$ be the set of paths starting at the identity in $\mathrm{Homeo}_0(\Sigma,\mu)$. We would like to define θ, in order to do that we will define first $\widetilde{\theta} : \widetilde{\mathrm{Homeo}_0}(\Sigma,\mu) \to H^1(\Sigma,\mathbb{R})$.

Let γ be a loop in Σ, and let $\{\phi_t\}$ a path in $\mathrm{Homeo}_0(\Sigma,\mu)$ connecting the identity to ϕ. We define $\widetilde{\theta}(\{\phi_t\})(\gamma) = \int_\sigma \mu$, where σ is the 2-chain

$$\sigma : [0,1]\times[0,1] \to \Sigma, (s,t) \mapsto \phi_s(\gamma(t)).$$

One can show that this definition descends to homology and that $\widetilde{\theta}(\phi_t)$ defines a cohomology class, moreover one can check that $\widetilde{\theta}$ is a homomorphism. Denote by Γ the image under $\widetilde{\theta}$ of the subset of loops based at Id in $\mathrm{Homeo}(\Sigma)$, then $\widetilde{\theta}$ descends naturally to $\theta : \mathrm{Homeo}(\Sigma,\mu) \to H^1(\Sigma,\mathbb{R})/\Gamma$. It is proved in [Lef] that if μ is induced by an symplectic form, then $\overline{\mathrm{Ham}}(\Sigma,\mu) = \mathrm{Ker}(\theta)$. Theorem 2 can thus be rewritten in terms of $\overline{\mathrm{Ham}}(\Sigma,\mu)$ whenever μ is given by a symplectic form.

3. Proof of some consequences of the C^0-fragmentation property

In this section we prove the Corollary 1 and 2 of Theorem 1 and 2. In order to prove Corollary 1 we will use the case where Σ is a disk, the following result is proved by Seyfaddini in [Sey13, Lemma 3.2], the proof relies on an ingenious use of Alexander's trick.

Lemma 3.1. *Suppose $\psi \in \mathrm{Ham}_c(B_r^{2n})$, then there exists a Hamiltonian $H : [0,1]\times B_r^{2n} \to \mathbb{R}$ such that $\psi = \phi_H^1$ and $\sup_t \|\phi_H^t\|_{C^0} \leq \|\psi\|_{C^0}$.*

With the help of Lemma 3.1, one can prove Corollary 1.

Proof of Corollary 1. Assume that we know that the statement of Corollary 1 is true on a C^0-neighborhood of Id in $\mathrm{Ham}(\Sigma)$. That is, there exists $\varepsilon > 0$ and a constant $C > 0$, such that for all $\|\phi\|_{C^0} \leq \varepsilon$, there exists a Hamiltonian H_t such that $\|\phi_H^t\|_{C^0}^{\text{path}} \leq C\|\phi\|_{C^0}$. We note $D := \mathrm{Diam}(\Sigma)$ the diameter of the surface, it is well-defined by compacity. Then $\|\phi_H^t\|_{C^0}^{\text{path}} \leq D$ for any Hamiltonian isotopy ϕ_H^t with $\phi_H^1 = \phi$. We can compute that if $\|\phi\|_{C^0} \geq \varepsilon$,

$$\|\phi_H^t\|_{C^0}^{\text{path}} \leq D = \frac{D}{\varepsilon}\varepsilon \leq \frac{D}{\varepsilon}\|\phi\|_{C^0}.$$

We can combine the two cases $\|\phi\|_{C^0} \leq \varepsilon$ and $\|\phi\|_{C^0} \geq \varepsilon$ to finally obtain that, for all $\phi \in \mathrm{Ham}(\Sigma)$ there exists a Hamiltonian H such that

$$\|\phi_H^t\|_{C^0}^{\mathrm{path}} \leq \max\left(\frac{D}{\varepsilon}, C\right)\|\phi\|_{C^0}.$$

Which is exactly the result we wanted.

It remains to prove the property of Corollary 1 on a well-chosen C^0-neighborhood of the identity. Take any finite covering $\mathcal{U} = (U_i)_{i=1}^m$ of Σ by disks U_i. We will take N and $C > 0$ respectively the neighborhood and the constant given by Theorem 1. Let $\phi \in \mathrm{Ham}(\Sigma)$, the C^0-fragmentation property tells us that it is possible to decompose $\phi = \phi_1 \circ \phi_2 \circ \cdots \circ \phi_m$, where each ϕ_i is a compactly supported Hamiltonian diffeomorphism of the disk U_i and $\|\phi_i\|_{C^0} \leq C\|\phi\|_{C^0}$. We now use Lemma 3.1, so there exists compactly supported Hamiltonian $H_i : [0,1] \times U_i \to \mathbb{R}$ such that

$$\phi_{H_i}^1 = \phi_i, \quad \|\phi_{H_i}^t\|_{C^0}^{\mathrm{path}} \leq \|\phi_i\|_{C^0}.$$

Let $H := H_1 \# H_2 \# \cdots \# H_m$, then H generates ϕ_H^t a Hamiltonian isotopy such that $\phi_H^t = \phi_{H_1}^t \circ \phi_{H_2}^t \circ \cdots \phi_{H_m}^t$, in particular $\phi = \phi_H^1$. Also, by the property of the C^0-norm and the previous estimates,

$$\begin{aligned}
\|\phi_H^t\|_{C^0}^{\mathrm{path}} &= \|\phi_{H_1}^t \circ \phi_{H_2}^t \circ \cdots \circ \phi_{H_m}^t\|_{C^0}^{\mathrm{path}} \\
&\leq \|\phi_1^t\|_{C^0}^{\mathrm{path}} + \|\phi_2^t\|_{C^0}^{\mathrm{path}} + \cdots + \|\phi_m^t\|_{C^0}^{\mathrm{path}} \\
&\leq \|\phi_1\|_{C^0} + \|\phi_2\|_{C^0} + \cdots + \|\phi_m\|_{C^0} \\
&\leq Cm\|\phi\|_{C^0}.
\end{aligned}$$

Which is the result we intended to prove. □

We also prove the Corollary 2 by using the same method we just need to adapt Seyfaddini's method to show the following lemma.

Lemma 3.2. *Suppose that $\psi \in \overline{\mathrm{Ham}_c}(B_r^{2n})$, then there exists a family ϕ^t of homeomorphism in $\overline{\mathrm{Ham}_c}(B_r^{2n})$ such that $\psi = \phi^0$, $\psi = \phi^1$ and $\sup_t \|\phi^t\|_{C^0} \leq \|\psi\|_{C^0}$.*

Proof. This lemma is actually easier to prove than Lemma 3.1 since we do not have to take care of the smoothness of the family ϕ^t. We assume first that $r = 1$. We want to show that the family

$$\phi^s(x) = \begin{cases} s\psi(\frac{x}{s}) \text{ if } |x| \leq s, \\ x \text{ otherwise} \end{cases}$$

satisfies the conclusion of Lemma 3.2. It is easily seen that $\|\phi^t\| \leq s\|\psi\|_{C^0}$ we now only need to check that at all time ψ^t is indeed in the group $\overline{\mathrm{Ham}_c}(B_r^{2n})$.

Let $(\psi_k)_{k\in\mathbb{N}}$ be a family of Hamiltonian diffeomorphisms generated by the Hamiltonians $(H_k)_k$ approximating ψ for the C^0-norm. Then, ϕ^s can be approximated by ψ_k^s, where ψ_k^s is defined as

$$\psi_k^s(x) = \begin{cases} s\psi_k(\frac{x}{s}) \text{ if } |x| \leq s, \\ x \text{ otherwise.} \end{cases}$$

However, for all $s > 0$, ψ_k^s is generated by the Hamiltonian

$$H_{k,s}(t,x) = \begin{cases} s^2 H_k(t, \frac{x}{s}) \text{ if } |x| \leq s, \\ 0 \text{ otherwise.} \end{cases}$$

Hence, proving that ϕ^s is indeed approximated by Hamiltonian diffeomorphisms and thus finishing the proof of the lemma. □

4. Proof of the C^0-fragmentation property

In this section we prove Theorem 1 and Theorem 2. In order to prove the C^0-fragmentation property it will be easier to work on a refinement of the given open covering $\mathcal{W}$, we will first prove the C^0-fragmentation property for this refinement and then go back to the initial covering by applying a trick that allows us to switch places of diffeomorphisms in the decomposition despite the lack of commutativity a priori. The refinement of the covering is obtained by associating an open set to each vertex, edge and face of a given triangulation T, if the triangulation is taken thin enough the resulting covering will indeed be a refinement. This construction is described in Section 4.1 and is due to Thurston and Banyaga. In Section 4.4 we will decompose any C^0-small diffeomorphism according to the new covering by defining the fragments we need in this order: around the vertices, the edges and then the faces. The construction of the fragments is where Lemma 1.3 will be useful. More precisely in Section 4.2 we will describe two slight modifications of the extension lemma, namely the area-preserving extension lemma for disks and the area-preserving extension lemma for rectangles. It is however not possible to apply naively the two extension lemmas, there is some obstruction to it given by the area between a curve and the image of it (see. the area condition in Lemma 1.3), this area represents an obstruction to the extension, we will take care of it in Section 4.3.

