Relativistic stochastic hydrodynamics from quantum nonlinear projection operator

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We systematically derive relativistic stochastic hydrodynamics based on the method of quantum nonlinear projection operator. Morozov's nonlinear projection operator is a generalization of the well-known linear Mori-Zwanzig projection operator method, from which one can account for the nonlinear interaction between macroscopic modes. The quantum generalized Fokker-Planck and Langevin equations are also obtained using this formalism, which are fundamentally important in non-equilibrium statistical physics. As an application, the relativistic stochastic hydrodynamic equations with Gaussian noises are derived, which are applicable in studying anomalous transport phenomena near critical points. The possible extension to include multiplicative noises is also discussed.

I. INTRODUCTION

In the phase diagram of quantum chromodynamics (QCD), the critical point (CP) is the end point of the first-order phase transition line in the temperature-baryon chemical potential plane. Its property is one of the most fascinating features of the QCD phase diagram and draws the extensive attention of the community of heavy-ion collisions all the time. Since the concept of the QCD critical point was first put forward, constant theoretical efforts have been put into the related studies. Nowadays its existence is predicted by various QCD effective models and suggested by lattice QCD simulations [1–10]. Based on these existing researches, the Beam Energy Scan (BES) program was launched by Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory with one physical goal of searching for this conjectured QCD critical point. Recent relativistic heavy-ion collision experiments at RHIC [11], Facility for Antiproton and Ion Research (FAIR) [12, 13], and Nuclotron-base Ion Collider fAcility (NICA) [14, 15] are devoted to finding the QCD critical point and have seen much experimental progress [16–34]. At this point very large fluctuation will occur, which could induce a critical phenomenon of divergent bulk viscosity [35] as a potential signal of the existence of the critical point in experiments. However, as pointed out in [36], the universal mechanism causing the critical divergence of the transport coefficients has not been elucidated clearly. In fact, it is the nonlinear fluctuations between the fluctuations of macroscopic variables that lead to the divergence: the transport coefficients thus can be separated into two parts, the rapidly relaxing part of microscopic time scale decided by a microscopic theory, such as the Boltzmann equation [37] and the long-time tail of macroscopic time scale. Only by taking into account nonlinear interactions between macroscopic modes can the divergent behavior of the transport coefficients be fully understood [38, 39], especially when the thermal fluctuations of long-wavelength modes are necessarily large near the critical point, which motivates the present study on the relativistic fluctuating hydrodynamics.

Another important motivation is the introduction of the method of projection operator for studying the dynamic evolution of some relativistic systems. The projection operator, initially proposed by Mori and Zwanzig [40, 41], has long been a powerful tool for the extraction of slow dynamics. In many cases of our interest, especially when the behavior of a system over long distances and time scales is concerned, there is a clear separation in the time scales between the relevant slow variables and the vast number of fast microscopic degrees of freedom. The distinction of what is slow or fast is not that arbitrary but dictated by the fundamental physical principles. For instance, the conservation laws give rise to slow variables. According to the Noether theorem, every continuous symmetry corresponds to a conserved observable that generates the symmetry. The dynamics of the local density associated with the conserved generator exhibits a slow behavior, as is manifestly shown in hydrodynamics [42]. Another main source of slow variables concerns the phase transition. As one passes into the ordered state in a phase transition, the slow variables associated with the breaking of a continuous symmetry appear known as Nambu–Goldstone modes [43, 44]. For instance, the structure of the fluctuation function of the transverse staggered fields in isotropic antiferromagnet is changed qualitatively by the broken symmetry and looks much like the hydrodynamic correlation functions in a fluid, indicating that the Nambu–Goldstone modes associated with the spontaneous symmetry breaking have the dynamical effect of simulating a conservation law [45]. A third example of a slow variable is the order parameter near a second-order phase transition, during which the system is experiencing a phenomenon called critical slowing down [46]. Besides, the time scale separation, as observed in Brownian motion [47, 48], typically results from the significant

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difference in masses between the pollen particle and the background fluid particles, which is also a very classical example. Note that the formalism of the projection operator has been widely applied to study slow dynamics listed above in the context of condensed matter physics and modern statistical physics, see [41, 45, 49–51] and the reference therein for a comprehensive overview. However, there are fewer related works in the context of relativistic statistics [52–54]. Therefore, it is worthwhile to give discussions on the application of the projection operator techniques to relativistic situations such as heavy-ion collisions, cosmology, and astrophysics, as we will devote our efforts to below.

In this paper, we utilize the nonlinear projection operator method developed by Morozov [55, 56] to derive the nonlinear evolution equations of relativistic fluctuating hydrodynamics in a fully quantum manner. Before passing into the next stage, it is necessary to pause and give comments on the adopted formalism. Mori's original projection operator is inherently linear in the relevant variables and when used to describe a fluid system, it is only applicable to systems in the linear regime [53]. To circumvent this problem, new types of projection operators were proposed with notable contributions in this regard [38, 57–59]. The Dirac delta functions with macrovariables as arguments are key elements for technically resolving these issues. As will be demonstrated in the main text, this elegant approach not only allows for the incorporation of macrovariable dynamics into a single equation but also facilitates the derivation of both the generalized Fokker-Planck equation and the generalized Langevin equation simultaneously, while maintaining exactness. The nonlinear projection operator proposed by Morozov is a quantum treatment including the non-commutativity of operators. In this case, the Dirac delta function is enhanced to the Dirac delta operator according to the Weyl correspondence principle [60, 61] in accordance with its classical form. In the meanwhile, the c-number symbols of operators is used to characterize the dynamical variables. We leave the details about the Weyl symbol description to the following sections.

This paper is organized as follows. In Sec. II, we present a brief review of Morozov's quantum nonlinear projection operator. In Sec. III and subsequent IV, the generalized Fokker-Planck equation and the generalized Langevin equation are derived based on the formalism of nonlinear projection operator. After finishing these, Sec. V serves as a quantum derivation of fluctuating hydrodynamics with Gaussian noise. The possible extension to include multiplicative noises in the present framework is discussed in Sec. VI. Finally, we give a summary and outlook in Sec. VII.

Natural units $\hbar = k_B = c = 1$ are used. The metric tensor here is given by $g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, while $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$ is the projection tensor orthogonal to the four-vector fluid velocity u^μ. In addition, we employ the symmetric shorthand notations:

$$X^{(\mu\nu)} \equiv (X^{\mu\nu} + X^{\nu\mu})/2, \tag{1}$$

$$X^{\langle\mu\nu\rangle} \equiv \left(\frac{\Delta^\mu_\alpha\Delta^\nu_\beta + \Delta^\nu_\alpha\Delta^\mu_\beta}{2} - \frac{\Delta^{\mu\nu}\Delta_{\alpha\beta}}{3}\right)X^{\alpha\beta}. \tag{2}$$

