

Conductivity of a nonlinear Luttinger liquid with a large-scaled random potential

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Abstract

We perform a calculation of the uniform conductivity of one-dimensional degenerate fermion system with some large-scaled disorder. We use the nonlinear Luttinger liquid model. The only important limitation is that the frequency has to be large enough. Apart from that and few other minor constraints, the calculation is fairly general. For instance, we neither assume the interaction to be point-like nor weak. As a result, we obtain conductivity across a wide range of parameters, the parameters are the disorder scale, frequency, interaction radius and temperature.

I. INTRODUCTION

The advancements in nanotechnology allow one to create an increasing variety of types of 1-D systems. Nowadays numerous types exist, including: quantum wires, the chiral edge states of quantum Hall bars, the edge states of a two-dimensional topological insulator, carbon nanotubes etc. [1]. A key aspect to note is that any point of these systems is accessible, allowing for direct measurement. The control of quasiparticles allows one to work with quantum information [2]. Additionally, given the similarity between a photon in an optical waveguide and a quasiparticle in a 1-D channel, the latter might parallel the role of quantum optical systems in basic research [3, 4].

Luttinger liquid is one of the prevalent models of 1-D systems in the limit of low energies [1, 5]. However, many applications require taking into account the nonlinear dispersion [6]. These applications include energy dissipation and relaxation [6, 7], charge fractionalization [1, 8, 9] and spin-charge fractionalization [10–12], the Coulomb drag effect [13–15]. Even the most subtle predictions of nonlinear Luttinger liquid theory are being confirmed experimentally [16, 17].

At zero temperature, nonlinear theory expands further, allowing the consideration of very high energy excitations. This expansion is the model of mobile impurity [6, 18, 19]. It was successfully used in Hall systems [20] in spin chain systems [21, 22], etc [6]. Nonetheless, doubts exist regarding the model's ability to provide general order of accuracy. [23]

The unusual nature of 1-D systems is also demonstrated by their unusual quantum hydrodynamics at low temperature [24]. Many hydrodynamic properties have been calculated

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from first principles such as viscosity [25]. The problems of local thermalization relaxation [26–28] and the thermal front propagation [29] are also actively studied. The model of mobile impurity also allows performing such calculations [18].

This work is concentrated on fermions, however, it is important to note that 1-D boson systems are also actively studied experimentally [30, 31], and theoretically [32, 33]

Transport properties of a Luttinger liquid have been the focus of considerable research due to their unusual nature compared to three dimensional systems. Owing to one-dimensionality, a small impurity radically changes the ideal ballistic character of the particles motion [34], and even a weak local cluster of impurities dramatically affects the general conductance as well as the tunneling conductance [35–37]. On the other hand, the fact that a relatively short system has electrical leads becomes crucial in one dimension [38]. Sometimes a one-dimensional problem is integrable, which has interesting mathematical reasons [39]. The case opposite to a single point-like impurity is the case of large-scaled random potential [38, 40]. It was shown that nonlinearity of the spectrum and forward scattering on the random potential are crucial in this case [40]. Additionally, this case is interesting because the effect of weak localization can take place [41, 42].

This is one of two planned papers by our colleagues and us considering the conductivity $\hat{\sigma}(\omega)$ of a one-dimensional degenerate liquid system of electrons with a finite mass m in a sample with some random potential $U(x)$, whose scale d is presumed large and the potential will be presumed weak $U(x) \ll E_F$, where E_F is the Fermi energy. In the present paper the bosonization technique is used. For our problem, this technique permits one to consider only frequencies bigger than some minimal limit, but on the other hand, it allows one to find the conductivity from the first principles with minimal other restrictions, for instance, the interaction between the particles does not have to be point-like nor weak, see Sec. IV. In the paper to be published by our colleagues, the method of kinetic equation is used. It does not limit the frequency from below, only from above and the interaction has to be weak. Within the area where both methods are applicable, the conductivity obtained with these methods coincide exactly.

In both cases a system with nonlinear dispersion is considered. The system is presumed long enough for the effect of the leads to be omitted. As usual for a Luttinger liquid $a \gg \lambda_F$, where a is the interaction radius and λ_F is the Fermi wavelength. Also, we presume that the random potential is large-scaled, meaning $d \gg a$ and, thereby, $d \gg \lambda_F$; this allows

us to neglect backward scattering from the random potential. Apart from these spatial values, there are two more: the thermal length $l_T = k_F/mT$ and the frequency length $l_\omega = \tilde{v}/\omega$, where T is the temperature, k_F is the Fermi momentum and \tilde{v} is a renormalized Luttinger velocity different from v_F , density variations move at this velocity. Different relations between these lengths correspond to different regimes (frequency and temperature dependence) of conduction. Throughout the paper, we presume that the temperature is low enough $\lambda_F \ll l_T$, hence the liquid is degenerate. Summing up, the conditions for length scales are

$$d \gg a \gg \lambda_F; \quad l_T \gg \lambda_F. \quad (1)$$

Note that some non-uniformity is necessary for the dissipation to be nonzero $\sigma(\omega) \neq 0$, when $\omega \neq 0$, see [6]. It can be explained by the fact that the response of a system to a uniform force field is purely inertial and independent of inner interactions between the particles. Indeed it is nothing but a center-of-mass motion caused by an applied force uniform in space [6].

The paper is organized as follows.

In Section II, we derive a general formula that expresses the conductivity through the electron liquid density and the random potential.

In Section III we describe the density correlator of a liquid without the external potential for a non-interacting gas as well as for a liquid with a finite radius interaction.

In Section IV we describe in great details the method area of applicability and, hence, when our results are applicable.

In Section V we consider a gas of non-interacting electrons and obtain expressions for the conductivity of our system under the classical consideration (subsection V A) and by the bosonization method (subsection V B). The obtained expressions, as expected, coincide.

In Section VI, we consider the gas of interacting electrons within the framework of the bosonization method and obtain specific expressions for the conductivity at different ratios between the spatial scales.

Finally, in Section VII, we briefly summarize the main results of the work.

II. A GENERAL FORMULA FOR THE CONDUCTIVITY

In this section using the bosonization method we obtain a general formula that expresses the conductivity through the electron liquid density and the random potential. In the case of a weak potential using the formula we derive a simpler one that defines the conductivity through the random potential correlator $W(x_2 - x_1) = \langle U(x_2)U(x_1) \rangle$ and through the electron liquid density correlator of a liquid without the external potential $\text{Im} \langle \rho \rho \rangle_{q,\omega}$.

