SYMPLECTIC DIFFERENTIAL REDUCTION ALGEBRAS AND SKEW-AFFINE GENERALIZED WEYL ALGEBRAS

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ABSTRACT. For a map $\varphi : U(\mathfrak{g}) \to A$ of associative algebras, $U(\mathfrak{g})$ the universal enveloping algebra of a (complex) finite-dimensional reductive Lie algebra, the representation theory of A is intimately tied to the representation theory of the A-subquotient known as the reduction algebra for $(A, \mathfrak{g}, \varphi)$. Herlemont and Ogievetsky studied differential reduction algebras for the general linear Lie algebra $\mathfrak{gl}(n)$ as the algebra of h-deformed differential operators formed from realizations of $\mathfrak{gl}(n)$ in the N-fold tensor product of the nth Weyl algebra. In this paper, we further the study of differential reduction algebras by presenting the symplectic differential reduction algebra $D\big(\mathfrak{sp}(4)\big)$, by generators and relations, and showing its connections to Bavula's generalized Weyl algebras (GWAs). In doing so, we determine a new class of GWAs we call *skew-affine* GWAs, of which $D\big(\mathfrak{gl}(2)\big)$ and $D\big(\mathfrak{sp}(4)\big)$ are examples. We conjecture that the differential reduction algebra of the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2n)$ is a twisted generalized Weyl algebra (TGWA) and that the relations for $D\big(\mathfrak{sp}(2n)\big)$ yield solutions to the dynamical Yang-Baxter equation (DYBE).

1. Introduction

1.1. **Background.** The reduction problem is a long-standing problem within representation theory arising from the composition of homomorphisms. Indeed, if $(\mathfrak{g}, \phi)$ is a Lie algebra representation, then composing ϕ with a Lie algebra map $\psi\colon \mathfrak{k} \to \mathfrak{g}$ provides a representation of $\mathfrak{k}$ as $\phi \circ \psi$, a so-called pullback representation. One can then ask if the additional perspective of the representation of $\mathfrak{k}$ provides information about the representation of $\mathfrak{g}$, and this has been a fruitful journey (one could begin with [23, 32, 36]) in many cases. In the situation of inclusion of a semisimple subalgebra $\mathfrak{k}$ into a finite-dimensional complex Lie algebra $\mathfrak{g}$, Mickelsson addressed the reduction problem with the introduction of step algebras [26], which are now known as Mickelsson algebras. Thus Mickelsson algebras belong to the class of reduction algebras, algebraic structures used to solve the reduction problem through their action on singular (or primitive) vectors. The previous statement was substantiated in [34] so that a partial answer was given to the reduction problem (particular branching rules) and a general algebraic philosophy found in the study of reduction algebras.

To preview the technical details, set a Lie algebra $\mathfrak{k}$ reductive in a Lie algebra $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ so that the $\mathfrak{k}$-module $\mathfrak{g}$ is completely reducible under the adjoint action of $\mathfrak{k}$. Now $\mathfrak{k}_\pm = \mathfrak{g}_\pm \cap \mathfrak{k}$. Also, let $Z = Z(\mathfrak{g}, \mathfrak{k})$ be the reduction algebra associated with $\mathfrak{g} \supset \mathfrak{k}$. For a $\mathfrak{k}_+$-locally finite[1] $\mathfrak{k}$-semisimple irreducible $\mathfrak{g}$-module V, we have the $\mathfrak{k}$-module decomposition

$$V = \bigoplus_{i=1}^{\dim(V_{\mathfrak{k}}^{+})} V_i,$$

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[1]We use the definition for locally finite meaning $\dim(U(\mathfrak{k}_+)v) < \infty$, for all $v \in V$.

where the space of $\mathfrak{k}_+$-singular vectors $V_{\mathfrak{k}}^+$ is the subspace $\{v \in V \mid ev = 0, \text{ for all } e \in \mathfrak{k}_+\}$ and the set of $\mathfrak{k}$-modules $\{V_i\}$ equals $\{U(\mathfrak{k}_-)k \mid k = z.w_0 \text{ for weight vectors } z \in Z\} \setminus \{\{0\}\}$ and w_0 is any nonzero weight vector in $V_{\mathfrak{k}}^+$.

Specific applications and developments relying on reduction algebras are found in the decomposition of tensor product modules via actions of diagonal reduction algebras [20, 10], higher order Fischer decompositions in harmonic analysis [7], conformal field theory [5, 6], and the construction of wave functions [1] in the realm of theoretical particle and nuclear physics, as examples. The last paper formalized extremal projectors, and Zhelobnenko [37] is credited with realizing the importance of extremal projectors in a scheme to determine generators and relations of localized reduction algebras. See [40] for an overview of the construction of generalized step algebras, what we equate to the class of reduction algebras.

We emphasize now two points with the first being that reduction algebras are associative algebras with their own lanes of study. There are a great number of results on reduction algebras with many summarized at the end of Tolstoy's commemorative paper [31]. Khoroshkin and Ogievetsky furthered results by giving a complete presentation of the diagonal reduction algebra [20] of type $\mathfrak{gl}$ in all degrees. Their care in the technical details sheds light on the role of reduction algebras in the theory of quantum matrix algebras [27, and references therein] and the related quantum inverse scattering method. Previously, Khoroshkin [19] had noted the application of extremal projectors in the quantum dynamical group picture and, with Ogievetsky, solidified [21] that the intertwining operators establishing dynamical quantum group theory appear as special cases of reduction algebras. With motivation drawn from the second author's dissertation [35] on finding explicit bases of reduced tensor product representations of the orthosymplectic Lie superalgebras $\mathfrak{osp}(1|2n)$, the authors gave a complete presentation [10, Section 3.5] of the diagonal reduction superalgebra $Z(\mathfrak{osp}(1|2) \times \mathfrak{osp}(1|2), \mathfrak{osp}(1|2))$ for $\mathfrak{osp}(1|2)$, the first recorded diagonal reduction algebra associated to a classical [18] Lie superalgebra. The sequel [11] classifies finite-dimensional and certain infinite-dimensional representations of $Z(\mathfrak{osp}(1|2) \times \mathfrak{osp}(1|2), \mathfrak{osp}(1|2))$.

The second matter is that homomorphisms to the nth Weyl algebra A_n (see the use of *canonical realizations* in [12, Definition 1]) yield examples of reduction algebras important to quantum deformations and noncommutative geometry. A particular example is the ring $\mathcal{D}_{\mathbf{h}}(n, N)$ of $\mathbf{h}$-deformed differential operators [14] as the reduction algebra defined from the diagonal map sending $\mathfrak{gl}(n)$ to the tensor product $U(\mathfrak{gl}(n)) \otimes A_{nN}$. In this paper we define $D\big(\mathfrak{sp}(4)\big)$ as the symplectic analogue to $\mathcal{D}_{\mathbf{h}}(2, 1)$ by leveraging the oscillator realization of $\mathfrak{sp}(4)$ in A_2.

Combing the two points of emphasis above, we initiate the study of symplectic differential reduction algebras and inspire results in the theory of generalized Weyl algebras (GWAs). There is quite a natural progression to this aim: In [22], Khoroshkin and Ogievetsky introduced the method of *cutting and stabilization* relating commutation relations in the diagonal reduction algebra of the Cartesian product $\mathfrak{gl}(m) \times \mathfrak{gl}(n)$ to that of $\mathfrak{gl}(m + n)$. A reasonable mathematical pursuit is the generalization of this method to more general pairs $\mathfrak{k} \to \mathfrak{g}$. For this, it is important to calculate examples. The sequence of Lie (super)algebras

$$\mathfrak{gl}(n) \to \mathfrak{sp}(2n) \to \mathfrak{osp}(1|2n)$$

is of particular interest to us. For $n = 2$, the Lie algebra $\mathfrak{sp}(4)$ plays a crucial role, and we expect the relations found in this paper to be important for future work on $\mathfrak{osp}(1|4)$ and higher rank cases. Specifically, we introduce skew-affine generalized Weyl algebras that will be important for higher n, as well as being of independent interest in noncommutative ring theory. We also state some related open problems which would be valuable to investigate. For the $\mathfrak{osp}(1|2n)$ case, we conjecture that the differential reduction superalgebra $D\big(\mathfrak{osp}(1|2n)\big)$ is a twisted generalized Weyl algebra (TGWA), as the family of TGWAs contains superalgebra versions of GWAs [9]. Finally, we expect solutions to the dynamical Yang-Baxter equation (DYBE) to arise from explicit computation of relations between generators of $D\big(\mathfrak{sp}(2n)\big)$.

The paper's structure is as follows: We provide preliminaries in Section 2. In particular, Section 2.3 reviews the work of Ogievetsky, Khoroshkin, and Herelemont on $\mathcal{D}_{\mathbf{h}}(2,1)$ to motivate our technique. Section 3 contains a proof of our presentation for the rank two symplectic differential reduction algebra $D\big(\mathfrak{sp}(4)\big)$. We supply in the concluding paragraphs open problems and conjectures. We end the current section with a summary of main results.

1.2. **Main Results.** The following result, proved in Section 3.4, is the main result of the paper.