II. NONLINEAR QUANTUM PROJECTION OPERATOR METHOD

In this section, we give a short review on Morozov's nonlinear projection operator method [55], which can be used elegantly to extract the slow dynamics from microscopic Hamiltonian dynamics. According to quantum mechanics, the evolution of an operator at time t, $\hat{B}(t) = e^{i\hat{H}t}\hat{B}(0)e^{-i\hat{H}t}$, is ruled by the Heisenberg equation,

$$\partial_t\hat{B}(t) = i[\hat{H}, \hat{B}(t)] \equiv iL\hat{B}(t), \tag{3}$$

where the Hamiltonian $\hat{H}$ encodes all the information of microscopic dynamics. Formally, the Liouville operator L is defined and $\hat{B}(t) = e^{iLt}\hat{B}(0)$. For the need of discussing a many-body statistical system, a density operator $\hat{\rho}(\Gamma, t)$ describing the phase space Γ distribution of a given ensemble system is indispensable. The Liouville equation governing the evolution of the density operator is written as

$$\partial_t\hat{\rho}(t) = -iL\hat{\rho}(t) \tag{4}$$

where the dependence on Γ has been omitted. Note as an aside, $\hat{\rho}(t)$ is sometimes called the nonequilibrium statistical operator. Formally, $\hat{\rho}(t)$ takes the form $\hat{\rho}(t) = e^{-iLt}\hat{\rho}(0)$. The average of $\hat{B}$ over the density operator $\rho(t)$ is defined as

$$\langle\hat{B}\rangle = \mathrm{Tr}(\hat{\rho}(t)\hat{B}) = \mathrm{Tr}(\hat{\rho}\hat{B}(t)\,) \tag{5}$$

and the second equality indicates that the same expectation value is given in both Schrödinger picture and Heisenberg picture. Eq.(5) also means that iL is an anti-self-adjoint operator in the sense of

$$\mathrm{Tr}\left(\hat{B}iL\hat{C}\right) = -\,\mathrm{Tr}\left(\hat{C}iL\hat{B}\right). \tag{6}$$

In the remainder of this text, we choose to work in Heisenberg picture, contrary to the one used in [55].

The physical contemplation must be initiated right from the outset, which lies in a sensible consideration about how to choose the basis variables that are projected onto. A first key point is the identification of the slow dynamic variables. Considering that the conservation laws can be naturally made a guiding principle, a set of coarse-grained collective variables for describing slowly-varying macroscopic processes in the system are chosen as the basis vectors $\{\hat{A}(t,\boldsymbol{x})\} = \{\hat{A}_1, \hat{A}_2, ..., \hat{A}_N\}$, namely, all the local densities of correspondent conserved quantities are included in the list, which are closely related to the slow modes as a result of their conserved property. As a supplement, these properly chosen collective variables are all Hermitian.

Mori's linear projection operator formalism possess inherent shortcomings and has no access to the nonlinear interactions between the slow modes. To overcome it, a Dirac delta operator is introduced

$$\hat{f}(a) = \delta(\hat{A}-a) = \frac{1}{(2\pi)^N}\int dx \exp(ix\cdot(\hat{A}-a)) \tag{7}$$

which is obtained according to Weyl correspondence rule [55, 60, 61] consistent with its corresponding classical form

$$f(a) = \delta(A-a) = \Pi_{n=1}^{N}\delta(A_n - a_n) \tag{8}$$

where N is the number of basis variables. With the Dirac delta operator at hand, an arbitrary operator can be represented in the form of the following identity with the help of the Weyl symbol,

$$g(\hat{A}) = \int da g(a)\delta(\hat{A}-a) \tag{9}$$

where $g(a)$ is the Weyl symbol of $g(\hat{A})$. A key observation should be notice that $\delta(\hat{A}-a)$ can generate all nonlinear couplings among the variables $\hat{A}_i$ effortlessly as long as $g(a)$ is nonlinear function of a_i.

In addition, we introduce

$$f(a,t) \equiv \mathrm{Tr}\left(\hat{\rho}(t)\hat{f}(a)\right), \tag{10}$$

which is the macroscopic probability density function in a space. Once it is integrated with $\hat{f}(a)$ over a, the distribution function of $\hat{A}$ will be obtained. For pedagogical reason, it is more convenient to rewrite $f(a,t)$ in a classical form

$$f(a,t) \equiv \int d\Gamma \delta(A(\Gamma)-a)\rho(\Gamma,t) \tag{11}$$

where Γ denotes the phase space variables and the integration covers the phase space points Γ lying on the compatible hypersurface with a constraint $A(\Gamma) = a$. One can also transfer to Heisenberg picture without doubts

$$f(a,t) = \mathrm{Tr}\left(\hat{\rho}\hat{f}(a,t)\right) \tag{12}$$

where $\hat{f}(a,t) = e^{iLt}\hat{f}(a,0) = e^{iHt}\hat{f}(a,0)e^{-iHt}$ is an operator in Heisenberg picture.

It is useful to recall the retarded Liouville equation taking the form

$$\left(\frac{\partial}{\partial t} + iL\right)\hat{\rho}(t) = -\epsilon(\hat{\rho}(t) - \hat{\rho}_q(t)) \tag{13}$$

where the quasiequilibrium statistical operator $\hat{\rho}_q$ is introduced and $\epsilon \to 0$ after taking thermodynamic limit [62, 63]. Compared to Eq.(4), the presence of the infinitesimal source ϵ in the right-hand side (rhs) of this equation breaks the time reversal symmetry of Liouville equation and selects causal and retarded solutions. The form of the quasi-equilibrium statistical operator $\hat{\rho}_q(t)$ is determined by the condition of maximum informational entropy and will be specified soon. In this script, we assume that $\hat{\rho}_q(t)$ and $\rho(t)$ are expressed through the distribution function of basic variables $f(a,t)$. Furthermore, the statistical operator $\hat{\rho}_q$ also needs to satisfy the self-consistency condition

$$f(a,t) = \mathrm{Tr}\left(\hat{\rho}\hat{f}(a,t)\right) = \mathrm{Tr}\left(\hat{\rho}_q\hat{f}(a,t)\right), \tag{14}$$

which is equivalent to the Landau matching conditions widely used in relativistic kinetic theory for defining the quasiequilibrium distribution function [37, 42]. Note we have shifted to the Heisenberg picture in Eq.(14).

Using the Weyl correspondence rule, $\hat{\rho}_q$ can be put in the following form

$$\hat{\rho}_q = \int da' G(a')\hat{f}(a'). \tag{15}$$

The task in this stage turns into the determination of the Weyl symbol of $\hat{\rho}_q$ based on the self-consistency condition Eq.(14). To make it, we'll multiply Eq.(15) by $\hat{f}(a)$ and use Eq.(14). As a result, we obtain

$$f(a,0) = \int da' W(a,a') G(a') \tag{16}$$

with the definition

$$W(a,a') = \mathrm{Tr}(\hat{f}(a)\hat{f}(a')\,) \tag{17}$$

possessing the following properties

$$\begin{aligned} W(a) &= \int da' W(a,a') = \mathrm{Tr}(\hat{f}(a)\,), \\ W(a') &= \int da W(a,a') = \mathrm{Tr}(\hat{f}(a')\,) \end{aligned} \tag{18}$$

where $W(a)$ denotes a quantum-mechanical generalization of the classical density of states with given values of the coarse-grained variables a.