4.1. **Covering associated to a triangulation.** In this section we associate to a triangulation $T = (\Delta_i^k)_{i\in I_k}$, $k=0,1,2$ three open coverings $\mathcal{V} = (V_i^k)_{i\in I_k}$, $\mathcal{V}' = (V_i^{'k})_{i\in I_k}$ and $\mathcal{U} = (U_i^k)_{i\in I_k}$, following the construction of Thurston and Banyaga [Ban78]. The covering $\mathcal{V}$ verify the fact that for each $k = 0,1,2$ the elements of the subset $(V_i)_{I_k}^k$ are pairwise disjoint, $\mathcal{U}$ and $\mathcal{V}'$ are obtained by successively thickening $\mathcal{V}$ and will also verify the same condition as $\mathcal{V}$.

We build the covering $\mathcal{V}$ by induction on the skeletons of the triangulation, an example is shown in Figure 1. The base case is easy, the disks V_i^0 are balls containing Δ_i^0 and such that $V_i^0 \cap V_j^0 \neq \emptyset$ whenever $i \neq j$, they are colored in red in Figure 1.

For the inductive step, let us assume that we already constructed the disks $(V_i^\ell)_{i\in I_\ell}$, $\ell = 0,1,\dots,k-1$ such that

$$\widetilde{\Delta}_i^k := \Delta_i^k - \bigcup_{\ell\leq k-1} V^\ell$$

is a contraction of Δ_i^1, where $V^\ell = \bigcup_{j\in I_\ell} V_j^\ell$. Let $\widehat{\Delta}_i^k$ a small thickening of $\widetilde{\Delta}_i^k$. Then V_i^k is defined as a tubular neighborhood of $\widehat{\Delta}_i^k$. If this tubular neighborhood is small enough the open sets will verify $V_i^k \cap V_j^k = \emptyset$ for all $i \neq j$. This finishes the proof by induction.

Now that $\mathcal{V}$ is defined we define $\mathcal{V}' = (V_i^{'k})_{i\in I_k}$ and $\mathcal{U} = (U_i^k)_{i\in I_k}$ two other open coverings obtained by thickening the V_i^k such that the following conditions hold:

$$\forall k \leq 2, i \in I_k,\ \overline{V_i^k} \subset V_i^{'k} \subset \overline{V_i^{'k}} \subset U_i^k \text{ and } V_i^{'k} \cap V_j^{'k} = U_i^k \cap U_j^k = \emptyset,$$

whenever $i \neq j$.

Furthermore, if T is chosen such that for any simplex σ in T, its *star* (the union of the simplex touching σ) is inside an open set of the covering $\mathcal{W}$, then the open coverings $\mathcal{V}$, $\mathcal{V}'$ and $\mathcal{U}$ are refinement of $\mathcal{W}$. We fix now, and for the rest of the paper, $T = (\Delta_i^k)_{i\in I_k}$ a triangulation with the above property along with $\mathcal{V} = (V_i^k)_{i\in I_k}$, $\mathcal{V}' = (V_i^{'k})_{i\in I_k}$ and $\mathcal{U} = (U_i^k)_{i\in I_k}$ the open coverings associated to T.

4.2. **Two corollaries of the area-preserving extension lemma for the annulus.** As announced before, in this section we will prove two corollaries of Lemma 1.3 that will suit better our needs for the proof of the C^0-fragmentation property. The corollaries are improved versions of Lemmas 2 and 3 in [EPP12, section 1.6.1] and we will mimic their proof.

Corollary 3 (Area-preserving extension lemma for disks)**.** *Let $D_1 \subset D_2 \subset D_3 \subset \mathbb{R}^2$ be closed disks such that $D_1 \subset$ Interior$(D_2) \subset D_2 \subset$ Interior(D_3). Let $\phi : D_2 \to D_3$ be a smooth area-preserving embedding. If ϕ is sufficiently C^0-small, then there exists $\psi \in \mathrm{Ham}(D_3)$ such that*

$$\psi|_{D_1} = \phi|_{D_1} \text{ and } \|\psi\|_{C^0} \leq C\|\phi\|_{C^0},$$

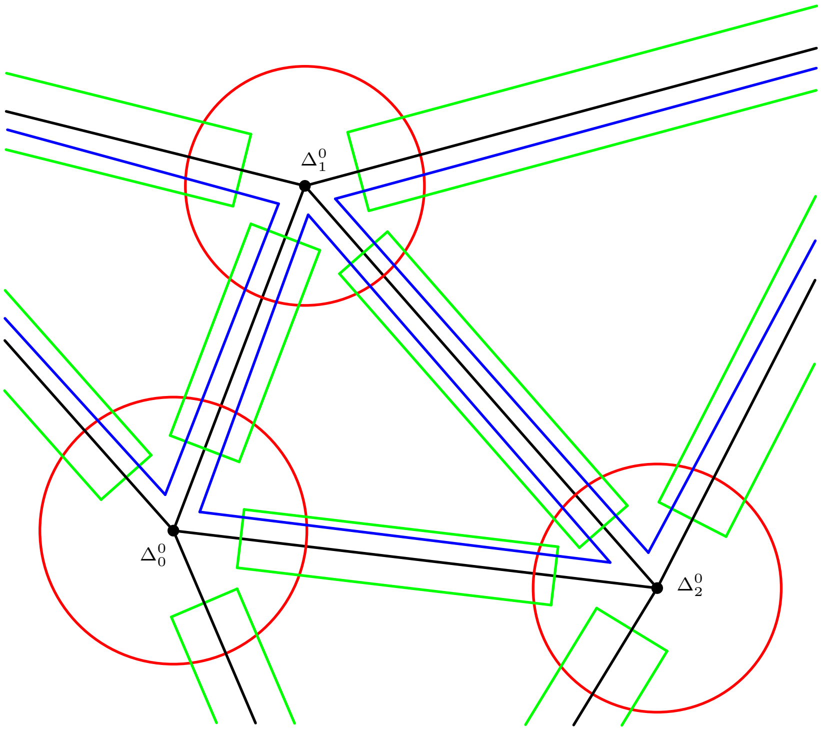

FIGURE 1. The triangulation T is represented in black, the boundary of the open sets of the form V_i^0 are painted in red, the boundary of the open sets of the form V_i^1 are painted in green and the boundary of the open sets of the form V_i^2 are painted in blue.

for some constant $C > 0$.

Remark 4.1. *The Corollary 3 can be completely adapted in the continuous setting if we ask to extend ϕ, a continuous measure-preserving embedding, by ψ, an element of* $\mathrm{Ker}(\theta)$*. We can then prove it by adapting the proof and using Lemma 1.5 instead of Lemma 1.3.*

Proof. We mimic the proof in [EPP12], adding only the sharper estimate of Lemma 1.3.

Up to replacing D_2 by a slightly smaller disk, we can assume that ϕ is defined in a neighborhood of D_2. Identify some small neighborhood of ∂D_2 with $\mathbb{A}_2$ so that ∂D_2 is identified with $S^1 \times 0 \subset \mathbb{A}_1 \subset \mathbb{A}_2$. If ϕ is C^0-small enough, we have $\phi(\mathbb{A}_1) \subset \mathrm{Interior}(\mathbb{A}_2) \subset \mathrm{Interior}(D_3) \backslash \phi(D_1)$. Denote $\delta := \|\phi\|_{C^0}$.