For the statement, we denote by R a homomorphic image in $D\big(\mathfrak{sp}(4)\big)$ of the universal enveloping algebra of the Cartan subalgebra of $\mathfrak{sp}(4)$ localized at the monoid generated by all $h-n$ where h is a coroot and $n \in \mathbb{Z}$. The algebra $D\big(\mathfrak{sp}(4)\big)$ is an R-ring with product denoted

$$\diamond : D\big(\mathfrak{sp}(4)\big) \otimes_R D\big(\mathfrak{sp}(4)\big) \to D\big(\mathfrak{sp}(4)\big).$$

Theorem 1.1 (Presentation). *$D\big(\mathfrak{sp}(4)\big)$ is generated as an R-ring by $\bar{x}_1, \bar{\partial}_1, \bar{x}_2, \bar{\partial}_2$ subject to relations, where $H_\gamma \in R$ denote the images of the coroots of $\mathfrak{sp}(4)$:*

$$\bar{x}_1 H_\alpha = (H_\alpha - 1)\bar{x}_1 \qquad \bar{x}_1 H_\beta = H_\beta x_1 \tag{1.1a}$$

$$\bar{\partial}_1 H_\alpha = (H_\alpha + 1)\bar{\partial}_1 \qquad \bar{\partial}_1 H_\beta = H_\beta \partial_1 \tag{1.1b}$$

$$\bar{x}_2 H_\alpha = (H_\alpha + 1)\bar{x}_2 \qquad \bar{x}_2 H_\beta = (H_\beta - 1)\bar{x}_2 \tag{1.1c}$$

$$\bar{\partial}_2 H_\alpha = (H_\alpha - 1)\bar{\partial}_2 \qquad \bar{\partial}_2 H_\beta = (H_\beta + 1)\bar{\partial}_2 \tag{1.1d}$$

$$\bar{x}_1 \diamond \bar{x}_2 = \Big(1 + \frac{1}{H_\alpha + 1}\Big)\bar{x}_2 \diamond \bar{x}_1 \qquad \bar{\partial}_2 \diamond \bar{\partial}_1 = \bar{\partial}_1 \diamond \bar{\partial}_2 \Big(1 + \frac{1}{H_\alpha + 1}\Big) \tag{1.1e}$$

$$\bar{x}_1 \diamond \bar{\partial}_2 = \Big(1 + \frac{1}{H_{\beta+\alpha} + 1}\Big)\bar{\partial}_2 \diamond \bar{x}_1 \qquad \bar{x}_2 \diamond \bar{\partial}_1 = \bar{\partial}_1 \diamond \bar{x}_2 \Big(1 + \frac{1}{H_{\beta+\alpha} + 1}\Big) \tag{1.1f}$$

$$\bar{x}_1 \diamond \bar{\partial}_1 = -1 + \frac{1}{H_\alpha + 1} + f_{11}\bar{\partial}_1 \diamond \bar{x}_1 + f_{12}\bar{\partial}_2 \diamond \bar{x}_2 \tag{1.1g}$$

$$\bar{x}_2 \diamond \bar{\partial}_2 = -1 + f_{21}\bar{\partial}_1 \diamond \bar{x}_1 + f_{22}\bar{\partial}_2 \diamond \bar{x}_2 \tag{1.1h}$$

where $f_{ij} = f_{ij}(H_\alpha, H_\beta) \in R$ are given by

$$f_{11} = \frac{(a+1)(a-1)(b+1)}{a^2 b} \qquad f_{12} = \frac{-(d+2)}{ac}$$
$$f_{21} = \frac{a(d-1) + c(d+1)}{acd} \qquad f_{22} = \frac{d+1}{d} \tag{1.2}$$

and we put $a = H_\alpha + 1$, $b = H_{\beta+2\alpha} + 1$, $c = H_{\beta+\alpha} + 1$, $d = H_\beta + 1$.

We remark that, if we formally send all $H_\gamma \to \infty$ we obtain the usual Weyl algebra relations. This is to be expected according to general reduction algebra principles. More precisely, consider the extension of scalars $D^*\big(\mathfrak{sp}(4)\big) = \mathbb{C}[\hbar, \hbar^{-1}] \otimes_{\mathbb{C}} D\big(\mathfrak{sp}(4)\big)$. There is an "integral form" $D^0\big(\mathfrak{sp}(4)\big)$ inside this algebra defined as the $\mathbb{C}[\hbar]$-subalgebra of $D^*\big(\mathfrak{sp}(4)\big)$ generated by

$$H'_\alpha = \hbar H_\alpha,\ H'_\beta = \hbar H_\beta,\ \bar{x}_1,\ \bar{x}_2,\ \bar{\partial}_1,\ \bar{\partial}_2.$$

Substituting $H_\gamma = \frac{1}{\hbar}H'_\gamma$ in all relations above (and multiplying (1.1a)–(1.1d) by $\hbar$), we see that the quotient algebra of $D^0\big(\mathfrak{sp}(4)\big)$ by the principal ideal $(\hbar)$ is isomorphic to the Weyl algebra $A_2(\mathbb{C})$ tensored (over $\mathbb{C}$) by the ring R.

Our second main theorem provides a connection to generalized Weyl algebras in the sense of Bavula [3].

Theorem 1.2 (GWA realization; see Theorem 4.5 for details)**.** *The normalized generators*

$$\widehat{x}_1 = x_1, \qquad \widehat{x}_2 = (H_\alpha + 2)\bar{x}_2, \qquad \widehat{\partial}_1 = \bar{\partial}_1(H_\alpha + 1)(H_{\beta+\alpha} + 1), \qquad \widehat{\partial}_2 = \bar{\partial}_2(H_{\beta+\alpha} + 1) \tag{1.3}$$

satisfy

$$[\widehat{x}_1, \widehat{x}_2]_\diamond = [\widehat{\partial}_1, \widehat{\partial}_2]_\diamond = [\widehat{x}_1, \widehat{\partial}_2]_\diamond = [\widehat{x}_2, \widehat{\partial}_1]_\diamond = 0, \tag{1.4}$$

where $[a, b]_\diamond = a \diamond b - b \diamond a$ is the diamond commutator. In fact, $D\big(\mathfrak{sp}(4)\big)$ is an example of a generalized Weyl algebra.

2. Preliminaries

The base field is $\mathbb{C}$ except when specified otherwise. Let $A_n(\mathbb{C})$ denote the nth Weyl algebra. More precisely, $A_n(\mathbb{C})$ is the complex associative algebra with generators $x_i, \partial_i = \frac{\partial}{\partial x_i}$, for $i = 1, 2, \ldots, n$, and relations

$$x_i x_j - x_j x_i = 0, \qquad \partial_i \partial_j - \partial_j \partial_i = 0, \qquad \partial_i x_j - x_j \partial_i = \delta_{ij}, \tag{2.1}$$

where δ_{ij} is the Kronecker delta. Relations 2.1 are expressed more compactly using the standard commutator $[a, b] = ab - ba$ within any ring. The Weyl algebra carries an order 2 anti-automorphism $\vartheta(x_i) = \partial_i$.

2.1. Reduction Algebras.

There is an extensive history [34, 37, 38, 21, 2, 16, 11] on the study of reduction algebras and their applications. We provide a brief overview of four constructions/perspectives for a complex finite-dimensional reductive Lie algebra $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ with opposite nilpotent parts $\mathfrak{g}_\pm$ and Cartan subalgebra $\mathfrak{h}$. See [10, Section 2] for details in the super case.

It is helpful to see these constructions as intertwined rather than separate notions.

(1) *Subquotient algebra*: Let A be an associative algebra and $\varphi\colon U(\mathfrak{g}) \to A$ a map of associative algebras. In particular, this makes A into a $U(\mathfrak{g})$-bimodule via multiplication. Let $I = A{\cdot}\mathfrak{g}_+$ (written as $A\mathfrak{g}_+$ from hereon) be the left ideal in A generated by the image of $\mathfrak{g}_+$ under φ composed with the natural inclusion. The reduction algebra Z associated to $(A, \mathfrak{g}, \varphi)$ is defined to be $Z = N_A(I)/I$, where $N_A(I)$ is the normalizer of I in A. To clarify, the *normalizer* $N_A(I)$ of the left ideal I in A is the maximal subalgebra of A with respect to containment of I as a two-sided ideal. Hence as an A-subquotient the reduction algebra Z inherits its product from A. Note that normalizers may be termed *idealizers* in general ring theory [with historical context found in 17, Section 2].

Remark 2.1. When φ is induced from a Lie algebra map $\mathfrak{g} \to \mathfrak{G}$, $\mathfrak{g}$ reductive in $\mathfrak{G}$, it is common to denote the reduction algebra $Z = Z(U(\mathfrak{g}), \mathfrak{g}, \varphi)$ as $Z = Z(\mathfrak{G}, \mathfrak{g})$.

(2) *Opposite endomorphism algebra*: Since $\big(\mathrm{End}_A(A)\big)^{\mathrm{op}}$ is isomorphic to A, one might hope for a natural realization of the A-subquotient Z as endomorphisms of an A-module. Indeed,

since $n \in N_A(I)$ implies $In \subset I$, right multiplication by $N_A(I)$ defines a right action on A/I such that $I \subset \mathrm{Ann}_{N_A(I)}(A/I)$. It follows that A/I is a right Z-module, in fact, an (A, Z)-bimodule. Then $\mathrm{End}_A(A/I)$ is a left Z-module isomorphic to the left Z-module Z, particularly since endomorphisms of A/I preserve Z and $1 + I \in Z$ generates A/I as an A-module. Now consider maps $1 + I \mapsto n + I$, $n \in N_A(I)$, which are precisely the elements of $\mathrm{End}_A(A/I)$ by the previous statement. Then opposite composition shows that the reduction algebra Z is isomorphic to $Z_{\mathrm{maps}} = \big(\mathrm{End}_A(A/I)\big)^{\mathrm{op}}$, the opposite algebra of left A-module endomorphisms on A/I.

(3) *Subspace composed of* $\mathfrak{g}_+$*-invariants*: For a left A-module V we have the subspace

$$V^+ = \{v \in V \mid \; ev = 0, \text{ for all } e \in \mathfrak{g}_+\}$$

of $\mathfrak{g}_+$-invariants. Then since $N_A(I)V^+ \subset V^+$ and $IV^+ = \{0\}$, there is a left Z action on V^+. Taking the case where $V = A/I$, we have that $Z = (A/I)^+$ since $\mathfrak{g}_+(a+I) = \{0+I\}$ if and only if $Ia \subset I$, the defining condition for $a \in N_A(I)$. We say then that A/I is the universal $\mathfrak{g}_+$-highest-weight A-module in the sense that given a pair (W, w) of a left A-module W containing a (singular) vector w (such as any $\mathfrak{g}_+$-highest-weight vector), there exists a unique A-module homomorphism from A/I to W sending $1 + I$ to w. Thus evaluation at $1 + I \in A/I$ determines completely and the left Z-module $\mathrm{Hom}_A(A/I, W)$ and W^+ are isomorphic as left Z-modules, as in the special case in the previous paragraph.