From Eq.(16), we get

$$G(a) = \int da' W_{-1}(a,a') f(a',0) \tag{19}$$

where the following identity is satisfied

$$\int da'' W(a,a'') W_{-1}(a'',a') = \delta(a-a'). \tag{20}$$

After substituting Eq.(19) into (15), the quasiequilibrium statistical operator is brought to the form

$$\hat{\rho}_q = \int da da' \hat{f}(a) W_{-1}(a,a') f(a',0). \tag{21}$$

Combining Eqs.(18) and (20), it is not hard to show that $W(a,a')$ and $W_{-1}(a,a')$ have singular parts that can be conveniently isolated. To that end, we assume

$$W(a,a') = W(a)\big(\delta(a-a') - R(a,a')\big) \tag{22}$$

$$W_{-1}(a,a') = W^{-1}(a')\big(\delta(a-a') + r(a,a')\big) \tag{23}$$

then the following relations should be satisfied by analyzing Eq.(18)

$$\int da W(a) R(a,a') = \int da' R(a,a') = 0, \tag{24}$$

and the substitution of Eqs.(22) and (23) into Eq.(20) leads to

$$r(a,a') = R(a,a') + \int da'' R(a,a'') r(a'',a'). \tag{25}$$

Given $R(a,a')$, $r(a,a')$ is obtained by iteration. The use of Eq.(24) gives

$$\int da W(a) r(a,a') = \int da' r(a,a') = 0, \tag{26}$$

where the last equality sign follows from the iteration of the last equality sign of Eq.(24). The regular terms appearing in Eqs.(22) and (23) originate from the non-commutativity of quantum operators.

If focusing on the derivation in [55], one will find that Eq.(21) will play a central role in getting the definition of the projection operator. The author utilizes this to calculate the evolution function of $\hat{\rho}_q(t)$ (in Schrödinger picture) and the projection operator is naturally introduced to rewrite the resulting equation in a compact form. Following [55], the projection operator P is defined as

$$P\hat{B} = \int da da' \hat{f}(a) W_{-1}(a,a')\, \mathrm{Tr}(\hat{B}\hat{f}(a')\,), \tag{27}$$

as can be seen clearly, this definition is independent of which picture one works in. If $W_{-1}(a, a') \sim \delta(a - a')$ is local, then P degenerates into a similar form to classical delta projection operator [64]. This matching is not coincidental: if one assumes the initial state is the microcanonical distribution $f(a', 0) = \delta(a' - a_0)$, then in the local approximation

$$\hat{\rho}_q = \delta(\hat{A} - a_0)/W(a_0) \tag{28}$$

which is nothing but Eq.(12) of [64].

It is easy to verify P is idempotent, i.e., $P^2 = 1$. Moreover, the projection operator satisfies

$$P\hat{f}(a) = \hat{f}(a), \tag{29}$$

$$P\hat{\rho} = \hat{\rho}_q, \tag{30}$$

$$\mathrm{Tr}(\hat{B}P\hat{C}) = \mathrm{Tr}(\hat{C}P\hat{B}), \tag{31}$$

$$PG(\hat{A}) = G(\hat{A}), \tag{32}$$

where the first three equalities are almost self-evident, $G(\hat{A})$ is a nonlinear function of $\hat{A}$ and a proof of the last one is given in Eq.(A9). The fourth equation Eq.(32) reveals the nonlinearity of P, which generalizes $P\hat{A} = \hat{A}$ to $PG(\hat{A}) = G(\hat{A})$ such that the neglected nonlinear components in linear projection operator is retrieved.

III. GENERALIZED FOKKER-PLANCK EQUATION

This section is devoted to the derivation of generalized Fokker-Planck equation. We start from

$$\frac{\partial \hat{f}(a,t)}{\partial t} = e^{iLt} iL\hat{f}(a). \tag{33}$$

Recall the Dyson-Duhamel identity [65]

$$e^{iLt} = e^{iLt}P + \int_0^t du e^{iLu} PiL(1-P)e^{iL(1-P)(t-u)} + (1-P)e^{iL(1-P)t}, \tag{34}$$

this identity can be derived via Laplace transformation just as we will show below. First, the following decomposition is employed:

$$\partial_t e^{iLt} = e^{iLt} iL = e^{iLt} PiL + e^{iLt}(1-P)iL. \tag{35}$$

Next, the Laplace transform of $\exp(iLt)$ lead us to,

$$\int_0^\infty dt e^{-zt} e^{iLt} = \frac{1}{z - iL}. \tag{36}$$

Then, a decomposition of Eq. (36) into

$$\begin{aligned}\frac{1}{z - iL} &= \frac{1}{z - iL}(z - (1-P)iL)\frac{1}{z - (1-P)iL} \\ &= \frac{1}{z - iL}(z - iL + PiL)\frac{1}{z - (1-P)iL} \\ &= \frac{1}{z - (1-P)iL} + \frac{1}{z - iL} PiL \frac{1}{z - (1-P)iL}\end{aligned} \tag{37}$$

is utilized. After performing the inverse Laplace transform, a crucial identity is reached,

$$e^{iLt} = e^{(1-P)iLt} + \int_0^t ds e^{iL(t-s)} PiL e^{(1-P)iLs}, \tag{38}$$

which follows from the derivation in [53]. Noticing that

$$(1-P)e^{iL(1-P)t} = e^{(1-P)iLt}(1-P), \tag{39}$$

the multiplication of eq.(38) by $(1-P)$ from the right will give us the desired Dyson-Duhamel identity. The proof has been completed.

It is time to apply Dyson-Duhamel identity to the rhs of Eq.(33)

$$\frac{\partial \hat{f}(a,t)}{\partial t} = e^{iLt}PiL\hat{f}(a) + \int_0^t du e^{iLu} PiL(1-P)e^{iL(1-P)(t-u)}iL\hat{f}(a) + (1-P)e^{iL(1-P)t}iL\hat{f}(a). \tag{40}$$

Then we introduce a useful definition

$$iL\hat{f}(a) \equiv -\frac{\partial \hat{J}_i(a)}{\partial a_i} \tag{41}$$

where Einstein summation conventions are respected hereafter and

$$\hat{J}(a) \equiv \frac{1}{(2\pi)^N}\int dx e^{ix(\hat{A}-a)}\int_0^1 d\tau e^{-i\tau x\hat{A}} iL\hat{A} e^{i\tau x\hat{A}}. \tag{42}$$

In deriving this equation, an important identity, i.e., Kubo identity [66]

$$[e^{\hat{B}}, \hat{H}] = e^{\hat{B}}\int_0^1 d\tau e^{-\tau\hat{B}}[\hat{B},\hat{H}]e^{\tau\hat{B}} \tag{43}$$

has been utilized.