Here we use the Kubo formula in the form [14]

$$\sigma(\omega) = \frac{\text{Im} \langle \partial_t j, \partial_t j \rangle_{q=0,\omega}}{\omega^3}, \quad (2)$$

where $j(x, t)$ is the current at point x . And by the brackets with subscripts of two operators $\langle AB \rangle_{q,\omega}$ we, henceforth, mean the retarded Green's function

$$\langle A, B \rangle_{q,\omega} = \iota \int \int \langle [A(x, t), B(0, 0)] \rangle \theta(t) e^{-\iota q x + \iota \omega t} dt dx,$$

where simple angle brackets denote quantum-thermodynamic averaging and averaging over realizations of the random potential. Immediately note that the terms in $\partial_t j(x, t)$, which are full derivatives on x , do not give the contribution to (2) by virtue of $q = 0$. Start with the Hamiltonian

$$H = \pi v_F \int [R^2(x) + L^2(x)] dx + \frac{2\pi^2}{3m} \left(\int [R^3(x) + L^3(x)] dx \right. \\ \left. + \frac{1}{2} \iint \rho(x) g(x-y) \rho(y) dx dy + \int U(x) \rho(x) dx \right), \quad (3)$$

here $g(x-y)$ is the interaction between particles. For large-scaled particle densities $R(x)$ for right- and $L(x)$ for left-moving particles, we presume $\rho(x) = R(x) + L(x)$, which means using bosonization approximation. The commutation relations between the operators are

$$[R(x), L(x')] = 0, \quad -[L(x), L(x')] = [R(x), R(x')] = \frac{i}{2\pi} \partial_x \delta(x - x').$$

To use the equation of continuity

$$\partial_t \rho + \partial_x j = 0 \quad (4)$$

for obtaining the particle flux, find a derivative of the density operators:

$$\partial_t R(x, t) = i[H, R] = -\partial_x \left[v_F R(x, t) + \frac{\pi}{m} R^2(x, t) \right. \\ \left. + \frac{1}{2\pi} \int g(x-y) \rho(y, t) dy + \frac{1}{2\pi} U(x) \right], \quad (5a)$$

$$\begin{aligned} \partial_t L(x, t) = i[H, L] = \partial_x [& v_F L(x, t) + \frac{\pi}{m} L^2(x, t) \\ & + \frac{1}{2\pi} \int g(x-y) \rho(y, t) dy + \frac{1}{2\pi} U(x)]. \end{aligned} \quad (5b)$$

Put (5) into (4) and obtain

$$j = v_F(R - L) + \frac{\pi}{m} (R^2 - L^2). \quad (6)$$

When calculating $\partial_t j(x, t) = i[H, j(x, t)]$ one should note that the first two terms of (6) give a full derivative with respect to x and, thus, can be omitted for the calculation of $\sigma(\omega)$, see eq. (5). Note that these terms correspond to Tomonaga-Luttinger linear theory. Furthermore, by straightforward calculations it is easy to show that the commutators of the current (6) and the first two terms in the Hamiltonian are also equal to the full derivatives of x and can also be omitted.

Ultimately we have

$$\partial_t j(x, t) = -\frac{1}{m} \left[\rho(x, t) \partial_x U(x) + \rho(x, t) \int \partial_x g(x-y) \rho(y, t) dy \right] \equiv I_1(x, t) + I_2(x, t). \quad (7)$$

This expression means that the time derivative of the current is proportional to the sum of two forces: the first one is the concentration multiplied by the local force created by the disorder; the second one is the force created by the other electrons.

Generally, the Fourier transform of the second term of (7) does not affect the commutator with $q = 0$ we are interested in, since $\iint \partial_x g(x-y) \rho(y, t) \rho(x, t) dy dx = 0$. Indeed, the derivative of the interaction potential is antisymmetric $\partial_x g(x-y) = -\partial_x g(y-x)$ and $\rho(y, t) \rho(x, t)$ is symmetric, since the density operators taken at the same time commute. So one has

$$\partial_t j(x, t) = -\frac{1}{m} \rho(x, t) \partial_x U(x).$$

Now calculate the conductivity using the Kubo formula in the form (2)

$$\sigma(\omega) = \frac{e^2}{\hbar m^2 \omega^3} \text{Im} \langle (\partial_x U) \rho, (\partial_x U) \rho \rangle_{q=0, \omega}. \quad (8)$$

Note that we have not made any additional assumption when deriving this formula. Now, we presume that the magnitude of the random potential $U(x)$ is small, and, in the lowest

order, the obtained conductivity reads

$$\sigma(\omega) = \frac{e^2}{\hbar m^2 \omega^3} \int_{-\infty}^{+\infty} q^2 W_q \text{Im} \langle \rho \rho \rangle_{q,\omega} \frac{dq}{2\pi}, \quad (9)$$

where the angle brackets denote thermodynamic averaging. Here $W_q \equiv \int_{-\infty}^{\infty} W(x) \cos(qx) dx$. We emphasize here again that expression (9) is related to the second term in the Hamiltonian (3), which is responsible for the nonlinearity of the electron spectrum. In the case of a linear spectrum $\sigma(\omega) = 0$ at $\omega \neq 0$.

Thus, the problem comes down to finding $\text{Im} \langle \rho \rho \rangle_{q,\omega}$ of the interacting gas without the random potential. In this paper we obtain $\text{Im} \langle \rho \rho \rangle_{q,\omega}$ for a non point-like interaction. The structure of $\text{Im} \langle \rho \rho \rangle_{q,\omega}$ for a point like interaction is known, especially for zero temperature [6, 14], it is detailed in Sec III.

Expression (9) limits the range of q that can contribute to the conductivity. The value W_q decreases exponentially with q after $1/d$, which sets a limit, namely $q \lesssim 1/d$. And hence $q \ll k_F$.

III. THE LIQUID DENSITY CORRELATOR $\text{Im} \langle \rho \rho \rangle_{q,\omega}$

To find the conductivity, as indicated by (9), it is enough to calculate the imaginary part of the retarded Green's function $\text{Im} \langle \rho \rho \rangle_{q,\omega}$. In the present section we obtain different formulas for it.

A. The case of no interaction

In the linear dispersion limit $1/m \rightarrow 0$ one has [15, 43] for a non-interacting gas

$$\text{Im} \langle \rho \rho \rangle_{q,\omega} = \frac{q}{2v_F} [\delta(q - \omega/v_F) - \delta(q + \omega/v_F)]. \quad (10)$$

In the case of nonlinear dispersion and zero temperature the liquid density correlator is finite at all q and has [6] a rectangular shape [44]; for a finite temperature it is a widened temperature function [14]

$$\text{Im} \langle \rho \rho \rangle_{q,\omega} = \frac{m}{4q} \frac{\sinh\left(\frac{\omega}{2T}\right)}{\cosh\left(\frac{v_F[2m\{\omega - qv_F\} - q^2]}{4Tq}\right) \cosh\left(\frac{v_F[2m\{\omega - qv_F\} + q^2]}{4Tq}\right)}, \quad (11)$$