Remark 2.2. Recasting W^+ as solutions to a system of equations puts Z as a symmetry algebra of the solution space. In particular, reduction algebras are useful in determining solutions to so-called extremal equations [38].

(4) *Quotient space composed of* $\mathfrak{g}_-$*-coinvariants*: There is a dual construction of reduction algebras based on $\mathfrak{g}_-$-coinvariants, the elements of $V_- = V/\mathfrak{g}_-V$ for a left A-module V (taking into account the action of $\mathfrak{g}_-$ induced by φ). We begin with the quotient space $(A/I)_- = (A/I)/\mathfrak{g}_-(A/I)$ that comprises the $\mathfrak{g}_-$-coinvariants of A/I. Note that A/I being a right Z-module and A being a $U(\mathfrak{g})$-bimodule implies $(A/I)_-$ is a right Z-module. By multiple applications of the fundamental theorem of homomorphisms, we identify $(A/I)_-$ with the double coset space $J\backslash A/I = A/\mathbb{II}$, where $\mathbb{II}$ is the sum of abelian groups $J + I$. Explicitly,

$$(a + I) + \mathfrak{g}_-(A/I) \mapsto a + (I + J).$$

Then we have $\pi \circ \iota \colon Z \to A/\mathbb{II}$ with the natural inclusion $\iota \colon Z \to A/I$ and the natural surjection $\pi \colon A/I \to A/\mathbb{II}$ of right Z-modules. When the extremal projector is available, there is an inverse to $\pi \circ \iota$ that induces an associative product on $A/\mathbb{II}$ such that $Z \cong A/\mathbb{II}$. See Proposition 2.3.

The aforementioned extremal projector is available under satisfaction of the *coroot condition* and locally nilpotent actions: If (the image under φ of) $h - n$ is invertible in A for any coroot h and integer n, and $\mathfrak{g}_+$ acts locally nilpotently on A, then the extremal projector of $\mathfrak{g}$ is available. To ensure the coroot condition is satisfied, we follow Zhelobenko [38] to localize A at the denominator set D generated by $\{h - n \mid h \in \mathfrak{h},\, n \in \mathbb{Z}\}$. One can localize A first and then consider subquotients or form the the subquotient Z and continue with a localized version. The non-localized version is termed the *Mickelsson algebra* and may be regarded as a subalgebra of the corresponding (localized) reduction algebra. With the motivation for localization, let $A' = D^{-1}U(\mathfrak{h}) \otimes_{U(\mathfrak{h})} A$, recognizing the coefficients as "dynamical scalars", rational expressions in $\mathfrak{h}$. We also cite Zhelobenko [39, p. 4.16] for the method to obtain normalized elements belonging to the non-localized reduction algebra (Mickelsson algebra) from the localized reduction algebra by clearing denominators. We make use of normalized generators of Z in specific calculations.

The thesis [14] of Herlemont (supervised by Ogivetsky) expands on the connection between extremal projectors and reduction algebras. A more geometric treatment is found in [28]. What immediately follows is a summary on extremal projectors which is developed for the specific foci of this paper in Section 3.

2.2. **Extremal Projectors.** Projection operators or extremal projectors were introduced in [24] and subsequently studied by Asherova, Smirnov, Tolstoy [1]. Each finite-dimensional reductive Lie algebra $\mathfrak{g}$ over the complex numbers admits an extremal projector as an element of $\widehat{U}(\mathfrak{g})$, denoting a certain completion of the localized (in the same sense as A above) $U(\mathfrak{g})$ in terms of formal power series and finitary conditions on exponents within terms. In particular, the extremal projector corresponding to $\mathfrak{g}$ was recalled in [30]. See [10] for further background and applications of the construction [29] in the super setting. As stated, we give only the main ingredients to prime readers for the explicit construction found in Section 3.

Recall that $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ and let Φ be the root system of $\mathfrak{g}$ with Φ_+ the positive roots. Choose a convex [in 30, "normal"]) order on the positive roots, and denote the corresponding ordered set by $\vec{\Phi}_+$ so that indexing over $\vec{\Phi}_+$ is in regard to the convex ordering.

The extremal projector P of $\mathfrak{g}$ (associated with the triangular decomposition) is independent of the choice of convex ordering [30]:

$$P = \prod_{\alpha \in \vec{\Phi}_+} P_\alpha \tag{2.2a}$$

where

$$P_\alpha = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} f_{\alpha,k}^{-1} x_{-\alpha}^k x_\alpha^k \tag{2.2b}$$

and

$$f_{\alpha,k} = \prod_{j=1}^{k} \big(h_\alpha + \rho(\alpha) + j\frac{(\alpha,\alpha)}{2}\big) \tag{2.2c}$$

for ρ the linear functional on $\mathfrak{h}^*$ determined by its image $\rho(\alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i)$ on simple roots α_i, and $(x_\alpha, x_{-\alpha}, h_\alpha)$ is an $\mathfrak{sl}(2)$-triple associated with $\alpha \in \Phi_+$.

Precisely, P is the unique element of $\widehat{U}$ of satisfying $x_\alpha P = 0$, $P x_{-\alpha} = 0$, $P \equiv 1 \pmod{\mathfrak{g}_- \widehat{U} + \widehat{U} \mathfrak{g}_+}$. For each locally $\mathfrak{g}_+$-finite $\widehat{U}$-module V, P provides an inverse to the linear map $Q_V : V^+ \to V_-$, from the space of $\mathfrak{g}_+$-invariants to the space of $\mathfrak{g}_-$-coinvariants. The map Q_V is the composition of π_V with ι_V, the natural projection and injection, respectively. In other words, the extremal projector P acts an endomorphism

$$P_V : V \xrightarrow{\pi_V} V_- \xrightarrow{Q_V^{-1}} V^+ \xrightarrow{\iota_V} V,$$

which satisfies:

$$P_V(v) = v, \text{ for all } v \in V^+, \tag{2.3}$$

$$P_V(v) \in v + \mathfrak{g}_- V \subset V, \text{ for all } v \in V. \tag{2.4}$$

The name extremal projector is appropriate in that the characterization above and Equations (2.3) and (2.4) imply $P_V^2 = P_V$. Thus, we have the decomposition $V = V^+ \oplus \mathfrak{g}_- V$, that is, P projects V to the space V^+ along the space $\mathfrak{g}_- V$. Emphasizing $Q_V^{-1} \circ \pi_V$, we have the isomorphism $V^+ \cong V/\mathfrak{g}_- V$ of vector spaces.

If we choose $(A, \mathfrak{g}, \varphi)$ such that the coroot condition is satisfied and A/I is a locally $\mathfrak{g}_+$-finite $U(\mathfrak{g})$-module, then we have $\mathrm{Im}(P_{A/I}) = \mathrm{Im}(Q_{A/I}^{-1}) = Z$, the reduction algebra. The map $Q_{A/I}^{-1}$ yields a meaningful way to study $Z = N_A(I)/I$ since computing the entire subalgebra $N_A(I)$ by definition is generally unwieldy.

Also, we recall facts from the previous subsection: We have the identification of $(A/I)_-$ with the double coset space $A/\mathrm{I\!I}$. Now both Z and $A/\mathrm{I\!I}$ are right Z-modules, and $Q_{A/I} : Z \to A/\mathrm{I\!I}$ is the composition of right Z-module homomorphism. Hence $Q_{A/I}$ and its assumed inverse are right Z-module isomorphisms.

Moreover, the following result was noticed by Khoroshkin and Ogievetsky.

Proposition 2.3 ([21, Equation 3.7])**.** *There is an associative binary product in $A/\mathrm{I\!I}$ that makes $Q_{A/I}^{-1} : A/\mathrm{I\!I} \to Z$ an isomorphism of algebras:*

$$\bar{x} \diamond \bar{y} = xPy + \mathrm{I\!I}, \qquad \bar{x} = x + \mathrm{I\!I},\ \bar{y} = y + \mathrm{I\!I},\ x, y \in A,$$

where we make sense of $xPy + \mathrm{I\!I}$ as the image of $xP_{A/I}(y+I) \in A/I$ under the canonical projection $A/I \to A/\mathrm{I\!I}$.

Proof. Having $Q_{A/I} : Z \to A/\mathrm{I\!I}$ and $Q_{A/I}^{-1} : A/\mathrm{I\!I} \to Z$ as isomorphisms of right Z-modules, the result follows from writing the product in $A/\mathrm{I\!I}$ to define an algebra homomorphism $Z \to A/\mathrm{I\!I}$:

$$\begin{aligned}
(x + \mathrm{I\!I}) \diamond (y + \mathrm{I\!I}) &= Q_{A/I}\left(Q_{A/I}^{-1}(x+\mathrm{I\!I})(Q_{A/I}^{-1}(y+\mathrm{I\!I})\right) && \text{by definition of } \diamond \\
&= Q_{A/I}\left(Q_{A/I}^{-1}(x+\mathrm{I\!I})\right)\left(Q_{A/I}^{-1}(y+\mathrm{I\!I})\right) && \text{since } Q_{A/I} \text{ is a right } Z\text{-module map} \\
&= (x+\mathrm{I\!I})\left(Q_{A/I}^{-1}(y+\mathrm{I\!I})\right) \\
&= (x+\mathrm{I\!I})P_{A/I}(y+I) \\
&= xPy + \mathrm{I\!I} && \text{by right action of } Z \text{ on } A/\mathrm{I\!I}.
\end{aligned}$$

We clarify that the right action of $n + I \in Z$ on $x + \mathrm{I\!I}$ is $xn + \mathrm{I\!I}$, which is equal to the image of $x(n+I) \in A/I$ under the canonical projection $A/I \to A/\mathrm{I\!I}$. ☐

From now on we use Pv to denote $P_V(v)$ unless additional context is needed.