Apply Lemma 1.3 and find $h \in \mathrm{Ham}_c(\mathbb{A}_2)$,

$$\|h\|_{C^0} \leq C\|\phi\|_{C^0},$$

for some constant $C > 0$ and so that $h|_{\mathbb{A}_{1-D\delta}} = \phi$. Set $\phi_1 := h^{-1} \circ \phi \in \mathrm{Ham}_c(D_3)$. Note that $\phi_1|_{D_1} = \phi$ and ϕ_1 is the identity on $\mathbb{A}_{1-D\delta}$. Therefore we can extend $\phi_1|_{D_2 \cup \mathbb{A}_1}$ to D_3 by the identity and get the required ψ. ∎

Corollary 4 (Area-preserving extension lemma for rectangles)**.** *Let $\Pi_3 = [0, R] \times [-c_3, c_3]$ be a rectangle and let $\Pi_1 \subset \Pi_2 \subset \Pi_3$ be two smaller rectangles of the form $\Pi_i = [0, R] \times [-c_i, c_i], (i = 1, 2)$, $0 < c_1 < c_2 < c_3$. Let $\phi : \Pi_2 \to \Pi_3$ be a smooth area-preserving embedding such that*

- *ϕ is the identity near $0 \times [-c_2, c_2]$ and $R \times [-c_2, c_2]$,*
- *The area in Π_3 bounded by the curve $[0, R] \times y$ and its image under ϕ is zero for some (and hence for all) $y \in [-c_2, c_2]$.*

If ϕ is sufficiently C^0-small, then there exists $\psi \in Ham(\Pi_3)$ such that

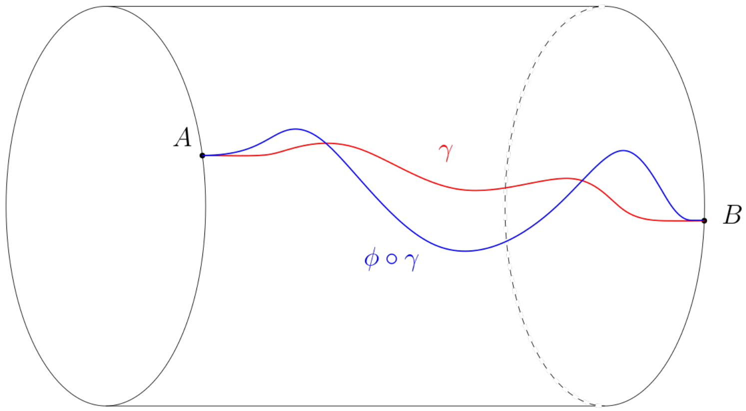

FIGURE 2. The obstruction $\mathcal{O}$ is the signed area between the red curve and the blue curve.

$$\psi|_{D_1} = \phi|_{D_1} \text{ and } \|\psi\|_{C^0} \leq C\|\phi\|_{C^0},$$

for some constant $C > 0$.

Remark 4.2. *Again Corollary 4 transposes completely in the continuous setting by taking* ϕ *being a continuous measure-preserving embedding and extending it by* $\psi \in \mathrm{Ker}(\theta)$.

Proof. The proof relies on the last assumption of Lemma 1.3, indeed we can identify the rectangle Π_3 with the subset $I \times [-2, 2] \subset \mathbb{A}_2$. □

4.3. **Definition of two obstructions.** In this section we fix (Σ, ω) a symplectic surface and T, $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{V}'$ a triangulation and 3 open coverings associated to it as defined in Section 4.1. We will define the flux for an annulus, we call this quantity

$$\mathcal{O} : \mathrm{Symp}_{0,c}(\mathbb{A}_1) \to \mathbb{R}.$$

Thus, $\mathcal{O}$ represents the obstruction for a diffeomorphism $\phi \in \mathrm{Symp}_c(\mathbb{A}_1)$ to belong to $\mathrm{Ham}_c(\mathbb{A}_1)$. We also define $\mathcal{A}_{i_1,i_2}(\phi)$ for i_1 and i_2 in I_0 which will represent the obstruction of the extension of a certain embedding of V_j^1 in $V_j'^1$ if the two vertices $\Delta^0_{i_1}$ and $\Delta^0_{i_2}$ are joined by an edge Δ^1_j (see Corollary 4). The obstruction $\mathcal{A}$ can be seen as a way to define $\mathcal{O}$ on a general symplectic surface.

Definition 4.3 (Definition of $\mathcal{O}$)**.** *Let* $\phi \in \mathrm{Symp}_{0,c}(\mathbb{A}_1)$*,* $A \in S^1 \times \{-1\}$ *and* $B \in S^1 \times \{1\}$ *two points on the boundary of* $\mathbb{A}_1$ *as in Figure 2 and* $\gamma = [0, 1] \to \mathbb{A}_1$*,* $t \mapsto \gamma(t)$ *an arc with endpoints* $\gamma(0) = A$ *and* $\gamma(1) = B$*. Then let* $h : [0, 1]_s \times [0, 1]_t \to \mathbb{A}_1$ *be a smooth homotopy from* γ *to* $\phi(\gamma)$ *such that* $h_{0,t} = \gamma(t)$ *and* $h_{1,t} = \phi(\gamma(t))$*. We can then define the quantity:*

$$\mathcal{O}(\phi) := \int_0^1 \int_0^1 \omega(\partial_s h_{s,t}, \partial_t h_{s,t})\, ds\, dt.$$

Remark 4.4. *We can also define* $\mathcal{O}$ *in a more general setting by defining it as the area of a 2-cell defined by a continuous homotopy* h *and consider* ω *as a measure, and even replace it by any Oxtoby-Ulam measure (see Section 1.1.2). This will be useful for Theorem 2.*

The next proposition gives some properties of $\mathcal{O}$, those properties can also be proven in a continuous setting. This proposition is classic and we do note give a proof.

Proposition 4.5. $\mathcal{O}$ *is well-defined, i.e. it does not depend on the choices of* h*,* γ *nor on the choice of the points* A *and* B*.*

Moreover, $\mathcal{O}$ *is exactly the obstruction for* $\phi \in \mathrm{Symp}_{0,c}(\mathbb{A}_1)$ *to be in* $\mathrm{Ham}(\mathbb{A}_1)$*, that is* $\phi \in \mathrm{Ham}(\mathbb{A}_1)$ *if and only if* $\mathcal{O}(\phi) = 0$.

We now state Lemma 4.6, it proves the surjectivity of this obstruction. We also give a bound on the norm of a well-chosen antecedent of a real number (it will be useful in the proof of Lemma 4.10 for example). We will prove Lemma 4.6 at the end of this proof.

Lemma 4.6 (Surjectivity of $\mathcal{O}$ and an estimate on the norm of a pre-image)**.** *With the previous definitions, the obstruction $\mathcal{O} : \mathrm{Symp}_c(\mathbb{A}_1) \to \mathbb{R}$, $\psi \mapsto \mathcal{O}(\psi)$ is a surjective function.*

Moreover, there exists a constant $C > 0$ such that for all $\varepsilon \in \mathbb{R}$, there exists $\psi_\varepsilon \in \mathrm{Symp}_c(\mathbb{A}_1)$ such that $\mathcal{O}(\psi_\varepsilon) = \varepsilon$ and

$$\|\psi_\varepsilon\|_{C^0} \leq C|\varepsilon|.$$

Proof of Lemma 4.6. Let $\varepsilon \in \mathbb{R}$ and define $H_s^\varepsilon(x, y) = \chi(y)\varepsilon$ a Hamiltonian-like function, where χ is a smooth function in $[-1, 1]$, satisfying $\chi(-1) = 0$ and $\chi(1) = 1$ and χ' is compactly supported in $[-1, 1]$. We let $C = \|\chi'\|_0$. We then define ϕ_s as the flow generated by this Hamiltonian-like function. That is, define the vector field X_s by the equation

$$\iota_{X_s}\omega = dH_s^\varepsilon = \chi'(y)\varepsilon\, dy.$$

So $X_s = \chi'(y)\varepsilon \frac{\partial}{\partial x}$ and $\phi_s(x, y) = (x + \varepsilon\chi'(y)s, y)$, where x is in $\mathbb{R}/\mathbb{Z}$ and $x + \varepsilon\chi'(y)s$ is taken in $\mathbb{R}/\mathbb{Z}$. Then $\|\phi\|_{C^0} \leq C|\varepsilon|$ and $\mathcal{O}(\phi) = \int_0^1 H_s(1)\, ds = \varepsilon$ as wanted. □

A seen in Remark 4.4 there is an analogous definition of $\mathcal{O}$ in the continuous setting, it is not hard to see that Proposition 4.5 transposes in the continuous setting once we work with $\mathrm{Ker}(\theta)$ instead of $\mathrm{Ham}(\mathbb{A}_1)$.

Definition 4.7 (Definition of $\mathcal{A}$)**.** *Let T, $\mathcal{U}$ and $\mathcal{V}$ be as in Section 4.1. Let $\phi \in \mathrm{Symp}_0(\Sigma)$ be such that for all $i \in I_0$, $\phi|_{V_i^0} = Id$. Given $\Delta_{i_1}^0$ and $\Delta_{i_2}^0$ two vertices in T linked by an edge Δ_j^1 parameterized by an arc γ with $\gamma(0) = \Delta_{i_1}^0$ and $\gamma(1) = \Delta_{i_2}^0$, we define:*

$$\mathcal{A}_{i_1,i_2}(\phi) = \int_0^1 \int_0^1 \omega(\partial_s h_{s,t}, \partial_t h_{s,t}),$$

where $h : [0, 1] \times [0, 1] \to U_j^1$ is an isotopy with fixed endpoints from γ to $\phi(\gamma)$.

Remark 4.8. *We can also define $\mathcal{A}$ via integration on 2-cells instead. Again it allows us to define $\mathcal{A}$ on the continuous case.*

We prove now several properties of this obstruction $\mathcal{A}$.