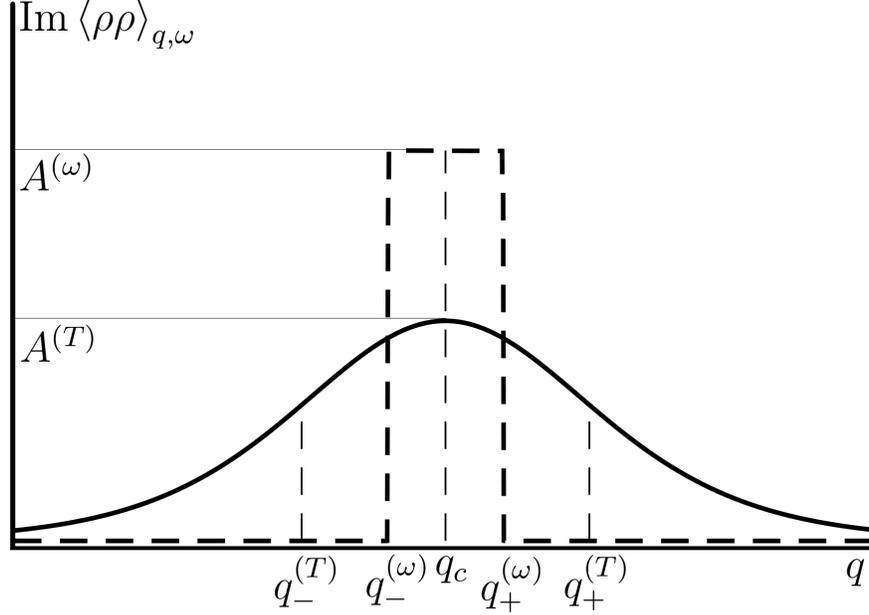


Figure 1. Correlator $\text{Im} \langle \rho \rho \rangle_{q, \omega}$ for a non-interacting gas with nonlinear dispersion as a function of q with a fixed ω for low (dashed line) temperature $T \ll \omega$ and relatively large (solid line) temperature $T \gg \omega$. The low temperature peak is concentrated between the values $q_{\pm}^{(\omega)}$ and the high temperature one between $q_{\pm}^{(T)}$. Correspondingly, the high of the peaks are denoted as $A^{(\omega)}$ and $A^{(T)}$

that is shown in fig. 1. The center of the distribution is $q_c = \omega/v_F$. Its width is easy to estimate as $q_{\pm}^{(\omega)} - q_c \approx \pm q_c^2/mv_F$ for low temperature $T \ll \omega$; for relatively large temperature $T \gg \omega$, the estimation is $q_{\pm}^{(T)} - q_c \approx \frac{q_c^2}{2mv_F} \frac{T}{\omega}$. Or, combining it together $q_+ - q_- \approx \frac{q_c^2}{2mv_F} \frac{\max(\omega, T)}{\omega}$. The high of the peak can be as $A \approx \frac{v_F m}{\max(\omega, T)}$.

B. Review of the interacting correlator structure

When there is an interaction in the system, the simple peak in fig. 1 warps in a complex way [6], it is now concentrated near $q_c = \omega/v$ between $q_{\pm} = q_c \pm \frac{q_c^2}{2mv} \frac{\max(\omega, T)}{\omega}$, its height A can be estimated as $A \approx \frac{v_F m}{2\max(\omega, T)}$ and still weakly depends on the interaction. Furthermore, in addition to the peak, long tails are formed as shown in fig. 2. The tails decrease slowly and as we will see below, for some sets of parameters give a major contribution into the overall conductivity (9), that is why they have to be taken into account.

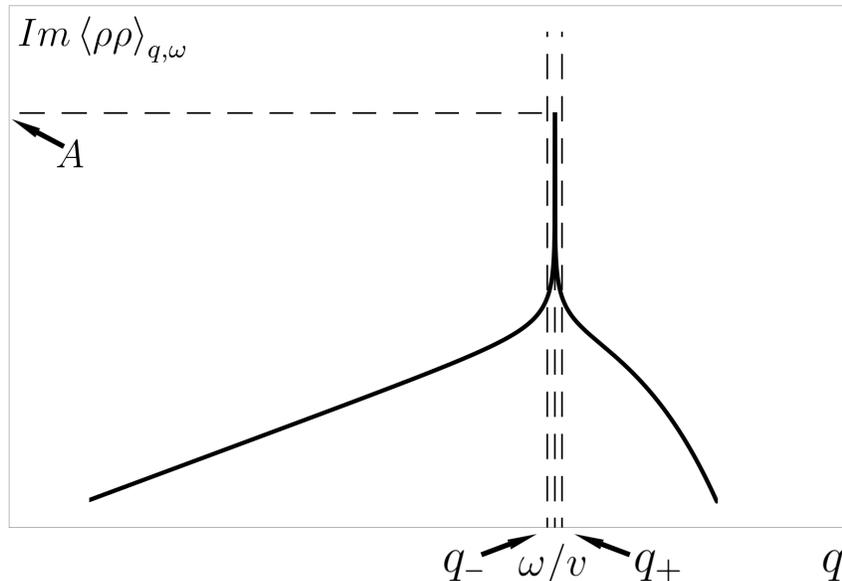


Figure 2. Correlator $\text{Im} \langle \rho \rho \rangle_{q,\omega}$ as a function of q with a fixed ω in log-log format. The temperature is chosen low $T \ll \tilde{v}/d$. The warped peak from fig. 1 is concentrated between q_{\pm} (see Sec. III A). Interaction induces long tails, that may decrease very slowly. This picture is drawn for a point-like interaction, for a finite range interaction the picture is similar.

C. The calculation of a finite radius interaction correlator.

As demonstrated at the end of Sec. II, the applicability condition for the Tomonaga-Luttinger model $q \ll k_F$ is always satisfied. Therefore, treating the nonlinearity as a small perturbation seems reasonable. Within this perturbation theory, the Tomonaga-Luttinger result is the leading term in a power series with the value $1/m$. Then the nonlinear term in the Hamiltonian is a small perturbation. Generally, this perturbative approach fails and the diagrams for $\langle \rho \rho \rangle_{q,\omega}$ diverge [6]. To get an adequate result that is not infinite, one has to provide some additional regularization [14] or use composite-fermion theory [6]. The mathematical reason here is that the linear approximation is only able to give a delta function for $\text{Im} \langle \rho \rho \rangle_{q,\omega}$, but the real correlator is a wrapped rectangle or a wide curve (11) of the width $\sim 1/m$, and no series is able to transform one into another [6].

However, when finding corrections to the imaginary part of Green's function $\text{Im} \langle \rho \rho \rangle_{q,\omega}$ away from point $q \approx \omega/v$, this perturbation theory is applicable and no regularization is needed [14, 45]. This is the approach we adopt in this work.