2.3. **The Differential Reduction Algebra $D(\mathfrak{gl}(2))$.** To illustrate the methods of Section 3, in this section we compute a presentation for the differential reduction algebra of $\mathfrak{gl}(n)$ with $n = 2$. For general n, these algebras have been investigated by [14, 16, 15]

Consider the Lie algebra homomorphism from $\mathfrak{gl}(2)$ to the algebra $A_2(\mathbb{C}) \otimes U(\mathfrak{gl}(2))$ localized at the multiplicative monoid generated by $\{H - n \mid n \in \mathbb{Z}\}$, where

$$\begin{aligned}
e &\mapsto E = x_1\partial_2 \otimes 1 + 1 \otimes e \\
f &\mapsto F = x_2\partial_1 \otimes 1 + 1 \otimes f \\
h &\mapsto H = (x_1\partial_1 - x_2\partial_2) \otimes 1 + 1 \otimes h
\end{aligned}$$

$$I_2 \mapsto I = (x_1\partial_1 + x_2\partial_2 + 1)\otimes 1 + 1 \otimes I_2$$

where I_2 is the 2×2 identity matrix and e,f,h the standard $\mathfrak{sl}(2)$ basis. Put

$$\widetilde{x}_i = x_i \otimes 1, \quad \widetilde{\partial}_i = \partial_i \otimes 1.$$

We have the following rules for moving dynamical scalars to the left of these generators:

$$\begin{aligned}
\widetilde{x}_1 H &= (H-1)\widetilde{x}_1, & \widetilde{x}_1 I &= (I-1)\widetilde{x}_1,\\
\widetilde{x}_2 H &= (H+1)\widetilde{x}_2, & \widetilde{x}_2 I &= (I-1)\widetilde{x}_2,\\
\widetilde{\partial}_1 H &= (H+1)\widetilde{\partial}_1, & \widetilde{\partial}_1 I &= (I+1)\widetilde{\partial}_1,\\
\widetilde{\partial}_2 H &= (H-1)\widetilde{\partial}_2, & \widetilde{\partial}_2 I &= (I+1)\widetilde{\partial}_2.
\end{aligned} \tag{2.5}$$

To calculate relations in the reduction algebra we use the extremal projector for $\mathfrak{sl}(2)$,

$$P = 1 + F\frac{-1}{H}E + \cdots . \tag{2.6}$$

When $[E,[E,b]]=0$ or $[[a,F],F]=0$ (which will be the case for all our generators a,b), we have modulo $\mathrm{I\!I}$:

$$aPb \equiv ab + [a,F]\frac{-1}{H}[E,b].$$

First, since E commutes with $\widetilde{x}_1$ and $\widetilde{\partial}_2$, and F commutes with $\widetilde{x}_2$ and $\widetilde{\partial}_1$, we have the following congruences modulo $\mathrm{I\!I}$:

$$\begin{aligned}
aP\widetilde{x}_1 &\equiv a\widetilde{x}_1, & \widetilde{x}_2 Pb &\equiv \widetilde{x}_2 b,\\
aP\widetilde{\partial}_2 &\equiv a\widetilde{\partial}_2, & \widetilde{\partial}_1 Pb &\equiv \widetilde{\partial}_1 b.
\end{aligned}$$

Next,

$$\widetilde{x}_1 P\widetilde{x}_2 \equiv \widetilde{x}_1\widetilde{x}_2 + [\widetilde{x}_1, F]\frac{-1}{H}[E,\widetilde{x}_2] = \frac{H+2}{H+1}\widetilde{x}_2 P\widetilde{x}_1$$

So, with $\bar{x}_i = \widetilde{x}_i + \mathrm{I\!I}$ and $\bar{\partial}_i = \widetilde{\partial}_i + \mathrm{I\!I}$:

$$\bar{x}_1\bar{x}_2 = \frac{H+2}{H+1}\bar{x}_2\bar{x}_1, \qquad \bar{\partial}_2\bar{\partial}_1 = \frac{H+2}{H+1}\bar{\partial}_1\bar{\partial}_2 \tag{2.7}$$

The second relation follows from the first by applying the involution Θ and using that H commutes with $\bar{\partial}_1\bar{\partial}_2$.

Next, we have $\widetilde{\partial}_2 P\widetilde{x}_1 \equiv \widetilde{\partial}_2\widetilde{x}_1 \equiv \widetilde{x}_1\widetilde{\partial}_2 \equiv \widetilde{x}_1 P\widetilde{\partial}_2$, so

$$\bar{\partial}_2\bar{x}_1 = \bar{x}_1\bar{\partial}_2, \qquad \bar{\partial}_1\bar{x}_2 = \bar{x}_2\bar{\partial}_1, \tag{2.8}$$

again using Θ to get the second from the first.

Lastly,

$$\widetilde{\partial}_2 P\widetilde{x}_2 \equiv \widetilde{\partial}_2\widetilde{x}_2 + [\widetilde{\partial}_2, F]\frac{-1}{H}[E,\widetilde{x}_2] \equiv 1 + \widetilde{x}_2 P\widetilde{\partial}_2 + \frac{-1}{H+1}\widetilde{\partial}_1 P\widetilde{x}_1.$$

Solving for $\widetilde{x}_2 P\widetilde{\partial}_2$, we get in the quotient

$$\bar{x}_2\bar{\partial}_2 = -1 + \frac{1}{H+1}\bar{\partial}_1\bar{x}_1 + \bar{\partial}_2\bar{x}_2. \tag{2.9}$$

Similarly,

$$\widetilde{x}_1 P\widetilde{\partial}_1 \equiv \widetilde{x}_1\widetilde{\partial}_1 + [\widetilde{x}_1, F]\frac{-1}{H}[E, \widetilde{\partial}_1] \equiv -1 + \widetilde{\partial}_1 P\widetilde{x}_1 + \frac{-1}{H+1}\widetilde{x}_2 P\widetilde{\partial}_2$$

That is,

$$\bar{x}_1\bar{\partial}_1 = -1 + \bar{\partial}_1\bar{x}_1 + \frac{-1}{H+1}\bar{x}_2\bar{\partial}_2.$$

Substituting (2.9) into this we get

$$\bar{x}_1\bar{\partial}_1 = -1 + \bar{\partial}_1\bar{x}_1 + \frac{-1}{H+1}\Big(-1 + \frac{1}{H+1}\bar{\partial}_1\bar{x}_1 + \bar{\partial}_2\bar{x}_2\Big)$$

Simplifying this, using the identity $1 + \frac{-1}{(H+1)^2} = \frac{H(H+2)}{(H+1)^2}$ we get

$$\bar{x}_1\bar{\partial}_1 = -1 + \frac{1}{H+1} + \frac{H(H+2)}{(H+1)^2}\bar{\partial}_1\bar{x}_1 + \frac{-1}{H+1}\bar{\partial}_2\bar{x}_2. \tag{2.10}$$

Now introduce normalized generators

$$\widehat{x}_1 = \bar{x}_1, \quad \widehat{x}_2 = (H+2)\bar{x}_2, \quad \widehat{\partial}_1 = (H+2)\bar{\partial}_1, \quad \widehat{\partial}_2 = \bar{\partial}_2. \tag{2.11}$$

Theorem 2.4. *Then one can check that*

$$[\widehat{x}_1, \widehat{x}_2] = [\widehat{\partial}_1, \widehat{\partial}_2] = [\widehat{\partial}_2, \widehat{x}_1] = [\widehat{\partial}_1, \widehat{x}_2] = 0. \tag{2.12}$$

Furthermore, using (2.9)*, one checks that*

$$\widehat{x}_2\widehat{\partial}_2 = -(H+2) + \frac{1}{H+1}\widehat{\partial}_1\widehat{x}_1 + \frac{H+2}{H+1}\widehat{\partial}_2\widehat{x}_2, \tag{2.13}$$

$$\widehat{x}_1\widehat{\partial}_1 = -H + \frac{H}{H+1}\widehat{\partial}_1\widehat{x}_1 + \frac{-1}{H+1}\widehat{\partial}_2\widehat{x}_2. \tag{2.14}$$

Corollary 2.5. *The set*

$$\{\widehat{x}_1^a\widehat{x}_2^b\widehat{\partial}_1^c\widehat{\partial}_2^d \mid a,b,c,d \in \mathbb{Z}_{\geq 0}\} \tag{2.15}$$

is a basis for $D(\mathfrak{gl}(2))$ as a left R-module.

Proof. The linear independence follow from the linear independence of the corresponding monomials in $\mathscr{A}$. The spanning property follows from Theorem 2.4. □

We note that clearing denominators (multiplying (2.13) and (2.14) from the left by $(H+1)$) we obtain a presentation for a subalgebra of (a kind of "order" in) the non-localized version of the reduction algebra.

3. Constructing a Differential Reduction Algebra of $\mathfrak{sp}(4)$

3.1. **Type C Lie Subalgebra of the Weyl Algebra.** As seen in the introduction of [13], for example, we realize the (complex) symplectic Lie algebra $\mathfrak{sp}(2n)$ as the Lie subalgebra of the Weyl algebra spanned by anti-commutators $\{a,b\} = ab + ba$. Let $V = \mathrm{Span}\{x_1, \partial_1, \ldots, x_n, \partial_n\} \subset A_n(\mathbb{C})$. Define

$$\mathfrak{sp}(2n) = \mathrm{Span}\{ab + ba \mid a, b \in V\} \tag{3.1}$$

Thus, $\mathfrak{sp}(2n)$ is spanned by the following elements:

$$a_{ij} = \frac{1}{2}\{x_i, \partial_j\} = x_i\partial_j + \frac{1}{2}\delta_{ij}, \qquad b_{ij} = \frac{1}{2}\{x_i, x_j\} = x_i x_j = b_{ji}, \qquad c_{ij} = \frac{1}{2}\{\partial_i, \partial_j\} = \partial_i\partial_j = c_{ji}$$