Proposition 4.9. *Let T, $\mathcal{U}$ and $\mathcal{V}$ given as in Section 4.1. We assume, as before, that i_1 and i_2 are two indices in I_0 such that $\Delta_{i_1}^0$ and $\Delta_{i_2}^0$ are linked in T by an edge Δ_j^1, then if ϕ is C^0-small enough and verifies that, for all $i \in I_0$, $\phi|_{V_i^0} = Id$. We have the following four properties:*

(i) *$\mathcal{A}_{i_1,i_2}(\phi)$ is well defined, i.e. it does not depends on the choice made in its definition.*

(ii) *The following identity holds:*

$$\mathcal{A}_{i_1,i_2}(\phi) = -\mathcal{A}_{i_2,i_1}(\phi).$$

(iii) *There exists a constant $C > 0$ that does not depend on ϕ such that,*

$$|\mathcal{A}_{i_1,i_2}(\phi)| \leq C\|\phi\|_{C^0}.$$

(iv) *If $\phi \in \mathrm{Ham}(\Sigma)$, then for a loop of the triangulation i.e. a set of indices $i_1, i_2, \ldots, i_m$ such that i_p and i_{p+1} are linked by an edge in T (with the convention that $i_{m+1} = i_1$), we have:*

$$\sum_{p=1}^{m} \mathcal{A}_{i_p,i_{p+1}}(\phi) = 0.$$

Proof. We are going to prove the claims in the order they appear.

(i) The proof here is the same as for Proposition 4.5.

(ii) If $h_{s,t}$ is an isotopy from γ to $\phi(\gamma(t))$, then $h_{s,1-t}$ is an isotopy from $\gamma(1-t)$ to $\phi(\gamma(1-t))$, so after a change of variable in the integral we have the identity:

$$\mathcal{A}_{i_1,i_2}(\phi) = \int_0^1 \int_0^1 \omega(\partial_s h_{s,t}, \partial_t h_{s,t}) = -\mathcal{A}_{i_2,i_1}(\phi).$$

(iii) This claim is immediate since $\mathcal{A}_{i_1,i_2}(\phi)$ measures the area between Δ^1_j and $\phi(\Delta^1_j)$ in U^1_j (if we do not work in U^1_j then the area would not be well-defined) and $\phi(\Delta^1_j)$ is stuck in the tubular neighborhood of Δ^1_j of radius $\|\phi\|_{C^0}$, hence we indeed have:

$$|\mathcal{A}_{i_1,i_2}(\phi)| \leq C\|\phi\|_{C^0},$$

for some constant $C > 0$.

(iv) Let γ be the piece-wise smooth path going through $\Delta^1_{j_1}, \Delta^1_{j_2}, \dots, \Delta^1_{j_m}$, where $\Delta^1_{j_p}$ links $\Delta^0_{i_p}$ to $\Delta^0_{i_{p+1}}$, the path should not go twice through the same vertex. Let ϕ_t be a Hamiltonian isotopy from Id to ϕ, then $\widetilde{\text{Flux}}(\phi_t) = 0$, this means that the area of the cylinder $\phi_t(\gamma)$ is zero. However, nothing tells us a priori that the area of the cylinder of $\phi_t(\gamma)$ is the same as the sum $\sum_{p=1}^m \mathcal{A}_{i_p,i_{p+1}}(\phi)$, obtained also as the area of some cylinder between γ and $\phi \circ \gamma$ but with support in $\bigcup U^1_{j_p}$. However, we can glue those two cylinders to obtain one closed 2-cycle σ_2 so

$$\int_{\sigma_2} \omega = \widetilde{\text{Flux}}(\phi_t) + \sum_{p=1}^m \mathcal{A}_{i_p,i_{p+1}}(\phi) = \sum_{p=1}^m \mathcal{A}_{i_p,i_{p+1}}(\phi)$$

has value in $\omega \cdot H_2(\Sigma, \mathbb{Z}) \simeq < \omega, [\Sigma] > \mathbb{Z} \subset \mathbb{R}$. If ϕ is C^0-small enough then by the third point of the proposition, $|\sum_{p=1}^m \mathcal{A}_{i_p,i_{p+1}}(\phi)| \leq mC\|\phi\|_{C^0} \leq |I_k|\|\phi\|_{C^0}$ is inside this subgroup so must be 0 for $\|\phi\|_{C^0}$ small enough. This finishes the proof of Proposition 4.9.

□

4.4. **Proof of Theorems 1 and 2.** We have now all the tools to prove Theorem 1 and Theorem 2.

Proof of Theorem 1. Let T be a triangulation such that the star of every vertices of T are included in one of the open sets of the subcovering of $\mathcal{W}$, we will say that such triangulation is *good*. We will consider the open coverings $\mathcal{V}$, $\mathcal{V}'$ and $\mathcal{U}$ associated to T. Then the three coverings $\mathcal{V}$, $\mathcal{V}'$ and $\mathcal{U}$ are finer than $\mathcal{W}$. We will prove the fragmentation theorem on $\mathcal{U}$ which will imply it for $\mathcal{W}$ by a simple argument described at the end of the proof. We recall that for all $k = 0, 1, 2$ and $i \in I_k$, $V^k_i \subset V'^k_i \subset U^k_i$.

Let us describe briefly an outline of the proof. In order to fragment a diffeomorphism ϕ we will proceed in 3 steps.

In Lemma 4.10 and Lemma 4.11 we will start the fragmentation by finding Hamiltonian diffeomorphisms compactly supported in (U^0_i), agreeing with ϕ on V^0_i and some additional condition that will be needed in order to prove Lemma 4.12. We will then define ϕ' as the Hamiltonian diffeomorphism ϕ were we pre-composed by the inverses of the diffeomorphisms we just constructed. Thus, ϕ' is a Hamiltonian diffeomorphism which is the identity on the sets (U^0_i).

In Lemma 4.12 we find Hamiltonian diffeomorphisms compactly supported in U^1_i and agreeing with ϕ' on V^1_i. with the same construction as above we define ϕ''.

In Lemma 4.13 we will finish the fragmentation without any difficulty since ϕ'' is actually naturally fragmented.

Lemma 4.10 (Fragmentation on the 0-skeleton). *Let $T = (\Delta^k_i)_{i \in I_k}$ be a good triangulation $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{V}'$ open coverings associated with T as described in Section 4.1. Let $\phi \in \text{Ham}(\Sigma)$ be a C^0-small diffeomorphism. Then we can find the following C^0-fragmentation:*

$$\phi = \phi^{(0)}_1 \circ \phi^{(0)}_2 \cdots \circ \phi^{(0)}_\ell \circ \phi',$$

where $\ell := |I_0|$, for all $i \in I_0$, $\phi^{(0)}_i \in \text{Ham}(U^0_i)$, verifies $\phi^{(0)}_i|_{V^0_i} = \phi$ and satisfies the estimate

$$\|\phi^{(0)}_p\|_{C^0} \leq C\|\phi\|_{C^0},$$

where $C > 0$ is a constant.

Moreover, $\mathcal{A}_{i,j}(\phi') = 0$ for all $i, j \in I_0$ linked by an edge in T.

The first step in order to prove Lemma 4.10 is to prove Lemma 4.11, it is a fragmentation on the open sets on the vertices but we do not ask for the obstruction $\mathcal{A}$ to be 0.

Lemma 4.11. *Let $T = (\Delta_i^k)_{i\in I_k}$ be a good triangulation $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{V}'$ open coverings associated with T as described in Section 4.1. Let $\phi \in \mathrm{Ham}_c(\Sigma)$ a C^0-small diffeomorphism, then we can find the following C^0-fragmentation:*

$$\phi = \phi_1^{(-1)} \circ \phi_2^{(-1)} \cdots \circ \phi_\ell^{(-1)} \circ \widetilde{\phi},$$

where $\ell = |I_0|$, for all $i \in I_0$, $\phi_i^{(-1)} \in \mathrm{Ham}(U_i^0)$, verifies $\phi_i^{(-1)}|_{V_i^0} = \phi$ and satisfies the estimate

$$\|\phi_i^{(-1)}\|_{C^0} \leq C\|\phi\|_{C^0},$$

where $C > 0$ is a constant depending on the choice of T and the open coverings associated to it. The Hamiltonian diffeomorphism $\widetilde{\phi}$ is then supported in $\Sigma\backslash V^0$ and its C^0-norm satisfy a Lipschitz bound in the norm $\|\phi\|_{C^0}$.

Proof. Let $i \in I_0$, by Corollary 3 and since ϕ is a C^0-small diffeomorphism and an area-preserving embedding of $V_i'^0$ in U_i^0, there exists $\phi_i^{(-1)} \in \mathrm{Ham}(U_i^0)$ such that $\phi_i^{(-1)}|_{V_i} = \phi$. Moreover, there exists a constant $D > 0$ such that $\|\phi_i^{(-1)}\|_{C^0} \leq D\|\phi\|_{C^0}$. Then there exists a diffeomorphism $\widetilde{\phi}$ such that

$$\phi = \bigcirc_{i\in I_k}\phi_i \circ \widetilde{\phi}$$

(there is no issue with the composition since the supports of the ϕ_i are disjoint). We then have that $\widetilde{\phi}$ is supported in $\Sigma\backslash V^0$ and is a Hamiltonian diffeomorphism of Σ. Also, $\|\widetilde{\phi}\|_{C^0} \leq \|\phi\|_{C^0} + \sum \|\phi_i^{(-1)}\|_{C^0} \leq C\|\phi\|_{C^0}$, for $C > 0$ a constant. □

It is time now to prove Lemma 4.10.