Now we see that the correlator naturally splits into two parts, the Tomonaga-Luttinger

correlator as a zero approximation and find the first correction to it $\text{Im} \langle \rho \rho \rangle_{q,\omega} \approx \text{Im} \langle \rho \rho \rangle_{q,\omega}^{(1)} + \text{Im} \langle \rho \rho \rangle_{q,\omega}^{(2)}$, where $\text{Im} \langle \rho \rho \rangle_{q,\omega}^{(2)} \sim 1/m^2$ and $\text{Im} \langle \rho \rho \rangle_{q,\omega}^{(1)}$ does not depend on $1/m$. The Tomonaga-Luttinger part $\text{Im} \langle \rho \rho \rangle_{q,\omega}^{(1)}$ approximately describes the peak $q_c = \omega/v$ as a delta function, which is enough for our purposes. The correction $\text{Im} \langle \rho \rho \rangle_{q,\omega}^{(2)}$ describes the long tails away from the point $q_c = \omega/v$. This correlator is shown in fig. 2.

We refer to the corresponding contributions of the correlator as as single-boson $\text{Im} \langle \rho \rho \rangle_{q,\omega}^{(1)}$ contribution and as a double-boson contribution $\text{Im} \langle \rho \rho \rangle_{q,\omega}^{(2)}$. This terminology is due to the fact that the Tomonaga-Luttinger bosons, the ones after Bogolubov transform, do not interact in the linear dispersion limit. When nonlinearity is taken into account, the processes of a boson splitting into two starts to be possible if the interaction between fermions is non-zero. The interaction between the bosons is proportional to $1/m^2$.

1. One-boson contribution to the correlator

Let us now write $\text{Im} \langle \rho \rho \rangle_{q,\omega}^{(1)}$ of the linear dispersion limit. A simple calculation, similar to that of (10) gives [6]

$$\text{Im} \langle \rho \rho \rangle_{q,\omega}^{(1)} = \frac{K_q q}{2\tilde{v}_q} [\delta(q - \omega/\tilde{v}_q) - \delta(q + \omega/\tilde{v}_q)], \quad (12)$$

where, as usual, $K_q = 1/\sqrt{1 + g_q/(\pi v_F)}$, $g_q = \int g(x)e^{-iqx} dx$ and $\tilde{v}_q = v_F/K_q$. In this notation, v is defined as \tilde{v}_0 .

2. Two-boson contribution to the correlator

Now let us calculate $\text{Im} \langle \rho \rho \rangle_{q,\omega}^{(2)}$ for an interaction with a finite range interaction a . In the Hamiltonian

$$\begin{aligned} H_0 = & \pi v_F \int dx (R(x)^2 + L(x)^2) + (4\pi^2/6m) \int (R(x)^3 + L(x)^3) \\ & + (1/2) \int dx_1 dx_2 g(x_1 - x_2) \rho(x_1) \rho(x_2), \end{aligned} \quad (13)$$

we presume the second term small and find correction of the correlator in the lowest order in $1/m$.

After using Bogolubov transformation

$$R_q = ch\theta_q \tilde{R}_q - sh\theta_q \tilde{L}_q \quad (14)$$

$$L_q = ch\theta_q \tilde{L}_q - sh\theta_q \tilde{R}_q \quad (15)$$

one obtains

$$\begin{aligned} H_0 = & (\pi/l) \sum_q \tilde{v}_q (\tilde{R}_q \tilde{R}_{-q} + \tilde{L}_q \tilde{L}_{-q}) \\ & + (2\pi^2/3ml^2) \sum_{q_1 q_2 q_3} [\Gamma_{q_1, q_2, q_3}^{(1)} (\tilde{R}_1 \tilde{R}_2 \tilde{R}_3 + \tilde{L}_1 \tilde{L}_2 \tilde{L}_3) \\ & + 3\Gamma_{q_1, q_2, q_3}^{(2)} (\tilde{R}_1 \tilde{R}_2 \tilde{L}_3 + \tilde{L}_1 \tilde{L}_2 \tilde{R}_3)], \end{aligned} \quad (16)$$

where

$$\begin{aligned} \Gamma_{q_1, q_2, q_3}^{(1)} &= ch\theta_{q_1} ch\theta_{q_2} ch\theta_{q_3} - sh\theta_{q_1} sh\theta_{q_2} sh\theta_{q_3}, \\ \Gamma_{q_1, q_2, q_3}^{(2)} &= sh\theta_{q_1} sh\theta_{q_2} ch\theta_{q_3} - ch\theta_{q_1} ch\theta_{q_2} sh\theta_{q_3} \end{aligned} \quad (17)$$

and $\tanh 2\theta_q = \frac{g_q}{g_q + 2\pi v_F}$ and l is the size of the system .

The first line of (16) is the linearised Hamiltonian of the Tomonaga-Luttinger model, the second and third lines correspond to the nonlinearity of the spectrum, and they will be considered as the perturbation. Our goal is to find the retarded Green's function (its imaginary part), which can be easily found from the Matsubara Green's function by substituting $i\omega_n \rightarrow \omega + i0$.

The ordinary correlator is expressed from the one for the transformed operators \tilde{R} and \tilde{L} in this way

$$\langle \rho\rho \rangle_{q, \omega} = (Ch\theta_q - Sh\theta_q)^2 \langle \tilde{\rho}\tilde{\rho} \rangle_{q, \omega} = (Ch\theta_q - Sh\theta_q)^2 \left\langle (\tilde{R} + \tilde{L})(\tilde{R} + \tilde{L}) \right\rangle_{q, \omega} \quad (18)$$

We will find the perturbation of the Green's function in the lowest (second) order in $1/m$. That means that we will calculate the diagrams of the type illustrated in fig 3. A similar calculation was done in [14], but we will generalize it to the case of non-point interaction between the particles. However, since our goal is to find the imaginary part $\text{Im} \langle \rho\rho \rangle_{q, \omega}^{(2)}$ away from the pole, all the diagrams will converge and no regularization will be needed. As it was noted in [14], the second line in (16) does not affect the second order perturbation of the Green's function imaginary part $\text{Im} \langle \rho\rho \rangle_{q, \omega}^{(2)}$.

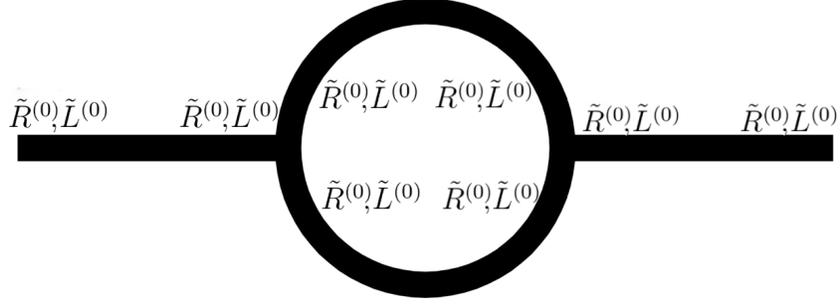


Figure 3. Boson diagram of the type $\langle \tilde{R}\tilde{R} \rangle$, $\langle \tilde{R}\tilde{L} \rangle$ etc. They corresponds to the first perturbation of the Green's function in the lowest order in $1/m$, that are calculated in App. B. Here a boson decays into two new ones and then they join into one boson again. With symbols we denote $\tilde{R}^{(0)}$ and $\tilde{L}^{(0)}$ the unperturbed operators.