These elements correspond to the size $2n \times 2n$ matrices of the form

$$\begin{bmatrix} A & B = B^t \\ C = C^t & -A^t \end{bmatrix}$$

Recall that ϑ denotes the $\mathbb{C}$-linear involutive anti-automorphism of the Weyl algebra A_n determined by $\vartheta(x_i) = \partial_i$. Note that ϑ preserves the subspace $\mathfrak{sp}(2n)$. We identify $\vartheta|_{\mathfrak{sp}(2n)}$ with the involutive Lie algebra anti-automorphism θ given by

$$\theta(a_{ij}) = a_{ji},\ \theta(b_{ij}) = c_{ij}.$$

The subspace $\mathfrak{h} = \mathrm{Span}\{a_{ii}\}_{i=1}^n$ is a Cartan subalgebra. The root system is

$$\Delta = \{\varepsilon_i - \varepsilon_j\}_{i \neq j} \cup \{\varepsilon_i + \varepsilon_j\}_{i,j} \cup \{-\varepsilon_i - \varepsilon_j\}_{i,j} \tag{3.2}$$

where $\varepsilon_i(a_{jj}) = \delta_{ij}$. A choice of positive roots is

$$\Delta_+ = \{\varepsilon_i - \varepsilon_j\}_{i<j} \cup \{\varepsilon_i + \varepsilon_j\}_{i,j} \tag{3.3}$$

and the corresponding simple roots are

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1}\ (i = 1, \ldots, n-1); \quad \alpha_n = 2\varepsilon_n. \tag{3.4}$$

For each positive root γ we fix an $\mathfrak{sl}(2)$-triple $(e_\gamma,\ f_\gamma,\ h_\gamma)$ as follows:

$$\begin{aligned} \gamma = \varepsilon_i - \varepsilon_j: &\quad (a_{ij},\ a_{ji},\ a_{ii} - a_{jj}) \\ \gamma = 2\varepsilon_i: &\quad (\frac{\sqrt{-1}}{2}b_{ii},\ \frac{\sqrt{-1}}{2}c_{ii},\ a_{ii}) \\ \gamma = \varepsilon_i + \varepsilon_j: &\quad (\sqrt{-1}b_{ij},\ \sqrt{-1}c_{ij},\ a_{ii} + a_{jj}) \end{aligned}$$

3.2. **The Differential Reduction Algebra of $\mathfrak{sp}(4)$.** In this section we obtain a symplectic extremal projector as defined in (2.2).

With the root data as give in the previous section, we now specialize to the case $n = 2$. Thus, $\mathfrak{g} = \mathfrak{sp}(4)$, the complex symplectic Lie algebra of rank two (type $C_2 \cong B_2$) with Cartan subalgebra $\mathfrak{h}$. The corresponding root system is shown in Figure 1.

From 3.2, we have the 8 roots

$$\pm(\varepsilon_1 - \varepsilon_2),\ \pm(\varepsilon_1 + \varepsilon_2),\ \pm 2\varepsilon_1,\ \pm 2\varepsilon_2$$

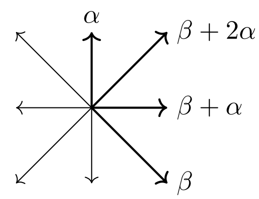

FIGURE 1. Root system of type of type C_2. Thick lines indicate a choice of positive roots.

For $n=2$ we denote the simple roots by

$$\alpha=\varepsilon_1-\varepsilon_2, \qquad \beta=2\varepsilon_2. \tag{3.5}$$

The corresponding simple root vectors of $\mathfrak{sp}(4)$ are

$$e_\alpha = e_{\varepsilon_1-\varepsilon_2} = a_{12} = x_1\partial_2 \tag{3.6a}$$
$$e_\beta = e_{2\varepsilon_2} = \frac{i}{2}x_2^2 \tag{3.6b}$$
$$e_{\beta+\alpha} = e_{\varepsilon_1+\varepsilon_2} = ix_1x_2 \tag{3.6c}$$
$$e_{\beta+2\alpha} = e_{2\varepsilon_1} = \frac{i}{2}x_1^2 \tag{3.6d}$$
$$h_\alpha = a_{11}-a_{22} = x_1\partial_1 - x_2\partial_2 \tag{3.6e}$$
$$h_\beta = a_{22} = x_2\partial_2 + \frac{1}{2} \tag{3.6f}$$
$$h_{\beta+\alpha} = a_{11}+a_{22} = x_1\partial_1 + x_2\partial_2 + 1 \tag{3.6g}$$
$$h_{\beta+2\alpha} = a_{11} = x_1\partial_1 + \frac{1}{2} \tag{3.6h}$$

The negative root vectors are obtained by applying the anti-automorphism θ. Also, note that $\operatorname{ad} h_{\beta+\alpha}$ is the total degree derivation on the Weyl algebra: $[h_{\beta+\alpha}, x_i] = x_i$, $[h_{\beta+\alpha}, \partial_i] = -\partial_i$.

Let $h_\alpha, h_\beta \in \mathfrak{h}$ be the simple coroots. Then $\alpha(h_\beta)=-1$, $\beta(h_\alpha)=-2$. The set of positive roots is

$$\Phi_+ = \{\beta, \beta+\alpha, \beta+2\alpha, \alpha\}. \tag{3.7}$$

There are exactly two convex orderings of the positive roots:

$$\beta < \beta+\alpha < \beta+2\alpha < \alpha, \tag{3.8a}$$
$$\alpha < \beta+2\alpha < \beta+\alpha < \beta. \tag{3.8b}$$

Let A_2 be the 2nd Weyl algebra over $\mathbb{C}$ with generators $x_1, \partial_1, x_2, \partial_2$. Let

$$\mathscr{A}' = A_2 \otimes U\big(\mathfrak{sp}(4)\big), \tag{3.9}$$

where $U(\mathfrak{sp}(4))$ is the universal enveloping algebra of $\mathfrak{sp}(4)$. Let $\{e_\alpha, h_\alpha, f_\alpha, e_\beta, h_\beta, f_\beta\}$ be Chevalley generators of $\mathfrak{sp}(4)$. We have a Lie algebra homomorphism

$$\varphi : \mathfrak{sp}(4) \to \mathscr{A}' \tag{3.10}$$

determined by ($i=\sqrt{-1}$):

$$e_\alpha \mapsto E_\alpha = x_1\partial_2\otimes 1 + 1\otimes e_\alpha \tag{3.11a}$$
$$f_\alpha \mapsto F_\alpha = x_2\partial_1\otimes 1 + 1\otimes f_\alpha \tag{3.11b}$$
$$h_\alpha \mapsto H_\alpha = [E_\alpha, F_\alpha] = (x_1\partial_1 - x_2\partial_2)\otimes 1 + 1\otimes h_\alpha \tag{3.11c}$$
$$e_\beta \mapsto E_\beta = \frac{i}{2}x_2^2\otimes 1 + 1\otimes e_\beta \tag{3.11d}$$
$$f_\beta \mapsto F_\beta = \frac{i}{2}\partial_2^2\otimes 1 + 1\otimes f_\beta \tag{3.11e}$$
$$h_\beta \mapsto H_\beta = [E_\beta, F_\beta] = \big(x_2\partial_2 + \frac{1}{2}\big)\otimes 1 + 1\otimes h_\beta \tag{3.11f}$$

Let

$$\mathscr{A} = S^{-1}\mathscr{A}' \tag{3.12}$$

where $S\subset\mathscr{A}'$ is the multiplicative submonoid generated by $\{\varphi(h)-n \mid h\in\mathfrak{h}-n\}$. Let

$$I_+ = \mathscr{A}E_\alpha + \mathscr{A}E_\beta \tag{3.13}$$

be the left ideal of $\mathscr{A}$ generated by the image of the positive nilpotent part of $\mathfrak{sp}(4)$. Let P be the extremal projector (see (2.2)) for $\mathfrak{sp}(4)$. Thus

$$P = P_\beta P_{\beta+\alpha}P_{\beta+2\alpha}P_\alpha = P_\alpha P_{\beta+2\alpha}P_{\beta+\alpha}P_\beta, \tag{3.14}$$

where

$$P_\gamma = 1 - \frac{1}{H_\gamma+2}F_\gamma E_\gamma + \cdots\,. \tag{3.15}$$

We will use the Chevalley integral basis

$$e_{\beta+\alpha} = [e_\alpha, e_\beta], \quad e_{\beta+2\alpha} = \frac{1}{2}[e_\alpha, e_{\beta+\alpha}],$$

$$f_{\beta+\alpha} = \theta(e_{\beta+\alpha}) = [f_\beta, f_\alpha], \quad f_{\beta+2\alpha} = \theta(e_{\beta+2\alpha}) = \frac{1}{2}[f_{\beta+\alpha}, f_\alpha],$$

where θ is the Chevalley involution of $\mathfrak{sp}(4)$. Then $\{e_\gamma, h_\gamma, f_\gamma\}$ is an $\mathfrak{sl}(2)$-triple for each positive root γ, where $h_\gamma := [e_\gamma, f_\gamma]$. Let

$$E_{\beta+\alpha} = [E_\alpha, E_\beta] = ix_1x_2\otimes 1 + 1\otimes e_{\beta+\alpha}, \tag{3.16a}$$
$$F_{\beta+\alpha} = i\partial_1\partial_2\otimes 1 + 1\otimes f_{\beta+\alpha}, \tag{3.16b}$$
$$E_{\beta+2\alpha} = \frac{i}{2}x_1^2\otimes 1 + 1\otimes e_{\beta+2\alpha}, \tag{3.16c}$$
$$F_{\beta+2\alpha} = \frac{i}{2}\partial_1^2\otimes 1 + 1\otimes f_{\beta+2\alpha}. \tag{3.16d}$$

The two crucial facts we need about these elements are that for each positive root γ, the set $\{E_\gamma, F_\gamma, H_\gamma = [E_\gamma, F_\gamma]\}$ is an $\mathfrak{sl}_2$ triple (we need this for the extremal projectors P_γ occurring in P), and that φ intertwines natural involutions (see Lemma below), which allows us to cut computations almost in half. Also we have

$$h_{\beta+\alpha} = h_\alpha + 2h_\beta, \qquad h_{\beta+2\alpha} = h_\alpha + h_\beta, \tag{3.17}$$

which can be verified by applying φ to both sides (φ is injective when restricted to $\mathfrak{h}$ since $\varphi(h_\alpha)$ and $\varphi(h_\beta)$ are linearly independent). Applying φ we obtain the same relation for the upper-case elements.