Proof of Lemma 4.10. We apply first Lemma 4.11 and we now want to do slight modifications on the diffeomorphisms $\phi_p^{(-1)}$ in order to vanish the obstruction $\mathcal{A}$ on each edge.

In order to do this, we first take a fixed index $i \in I_0$ and its corresponding vertex Δ_i^0. We will define for a vertex $j \in I_0$ the real number $C(j)$ by

$$C(j) := \sum_{p=0}^{m-1} \mathcal{A}_{i_p,i_{p+1}}((\widetilde{\phi})),$$

where $i_0 = i, i_1, \ldots, i_m = j$ is a sequence of indices such that they are all linked by an edge in T. Using property (iv) of Proposition 4.9 we see that this value does not depend on the sequence (i_p) we take. We define now $\phi_j^{(0)} = \rho_j \circ \phi_j^{(-1)}$ where ρ_j is a compactly supported diffeomorphism from the annulus $U_j^0\backslash V_j^0$ to itself, and such that $\mathcal{O}(\rho_j) = C(j)$ (we identify ∂V_j^0 with $S^1 \times \{-1\}$ to match Definition 4.3), note that Lemma 4.6 allows us to take ρ_j with a Lipschitz estimate. Then the diffeomorphisms $\phi_j^{(0)}$ are the fragments needed in Lemma 4.10, and

$$\phi' = \bigcirc_j(\rho_j^{(-1)})^{-1}\widetilde{\phi}$$

satisfy all the conditions of the conclusion of Lemma 4.10.

□

We now work on ϕ' and fragment it in Lemma 4.12.

Lemma 4.12 (Fragmentation on the 1-skeleton)**.** *Let ϕ' be the resulting Hamiltonian diffeomorphism after applying Lemma 4.10, then there exists a fragmentation of ϕ',*

$$\phi' = \phi_1^{(1)} \circ \phi_2^{(1)} \cdots \circ \phi_m^{(1)} \circ \phi'',$$

where $m = |I_1|$, for all $i \in I_1$, $\phi_i^{(1)} \in \mathrm{Ham}(U_i^1)$, $\phi_i^{(1)}|_{V_i^1} = \phi'$ and the following estimate is true

$$\|\phi_i^{(1)}\|_{C^0} \leq C\|\phi'\|_{C^0},$$

where $C > 0$ is a constant (that depends on T). The resulting ϕ'' is then supported in $\Sigma\backslash(V^0 \cup V^1)$ and satisfy a Lipschitz estimate with respect to $\|\phi'\|_{C^0}$ and thus with respect to $\|\phi\|_{C^0}$.

Proof. Let $i \in I_1$, and i_1 and i_2 are the vertices of the edge Δ_i^1. Then, since ϕ' is C^0-small ϕ' is an area-preserving embedding of $V_i'^1$ in U_i^1 being equal to the identity on $V_i^1 \cap V_{i_1}^0$ and $V_i^1 \cap V_{i_2}^0$, also the condition $\mathcal{A}_{i_1,i_2}(\phi') = 0$ is exactly the condition we need to apply Corollary 4. We have now $\phi_i^{(1)} \in \mathrm{Ham}(U_i^1)$ such that $\phi_i^{(1)}|_{V_i^1} = \phi'$ and $\|\phi_i^{(1)}\|_{C^0} \leq D\|\phi'\|_{C^0}$. So there exists a diffeomorphism ϕ'' such that $\phi' = \bigcirc_{i\in I_1} \phi_i^{(1)} \circ \phi''$. Then ϕ'' is a Hamiltonian diffeomorphism of Σ itself, it is compactly supported in $\Sigma \backslash (V_0 \cup V_1)$ and satisfy $\|\phi''\|_{C^0} \leq \|\phi'\|_{C^0} + \sum_{i\in I_1} \|\phi_i^{(1)}\|_{C^0} \leq C\|\phi\|_{C^0}$. □

Lemma 4.13 (Fragmentation on the 2-skeleton)**.** *Let ϕ'' be the resulting Hamiltonian diffeomorphism from Lemma 4.12, we can fragment it*

$$\phi'' = \phi_1^{(2)} \circ \phi_2^{(2)} \cdots \circ \phi_n^{(2)},$$

where $n = |I_2|$, for all $i \in I_2$, $\phi_i^{(2)} \in \mathrm{Ham}(U_i^2)$ and

$$\|\phi_i^{(2)}\|_{C^0} \leq C\|\phi''\|_{C^0},$$

where $C > 0$ is a constant.

Proof. ϕ'' has now support in $\Sigma \backslash (V^0 \cup V^1) \subset \bigcup U_i^2$. So it can be decomposed in Hamiltonian diffeomorphism with support in the U_i^2 (pairwise disjoint) and the bound is also immediate. □

Combining Lemmas 4.10, 4.12 and 4.13 we obtain a fragmentation with a finitely bounded number of fragments. We describe now the procedure to give the result in the shape of Theorem 1.

We want to swap the support of the diffeomorphisms, let $\psi = fg$ for two diffeomorphisms f and g such that $\mathrm{supp}(f) \subset B_1' \subset B_1$ and $\mathrm{supp}(g) \subset B_2$. Then $\psi = g(g^{-1}fg)$ and we want to show now that $g^{-1}fg$ is supported inside B_1'' the set of points whose distance to B_1' is smaller than $\|g\|_{C^0}$ if f and g have a small C^0-norm. Indeed, if $d(x, B_1') \geq \|g\|_{C^0}$ then $g^{-1}(f(g(x))) = g^{-1}(g(x)) = x$. This procedure is the kind of commutation we needed

$$\psi = \underbrace{f}_{\text{support in } B_1} \underbrace{g}_{\text{support in } B_2} = \underbrace{g}_{\text{support in } B_2} \underbrace{g^{-1}fg}_{\text{support in } B_1}.$$

We need to show that we can apply this procedure repeatedly. Note that one needs to apply it only a finite number of time since there are at most $N := \ell+m+n$ fragments in the fragmentation we have for now and one can obtain any permutation with N elements with at most $N!$ transpositions. If we have $\|\phi\|_{C^0}$ small enough, then the successive thickenings of B_1' in the operation described above will stay inside B_1. We thus obtain a fragmentation $\phi = \phi_1\phi_2\cdots\phi_m$ with $\phi_i \in \mathrm{Ham}(W_i)$ and $\|\phi_i\|_{C^0} \leq C\|\phi\|_{C^0}$ for some constant $C > 0$.

□

Proof of Theorem 2. The proof transposes for $\mathrm{Ker}(\theta)$ by adapting directly Corollaries 3 and 4, then the obstruction $\mathcal{O}$ and $\mathcal{A}$ and finally Lemma 4.10, 4.11, 4.12 and 4.13. □

5. Proof of the area-preserving extension lemma

5.1. **Preliminaries.** We need to state three propositions about the area forms on a surface in order to carry out the proof of Lemma 1.3. The two first propositions are already well-known. The third one is new and is the key piece that allows us to go from a Hölder estimate for the area-preserving extension lemma to the Lipschitz estimate of Lemma 1.3.

We recall that a Borel measure μ on a compact manifold X is said to be an *OU (Oxtoby-Ulam) measure* if it is nonatomic, of full support and is zero on the boundary. The first proposition is proven in [OU41].

Proposition 5.1. *Let μ and ν be two OU measures on a rectangular r-cell R such that $\mu(R) = \nu(R)$, then there exists a homeomorphism h which restricts to the identity on the boundary of B such that $h^*\nu = \mu$.*

We recall here some consequences of Moser's trick [Mos65] as described in [Sey13, Section 3.4.1].

Proposition 5.2 (Moser's trick). *Let M be a compact connected oriented manifold of dimension n, possibly with a non-empty boundary ∂M, and let ω_1, ω_2 be two volume forms on M. Assume that $\int_M \omega_1 = \int_M \omega_2$. If $\partial M \neq 0$, we also assume that the forms ω_1 and ω_2 coincide on ∂M.*

Then there exists a diffeomorphism $f : M \to M$, isotopic to the identity, such that $f^*\omega_2 = \omega_1$. Moreover, f can be chosen to satisfy the following properties:

(i) If $\partial M \neq \emptyset$, then f is the identity on ∂M, and if ω_1 and ω_2 coincide near ∂M, then f is the identity near ∂M.