Now use a simple formula known from the perturbation theory with Hamiltonian (16) in the form $H = H_0 + V(\tau)$ to find the second order correction in $V(\tau) \sim \frac{1}{m}$.

$$\langle T\rho_q(\tau)\rho(0,0) \rangle = \langle T\rho_q(\tau)\rho(0,0) \rangle^{(1)} - \int_0^\beta \int_0^\beta d\tau_1 d\tau_2 \langle T\rho_q(\tau)\rho(0,0)V(\tau_1)V(\tau_2) \rangle, \quad (19)$$

Here only connected diagrams are counted. The first term when put into (9), gives the one-particle contribution to the conductivity σ_1 , i.e. (27) and the second term gives σ_2 .

The calculation of the imaginary part of the second term in (19), which is the two-boson perturbation of the density correlators is given in Appendix B. The general result is

$$\begin{aligned} \text{Im} \langle \rho\rho \rangle_{q,\omega}^{(2)} = & K_q \frac{\hbar^2 q^2 (\omega^2 - (q\tilde{v})^2)}{m^2 2^5 \tilde{v}^3} \frac{\sinh(\hbar\omega/2T)}{\sinh(\hbar[\omega + q\tilde{v}]/4T) \sinh(\hbar[\omega - q\tilde{v}]/4T)} \\ & \times \left(\frac{\Gamma_{q, \frac{\omega}{2\tilde{v}} - \frac{q}{2}, \frac{\omega}{2\tilde{v}} + \frac{q}{2}}^{(2)}}{q\tilde{v} + \omega} + \frac{\Gamma_{q, \frac{\omega}{2\tilde{v}} + \frac{q}{2}, \frac{\omega}{2\tilde{v}} - \frac{q}{2}}^{(2)}}{q\tilde{v} - \omega} \right)^2 \end{aligned} \quad (20)$$

for q away from $\pm\omega/\tilde{v}$. This result gives conductivity when substituted into (9). Note that the conductivity is determined by the integral of (20) over the area away from its poles $\omega = \pm q\tilde{v}$ (refer to (28)).

The correlator formula for a point-like interaction is known [14]. To make the interaction from (20) point-like, put $a \rightarrow 0$, then one has $(\Gamma_{q, \frac{\omega}{2\tilde{v}} + \frac{q}{2}, \frac{\omega}{2\tilde{v}} - \frac{q}{2}}^{(2)})^2 \approx (\Gamma_{0,0,0}^{(2)})^2 = \frac{4\gamma}{K_0^2}$, where $\gamma = K_0(K_0^2 - 1)^2/64$ and $K_0 = K_q|_{q=0}$. Then, the expression (20) reduces to the corresponding one from [14].

IV. ON THE LIMITATIONS AND SIMPLIFICATIONS DUE TO THE SETS OF PARAMETERS CHOSEN

1. *The low frequency limit*

If the scale of the random potential is large $d \gg \lambda_F$, then the probability of backscattering of electrons on the potential is exponentially small, $\sim \exp(-2k_F d)$, where k_F is the Fermi momentum. One might think that it makes energy dissipation and hence finite conductivity impossible. However a time variable field is able to transfer its energy to the liquid even through a single particle due to forward scattering and this mechanism contributes to the conductivity. In this paper we refer to this mechanism as one-boson conductivity. This effect is valid even in classical physics and, because of it, even a liquid with no particle interaction has a finite conductivity, which will be shown in Sec. V A.

The fact that the probability of backscattering is small is especially important in the low-frequency limit, $\omega \rightarrow 0$, when the conductivity becomes proportional to the length of the electron's path relative to the backscattering. In the low-frequency limit, there is only one alternative mechanism of momentum relaxation - the collision of two or more electrons with simultaneous forward scattering of one of them on a random potential. The momentum transferred to the electronic system in this process is of the order of $1/d \ll k_F$, which at $\omega \rightarrow 0$ does not directly lead to a finite resistance. However, these collisions make possible the process of diffusion of an electron in momentum space from vicinity of point k_F to vicinity of point $-k_F$ through the bottom of the spectrum with elementary step of order $1/d$, which leads to finite resistance [28]. The momentum transfer to the diffusing electron occurs as it collides with the electrons of the temperature band. Deep below the Fermi level the fraction of unoccupied states is proportional to $\exp(-E_F/T)$, hence the probability of such a diffusion transition is proportional this exponent and the conductivity is proportional to $\exp(-E_F/T)$. Comparing this probability with the above mentioned probability of the backward scattering, we conclude that the diffusion process is more effective under the condition $v_F/T \ll d$ and leads to conductivity proportional to $\exp(-E_F/T)$. If the inverse inequality is satisfied, the conductivity at zero frequency is proportional to $\exp(2k_F d)$. A rigorous analytical theory of the low-frequency region, based on the solution of the kinetic equation, will be given in another article by our colleagues.

2. *On the parameter of the perturbation theory*

Now some details on the value $1/m$, it should be small, or more specifically, the non-linearity is small enough $v_F q \gg q^2/2m$. Here $q \lesssim 1/d$ (see the end of Sec. II), hence $\hbar/mdv_F \ll 1$. It also means the simple inequity mentioned above $d \gg \lambda_F$. Note that when taking the linear limit, it should be $m \rightarrow \infty$ but $v_F = \text{const}$, so the result of the linear consideration (10) remains constant nonlinear result (11) transforms into (10).

V. A NON-INTERACTING GAS CONDUCTIVITY

A. The classical calculation of the non-interacting gas conductivity

To understand the problem better, first consider classically a non-interacting Fermi gas.