Let $\mathrm{II} = I_- + I_+$, where $I_- = F_\alpha\mathscr{A} + F_\beta\mathscr{A}$. Define

$$D\big(\mathfrak{sp}(4)\big) = \mathscr{A}/\mathrm{II} \tag{3.18}$$

On $D\big(\mathfrak{sp}(4)\big)$ we have the diamond product

$$\bar{a} \diamond \bar{b} = aPb + \mathrm{II}, \quad \forall \bar{a} = a + \mathrm{II},\ \bar{b} = b + \mathrm{II}. \tag{3.19}$$

This makes $D\big(\mathfrak{sp}(4)\big)$ an associative algebra isomorphic to the Zhelobenko reduction algebra $N_{\mathscr{A}}(I_+)/I_+$. Let

$$R = S^{-1}\varphi(U(\mathfrak{h})) \tag{3.20}$$

and introduce the following elements of $D\big(\mathfrak{sp}(4)\big)$:

$$\bar{x}_i = x_i \otimes 1 + \mathrm{II}, \quad i = 1, 2, \tag{3.21}$$

$$\bar{\partial}_i = \partial_i \otimes 1 + \mathrm{II}, \quad i = 1, 2. \tag{3.22}$$

Lemma 3.1. $D\big(\mathfrak{sp}(4)\big)$ *carries an order two anti-automorphism* Θ *determined by*

$$\Theta(\bar{x}_i) = \bar{\partial}_i, \qquad i = 1, 2. \tag{3.23}$$

Proof. Let θ be the Chevalley involution of $\mathfrak{sp}(4)$, which extends to an order two algebra anti-automorphism of $U(\mathfrak{sp}(4))$ determined by $\theta(e_\gamma) = f_\gamma$ for $\gamma \in \{\alpha, \beta\}$. Let $\vartheta : A_2 \to A_2$ be the order two anti-automorphism determined by $\vartheta(x_i) = \partial_i$ for $i = 1, 2$. Then we have

$$\varphi(\theta(a)) = (\vartheta \otimes \theta) \circ \varphi(a), \quad a \in \mathfrak{sp}(4). \tag{3.24}$$

The anti-automorphism $\vartheta \otimes \theta$ of $A_2 \otimes U(\mathfrak{sp}(4))$ uniquely extends to the localization $\mathscr{A}$ and preserves the double coset II and fixes the extremal projector P, hence induces an anti-automorphism Θ of the required form. □

3.3. A Computational Lemma for $D\big(\mathfrak{sp}(4)\big)$.

In this section we provide some computations that will be needed in proof of Theorem 1.1.

In $\mathscr{A}$ we have for any positive root γ:

$$\begin{aligned} aP_\gamma b &\equiv ab + aF_\gamma \frac{-1}{H_\gamma} E_\gamma b + \cdots \\ &\equiv ab + aF_\gamma \frac{-1}{H_\gamma} [E_\gamma, b] + \cdots \\ &\equiv ab + [a, F_\gamma] \frac{-1}{H_\gamma} E_\gamma b + \cdots \\ &\equiv ab + [a, F_\gamma] \frac{-1}{H_\gamma} [E_\gamma, b] + \cdots \qquad (\mathrm{mod}\ \mathrm{II}) \end{aligned} \tag{3.25}$$

where $\cdots$ indicate higher order terms from the extremal projector. However, we will only need the constant and linear terms in the calculations of this paper.

Lemma 3.2. *In $\mathscr{A}$, the following relations hold.*

$$[E_\alpha, \widetilde{\partial}_1] = -\widetilde{\partial}_2, \qquad [\widetilde{x}_1, F_\alpha] = -\widetilde{x}_2, \tag{3.26a}$$

$$[E_\alpha, \widetilde{x}_2] = \widetilde{x}_1, \qquad [\widetilde{\partial}_2, F_\alpha] = \widetilde{\partial}_1, \tag{3.26b}$$

$$[E_{\beta+2\alpha}, \widetilde{\partial}_1] = -i\widetilde{x}_1, \qquad [\widetilde{x}_1, F_{\beta+2\alpha}] = -i\widetilde{\partial}_1, \tag{3.26c}$$

$$[E_{\beta+\alpha}, \widetilde{\partial}_1] = -i\widetilde{x}_2, \qquad [\widetilde{x}_1, F_{\beta+\alpha}] = -t\widetilde{\partial}_2, \tag{3.26d}$$

$$[E_{\beta+\alpha}, \widetilde{\partial}_2] = -i\widetilde{x}_1, \qquad [\widetilde{x}_2, F_{\beta+\alpha}] = -i\widetilde{\partial}_1, \tag{3.26e}$$

$$[E_\beta, \widetilde{\partial}_2] = -i\widetilde{x}_2, \qquad [\widetilde{x}_2, F_\beta] = -i\widetilde{\partial}_2, \tag{3.26f}$$

$$[E_\gamma, z] = 0 \text{ in remaining cases}, \qquad [w, F_\gamma] = 0 \text{ in remaining cases}, \tag{3.26g}$$

$$H_\alpha\widetilde{x}_1 = \widetilde{x}_1(H_\alpha + 1), \qquad H_\alpha\widetilde{\partial}_1 = \widetilde{\partial}_1(H_\alpha - 1), \tag{3.27a}$$

$$H_\alpha\widetilde{x}_2 = \widetilde{x}_2(H_\alpha - 1), \qquad H_\alpha\widetilde{\partial}_2 = \widetilde{\partial}_2(H_\alpha + 1), \tag{3.27b}$$

$$H_{\beta+2\alpha}\widetilde{x}_1 = \widetilde{x}_1(H_{\beta+2\alpha} + 1), \qquad H_{\beta+2\alpha}\widetilde{\partial}_1 = \widetilde{\partial}_1(H_{\beta+2\alpha} - 1), \tag{3.27c}$$

$$H_{\beta+\alpha}\widetilde{x}_i = \widetilde{x}_i(H_{\beta+\alpha} + 1), \qquad H_{\beta+\alpha}\widetilde{\partial}_i = \widetilde{\partial}_i(H_{\beta+\alpha} - 1), \tag{3.27d}$$

$$H_\beta\widetilde{x}_2 = \widetilde{x}_2(H_\beta + 1), \qquad H_\beta\widetilde{\partial}_2 = \widetilde{\partial}_2(H_\beta - 1), \tag{3.27e}$$

$$H_\gamma z = zH_\gamma \text{ in remaining cases.} \tag{3.27f}$$

Proof. Using the map (3.10), computations in $\mathscr{A}'$ verify the the left column of commutator relations (3.26) and shift relations (3.27). Applying the anti-automorphism Θ defined in Lemma 3.1 yields the relations in the right column. □

Lemma 3.3. *In $\mathscr{A}$, we have the following congruences modulo* II*:*

$$\widetilde{\partial}_1 P\widetilde{x}_1 \equiv \widetilde{\partial}_1\widetilde{x}_1 \tag{3.28}$$

$$\widetilde{\partial}_2 P\widetilde{x}_2 \equiv \widetilde{\partial}_2\widetilde{x}_2 + \frac{-1}{H_\alpha + 1}\widetilde{\partial}_1\widetilde{x}_1 \tag{3.29}$$

$$\widetilde{x}_2 P\widetilde{\partial}_2 \equiv -1 + \frac{H_\beta}{(H_\beta + 1)(H_{\beta+\alpha} + 1)}\widetilde{\partial}_1\widetilde{x}_1 + \Big(1 + \frac{1}{H_\beta + 1}\Big)\widetilde{\partial}_2\widetilde{x}_2 \tag{3.30}$$

$$\widetilde{x}_1 P\widetilde{\partial}_1 \equiv -1 + \frac{1}{H_\alpha + 1} + \Big(1 + \frac{H_\alpha H_{\beta+\alpha} + H_{\beta+2\alpha} + 1}{(H_\alpha + 1)(H_{\beta+\alpha} + 1)(H_{\beta+2\alpha} + 1)}\Big)\widetilde{\partial}_1\widetilde{x}_1 + \frac{H_\alpha - H_{\beta+\alpha} - 2}{(H_\alpha + 1)(H_{\beta+\alpha} + 1)}\widetilde{\partial}_2\widetilde{x}_2 \tag{3.31}$$

Proof. The congruence (3.28) follows immediately from the fact that $[E_\gamma, \widetilde{x}_1] = 0$ for all positive roots γ.

For the proof of (3.29), we have:

$$\begin{aligned}
\widetilde{\partial}_2 P\widetilde{x}_2 &\equiv \widetilde{\partial}_2 P_\alpha P_{\beta+2\alpha} P_{\beta+\alpha} P_\beta \widetilde{x}_2 && \text{def. of } P \\
&\equiv \widetilde{\partial}_2 P_\alpha \widetilde{x}_2 && \text{by } [E_{\beta+k\alpha}, \widetilde{x}_i] = 0 \\
&\equiv \widetilde{\partial}_2\widetilde{x}_2 + [\widetilde{\partial}_2, F_\alpha]\frac{-1}{H_\alpha}[E_\alpha, \widetilde{x}_2] && \text{by (3.25)} \\
&\equiv \widetilde{\partial}_2\widetilde{x}_2 + \widetilde{\partial}_1\frac{-1}{H_\alpha}\widetilde{x}_1 && \text{by (3.26b)}
\end{aligned}$$

$$\equiv \widetilde{\partial}_2\widetilde{x}_2 + \frac{-1}{H_\alpha+1}\widetilde{\partial}_1\widetilde{x}_1 \qquad \text{by (3.27a)}$$

This proves (3.29).