(ii) If M is partitioned into polyhedra (with piece-wise smooth boundaries), so that $\omega_1 - \omega_2$ is zero on the $(n-1)$-skeleton Γ of the partition and the integral of ω_1 and ω_2 on each of the polyhedra are equal, then f can be chosen to be the identity on Γ.

(iii) Suppose that $\omega_2 = \chi\omega_1$ for a function χ, then the diffeomorphism f can be chosen to satisfy the following estimate:

$$\|f\|_{C^0} \leq C\|\chi - 1\|_{C^0},$$

for some $C > 0$. Here, $\|\cdot\|_{C^0}$ denotes the standard sup norm on functions.

We describe now a lemma which allow us to adjust two volume forms by a C^0-small diffeomorphism if they disagree on a small strip only. This is the new idea that allowed us to improve the continuity property for the extension lemma.

Proposition 5.3. *Let $C \in \mathbb{R}$ be a constant and M^{n-1} a compact manifold equipped with a volume form ω' and a distance d', we also define d the product distance on $M^{n-1} \times [-1,1]$. Let $\omega = \omega' \wedge dz$ and Ω two volume forms on $M^{n-1} \times [-1,1]$, where z denote the coordinate on $[-1,1]$. Let χ be the function such that $\omega = (1+\chi)\Omega$. We assume that:*

- $\int_{M^{n-1}\times[-1,1]} \omega = \int_{M^{n-1}\times[-1,1]} \Omega$.
- $\|\chi\|_{C^0} \leq C$.
- *There exists $\delta > 0$ such that $Supp(\chi) \subset M^{n-1} \times [0,\delta]$.*

Then, there exists a constant $D \in \mathbb{R}$ independent of δ such that we can find $f \in Diff_c(M^{n-1} \times [-1,1])$, $f \equiv Id$ on $M^{n-1} \times [-1,0]$, $f^*\Omega = \omega$ and $\|f\|_{C^0} \leq D\delta$.

Proof. We will denote by z the last coordinate of the manifold $M^{n-1} \times [-1,1]$ and x the coordinate on M^{n-1}. Let $N^n := M^{n-1} \times \mathbb{R}$ we will describe two diffeomorphisms on N^n that will combine to give what we want on $M^{n-1} \times [-1,1]$.

Define $\rho_\delta : \mathbb{R} \to \mathbb{R}, x \mapsto 2\delta x$. We can then define two functions from N^n to N^n by:

$$\Psi_1(x,z) := \left(x, \int_0^z (1+\chi(x,t))\,dt\right)$$

and

$$\Psi_2(x,z) := \left(x, \int_0^z (1+\rho_\delta'(t)\chi(x,\rho_\delta(t)))\,dt\right).$$

We can then compute

$$d\Psi_1(x,z) := \begin{pmatrix} Id & * \\ 0 & 1+\chi(x,z) \end{pmatrix},$$

and

$$d\Psi_2(x,z) := \begin{pmatrix} Id & * \\ 0 & 1+\rho_\delta'\chi(x,\rho_\delta(z)) \end{pmatrix}.$$

Since $1+\chi > 0$ the function Ψ_1 is a diffeomorphism, in the same manner Ψ_2 is also a diffeomorphism. A simple computation also shows that $(\Psi_1)^*\omega = \det(d\Psi_1)\omega = (1+\chi(x,z))\omega = \Omega$ and also $(\Psi_2)^*\omega = (1+\rho_\delta'\chi(x,\rho_\delta(z)))\omega$ which gives

$$\left(((\Psi_1)^{-1} \circ \Psi_2)^*\Omega\right)_{(x,z)} = (1+\rho_\delta'(z)\chi(x,\rho_\delta(z)))\omega_{(x,z)}.$$

Moreover,

$$d\left((x,z),\Psi_1(x,z)\right) = \left|\int_0^z \chi(x,\rho_\delta(t))\,dt\right| \leq \|\chi\|_{C^0}\delta \leq C\delta$$

so $\|\Psi_1\|_{C^0} \leq C\delta$ and similarly $\|\Psi_2\|_{C^0} \leq C\delta$.

If $z > \delta$, since χ has support in a strip we have $\Psi_1(x,z) = (x, z + c(x))$, where $c(x)$ is a function independent of z and whose value is $c(x) = \int_0^\delta \chi(x,t)\,dt$. If $z > 0.5$, we have, by the same argument, that $\Psi_2(x,z) = (x, z + d(x))$, where $d(x) = \int_0^{0.5} \rho'_\delta(t)\chi(x, \rho_\delta(t))\,dt$.

It follows that those two diffeomorphisms aren't compactly supported in $M^{n-1} \times [-1,1]$, however since $c(x) = d(x)$ (simple change of variable in the integral), $\Psi := (\Psi_1)^{-1} \circ \Psi_2$ can be restricted on $M^{n-1} \times [-1,1]$ to a compactly supported diffeomorphism. Moreover,

$$(\Psi^*\Omega)_{(x,z)} = \left(((\Psi_1)^{-1} \circ \Psi_2)^*\Omega\right)_{(x,z)} = (1 + \rho'_\delta(z)\chi(x, \rho_\delta(z)))\omega_{(x,z)}$$

and

$$\|\Psi\|_{C^0} \leq \|\Psi_1\|_{C^0} + \|\Psi_2\|_{C^0} \leq 2C\delta.$$

By Proposition 5.2 one can find a function $h \in \mathrm{Diff}_c(M^{n-1} \times [-1,1])$ such that $h^*(\Psi^*(\Omega)) = \omega$ and there exists a constant C' independent of ω and Ω such that $\|h\|_{C^0} \leq C'\|\rho'_\delta\chi(\cdot, \rho_\delta(\cdot))\|_{C^0} \leq 2C'C\delta$. It then follows that Ψh is the diffeomorphism we needed.

□

5.2. **Proof of Lemma 1.3 and Lemma 1.5.** The proof of Lemma 1.3 (resp. Lemma 1.5) will go as follows we first extend the area-preserving embedding ϕ to a C^0-small diffeomorphism (resp. homeomorphism) f not necessarily preserving the area form (in the continuous case we still need that $f^*\omega$ is an area-form). The diffeomorphism (resp. homeomorphism) f will verify that its C^0-norm is smaller than $C_1\delta$ for some constant $C_1 > 0$ independent of δ. Then working on the area form $f^*\omega$, one can find g such that:

- the C^0-norm of g is bounded by $C_2\delta$ for some constant $C_2 > 0$ independent of δ;
- we have $g \in \mathrm{Diff}_{0,c}(\mathbb{A}_{1+D_2\delta})$ for some constant D_2 independent of δ;
- if χ is such that $g^*\omega = (1+\chi)\omega$, then $\|\chi\|_{C^0} \leq E_1$ for some constant $E_1 > 0$ independent of δ.

We will then be able to apply Proposition 5.3 in order to finish the proof.

Proof of Lemma 1.3. In what follows, the C_i's, D_i's and E_i's are going to be positive constants independent of δ, the letter C will be used for a control of the C^0-norm, the letter D for a control of the support and the letter E for a control on an area-form. We can assume without loss of generality that the area form ω is $dx \wedge dy$, where x denote the angular coordinate on S^1 and y the radius along $\mathbb{A}_3$. We start the proof as in [EPP12] by using a non area-preserving diffeomorphism f extending ϕ as stated in Lemma 5.4, we will discuss this lemma in Section 6.

Lemma 5.4. *Let ϕ be a smooth embedding of an open neighborhood of $\mathbb{A}_1$ into $\mathbb{A}_2$, isotopic to the identity, such that $\|\phi\|_{C^0} \leq \delta$ for some $\delta > 0$ small enough. Then there exists constants $C_1 > 0$ and $D_1 > 0$ that do not depend of δ and $f \in \mathrm{Diff}_{0,c}(\mathbb{A}_2)$ such that f is supported in $\mathbb{A}_{1+D_1\delta}$, satisfies*

$$f|_{\mathbb{A}_{1-D_1\delta}} = \phi|_{\mathbb{A}_{1-D_1\delta}}$$

and

$$\|f\|_{C^0} \leq C_1\delta.$$

Moreover, if $\phi = Id$ outside a quadrilateral $I \times [-1,1]$ and $f(I \times [-1,1]) \subset I \times [-2,2]$ for some arc $I \subset S^1$, then f can be chosen to be the identity outside $I \times [-2,2]$.