The first term in Eq.(40) is

$$\begin{aligned} e^{iLt}PiL\hat{f}(a) &= -\int da'' da' W_{-1}(a'',a')\,\mathrm{Tr}(\frac{\partial \hat{J}_i(a)}{\partial a_i}\hat{f}(a')\,)\hat{f}(a'',t) \\ &= -\frac{\partial}{\partial a_i}\int da' v_i(a,a')\hat{f}(a',t) \end{aligned} \tag{44}$$

with the nonlocal streaming velocity

$$v_i(a,a') \equiv \int da'' W_{-1}(a',a'')\,\mathrm{Tr}(\hat{J}_i(a)\hat{f}(a'')\,). \tag{45}$$

Define

$$\hat{X}_i(a) = (1-P)\hat{J}_i(a), \tag{46}$$

then the noise term is

$$(1-P)e^{iL(1-P)t}iL\hat{f}(a) = -\frac{\partial}{\partial a_i}\hat{X}_i(a,t). \tag{47}$$

The remaining diffusion term is

$$\begin{aligned} PiL(1-P)e^{iL(1-P)(t-u)}iL\hat{f}(a) &= PiLe^{i(1-P)L(t-u)}(1-P)iL\hat{f}(a) \\ &= \frac{\partial}{\partial a_i}\int da' K_{ij}(a,a',t-u)\frac{\partial}{\partial a'_j}\int da''\hat{f}(a'')W_{-1}(a'',a'). \end{aligned} \tag{48}$$

where $K_{ij}(a,a',t) \equiv \mathrm{Tr}(\hat{X}_i(a,t)\hat{X}_j(a')\,)$ and $\hat{X}(a,t) \equiv e^{(1-P)iLt}\hat{X}(a)$. The relation between the diffusion kernel K and noise function X is known as a kind of generalized fluctuation-dissipation theorem, which is of great theoretical significance in statistical physics. In the process of deriving the diffusion kernel K_{ij}, the properties of P listed in Eqs.(29) to (31) are frequently used.

Putting them together, we are reaching the quantum Fokker-Planck equation in operator form

$$\begin{aligned} \frac{\partial \hat{f}(a,t)}{\partial t} =& -\frac{\partial}{\partial a_i}\int da' v_i(a,a')\hat{f}(a',t) + \int_0^t du\frac{\partial}{\partial a_i}\int da' K_{ij}(a,a',t-u)\frac{\partial}{\partial a'_j}\int da''\hat{f}(a'',u)W_{-1}(a'',a') \\ &-\frac{\partial}{\partial a_i}\hat{X}_i(a,t), \end{aligned} \tag{49}$$

which is consistent with the Fokker-Planck equation obtained in a fully classical way [38]. The fluctuating force $\frac{\partial}{\partial a_i}\hat{X}_i(a,t)$ is uncorrelated with an arbitrary function $G(\hat{A})$ of $\hat{A}$,

$$\mathrm{Tr}(G(\hat{A})\hat{X}(a,t)) = 0. \tag{50}$$

A comment on these three terms in the rhs of Eq.(49) will be left where the nonlinear Langevin equations are derived.

Then we calculate the average of Eq.(49) with respect to $\hat{\rho}$,

$$\frac{\partial f(a,t)}{\partial t} = -\frac{\partial}{\partial a_i}\int da' v_i(a,a')f(a',t) + \int_0^t du \frac{\partial}{\partial a_i}\int da' K_{ij}(a,a',t-u)\frac{\partial}{\partial a'_j}\int da'' f(a'',u)W_{-1}(a'',a') \tag{51}$$

which is in agreement with the Fokker-Planck equation obtained in [55] but in a rather different way (directly solve $\hat{\rho}(t)$ from retarded Liouville equation and next construct $f(a,t)$ based on it). Identical results are given in two distinct quantum pictures, as it should be. The similar Fokker-Planck equation has been extensively studied with different methods [64]. There are also a large number of other works devoting to this, but due to space limitations, the reference here may not cover them. In this script, we name the equation as the Fokker-Planck equation in function form. A quick observation can be made that the noise term does not survive the expectation because

$$\mathrm{Tr}[\hat{\rho}(1-P)e^{iL(1-P)t}iL\hat{f}(a)] = \mathrm{Tr}[\hat{\rho}_q(1-P)e^{iL(1-P)t}iL\hat{f}(a)] = \mathrm{Tr}[((1-P)\hat{\rho}_q)e^{iL(1-P)t}iL\hat{f}(a)] = 0. \tag{52}$$

Recall that in Heisenberg picture $\hat{\rho} = \hat{\rho}_q$ if the system is initially in equilibrium, as is implicitly prescribed consistent with the retarded Liouville equation, see Eq.(4.3) of ref [55]. Due to this identical setting, Eq.(51) matches the resulting equation derived in [55].

Let us pause for a while to find a very useful relation. The trace over $\hat{f}(a,t)$ should be independent of time

$$\mathrm{Tr}(\hat{f}(a,t)) = \mathrm{Tr}(\hat{f}(a)) = W(a) \tag{53}$$

owing to the cyclical symmetry of trace. One should not confuse $f(a,t) = \mathrm{Tr}(\hat{\rho}\hat{f}(a,t))$ with $W(a) = \mathrm{Tr}(\hat{f}(a,t))$. Then the trace over the whole Fokker-Planck equation (49) is

$$\begin{aligned}\frac{\partial W(a)}{\partial t} =& 0 = -\frac{\partial}{\partial a_i}\int da' v_i(a,a')W(a') + \int_0^t du \frac{\partial}{\partial a_i}\int da' K_{ij}(a,a',t-u)\frac{\partial}{\partial a'_j}\int da'' W(a'')W_{-1}(a'',a') \\ &- \frac{\partial}{\partial a_i}\mathrm{Tr}(\hat{X}_i(a,t)).\end{aligned} \tag{54}$$

Now the remaining task awaiting us is to prove the second equality sign to see whether it holds naturally or offers extra constraints.

The first term in the rhs is nothing but the divergence condition [67],

$$\begin{aligned}&\frac{\partial}{\partial a_i}\int da' v_i(a,a')W(a') = \frac{\partial}{\partial a_i}\int da' \int da'' \mathrm{Tr}(\hat{J}_i(a)\hat{f}(a''))W_{-1}(a',a'')W(a') \\ =&\frac{\partial}{\partial a_i}\int da' \mathrm{Tr}(\hat{J}_i(a)\hat{f}(a')) = \frac{\partial}{\partial a_i}\mathrm{Tr}(\hat{J}_i(a)) = -\mathrm{Tr}(iL\hat{f}(a)) = i\,\mathrm{Tr}([\hat{f}(a),\hat{H}]) = 0.\end{aligned} \tag{55}$$

Because this one holds unconditionally, we should promote it to the divergence theorem instead of treating it as a condition. The same conclusion is also drawn in the context of classical statistical [45]. The second one vanishes for

$$\int da'' W(a'')W_{-1}(a'',a') = 1 + W^{-1}(a')\int da'' W(a'')r(a'',a') = 1 \tag{56}$$

is a constant, where the condition Eq.(26) is used.

As for the last noise term, it also vanishes as exhibited in appendix.B. To conclude, $\frac{\partial W(a)}{\partial t} = 0$ does not produce extra constraints. These extra constraints are very key in the construction of fluctuating nonlinear hydrodynamics for normal fluids in a classical context [67]. We shall comment on this when talking about multiplicative noises.