Write the classical Liouville's equation for the right electrons

$$\frac{\partial f_R}{\partial t} + \frac{\partial f_R}{\partial x} \frac{\partial H}{\partial p} - \frac{\partial f_R}{\partial p} \frac{\partial H}{\partial x} = 0, \quad (21)$$

where $H = \frac{p^2}{2m} + U(x) + eE_0 x \cos(\omega t)$. Rewrite the equation through new variables $x_{\text{new}} = x_{\text{old}}$, $t_{\text{new}} = t_{\text{old}}$ and $E = p^2/2m + U(x_{\text{old}})$, after transformations one has exact equation

$$\frac{\partial f_R}{\partial t} + \sqrt{\frac{2}{m}(E - U(x))} \frac{\partial f_R}{\partial x} = eE_0 \exp(i\omega t) \sqrt{\frac{2}{m}(E - U(x))} \frac{\partial f_R}{\partial E} \quad (22)$$

presuming that the field is weak one has to replace f_R in the right part with $f_{F,R}$, where $f_{F,R}$ is the Fermi distribution function of the right electrons and f_R now means deviation. The equation for the left electrons only differs in the sign of $\sqrt{\frac{2}{m}(E - U(x))}$, also f_R should be substituted with f_L . The equation is solved in Appendix A for $T = 0$. The real part of the obtained classical conductivity is

$$\sigma_{\text{classical}}(\omega) = \frac{e^2 v_F}{\pi \hbar} \left\{ 1 - \frac{3W(x=0)}{2m^2 v_F^4} \right\} \delta(\omega) + \frac{e^2 W(\omega/v_F)}{2\pi \hbar m^2 v_F^4}. \quad (23)$$

The physical explanation of why there is nonzero conductivity at $\omega \neq 0$ for a noninteracting liquid is the following. In the absence of disorder, under the influence of an oscillating uniform electric field, the electron velocity oscillates with the same frequency and with $\frac{\pi}{2}$ phase shift. As a result, the work of the field during the period becomes zero, which means $\sigma(\omega) = 0$. Under the influence of disorder, the phase shift between the field and the velocity becomes irregular and the conductivity turns out to be finite.

B. The quantum calculation of the non-interacting gas conductivity

To illustrate the general formula (9), let us find the conductivity of a non-interacting one-dimensional gas with nonlinear dispersion. Generally, one has to use the full nonlinear correlator (11) and put it into (9).

As it was stated in Sec. III A, to be able to substitute the complex peak (11) concentrated between q_+ and q_- with simple delta function (10), the scale of the external potential should not depend on the fine structure, i.e. $1/d \gg q_+ - q_-$. This yields weak condition

$$\omega \ll \frac{v_F}{d} \frac{mv_F^2}{\max(T, \omega)}. \quad (24)$$

After integrating (9) with (12) one has

$$\sigma_1(\omega) = \frac{e^2 W_{\omega/v_F}}{2\pi \hbar m^2 v_F^4}. \quad (25)$$

This expression ($\omega \neq 0$) exactly coincides with the classical result (23). The explanation is simple: the quantum parameter λ_F is much smaller than the scale of the disorder $d \gg \lambda_F$, which means applicability of the quasi-classical approach. The area of applicability of these quantum calculations is discussed in Section IV.

VI. THE INTERACTING ONE-DIMENSIONAL LIQUID CONDUCTIVITY

A. One-boson contribution to the correlator

First, calculate one-boson contribution. To do that put (12) into (9). As in the previous section, when condition $1/d \gg q_+ - q_-$ is satisfied, the inner structure of the peak can be neglected and substituted by delta function of the linear limit $1/m \rightarrow 0$. This yields

$$\omega \ll \frac{\tilde{v}}{d} \frac{m\tilde{v}^2}{\max(T, \omega)} \quad (26)$$

with the renormalized \tilde{v} .

Note that (12) gives $q = \omega/\tilde{v}_q$, where \tilde{v}_q depends on q itself. To simplify the final result, we apply some weak condition. Specifically, we use $\omega/v_F \ll 1/a$, then, after using (9) we obtain

$$\sigma_1(\omega) = \frac{e^2 K_{\omega/v_F} W_{\omega/v_F}}{2\pi \hbar m^2 \tilde{v}_{\omega/v_F}^4} \quad (27)$$

This result is rather similar to the non-interacting one (25), except for some renormalization. Note, that the one-boson contribution does not depend on temperature when (26).

As noted in the Introduction, the problem of calculating the conductivity in a one-dimensional system without backscattering was discussed earlier in [40]. In this paper, under the condition $0 < \omega \ll v_F/d$ a formula coinciding with (23) and (27) was obtained, in which, however, there is no δ -function at ω . This is due to the fact that the authors [40] in expression (7) for the conductivity omitted in the denominator of the infinitesimal imaginary additive $i0$, which could be done at all frequencies except zero.

B. Two-boson contribution to the correlator

Second, calculate two-boson contribution, that correspond to the long tails from fig. 2. If $1/d \ll v/T$ or $\omega \gg \tilde{v}/d$, then the section of the long tails near $q \approx 1/d$ contributes most significantly to the conductivity. We integrate these two contributions separately. However, this approach leads to a difficulty when $\omega/\tilde{v} \approx 1/d$ and we cannot predict the conductivity in this band, but there appear to be no singularities for this set of parameters. Specifically, it should be $|\omega - \tilde{v}/d| \gg q_+ - q_-$, or

$$|\omega - \frac{\tilde{v}}{d}| \gg \frac{\max(T, \frac{\tilde{v}}{d})}{mv_F d}. \quad (28)$$

The complex general result (9) with correlator (20) can be simplified if particular cases are considered. When integrating, it should be noted that (20) is obtained for q away from the apparent pole $\pm\omega/\tilde{v}$. [46]

The results for $T \gg \tilde{v}/d$ are

$$\sigma_2(\omega) = \frac{e^2 T}{8m^4 \tilde{v}^5 \omega^2} \int_0^{+\infty} q^2 W_q K_q \left[4\Gamma_{q, \frac{q}{2}, \frac{q}{2}}^{(2)} \right]^2 \frac{dq}{2\pi} \quad (T \gg \tilde{v}/d \gg \omega), \quad (29)$$

$$\sigma_2(\omega) = \frac{e^2 T}{8m^4 \tilde{v} \omega^6} \int_0^{+\infty} q^6 W_q K_q \left[4\Gamma_{q, \frac{\omega}{2\tilde{v}}, \frac{\omega}{2\tilde{v}}}^{(2)} \right]^2 \frac{dq}{2\pi} \quad (T \gg \omega \gg \tilde{v}/d) \quad (30)$$

$$\begin{aligned} \sigma_2(\omega) &= \frac{\hbar e^2}{32m^4 \tilde{v} \omega^5} \int_0^{+\infty} q^6 W_q K_q \left[4\Gamma_{q, \frac{\omega}{2\tilde{v}}, \frac{\omega}{2\tilde{v}}}^{(2)} \right]^2 \frac{dq}{2\pi} \\ &\approx \frac{\hbar e^2 W_0 K_0 \left[4\Gamma_{0,0,0}^{(2)} \right]^2}{7 \cdot 64\pi m^4 \tilde{v} \omega^5 d^7} \quad (\omega \gg T \gg \tilde{v}/d) \end{aligned} \quad (31)$$