The proof of the third congruence, (3.30), is more involved:

$$\begin{aligned}
&\widetilde{x}_2 P \widetilde{\partial}_2 \\
&\equiv \widetilde{x}_2 P_\alpha P_{\beta+2\alpha} P_{\beta+\alpha} P_\beta \widetilde{\partial}_2 && \text{def. of } P \\
&\equiv \widetilde{x}_2 P_{\beta+\alpha} P_\beta \widetilde{\partial}_2 && \text{by (3.26g)} \\
&\equiv \big(\widetilde{x}_2 + [\widetilde{x}_2, F_{\beta+\alpha}]\frac{-1}{H_{\beta+\alpha}}E_{\beta+\alpha}\big)\big(\widetilde{\partial}_2 + F_\beta \frac{-1}{H_\beta}[E_\beta, \widetilde{\partial}_2]\big) && \text{by (3.25)} \\
&\equiv \widetilde{x}_2\widetilde{\partial}_2 + (-i\widetilde{\partial}_1)\frac{-1}{H_{\beta+\alpha}}[E_{\beta+\alpha}, \widetilde{\partial}_2] + [\widetilde{x}_2, F_\beta]\frac{-1}{H_\beta}(-i\widetilde{x}_2) \\
&\quad + (-i\widetilde{\partial}_1)\frac{-1}{H_{\beta+\alpha}}E_{\beta+\alpha}F_\beta\frac{-1}{H_\beta}(-i\widetilde{x}_2) && \text{by (3.26e),(3.26e)} \\
&\equiv \widetilde{x}_2\widetilde{\partial}_2 + \widetilde{\partial}_1\frac{1}{H_{\beta+\alpha}}\widetilde{x}_1 + \widetilde{\partial}_2\frac{1}{H_\beta}\widetilde{x}_2 \\
&\quad - \widetilde{\partial}_1\frac{1}{H_{\beta+\alpha}}[E_{\beta+\alpha}, F_\beta]\frac{1}{H_\beta}\widetilde{x}_2 && \text{by } \widetilde{\partial}_1\frac{1}{H_{\beta+\alpha}}F_\beta = F_\beta\widetilde{\partial}_1\frac{1}{H_{\beta+\alpha}-2} \\
& && \text{and (3.26e)} \\
&\equiv \widetilde{x}_2\widetilde{\partial}_2 + \frac{1}{H_{\beta+\alpha}+1}\widetilde{\partial}_1\widetilde{x}_1 + \frac{1}{H_\beta+1}\widetilde{\partial}_2\widetilde{x}_2 \\
&\quad - \frac{1}{(H_{\beta+\alpha}+1)(H_\beta+1)}\widetilde{\partial}_1[E_\alpha, \widetilde{x}_2] && \text{by (3.27d),(3.27e),(3.27f)} \\
& && \text{and } [E_{\beta+\alpha}, F_\beta] = E_\alpha, \\
& && \text{and } H_\beta E_\alpha = E_\alpha(H_\beta - 1) \\
&\equiv \widetilde{x}_2\widetilde{\partial}_2 + \frac{H_\beta}{(H_\beta+1)(H_{\beta+\alpha}+1)}\widetilde{\partial}_1\widetilde{x}_1 + \frac{1}{H_\beta+1}\widetilde{\partial}_2\widetilde{x}_2 && \text{by (3.26b).}
\end{aligned}$$

Using that $\widetilde{x}_2\widetilde{\partial}_2 = -1 + \widetilde{\partial}_2\widetilde{x}_2$ and collecting terms we obtain (3.30).

The last case is the hardest one, as we need to use three of the four root-vector projectors P_γ. First, we have:

$$\begin{aligned}
&\widetilde{x}_1 P \widetilde{\partial}_1 \\
&\equiv \widetilde{x}_1 P_\beta P_{\beta+\alpha} P_{\beta+2\alpha} P_\alpha \widetilde{\partial}_1 && \text{def. of } P \\
&\equiv \widetilde{x}_1 P_{\beta+\alpha} P_{\beta+2\alpha} P_\alpha \widetilde{\partial}_1 && \text{by (3.26g) and (3.25)} \\
&\equiv \big(\widetilde{x}_1 + [\widetilde{x}_1, F_{\beta+\alpha}]\frac{-1}{H_{\beta+\alpha}}E_{\beta+\alpha}\big)P_{\beta+2\alpha}\big(\widetilde{\partial}_1 + F_\alpha\frac{-1}{H_\alpha}[E_\alpha, \widetilde{\partial}_1]\big) && \text{by (3.25)} \\
&\equiv \big(\widetilde{x}_1 + (-i\widetilde{\partial}_2)\frac{-1}{H_{\beta+\alpha}}E_{\beta+\alpha}\big)P_{\beta+2\alpha}\big(\widetilde{\partial}_1 + F_\alpha\frac{-1}{H_\alpha}(-\widetilde{\partial}_2)\big) && \text{by (3.26d), (3.26a)} \\
&\equiv \widetilde{x}_1 P_{\beta+2\alpha}\widetilde{\partial}_1 && (3.32a)
\end{aligned}$$

$$+\widetilde{x}_1 P_{\beta+2\alpha} F_\alpha \frac{1}{H_\alpha}\widetilde{\partial}_2 \tag{3.32b}$$

$$+\widetilde{\partial}_2 \frac{i}{H_{\beta+\alpha}} E_{\beta+\alpha} P_{\beta+2\alpha}\widetilde{\partial}_1 \tag{3.32c}$$

$$+\widetilde{\partial}_2 \frac{i}{H_{\beta+\alpha}} E_{\beta+\alpha} P_{\beta+2\alpha} F_\alpha \frac{1}{H_\alpha}\widetilde{\partial}_2 \tag{3.32d}$$

Then we consider each of these four terms in separate computations:

The first term:

$$\begin{aligned}
(3.32\text{a}) &\equiv \widetilde{x}_1 P_{\beta+2\alpha}\widetilde{\partial}_1 \\
&\equiv \widetilde{x}_1\widetilde{\partial}_1 + [\widetilde{x}_1, F_{\beta+2\alpha}]\frac{-1}{H_{\beta+2\alpha}}[E_{\beta+2\alpha},\widetilde{\partial}_1] \\
&\equiv \widetilde{x}_1\widetilde{\partial}_1 + (-i\widetilde{\partial}_1)\frac{-1}{H_{\beta+2\alpha}}(-i\widetilde{x}_1) && \text{by (3.26c)} \\
&\equiv \widetilde{x}_1\widetilde{\partial}_1 + \frac{1}{H_{\beta+2\alpha}+1}\widetilde{\partial}_1\widetilde{x}_1 && \text{by (3.27c).}
\end{aligned}$$

We could use $\widetilde{\partial}_1\widetilde{x}_1 = \widetilde{x}_1\widetilde{\partial}_1 + 1$ to rewrite this but we hold off until we have all four terms.

The second term:

$$\begin{aligned}
&(3.32\text{b}) \equiv \widetilde{x}_1 P_{\beta+2\alpha} F_\alpha \frac{1}{H_\alpha}\widetilde{\partial}_2 \\
&\equiv \big(\widetilde{x}_1 + [\widetilde{x}_1, F_{\beta+2\alpha}]\frac{-1}{H_{\beta+2\alpha}} E_{\beta+2\alpha}\big) F_\alpha \frac{1}{H_\alpha}\widetilde{\partial}_2 && \text{by (3.25)} \\
&\equiv [\widetilde{x}_1, F_\alpha]\frac{1}{H_\alpha}\widetilde{\partial}_2 + \widetilde{\partial}_1 \frac{i}{H_{\beta+2\alpha}}[E_{\beta+2\alpha}, F_\alpha]\frac{1}{H_\alpha}\widetilde{\partial}_2 && \text{by (3.26c),(3.26g)} \\
&\equiv (-\widetilde{x}_2)\frac{1}{H_\alpha}\widetilde{\partial}_2 + \widetilde{\partial}_1 \frac{i}{H_{\beta+2\alpha}}(-E_{\beta+\alpha})\frac{1}{H_\alpha}\widetilde{\partial}_2 && \text{by (3.26a) and } [E_{\beta+2\alpha}, F_\alpha] = -E_{\beta+\alpha} \\
&\equiv \frac{-1}{H_\alpha+1}\widetilde{x}_2\widetilde{\partial}_2 + \frac{-i}{(H_\alpha+1)(H_{\beta+2\alpha}+1)}\widetilde{\partial}_1[E_{\beta+\alpha},\widetilde{\partial}_2] && \text{by (3.27b),(3.27c),(3.27a)} \\
& && \text{and } [H_\alpha, E_{\beta+\alpha}] = -2E_{\beta+\alpha} \\
&\equiv \frac{-1}{H_\alpha+1}\widetilde{x}_2\widetilde{\partial}_2 + \frac{-1}{(H_\alpha+1)(H_{\beta+2\alpha}+1)}\widetilde{\partial}_1\widetilde{x}_1 && \text{by (3.26e)}
\end{aligned}$$

The third term:

$$\begin{aligned}
&(3.32\text{c}) \equiv \widetilde{\partial}_2 \frac{i}{H_{\beta+\alpha}} E_{\beta+\alpha} P_{\beta+2\alpha}\widetilde{\partial}_1 \\
&\equiv \frac{i}{H_{\beta+\alpha}+1}\widetilde{\partial}_2 E_{\beta+\alpha}\Big(\widetilde{\partial}_1 + F_{\beta+2\alpha}\frac{-1}{H_{\beta+2\alpha}}(-i\widetilde{x}_1)\Big) && \text{by (3.27d),(3.25),(3.26c)} \\
&\equiv \frac{i}{H_{\beta+\alpha}+1}\Big(\widetilde{\partial}_2[E_{\beta+\alpha},\widetilde{\partial}_1] + \widetilde{\partial}_2[E_{\beta+\alpha}, F_{\beta+2\alpha}]\frac{i}{H_{\beta+2\alpha}}\widetilde{x}_1\Big) && \text{by (3.26g)}
\end{aligned}$$

$$\equiv \frac{i}{H_{\beta+\alpha}+1}\Big(\widetilde{\partial}_2(-i\widetilde{x}_2)+\widetilde{\partial}_2 F_\alpha \frac{i}{H_{\beta+2\alpha}}\widetilde{x}_1\Big) \qquad \text{by (3.26d) and } [E_{\beta+\alpha},F_{\beta+2\alpha}]=F_\alpha$$

$$\equiv \frac{1}{H_{\beta+\alpha}+1}\Big(\widetilde{\partial}_2\widetilde{x}_2+\widetilde{\partial}_1 \frac{-1}{H_{\beta+2\alpha}}\widetilde{x}_1\Big) \qquad \text{by (3.26b)}$$

$$\equiv \frac{1}{H_{\beta+\alpha}+1}\widetilde{\partial}_2\widetilde{x}_2+\frac{-1}{(H_{\beta+\alpha}+1)(H_{\beta+2\alpha}+1)}\widetilde{\partial}_1\widetilde{x}_1 \qquad \text{by (3.27c).}$$