Denote $\Omega := f^*\omega$ and define $\mathbb{A}_- := S^1 \times [-2,0]$ and $\mathbb{A}_+ := S^1 \times [0,2]$. By the condition on the area between the curves $S^1 \times y$ and $\phi(S^1 \times y)$ the following equalities hold:

$$\int_{\mathbb{A}_-} \omega = \int_{\mathbb{A}_-} \Omega, \quad \int_{\mathbb{A}_+} \omega = \int_{\mathbb{A}_+} \Omega.$$

We are going to adjust f by constructing $h \in \mathrm{Diff}_{0,c}(\mathbb{A}_2)$ such that $h|_{\mathbb{A}_{1-D_3\delta}} = Id$, $h^*\Omega = \omega$ and $\|h\|_{C^0} \leq C_3\|\phi\|_{C^0}$, for some $D_3, C_3 > 0$. To do so we just need to provide an extension on $\mathbb{A}_+$. Indeed, by symmetry, it would also give an extension on $\mathbb{A}_-$. Now define a symplectomorphism Ψ by gluing the two extension along $\mathbb{A}_{1-D_3\delta}$. The symplectomorphism Ψ might not be a Hamiltonian diffeomorphism but after a Lipschitz C^0-adjustment (given in Lemma 4.6), we can make sure that the resulting symplectomorphism is a Hamiltonian diffeomorphism. This would finish the proof of the extension lemma.

Adjusting the area of the squares

Divide the annulus $S^1 \times [1-(1+C_1+D_1)\delta, 1+(1+C_1+D_1)\delta]$ in N squares $R_1, \dots, R_N$ for some integer N such that the squares have side length $2(1+C_1+D_1)\delta$. We denote Γ the 1-skeleton of the partition by squares. Then we can find $h_1 \in \mathrm{Diff}_c(\mathbb{A}_2)$ a C^0-small diffeomorphism, which has C^0-norm as small as we want such that $h_1^*\Omega$ is equal to ω on Γ (for the construction of h_1 we refer to the paragraph **Adjusting Ω on Γ** in [EPP12]). Denote

$$\Omega' := h_1^*\Omega,$$

and assume that $\|fh_1\|_{C^0} \leq (1+C_1)\delta$ by asking $\|h_1\|_{C^0} \leq \delta$. Note that here we have

$$\int_{\mathbb{A}_+} \Omega' = \int_{\mathbb{A}_+} \omega.$$

We will use the same method as in the paragraph **Adjusting the areas of the squares** of [EPP12] but modify it in order to apply Proposition 5.3 as we wish. On a square R_i we obtain the following inequality by considering the fact that $\|fh_1\|_{C^0} \leq (1+C_1)\delta$ so the image of R_i by fh_1 is inside a square of side length $(4+4C_1+2D_1)\delta$ and also such that the square of side length $2D_1\delta$ is inside of the image of R_i, this gives the following estimates

$$\frac{4(D_1)^2\delta^2}{4(1+C_1+D_1)^2\delta^2} \leq \frac{\int_{R_i}\Omega'}{\int_{R_i}\omega} \leq \frac{4(2+2C_1+D_1)^2\delta^2}{4(1+C_1+D_1)^2\delta^2}$$

and after simplification,

$$\frac{(D_1)^2}{(1+C_1+D_1)^2} \leq \frac{\int_{R_i}\Omega'}{\int_{R_i}\omega} \leq \frac{(2+2C_1+D_1)^2}{(1+C_1+D_1)^2}.$$

By renaming the constant on the left-hand side and right-hand side of the inequality, and setting $s_i := \int_{R_i}\Omega'$, $r_i := \int_{R_i}\omega$ we can eventually rewrite this as:

$$0 < 1 - A \leq \frac{s_i}{r_i} \leq 1 + A, \tag{1}$$

for some constant $1 > A > 0$ independent of δ. Set $t_i := \frac{s_i}{r_i} - 1$. By (1),

$$|t_i| \leq A < 1. \tag{2}$$

For each i choose a non-negative function $\overline{\rho}_i$ supported in the interior of R_i such that $\int_{R_i}\overline{\rho}_i\omega = r_i$ and

$$\|\overline{\rho}_i\|_{C^0} \leq E_2 < \frac{1}{A}, \tag{3}$$

for some constant E_2 independent of δ. Define a function ϱ on $\mathbb{A}_{1+C_1+D_1}$ by

$$\varrho = 1 + \sum_i t_i\overline{\rho}_i.$$

We denote $D_2 := 1 + C_1 + D_1$. By (2) and (3), we see that ϱ is positive. Moreover, ϱ is equal to 1 over Γ and the two area forms $\varrho\omega$ and Ω' have the same integral on each square R_i. We can thus apply part (iii) of Proposition 5.2 to the forms Ω' and $\varrho\omega$ on $S^1 \times [1-D_2\delta, 1+D_2\delta]$ and the skeleton Γ: these forms coincides near the boundary of $S^1 \times [1-D_2\delta, 1+D_2\delta]$, therefore there exists a diffeomorphism h_2 with compact support in $S^1 \times [1-D_2\delta, 1+D_2\delta]$ such that $h_2^*\Omega = \varrho\omega$ and $\|h_2\|_{C^0} \leq 2\sqrt{2D_2}\delta$.

Note that

$$\int_{\mathbb{A}_+} \varrho\omega = \int_{\mathbb{A}_+} \Omega' = \int_{\mathbb{A}_+} \omega.$$

In conclusion, we constructed a diffeomorphism $fh_1h_2 := g \in \mathrm{Diff}_c(\mathbb{A}_{1+D_2\delta})$ such that:

- the C^0-norm is bounded, $d(fh_1h_2, Id) \leq (1 + C_1 + 2\sqrt{2D_2})\delta =: C_2\delta$,
- the support is controlled, $fh_1h_2|_{\mathbb{A}_{1-D_2\delta}} = \phi$,
- the pullback of the area form is controlled by $\varrho\omega = (fh_1h_2)^*\omega = (1+\chi)\omega$ and $\|\chi\|_{C^0} \leq E_1 := E_2A$.

We define
$$\Omega'' := g^*\omega.$$
We can now apply Proposition 5.3 with $M = S^1$ and the two area forms ω and Ω'' in order to find h and finish the proof. $\square$

To adapt the extension lemma in the continuous case, we will only describe the changes in the previous proof. The overall idea stays the same in both proofs.

Proof of Lemma 1.5. We use Proposition 5.1 in order find first a homeomorphism that allows us to restrict to the case where μ is a symplectic form ω. We extend the continuous area-preserving embedding ϕ to a global homeomorphism of $\mathbb{A}_2$ with the help of Lemma 5.5 (analogue of Lemma 5.4).

Lemma 5.5. *Let ϕ be a continuous area-preserving embedding of an open neighborhood of $\mathbb{A}_1$ into $\mathbb{A}_2$, isotopic to the identity, such that $\|\phi\|_{C^0} \leq \delta$ for some $\delta > 0$ small enough. Then there exists constants $C_1 > 0$ and $D_1 > 0$ that do not depend on δ and $f \in Homeo_{0,c}(\mathbb{A}_2)$ such that f is supported in $\mathbb{A}_{1+D_1\delta}$, satisfies*
$$f|_{\mathbb{A}_{1-D_1\delta}} = \phi|_{\mathbb{A}_{1-D_1\delta}}$$
and
$$\|f\|_{C^0} \leq C_1\delta.$$
Moreover, if $f = Id$ outside a quadrilateral $I \times [-1, 1]$ and $f(I \times [-1, 1]) \subset I \times [-2, 2]$ for some arc $I \subset S^1$, then f can be chosen to be the identity outside $I \times [-2, 2]$.

Denote $\nu := f^*\omega$ (ν is then an OU measure) then, by the condition on nullity of the area between $S^1 \times y$ and $\phi(S^1 \times y)$ in Lemma 1.5, the following equalities hold:
$$\omega(\mathbb{A}_-) = \nu(\mathbb{A}_-), \quad \omega(\mathbb{A}_+) = \nu(\mathbb{A}_+).$$

We proceed as before and we claim that the paragraph **Modifying the area form Ω to find a constant estimate** is actually now easier. Indeed everything transpose at one exception, since it is not mandatory to obtain a diffeomorphism we do not have to adjust the Ω on the skeleton Γ. We just have to apply Proposition 5.1 directly on squares $R_1, \ldots, R_n$ alongside the neighborhood of $\mathbb{A}_1$. The resulting C^0-small homeomorphism h_1 is such that $h_1^*\nu = \varrho\omega$ is now an area-form, this means that we can apply Proposition 5.3 and finish the proof of Lemma 1.5. $\square$

6. Proof of the extension lemmas

The proof of Lemma 5.4 can be found, after some small adjustment, in [EPP12]. We present instead a proof of Lemma 5.5, the proof is an adaptation to the continuous case of Lemma 5 in [EPP12, section 1.6.3]. For this we will need an adaptation of the appendix of Michael Khanevsky from the same paper.

Lemma 6.1. *Set $L := S \times 0$ in $\mathbb{A}_1$. Assume that ϕ is a continuous embedding of an open neighborhood of L in $\mathbb{A}_1$, so that L is homotopic to $\phi(L)$ and $\|\phi\|_{C^0} \leq \varepsilon$ for some ε small enough.*

Then there exists a homeomorphism ψ of $\mathbb{A}_{D\varepsilon}$ such that $\psi = \phi$ on L and $\|\psi\|_{C^0} \leq C\varepsilon$ for some $D, C > 0$ independent on ε.