IV. GENERALIZED LANGEVIN EQUATION

In this section, we derive the generalized Langevin equation, which can be derived from the generalized Fokker-Planck equation. Multiply Eq.(49) with a and then integrate a, and we will find

$$\begin{aligned}\frac{\partial \hat{A}_\alpha(t)}{\partial t} =& -\int da a_\alpha \frac{\partial}{\partial a_i}\int da' v_i(a,a')\hat{f}(a',t) + \int da a_\alpha \int_0^t du \frac{\partial}{\partial a_i}\int da' K_{ij}(a,a',t-u) \\ &\times \frac{\partial}{\partial a'_j}\int da'' \hat{f}(a'',u)W_{-1}(a'',a') - \int da a_\alpha \frac{\partial}{\partial a_i}\hat{X}_i(a,t) \\ =& \int da \int da' v_\alpha(a,a')\hat{f}(a',t) + \int_0^t du \int da' (\frac{\partial}{\partial a'_j}K_{\alpha j}(a',t-u)\,) \int da'' \hat{f}(a'',u)W_{-1}(a'',a') + \hat{R}_\alpha(t) \end{aligned} \tag{57}$$

where $\hat{R}_k(t) \equiv e^{(1-P)iLt}(1-P)iL\hat{A}_k$, and recalling that

$$\hat{X}_i(a) = (1-P)\hat{J}_i(a) = \frac{1}{(2\pi)^N}(1-P)\int dx \int_0^1 d\tau e^{ix(\hat{A}-a)}e^{-i\tau x\hat{A}}(iL\hat{A}_i)e^{i\tau x\hat{A}}, \tag{58}$$

then we have

$$\begin{aligned} K_{ij}(a',t) =& \int da K_{ij}(a,a',t) = \int da\, \mathrm{Tr}(\hat{X}_i(a,t)\hat{X}_j(a')\,) \\ =& \frac{1}{(2\pi)^{2N}}\int da \int dx dx' \int d\tau d\tau'\, \mathrm{Tr}(e^{i(1-P)Lt}(1-P)e^{ix(\hat{A}-a)}e^{-i\tau x\hat{A}}(iL\hat{A}_i)e^{i\tau x\hat{A}} \\ &\times (1-P)e^{ix'(\hat{A}-a')}e^{-i\tau' x'\hat{A}}(iL\hat{A}_j)e^{i\tau' x'\hat{A}}\,) \\ =& \frac{1}{(2\pi)^N}\int dx' \int_0^1 d\tau'\, \mathrm{Tr}\left((e^{i(1-P)Lt}(1-P)iL\hat{A}_i)\ (1-P)e^{ix'(\hat{A}-a')}e^{-i\tau' x'\hat{A}}(iL\hat{A}_j)e^{i\tau' x'\hat{A}}\,\right) \\ =& \mathrm{Tr}(\hat{R}_i(t)\hat{X}_j(a')). \end{aligned} \tag{59}$$

Because of Eq.(50), the Langevin fluctuating force is also statistically independent of the collective variables, namely,

$$\mathrm{Tr}(G(\hat{A})\hat{R}_i(a,t)) = 0. \tag{60}$$

Eq.(57) is the full generalized quantum Langevin equation without any approximations. The first term in the rhs is the drift term characterized by a nonlocal streaming velocity $v(a,a')$, whose nonlocality stems from the non-commutativity of quantum operators and thus is a quantum effect absent in the classical Langevin equation. The second one is a diffusion term with non-Markov memory effects. If looking at the specific form of the diffusion or memory kernel $K_{ij}(a,t)$, one can also find that the quantum non-commutativity is involved hindering the direct comparison with its classical correspondence. The third one represents fast motion of noise, whose relation with the diffusion kernel is so-called fluctuation-dissipation theorem but not that evident at a simple glance. In order to correctly account for the physical effects brought by the nonlinear fluctuations, the stochastic noise terms, trigger by the hydrodynamic excitations generated by thermal fluctuations within the fluid, are indispensable [68].

In order to compare with the existing classical results, we have to turn to the local approximation, that is,

$$v(a,a') = v(a)\delta(a-a'), \quad W_{-1}(a'',a') = W^{-1}(a')\delta(a'-a'') \tag{61}$$

with a local averaged velocity

$$v(a) = \mathrm{Tr}(\hat{f}(a)iL\hat{A})W^{-1}(a). \tag{62}$$

In fact, a regular part can also be separated from $v(a,a')$ in the same fashion to $W(a,a')$ and $W_{-1}(a,a')$ of which the details can be found in [55]. Then the full Langevin equation becomes

$$\frac{\partial \hat{A}_\alpha(t)}{\partial t} = v_\alpha(\hat{A},t) + \int_0^t du \int da' (\frac{\partial}{\partial a'_j}K_{\alpha j}(a',t-u)\,) W^{-1}(a')\hat{f}(a',u) + \hat{R}_\alpha(t), \tag{63}$$

which matches the result of [38]. Furthermore, if we introduce the definition $\tilde{K}_{ij}(a,t) \equiv K_{ij}(a,t)/W(a)$, then the equation turns into

$$\begin{aligned}\frac{\partial \hat{A}_\alpha(t)}{\partial t} =& v_\alpha(\hat{A},t) + \int_0^t du \int da \frac{\partial}{\partial a_j}(\tilde{K}_{\alpha j}(a,t-u)W(a))W^{-1}(a)\hat{f}(a,u) + \hat{R}_\alpha(t) \\ =& v_\alpha(\hat{A},t) + \int_0^t du \int da (\frac{\partial}{\partial a_j}\tilde{K}_{\alpha j}(a,t-u))\hat{f}(a,u) + \int_0^t du \int da \tilde{K}_{\alpha j}(a,t-u)F_j(a)\hat{f}(a,u) + \hat{R}_\alpha(t)\end{aligned} \tag{64}$$

in good agreement with the form present in [64]. Here the conjugate variable of a_k is defined, in a conventional way, as the derivative of $\ln W(a)$ with respect to a_k

$$F_k(a) = \frac{\partial}{\partial a_k}\ln W(a). \tag{65}$$

Since $W(a)$ is the structure function providing a complete thermodynamic description of a system in thermal equilibrium subject to constraints giving rise to the specified values "$\hat{A}=a$", it can be seen as a partition function of the microcanonical ensemble. More intuitively, Eq.(62) can be rewritten as

$$v(a) = \mathrm{Tr}(\hat{f}(a)iL\hat{A})W^{-1}(a) = \mathrm{Tr}(\hat{\rho}(a)iL\hat{A}) \ . \tag{66}$$

Note $\mathrm{Tr}\,\hat{\rho}(a) = 1$ and $\mathrm{Tr}(\hat{\rho}(a)\hat{A}_i) = a_i$, then $\hat{\rho}(a)$ acts like a density operator in microcanonical ensemble with fixed values of the coarse-grained variables. In the case where the coarse-grained variables are the densities of thermodynamic quantities, one can take advantage of the equivalence of statistical ensembles and replace the averaging in Eq.(62) with a canonical or grand-canonical ensemble. Notably, the third term in the last line of Eq.(64) resembles the famous Onsager relation and can be viewed as a generalized Onsager relation in a convolution form with memory effects.