And for $\tilde{v}/d \gg T$

$$\sigma_2(\omega) = \frac{12e^2 T^4 W_0 K_0 \left[4\Gamma_{0,0,0}^{(2)} \right]^2}{\hbar^3 m^4 \tilde{v}^8 \omega^2 \pi} \quad (\tilde{v}/d \gg T \gg \omega) \quad (32)$$

$$\sigma_2(\omega) \approx \frac{\hbar e^2 W_0 K_0 \left[4\Gamma_{0,0,0}^{(2)} \right] \omega^2}{28\pi m^4 \tilde{v}^8} \quad (\tilde{v}/d \gg \omega \gg T). \quad (33)$$

$$\sigma_2(\omega) \approx \frac{\hbar e^2 W_0 K_0 \left[4\Gamma_{0,0,0}^{(2)} \right]^2}{7 \cdot 64\pi m^4 \tilde{v} \omega^5 d^7} \quad (\omega \gg \tilde{v}/d \gg T) \quad (34)$$

Note that (31) and (34) are equal, but correspond to different frequency bands. Results (31), (33) and (34) are the low temperature limit $T \rightarrow 0$, when the temperature dependent factor in (20) turns into a step function, which equals zero when $\omega < q\tilde{v}$, compare with [19]. The integrals in (32) and (33) are truncated at $q \approx 1/l_T$ in the other cases at $q \approx 1/d$.

When the interaction is weak $g_0/\pi v_F \ll 1$, this formulas become simpler $4\Gamma_{q, \frac{q}{2}, \frac{q}{2}}^{(2)} \rightarrow g_{\frac{q}{2}}/\pi v_F$ and $4\Gamma_{q, \frac{\omega}{2\tilde{v}}, \frac{\omega}{2\tilde{v}}}^{(2)} \rightarrow g_{\frac{\omega}{2\tilde{v}}}/\pi v_F$ correspondingly, within this approximation one also has $K_q \rightarrow 1$ and $\tilde{v} \rightarrow v_F$.

Resulting conductivity consists of two contributions σ_1 and σ_2 and depending on set of parameters one of them dominate, which means that both contributions are important, see fig. 4.

As an example let us now consider one particular case of expressions (29) - (34) simplification. This case is considered in our colleagues' paper by the method of kinetic equations. The first obligate condition is $\omega \ll \tilde{v}/d$; this condition is necessary when using a kinetic equation. There also should be $\omega \ll T$ for the applicability of this approach. Also the interaction should be weak.

This leaves two options: result (29) and result (32). For example, in particular case $T \gg \tilde{v}/d$, the conductivity has formula (29), which simplifies to

$$\sigma_2^{(\text{boz})}(\omega) = \frac{e^2 T}{16\pi m^4 v_F^5 \omega^2} \int_0^\infty dq q^2 W_q \left(\frac{g_{q/2}}{\pi v_F} \right)^2. \quad (35)$$

This two-boson conductivity formula coincide with the one obtained by using the kinetic equation. The same can be shown for the formula (32).

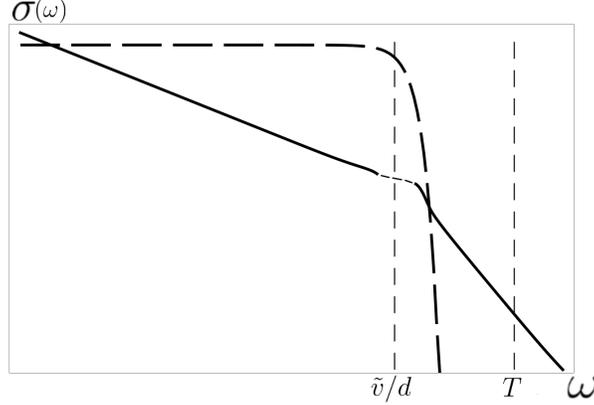


Figure 4. One-boson σ_1 (dashed line) and two-boson σ_2 (solid line) conductivities as a function of ω in log-log format. The temperature is chosen high $T \gg \tilde{v}/d$. Conductivity σ_2 decreases as a power of $1/\omega$, while σ_1 does as W_{ω/v_F} . Since W_q dependence is exponential, one can see that there are two areas of frequencies where $\sigma_2 > \sigma_1$ and both of them are valid within boson approximation. Note that there is a area of ω in σ_2 near point \tilde{v}/d , that our method cannot describe, see condition of applicability (28).

VII. CONCLUSIONS

Within the framework of the bosonization method, we obtain a general formula for the conductivity of an electron system in a sample with a random impurity potential, then we apply the formula to the case of a large-scaled potential $d \gg a$, when backward scattering can be neglected. Then we present several specific limiting cases of the formula.

The calculations are done for a non-interacting gas as well as for a liquid with a finite radius interaction. We compare the results to the ones for a classical gas with no particle interaction, providing insights into the physical mechanism behind the conductivity and energy dissipation in this system.

Throughout the paper we use the bosonization technique, which imposes the only important limitation: the frequency must significantly be high enough. Despite minor constraints, the bosonization technique allows us to avoid some common constraints. For instance, we do not presume that the interaction is either point-like or weak.

The overall conductivity consists of two contributions $\sigma_1(\omega)$ and $\sigma_2(\omega)$ that we refer to as a one-boson and a two-boson contribution correspondingly. Conductivity $\sigma_2(\omega)$ decreases with frequency as a power of $1/\omega$ and $\sigma_1(\omega)$ decreases exponentially with ω . Additionally,

one-boson conductivity $\sigma_2(\omega)$ depends strongly on temperature, frequency and interaction, while $\sigma_1(\omega)$ depends not on temperature and weakly on the other two. Hence there always exist areas where $\sigma_2(\omega) > \sigma_1(\omega)$ or vice versa, which means that both contributions are important.

The two-boson contribution σ_2 is particularly sensitive to the nonlinearity of the spectrum, emphasizing the importance of nonlinearity in energy dissipation and conductivity. Indeed, the simple linear Luttinger liquid cannot dissipate energy even in the presence of the random potential.

Remarkably, the two-boson contribution σ_2 demonstrates distinctive dependence on the frequency such as a $1/\omega^5$ dependence in (31) and (34); a $1/\omega^6$ dependence in (30); and, within a specific interval, it even exhibits an ω^2 dependence in (33).

Appendix A: One-particle contribution obtained with the classical kinetic equation

For the sake of brevity, we omit index 'classical' in this appendix. Since we consider the linear response of the classical system it is possible to replace f_R in the right part of (22) with $f_{F,R}$, where $f_{F,R}$ is the Fermi distribution function of the right electrons and f_R now means deviation.