Moreover, if $\phi = Id$ outside some arc $I \subset L$ and $\phi(I) \subset I \times [-1, 1]$, then we can also chose ψ with ψ being the identity outside $I \times [-1, 1]$.

Remark 6.2. *We added an extra result on the support of ψ that was not made by Michael Khanevsky but does not require more effort and is needed in the proof of the sharp area-preserving extension lemma.*

We recall the following corollary of the Jordan-Schönflies Theorem presented in [Tho92].

Corollary 5. *Let Γ and Γ' be two 2-connected plane graphs (planar graphs embedded in $\mathbb{R}^2$) and g a homeomorphism and plane-homeomorphism (i.e. g is an isomorphism such that a cycle in Γ is a face boundary of Γ if and only if the corresponding is a face boundary of Γ') of Γ onto Γ'. Then g can be extended to a homeomorphism of the whole plane.*

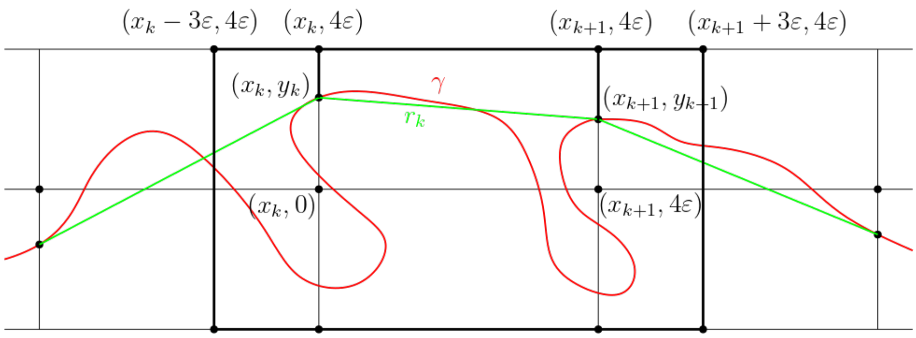

FIGURE 3. Illustration of the proof of the extension lemma.

Proof of Lemma 6.1. We start by defining some notations and preliminary results, let $K := \phi(L)$, and let K be parameterized by $\gamma : \mathbb{S}^1 \to \mathbb{A}_1$, $t \mapsto \phi(t, 0)$. Thus γ can be viewed as a map from L to K and this map has a C^0-norm smaller than ε.

Let $(x_1, 0), (x_2, 0), \dots, (x_{2n}, 0)$ points on L in trigonometric order around L such that the distance between two consecutive points is 8ε (more precisely the distance is $8\varepsilon + O(\varepsilon)$). We denote S_k the square whose vertices are $(x_k, -4\varepsilon), (x_k, 4\varepsilon), (x_{k+1}, 4\varepsilon)$ and $(x_{k+1}, -4\varepsilon)$. We also denote R_k the rectangle whose vertices are $(x_k - 3\varepsilon, -4\varepsilon), (x_k - 3\varepsilon, 4\varepsilon), (x_{k+1} + 3\varepsilon, 4\varepsilon)$ and $(x_{k+1} + 3\varepsilon, -4\varepsilon)$. The first observation is that since $\|\phi\|_{C^0} \leq \varepsilon$, the curve K is strictly inscribed inside the union of the squares S_k.

We define $y_k := \sup\{y \in \mathbb{R}, (x_k, y) \in K\}$, that is (x_k, y_k) is the highest point on the side of the square R_k that is also in K. Let r_k the line segment joining (x_k, y_k) and (x_{k+1}, y_{k+1}). Our goal is to find ψ a C^0-small homeomorphism such that $\psi(K)$ is the union of the segments r_k. We also define $t_k := \gamma^{-1}(x_k, y_k)$. The second observation we make is that $\gamma|_{[t_k, t_{k+1}]}$ lies inside R_k, indeed we must have $[t_k, t_{k+1}] \subset [x_k - \varepsilon, x_{k+1} + \varepsilon]$, and $\pi_x(\gamma([t_k, t_{k+1}])) \subset [x_k - 2\varepsilon, x_{k+1} + 2\varepsilon]$, where π_x denotes the projection onto $S^1 \times \{0\}$.

Step 1. We define now, for k odd, Γ_k the 2-connected graph whose vertices are the points $(x_k, 4\varepsilon)$ and $(x_{k+1}, 4\varepsilon)$ and whose edges are $[(x_k, 4\varepsilon)(x_k, y_k)] \cup \gamma|_{[t_k, t_{k+1}]} \cup [(x_{k+1}, y_{k+1})(x_{k+1}, 4\varepsilon)]$ and R_k, this is indeed a plane graph since $\gamma|_{[t_k, t_{k+1}]}$ lies strictly inside R_k. We also define Γ'_k the graph with the same edges except for $\gamma|_{[t_k, t_{k+1}]}$ that is replaced by r_k. Now we define a homeomorphism g_k between Γ_k and Γ'_k by letting g be the identity on their common edges and any homeomorphism between $\gamma|_{[t_k, t_{k+1}]}$ and r_k. We can then apply the Corollary 5 of the Jordan-Schönflies Theorem. We obtain the homeomorphism ψ_k, since the extension is made connected component by connected component we can ask for ψ_k to be the identity on the unbounded connected component of Γ_k and Γ'_k. Since the support of ψ_k is inside R_k we have then $\|\psi_k\|_{C^0} \leq 17\varepsilon$. Now $\psi' := \Pi_{k \text{ odd}} \psi_k$ has good properties, namely, $\|\psi'\|_{C^0} \leq 17\varepsilon$, and $\psi'(K)$ coincides with r_k for k odd.

Step 2. We have to do the same thing for k even. We define $M_k := ((x_k + x_{k+1})/2, (y_k + y_{k+1})/2)$ and $N_k := ((x_k + x_{k+1})/2, -4\varepsilon)$.

For this we split each segments r_k in two parts in the middle, $p_k := [(x_k, y_k)M_k]$ and $q_{+1} := [M_k, (x_{k+1}, y_{k+1})]$. For k odd we link M_k and N_k by a polygonal line segment s_k inside R_k and not crossing $\gamma|_{[t_{k-1}, t_k]}$ nor $\gamma|_{[t_k, t_{k+1}]}$ (this can be done by the mean of Lemma 2.1 of [Tho92] for example). We define now, for k even, R'_k the polygon whose sides are the same side as of R_k but we replace the segment $[(x_k, y_k)(x_k, -4\varepsilon)]$ by $q_k \cup s_{k-1} \cup [N_{k-1}(x_k, -4\varepsilon)]$ and the line segment $[(x_{k+1}, y_{k+1})(x_{k+1}, -4\varepsilon)]$ by $p_{k+1} \cup s_{k+1} \cup [N_{k+1}(x_{k+1}, -4\varepsilon)]$.

We can now apply the Corollary 5 to the 2-connected plane graphs $\Gamma_k := R'_k \cup \gamma|_{[t_k, t_{k+1}]}$ and $\Gamma'_k := R'_k \cup r_k$ and the homeomorphism g being the identity on R'_k, we obtain ψ_k a homeomorphism such that $\psi_k(\gamma|_{[t_k, t_{k+1}]}) = r_k$ and ψ_k is the identity outside R'_k. We denote $\psi'' = \Pi_{k \text{ even}}$, since the diameter of R'_k is smaller than $C\varepsilon$ for C a positive real number we have $\|\psi_k\|_{C^0} \leq C\varepsilon$ and $\|\psi'\|_{C^0} \leq C\varepsilon$ and $\psi''(\psi'(K))$ is now the union of the segments r_k.

Step 3. Once we have the union of the segments r_k it is easier to finish, by taking for example the flow of an appropriate vector field (the cut-off of a constant vector field on the fibers $(x, \cdot)$ for example). We denote ψ''' the last homeomorphism we obtain.

Step 4. There is one last thing to do is a small perturbation of $\psi''' \circ \psi'' \circ \psi'$ such that it coincides on L with the homeomorphism ϕ. This finishes the proof.

□

This is the tool we needed to prove Lemma 5.5. Notice here that in contrary of the proof of Lemma 6.6 in [EPP12] the proof is very short, indeed we do not need to care more about the extension since we care about obtaining a diffeomorphism and not only a homeomorphism.

Proof of Lemma 5.5. We apply Lemma 6.1 to the curves $S^1 \times \{\pm 1\}$ in $\mathbb{A}_2$ and their images under ϕ. We can find $\psi \in \mathrm{Homeo}_{0,c}(\mathbb{A}_2)$ supported in $\mathbb{A}_{1-D_1\delta, 1+D_1\delta}$ and agreeing with ϕ^{-1} on $\phi(S \times \{\pm 1\})$. Now when we extend $\psi' = \psi\phi$ by the identity outside of $\mathbb{A}_1$ we get the required result. □

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