The second term of Eq.(64) is key and its presence indicates that the diffusion kernel $\tilde{K}$ has in general nontrivial dependence on the coarse-grained variables, which will be commented and stressed afterwards. In this place, we want to show a simplified but very helpful form of the generalized Langevin equation. By replacing the diffusion kernel with its average

$$\tilde{K}_{\alpha j}(a,t-u) \to \int da W(a)\tilde{K}_{\alpha j}(a,t-u) = \int da K_{\alpha j}(a,t-u) = \mathrm{Tr}(\hat{R}_\alpha(t-u)\hat{R}_j), \tag{67}$$

where $W(a)$ is taken as a equilibrium distribution function of a up to a normalized factor

$$\int da W(a) = \int da\, \mathrm{Tr}(\hat{f}(a)) = \mathrm{Tr}(1) = \mathrm{const.} \tag{68}$$

This constant can be completely absorbed into the definition of other coefficients so there is no need to worry about it. Such a procedure is also implemented in [38] to seek a simpler equation. After finishing the replacement, we will get

$$\frac{\partial \hat{A}_\alpha(t)}{\partial t} = v_\alpha(\hat{A},t) + \int_0^t du\, \mathrm{Tr}(\hat{R}_\alpha(t-u)\hat{R}_j)F_j(\hat{A}(u)) + \hat{R}_\alpha(t) \tag{69}$$

which recovers our familiar Langevin equation irrespective of the memory structure. Moreover, the diffusion kernel is expressed as a two-point time correlation function of the noise term.

As is seen from above, the local approximation is very crucial to the simplification operations. At least, this is valid for classical systems. Furthermore, the validity may also hold for some quantum systems. For example, the contribution of the integral terms in the Fokker-Planck and Langevin equations can be minor as long as the commutators of the basis operators are parametrically small. In this case, the nonlocality represented by $r(a,a')$ are expressed in terms of a series expansion in the small parameter, which is also the same case for the nonlocality of $v(a,a')$. On the other hand, if the non-equilibrium state of the system is such that the distribution function $f(a,t)$ and the corresponding density of states $W(a)$ change slowly when varying the distance $|a-a'|$ compared to the variation of $r(a,a')$, then in the first approximation, they can be considered constant when we calculate the integral term in the Fokker-Planck and Langevin equations. In this case, the contribution of regular parts becomes minor due to Eq.(26). Especially, in some physical system such as the relativistic heavy ion collisions of our interest, the created physical environment is a super hot medium. In this case, the quantum effects are suppressed by thee scale of high temperature. In this case, the quantum effects are often neglected and the local approximation is acceptably useful.

V. A QUANTUM DERIVATION OF FLUCTUATING HYDRODYNAMICS WITH GAUSSIAN NOISE

A. Markov approximation

When casting the generalized Langevin equation into fluctuating hydrodynamics, the Markov approximation is necessary to remove the delay effects appearing in the integral term. Considering that the operator of projection $1-P$ in the diffusion kernel $K(a,t)$ or $K(a,a',t)$ eliminates the slow evolution associated with the coarse-grained variables, the effect of time delays can be neglected in the evolution equations when a clear separation of time scales exists. This means that the time scales for an appreciable change of $\hat{A}(t)$ are distinctly larger than the typical time scales involved in the Langevin fluctuating force $\hat{R}_k(t)$. The simplified Langevin equation is taken as the starting point

$$\frac{\partial \hat{A}_\alpha(t)}{\partial t} = v_\alpha(\hat{A}(t)) + \int_0^\infty du\, \mathrm{Tr}(\hat{R}_\alpha(u)\hat{R}_j)F_j(\hat{A}(t)) + \hat{R}_\alpha(t) \tag{70}$$

where the time hierarchy $t \gg u$ is assumed. A further transformation leads to

$$\frac{\partial \hat{A}_\alpha(t)}{\partial t} = v_\alpha(\hat{A}(t)) + \gamma_{\alpha j}F_j(\hat{A}(t)) + \hat{R}_\alpha(t) \tag{71}$$

where γ is the bare kinetic coefficient defined as

$$\gamma_{ij} \equiv \int_0^\infty du\, \mathrm{Tr}(\hat{R}_i(u)\hat{R}_j). \tag{72}$$

The first two terms in the rhs of Eq.(70) generally contain nonlinear coupling of $\hat{A}$ due to the presence of $v(\hat{A})$ and $F(\hat{A})$ while $\hat{R}_\alpha(t)$ is a fluctuating force satisfying

$$\mathrm{Tr}(\hat{R}_i(t)\hat{R}_j(t')\,) = 2\gamma_{ij}\delta(t-t'). \tag{73}$$

It can be shown in [38] that the nonlinear couplings contained in the drift term will lead to the renormalization of the bare kinetic coefficient. Last but not least, if choosing to write the $\boldsymbol{x}$ dependence explicitly, we get

$$\frac{\partial \hat{A}_\alpha(t,\boldsymbol{x})}{\partial t} = v_\alpha(\hat{A}(t,\boldsymbol{x})) + \frac{1}{V}\int d\boldsymbol{x}'\gamma_{\alpha j}(\boldsymbol{x}-\boldsymbol{x}')F_j(\hat{A}(t,\boldsymbol{x}')) + \hat{R}_\alpha(t,\boldsymbol{x}) \tag{74}$$

with

$$\gamma_{ij}(\boldsymbol{x}-\boldsymbol{x}') \equiv \int_0^\infty du\, \mathrm{Tr}(\hat{R}_i(u,\boldsymbol{x})\hat{R}_j(0,\boldsymbol{x}')\,). \tag{75}$$

B. Relativistic hydrodynamics

The evolution of a relativistic fluid is governed by the continuity equations

$$\partial_\mu T^{\mu\nu} = 0\,, \tag{76}$$
$$\partial_\mu N^{\mu} = 0\,, \tag{77}$$

where $T^{\mu\nu}, N^\mu$ are the energy momentum tensor and conserved current respectively. In this subsection, the hat symbol of the operators is temporarily neglected.

By utilizing symmetry analysis and the second law of thermodynamics, $T^{\mu\nu}$ and N^μ can be conveniently written as in Landau frame

$$T^{\mu\nu} = eu^\mu u^\nu - p\Delta^{\mu\nu} + \tau^{\mu\nu}\,, \tag{78}$$
$$N^\mu = nu^\mu + j^\mu\,, \tag{79}$$

with e, n and p being the energy density, the charge density and the pressure. The metric tensor here is given by $g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, while $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$ is the projection tensor orthogonal to the four-vector fluid velocity u^μ. Also, the dissipative terms, π and j, are given by

$$\tau^{\mu\nu} = -2\eta\nabla^{\langle\mu}u^{\nu\rangle} - \zeta\theta\Delta^{\mu\nu}, \tag{80}$$
$$j^\mu = -\frac{\sigma T(e+p)}{n}\nabla^\mu\frac{\mu}{T}, \tag{81}$$

where μ is the chemical potential conjugate the conserved charge density n and $\nabla_\mu \equiv \Delta_{\mu\nu}\partial^\nu$ is the spatial derivative. A shorthand for the thermodynamic force $\theta = \nabla \cdot u$ denotes the expansion rate. Three transport coefficients η, ζ and κ are the bare shear, bulk viscosities and the bare thermal conductivity, respectively.