The Green's function of the left part of equation (22) then reads

$$G(x, E, t | x', E', t') = \theta(t - t') \delta(E - E') \frac{1}{\sqrt{2(E' - U(x'))/m}} \delta\left(t - t' - \int_{x'}^x \frac{dx_1}{\sqrt{2(E' - U(x_1))/m}}\right). \quad (\text{A1})$$

When the temperature is zero $f_F(E) = \theta(\mu - E)$, so

$$\frac{\partial f_{F,R}}{\partial E} = -\delta(\mu - E). \quad (\text{A2})$$

Expressions (22), (A1) and (A2) give the correction of $f_{F,R}$ induced by the variable field

$$g_R(x, t) = -eE_0\delta(E - \mu) \int_{-\infty}^x dx' \exp\left\{- (i\omega + \gamma) \int_{x'}^x \frac{dx_1}{\sqrt{2(E' - U(x_1))/m}} + i\omega t\right\}$$

where $\gamma \rightarrow +0$. The flux of the right particles at x can be calculated if one multiplies $g_R(x, t)$ by $-e$ and $V = \sqrt{2(E - U(x))/m}$ and integrates the result over energy with the density of states $v(E, x) = \frac{\theta(E - U(x))}{2\pi\hbar\sqrt{2(E - U(x))/m}}$.

$$j_R(x, t) = \frac{e}{2\pi\hbar} \int_{-\infty}^x dx' \exp\left\{- (i\omega + \gamma) \int_{x'}^x \frac{dx_1}{\sqrt{2(\mu - U(x_1))/m}}\right\} E_0 \exp(i\omega t)$$

After averaging this expression over potential realisations the coordinate dependence vanishes, so one can write the expression for the particle flux

$$\langle j_R(x, t) \rangle = \langle j_R(0, t) \rangle = \frac{e}{2\pi\hbar} \int_{-\infty}^0 dx \left\langle \exp \left\{ -(i\omega + \gamma) \int_x^0 \frac{dx_1}{\sqrt{2(\mu - U(x_1))/m}} \right\} \right\rangle E_0 \exp(i\omega t)$$

Left electrons give the same contribution to the current, so the conductivity reads

$$\sigma(\omega) = \frac{e^2}{\pi\hbar} \int_{-\infty}^0 dx \left\langle \exp \left\{ -(i\omega + \gamma) \int_x^0 \frac{dx_1}{\sqrt{2(\mu - U(x_1))/m}} \right\} \right\rangle. \quad (\text{A3})$$

Integrate it over frequency and get the sum rule

$$\int_{-\infty}^{+\infty} \sigma(\omega) d\omega = \frac{e^2}{\hbar} \langle \sqrt{2[\mu - U(0)]/m} \rangle = \frac{e^2 \pi \langle n \rangle}{2m} \approx \frac{e^2 v_F}{\hbar} (1 - \langle U^2(0) \rangle / 8\mu^2), \quad (\text{A4})$$

where $\langle n \rangle$ is the average concentration. Here we take into account that the delta function $\delta(x)$ which appears during the calculation gives 1/2 because of the integration limits. The last approximation is valid when $U(x) \ll \mu$.

It is impossible to perform direct averaging in (A3). To do it, take $U(x) \ll \mu$ in (A3) and keep only the first and second order of U/μ when expanding in the row. The result is

$$\sigma(\omega) \approx \frac{e^2}{\pi\hbar} \int_{-\infty}^0 dx \left\langle \exp \left\{ -(i\omega + \gamma) \int_x^0 \frac{dx_1}{v_F} \left(1 + \frac{U(x_1)}{2\mu} + \frac{3}{8} \left(\frac{U(x_1)}{\mu} \right)^2 \right) \right\} \right\rangle \quad (\text{A5})$$

Expanding the exponent and averaging one obtains

$$\sigma(\omega) = \frac{e^2}{\pi\hbar} \int_0^{\infty} dx \exp \left\{ -(i\omega + \gamma) \frac{x}{v_F} \right\} \left\{ 1 - i\omega \int_0^x \frac{3W(0)}{8v_F\mu^2} - \omega^2 \int_0^x dx_1 \int_0^{x_1} dx_2 \frac{W(x_1 - x_2)}{4v_F^2\mu^2} \right\}, \quad (\text{A6})$$

where $W(x_2 - x_1) = \langle U(x_2)U(x_1) \rangle$. After integration over x it reads

$$\sigma(\omega) = \frac{e^2 v_F}{\pi\hbar} \left\{ \frac{-i}{\omega - i\gamma} + \frac{3W(0)}{8\mu^2} \frac{i}{\omega - i\gamma} + \frac{1}{4v_F\mu^2} \int_0^{\infty} \exp(i\omega x/v_F) W(x) dx \right\}.$$

The sum rule is satisfied since

$$\int_{-\infty}^{+\infty} \sigma(\omega) d\omega \approx \frac{e^2 v_F}{\hbar} (1 - W(0)/8\mu^2),$$

which coincides with (A4).

The real part of this expression is (23).

Appendix B: Two-boson perturbations of the density correlators

Here we write the perturbations of the correlators obtained by using (19)

$$\begin{aligned}
\text{Im} \left\langle \tilde{R}\tilde{R} \right\rangle_{q,\omega}^{(2)} &= \frac{e^2 q^2 (\omega^2 - (q\tilde{v})^2)}{m^2 2^6 \tilde{v}^3} \frac{\sinh(\omega/2T)}{2 \sinh([\omega + q\tilde{v}]/4T) \sinh([\omega - q\tilde{v}]/4T)} \frac{4 \left(\Gamma_{q,q',q'-q}^{(2)} \right)^2}{(q\tilde{v} - \omega)^2}, \\
\text{Im} \left\langle \tilde{L}\tilde{L} \right\rangle_{q,\omega}^{(2)} &= \frac{e^2 q^2 (\omega^2 - (q\tilde{v})^2)}{m^2 2^6 \tilde{v}^3} \frac{\sinh(\omega/2T)}{2 \sinh([\omega + q\tilde{v}]/4T) \sinh([\omega - q\tilde{v}]/4T)} \frac{4 \left(\Gamma_{q,q'-q,q'}^{(2)} \right)^2}{(q\tilde{v} + \omega)^2}, \\
\text{Im} \left\langle \tilde{R}\tilde{L} \right\rangle_{q,\omega}^{(2)} &= \frac{e^2 q^2 (\omega^2 - (q\tilde{v})^2)}{m^2 2^6 \tilde{v}^3} \frac{\sinh(\omega/2T)}{2 \sinh([\omega + q\tilde{v}]/4T) \sinh([\omega - q\tilde{v}]/4T)} \frac{4 \Gamma_{q,q',q'-q}^{(2)} \Gamma_{q,q'-q,q'}^{(2)}}{(q\tilde{v} - \omega)(q\tilde{v} + \omega)}, \\
\text{Im} \left\langle \tilde{L}\tilde{R} \right\rangle_{q,\omega}^{(2)} &= \text{Im} \left\langle \tilde{R}\tilde{L} \right\rangle_{q,\omega}^{(2)}, \tag{B1}
\end{aligned}$$

where $q' = \frac{\omega}{2\tilde{v}} + \frac{q}{2}$. Using these results and (14), one obtains the two-particle part of the correlator (20).

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