In the linear regime, hydrodynamic modes can be sought via the linearized equations. A linear mode analysis is carried out on top of the background of thermal equilibrium. In a relativistic fluid, the quiescent equilibrium system is perturbed according to

$$\begin{aligned}
e(t,\boldsymbol{x}) &= e_0 + \delta e(t,\boldsymbol{x}), \quad p(t,\boldsymbol{x}) = p_0 + \delta p(t,\boldsymbol{x}),\\
n(t,\boldsymbol{x}) &= n_0 + \delta n(t,\boldsymbol{x}), \quad u^\mu(t,\boldsymbol{x}) = (1,0) + (0,\delta v^i(t,\boldsymbol{x})),\\
T(t,\boldsymbol{x}) &= T_0 + \delta T(t,\boldsymbol{x}), \quad \frac{\mu}{T} = \frac{\mu_0}{T_0} + \delta(\frac{\mu}{T}).
\end{aligned} \tag{82}$$

Due to $u \cdot u = 1$, $(\delta v)^2 \ll 1$, the terms of $O(\delta v^2)$ is neglected afterward. Then we have

$$\frac{\partial \delta n}{\partial t} = -n_0 \nabla \cdot \delta v - \frac{\sigma T_0 (e_0 + p_0)}{n_0} \nabla^2 \delta\left(\frac{\mu}{T}\right), \tag{83}$$

$$\frac{\partial \delta e}{\partial t} = -(e_0 + p_0)\nabla \cdot \delta v, \tag{84}$$

$$\frac{\partial \pi_i}{\partial t} = -\nabla_i \delta p - (\zeta + \frac{1}{3}\eta)\nabla_i(\nabla \cdot \delta v) - \eta \nabla^2 \delta v_i. \tag{85}$$

where $\pi^i \equiv \delta T^{0i}$.

C. Relativistic fluctuating hydrodynamics

The nonlinear interactions among these hydrodynamic modes is absent in conventional hydrodynamics in the linear regime [53], more importantly, without stochastic noises, conventional hydrodynamics ignores the inherent hydrodynamic excitations triggered by thermal fluctuations. When the thermodynamic fluctuations are large, the effects of nonlinear couplings become significantly enhanced, which is the case in the critical regime. The impacts of nonlinear fluctuations of the coarse-grained variables can be properly accounted for in the framework of the Langevin equation. In this subsection, we derive the relativistic fluctuating hydrodynamics from the Langevin equation step by step.

Inspire by the relativistic hydrodynamic equations, the coarse-grained slow variables set is taken to be $\{\hat{A}(t,\boldsymbol{x})\} = \{\delta e(t,\boldsymbol{x}), \delta n(t,\boldsymbol{x}), \pi_i(t,\boldsymbol{x})\}$. First, we need to figure out the streaming term

$$v(a) = \mathrm{Tr}(\hat{\rho}(a) iL\hat{A}(t,\boldsymbol{x})) = \mathrm{Tr}(\hat{\rho}(a)\partial_t \hat{A}(t,\boldsymbol{x})). \tag{86}$$

If looking at the continuity equation,

$$\partial_t \hat{A}(t,\boldsymbol{x}) = \nabla \cdot \hat{J}_A, \tag{87}$$

the derivative of $\hat{A}$ with respect to t gives exactly the related reversible currents and dissipative currents vanishes when averaged over a thermodynamic ensemble shown in the rhs of Eq.(86). Therefore, we can directly read the streaming velocities,

$$v_n = -\nabla \cdot (\hat{n}\delta\hat{v}), \quad v_e = -\nabla \cdot \hat{\pi}, \quad v_\pi = -\nabla_i \delta \hat{p}, \tag{88}$$

where the background is set to be rest and the nonlinear couplings between fluctuations is manifestly shown in v_n and v_π (noticing that $\hat{p}$ may contain nonlinear couplings in terms of these basis fluctuations, see also [45]).

Next, recalling the given definition of the thermodynamic force,

$$F_k(\hat{A}) = \frac{\delta}{\delta \hat{A}_k} \ln W(\hat{A}) \tag{89}$$

we can translate $\delta(\mu/T)$ and δv into $\frac{1}{V}\frac{\delta \ln W(\hat{A})}{\delta \hat{n}}$ and $\frac{T_0}{V}\frac{\delta \ln W(\hat{A})}{\delta \hat{\pi}}$ according to Boltzmann relation $S \equiv \ln W(\hat{A})$ with S, V being the total entropy and volume. The factor $1/V$ comes without surprise because it counteracts the same factor in Eq.(74).

Comparing the relativistic hydrodynamic equation Eqs.(76) to (81) with the Langevin equation, the kinetic coefficients can also be read without efforts

$$\gamma_{nn} = -\frac{\sigma T_0(e_0 + P_0)}{n_0}\nabla^2, \tag{90}$$

$$\gamma_{\pi_i\pi_j} = -\big((\zeta + \frac{1}{3}\eta)\nabla_i\nabla_j + \eta\delta_{ij}\nabla^2\big), \tag{91}$$

which demonstrates that

$$\mathrm{Tr}(\hat{R}_n(t,\boldsymbol{x})\hat{R}_n(t',\boldsymbol{x}')\,) = -\frac{2\sigma T_0(e_0 + P_0)}{n_0}\nabla^2\delta(t-t')\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{92}$$

$$\mathrm{Tr}(\hat{R}_{\pi_i}(t,\boldsymbol{x})\hat{R}_{\pi_j}(t',\boldsymbol{x}')\,) = -2\big((\zeta + \frac{1}{3}\eta)\nabla_i\nabla_j + \eta\delta_{ij}\nabla^2\big)\delta(t-t')\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{93}$$

where the spatial coordination $\boldsymbol{x}$ dependence is explicitly recovered. Note through out the script, $\boldsymbol{x}$ dependence is frequently suppressed for compactness sometimes.

The parameterization given in Eq.(90) and (91) is expected to successfully reproduce the Ornstein–Zernike result [36, 45]. Attention is supposed to be paid to the vanishing of the noise term R_e and its corresponding diffusion kernel. This is a nontrivial consistency cross check. The reason for its vanishing is that its reversible current is also a local density of the conserved quantity, i.e., $\delta\hat{T}^{0i}$, which is also in the list of the chosen coarse-grained slow variables,

$$\hat{R}_e \equiv (1-P)iL\hat{e} = (1-P)\partial_i\hat{\pi}_i = \partial_i(1-P)\hat{\pi}_i = 0, \tag{94}$$

where $P\hat{A} = \hat{A}$ is proven in the appendix.A.

The conclusion that the energy density does not dissipate depends on the choice of the definition of the fluid velocity, or the choice of the set of coarse-grained slow variables in the language of projection operator . If working in Eckart frame, the charge density, rather than the energy density, does not dissipate .

After all the dust has set down, we eventually obtain

$$\begin{aligned}
\frac{\partial\delta\hat{n}}{\partial t} &= -\nabla\cdot(\hat{n}\delta\hat{v}) - \frac{\sigma T_0(e_0+P_0)}{n_0}\nabla^2\frac{\delta\ln W(\hat{A})}{\delta\hat{n}} + \hat{R}_n(t,\boldsymbol{x}), && (95)\\
\frac{\partial\delta\hat{e}}{\partial t} &= -\nabla\cdot\hat{\pi}, && (96)\\
\frac{\partial\hat{\pi}_i}{\partial t} &= -\nabla_i\delta\hat{p} - (\zeta+\frac{1}{3}\eta)\nabla_i(\nabla\cdot\frac{\delta\ln W(\hat{A})}{\delta\hat{\pi}_i}) - \eta\nabla^2\frac{\delta\ln W(\hat{A})}{\delta\hat{\pi}_i} + \hat{R}_{\pi_i}(t,\boldsymbol{x}). && (97)
\end{aligned}$$

Note our results here are almost identical to the ones derived in [36] in a classical way except the streaming term of Eq.(97), where the author neglects δp and uses a potential condition to rewrite that streaming term. Besides, $\nabla\cdot\pi$ is replaced by $\nabla\cdot((\hat{e}+\hat{p})\delta\hat{v})$ therein. But we prefer $\nabla\cdot\pi$ because the linearity in the basis fluctuations is unambiguously revealed without causing extra confusion.

VI. POSSIBLE EXTENSION TO INCLUDE MULTIPLICATIVE NOISES

In this section, we comment on the possible extension to include multiplicative noises. The kinetic coefficients in Eqs.(95) to (97) is independent of the fluctuating fields, therefore, the noise $\hat{R}$ is so-called Guassian noise, which is not always the case. In view of the fact that the kinetic coefficients characterize the intrinsic properties of the system with given thermodynamic states $(e_0, p_0\cdots)$. If the perturbation is not too large to change the intrinsic properties, the description with fluctuation-independent kinetic coefficients suffices, in line with the logic hidden in the linear response theory. However, if the perturbation is strong enough to fundamentally alter the intrinsic properties and the thermodynamic state of the system, for instance, when the critical behavior in vicinity of the QCD critical point is studied, the critical fluctuation is so large that the Langevin equation with multiplicative noises and field-dependent kinetic coefficients is essentially needed.

Such a fluctuating nonlinear hydrodynamics for normal fluids is given in the context of classical nonrelativistic statistics [67]. The authors construct a phenomenological model to achieve this. In their derivation, a very alike formula $\frac{\partial W(a)}{\partial t} = 0$ appears and provides three nontrivial constraint conditions, among which there is a detailed balance condition. The detailed balance condition plays an essential role in the construction of the fluctuating nonlinear hydrodynamics, from which the multiplicative noises and the diffusion kernel can be worked out just by "reading" as is done in the last subsection.

Just as is commented around Eq.(54), there are no constraint conditions coming from Eq.(54) according to our derivation step by step, which is also confirmed by one of the authors of [67] in a later textbook [45] using a classical treatment. Therefore, there is no shortcut to the possible extension with multiplicative noises, we have to literally work out Eq.(64), which is left as a future work in progress.

VII. SUMMARY AND OUTLOOK

In this work, we present a systematic derivation of relativistic fluctuating hydrodynamics employing the quantum nonlinear projection operator method. Morozov's nonlinear projection operator, as an extension of the well-established linear Mori-Zwanzig projection operator, enables the consideration of nonlinear interactions among macroscopic modes. This approach yields the quantum Fokker-Planck and Langevin equations, which are key in investigating anomalous transport phenomena in the vicinity of critical points. We then apply these equations to the coarse-grained hydrodynamic variables, resulting in the formulation of the relativistic fluctuating hydrodynamics with Gaussian noise. Additionally, the potential extension of this framework to incorporate the effects of multiplicative noises is also discussed.

There are some possible extensions which deserve to be focused on in future. The first one, as noted above, is to bring the multiplicative noises in and re-derive the relativistic fluctuating hydrodynamics. The multiplicative noises shall play an important role in studying the critical behavior near the QCD critical point and may shed light on the research for QCD critical point in the experimental aspect. A comprehensive analysis by the dynamic renormalization group based on the relativistic fluctuating hydrodynamics can also be sought, which is a mature technique in studying the critical behavior in condensed matter physics. In addition, the obtained generalized Fokker-Planck and Langevin equations with nonlocal quantum effects can be helpful in condense matter physics and low-temperature physics, where quantum effects are non-negligible and of much research interest.

VIII. ACKNOWLEDGMENTS

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Appendix A: The proof of useful identities

In this appendix, we prove some useful identities associated with the nonlinear projection operator. The first one is $P\hat{A} = \hat{A}$. We have

$$\begin{aligned} P\hat{A} &= \int dada' \hat{f}(a) W_{-1}(a,a') \,\mathrm{Tr}(\hat{A}\hat{f}(a')\,) \\ &= \int dada'a' \hat{f}(a) W_{-1}(a,a') W(a') \\ &= \int dada'a' \hat{f}(a) (\delta(a-a') + r(a,a')\,) \\ &= \hat{A} + \int da \hat{f}(a) \int da'a' r(a,a'), \end{aligned} \tag{A1}$$

and

$$\int da'a' W(a,a') = \int da'a' \,\mathrm{Tr}(\hat{f}(a)\hat{f}(a')\,) = \mathrm{Tr}(\hat{f}(a)\hat{A}\,) = aW(a). \tag{A2}$$

On the other hand

$$\int da'a' W(a,a') = \int da'a' W(a)(\delta(a-a') + R(a,a')\,) = aW(a) \to \int da'a' R(a,a') = 0. \tag{A3}$$

Now we get

$$a'r(a,a') = a'R(a,a') + \int da'' R(a,a'')a'r(a'',a'). \tag{A4}$$

By iterating Eq.(A4), we reach

$$\int da'a'r(a,a') = 0, \tag{A5}$$

which follows from the same reason for getting the second identity in Eq.(26). Finally, we obtain $P\hat{A} = \hat{A}$.

This can be also derived in a simpler way, noticing that

$$P\hat{f}(a) = \hat{f}(a), \tag{A6}$$

then a integral over a constructed as

$$\int daG(a)P\hat{f}(a) = \int daG(a)\hat{f}(a). \tag{A7}$$

Because P only act upon the operator, it can be safely factorized out

$$\int daG(a)P\hat{f}(a) = P\int daG(a)\hat{f}(a), \tag{A8}$$

this eventually gives us

$$PG(\hat{A}) = G(\hat{A}). \tag{A9}$$

Here $G(\hat{A})$ is a nonlinear function of $\hat{A}$.

Appendix B: The vanishing of noise contribution

The trace of the noise term is

$$\begin{aligned}
\mathrm{Tr}[(1-P)e^{iL(1-P)t}iL\hat{f}(a)] &= \mathrm{Tr}[e^{iL(1-P)t}iL\hat{f}(a)] - \mathrm{Tr}[Pe^{iL(1-P)t}iL\hat{f}(a)] \\
&= \mathrm{Tr}[\hat{Y}(a,t)] - \mathrm{Tr}[\int da''da'\hat{f}(a'')W_{-1}(a'',a')\,\mathrm{Tr}(\hat{Y}(a,t)\hat{f}(a')\,)] \\
&= \mathrm{Tr}[\hat{Y}(a,t)] - \int da''da'W_{-1}(a'',a')\,\mathrm{Tr}\left(\hat{f}(a'')\right)\mathrm{Tr}\left(\hat{Y}(a,t)\hat{f}(a')\right) \\
&= \mathrm{Tr}[\hat{Y}(a,t)] - \int da''da'W_{-1}(a'',a')W(a'')\,\mathrm{Tr}\left(\hat{Y}(a,t)\hat{f}(a')\right) \\
&= \mathrm{Tr}[\hat{Y}(a,t)] - \mathrm{Tr}[\hat{Y}(a,t)] = 0
\end{aligned} \tag{B1}$$

where the shorthand notation is $\hat{Y}(a,t) \equiv e^{iL(1-P)t}iL\hat{f}(a)$ and Eq.(26) has been used.

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