<u>The fourth term:</u>

$$\text{(3.32d)} \equiv \frac{i}{(H_{\beta+\alpha}+1)(H_\alpha+1)}\widetilde{\partial}_2 E_{\beta+\alpha}P_{\beta+2\alpha}F_\alpha\widetilde{\partial}_2 \qquad \text{by } \begin{cases}\text{(3.27d), (3.27b)}\\ H_\alpha F_\alpha = F_\alpha(H_\alpha-2)\\ H_\alpha E_{\beta+\alpha}=E_{\beta+\alpha}H_\alpha\end{cases}$$

For brevity, we omit the prefactor and compute:

$$\widetilde{\partial}_2 E_{\beta+\alpha}P_{\beta+2\alpha}F_\alpha\widetilde{\partial}_2$$

$$\equiv \widetilde{\partial}_2\Big(E_{\beta+\alpha}F_\alpha+[E_{\beta+\alpha},F_{\beta+2\alpha}]\frac{-1}{H_{\beta+2\alpha}}[E_{\beta+2\alpha},F_\alpha]\Big)\widetilde{\partial}_2 \qquad \text{by (3.26g)}$$

$$\equiv \widetilde{\partial}_2\Big(F_\alpha E_{\beta+\alpha}-2E_\beta+F_\alpha\frac{-1}{H_{\beta+2\alpha}}(-E_{\beta+\alpha})\Big)\widetilde{\partial}_2 \qquad \text{by } \begin{cases}[E_{\beta+\alpha},F_\alpha]=-2E_\beta\\ [E_{\beta+\alpha},F_{\beta+2\alpha}]=F_\alpha\\ [E_{\beta+2\alpha},F_\alpha]=-E_{\beta+\alpha}\end{cases}$$

$$\equiv [\widetilde{\partial}_2,F_\alpha][E_{\beta+\alpha},\widetilde{\partial}_2]-2\widetilde{\partial}_2[E_\beta,\widetilde{\partial}_2]+[\widetilde{\partial}_2,F_\alpha]\frac{1}{H_{\beta+2\alpha}}[E_{\beta+\alpha},\widetilde{\partial}_2] \qquad \text{by congruence mod II}$$

$$\equiv \widetilde{\partial}_1(-i\widetilde{x}_1)-2\widetilde{\partial}_2(-i\widetilde{x}_2)+\widetilde{\partial}_1\frac{1}{H_{\beta+2\alpha}}(-i\widetilde{x}_1) \qquad \text{by (3.26b),(3.26e),(3.26f)}$$

$$\equiv (-i)\Big(\frac{H_{\beta+2\alpha}+2}{H_{\beta+2\alpha}+1}\widetilde{\partial}_1\widetilde{x}_1-2\widetilde{\partial}_2\widetilde{x}_2\Big) \qquad \text{by (3.27c).}$$

With the prefactor we thus get:

$$\text{(3.32d)} \equiv \frac{1}{(H_\alpha+1)(H_{\beta+\alpha}+1)}\Big(\frac{H_{\beta+2\alpha}+2}{H_{\beta+2\alpha}+1}\widetilde{\partial}_1\widetilde{x}_1-2\widetilde{\partial}_2\widetilde{x}_2\Big) \tag{3.33}$$

Adding up the four terms, and using that $\widetilde{x}_i\widetilde{\partial}_i=-1+\widetilde{\partial}_i\widetilde{x}_i$ and collecting coefficients of $\widetilde{\partial}_1\widetilde{x}_1$ and $\widetilde{\partial}_2\widetilde{x}_2$ we get:

$$\begin{aligned}\widetilde{x}_1 P\widetilde{\partial}_1 &\equiv \widetilde{x}_1\widetilde{\partial}_1+\frac{1}{H_{\beta+2\alpha}+1}\widetilde{\partial}_1\widetilde{x}_1\\ &+\frac{-1}{H_\alpha+1}\widetilde{x}_2\widetilde{\partial}_2+\frac{-1}{(H_\alpha+1)(H_{\beta+2\alpha}+1)}\widetilde{\partial}_1\widetilde{x}_1\\ &+\frac{1}{H_{\beta+\alpha}+1}\widetilde{\partial}_2\widetilde{x}_2+\frac{-1}{(H_{\beta+\alpha}+1)(H_{\beta+2\alpha}+1)}\widetilde{\partial}_1\widetilde{x}_1\\ &+\frac{1}{(H_\alpha+1)(H_{\beta+\alpha}+1)}\Big(\frac{H_{\beta+2\alpha}+2}{H_{\beta+2\alpha}+1}\widetilde{\partial}_1\widetilde{x}_1-2\widetilde{\partial}_2\widetilde{x}_2\Big)\end{aligned}$$

$$\begin{aligned}&\equiv -1+\frac{1}{H_\alpha+1}+\Big(1+\frac{1}{H_{\beta+2\alpha}+1}+\frac{-1}{(H_\alpha+1)(H_{\beta+2\alpha}+1)}+\frac{-1}{(H_{\beta+\alpha}+1)(H_{\beta+2\alpha}+1)}\\&\qquad+\frac{H_{\beta+2\alpha}+2}{(H_\alpha+1)(H_{\beta+\alpha}+1)(H_{\beta+2\alpha}+1)}\Big)\widetilde{\partial}_1\widetilde{x}_1\\&\quad+\Big(\frac{-1}{H_\alpha+1}+\frac{1}{H_{\beta+\alpha}+1}+\frac{-2}{(H_\alpha+1)(H_{\beta+\alpha}+1)}\Big)\widetilde{\partial}_2\widetilde{x}_2\end{aligned}$$

This can be simplified to (3.31). □

3.4. Presentation of $D\big(\mathfrak{sp}(4)\big)$.

In this section we prove Theorem 1.1.

Proof. Relations (1.1a)–(1.1d) follow from the corresponding identities in $\mathscr{A}$. For example:

$$\begin{aligned}\bar{x}_1H_\alpha &= (\widetilde{x}_1+\mathrm{I\!I})H_\alpha=\widetilde{x}_1H_\alpha+\mathrm{I\!I}=(x_1\otimes 1)\big((x_1\partial_1-x_2\partial_2)\otimes 1+1\otimes h_\alpha\big)+\mathrm{I\!I}\\&=x_1(x_1\partial_1-x_2\partial_2)\otimes 1+x_1\otimes h_\alpha+\mathrm{I\!I}\\&=(x_1\partial_1-x_2\partial_2-1)x_1\otimes 1+x_1\otimes h_\alpha+\mathrm{I\!I}=\cdots=(H_\alpha-1)\bar{x}_1\end{aligned}$$

where we used $x_1\partial_1=\partial_1x_1-1$.

Next, we prove (1.1e). The second relation follows from the first by applying Θ. For the first relation, observe that neither of E_α and E_β involve ∂_1 in their expressions (see (3.11a), (3.11d)). Therefore, they both commute with $\widetilde{x}_1$ and hence

$$[E_\gamma,\widetilde{x}_1]=0,\quad \forall\gamma\in\Phi_+. \tag{3.34}$$

In particular,

$$\widetilde{x}_2P\widetilde{x}_1\equiv\widetilde{x}_2\widetilde{x}_1 \ (\mathrm{mod}\ \mathrm{I\!I}). \tag{3.35}$$

Similarly, none of $E_\beta,E_{\beta+\alpha},E_{\beta+2\alpha}$ involve ∂_2 which implies that

$$[E_\gamma,\widetilde{x}_2]=0,\quad \forall\gamma\in\Phi_+\setminus\{\alpha\} \tag{3.36}$$

$$[E_\alpha,\widetilde{x}_2]=[x_1\partial_2\otimes 1+1\otimes e_\alpha,x_2\otimes 1]=x_1\otimes 1=\widetilde{x}_1. \tag{3.37}$$

Similarly

$$[\widetilde{x}_1,F_\alpha]=-\widetilde{x}_2. \tag{3.38}$$

Consequently, using $\widetilde{x}_1H_\alpha=(H_\alpha-1)\widetilde{x}_1$, we have ($\equiv$ means congruence mod $\mathrm{I\!I}$):

$$\begin{aligned}\widetilde{x}_1P\widetilde{x}_2&\equiv\widetilde{x}_1P_\alpha P_{\beta+2\alpha}P_{\beta+\alpha}P_\beta\widetilde{x}_2\\&\equiv\widetilde{x}_1P_\alpha\widetilde{x}_2\\&\equiv\widetilde{x}_1\widetilde{x}_2-\frac{1}{H_\alpha+1}[\widetilde{x}_1,F_\alpha][E_\alpha,\widetilde{x}_2]\\&\equiv\widetilde{x}_1\widetilde{x}_2+\frac{1}{H_\alpha+1}\widetilde{x}_2\widetilde{x}_1\\&\equiv\frac{H_\alpha+2}{H_\alpha+1}\widetilde{x}_2\widetilde{x}_1.\end{aligned}$$

This computation along with (3.35) proves the first relation in (1.1e).

Next we prove (1.1f). Again, the second follows from the first by applying Θ, so we focus on the first one. We use $P = P_\beta P_{\beta+\alpha} P_{\beta+2\alpha} P_\alpha$. We have

$$[\widetilde{x}_1, F_\beta] = [x_1, \frac{i}{2}\partial_2^2] \otimes 1 = 0,$$

and, similarly,

$$[E_\alpha, \widetilde{\partial}_2] = 0, \quad [E_{\beta+2\alpha}, \widetilde{\partial}_2] = 0.$$

Thus, computing mod $\mathrm{I\!I}$, we have

$$\begin{aligned}
\widetilde{x}_1 P \widetilde{\partial}_2 &\equiv \widetilde{x}_1 P_{\beta+\alpha} \widetilde{\partial}_2 \\
&\equiv \widetilde{x}_1 \widetilde{\partial}_2 + [\widetilde{x}_1, F_{\beta+\alpha}] \frac{-1}{H_{\beta+\alpha}} [E_{\beta+\alpha}, \widetilde{\partial}_2] \\
&\equiv \widetilde{x}_1 \widetilde{\partial}_2 + [\widetilde{x}_1, i\widetilde{\partial}_1 \widetilde{\partial}_2] \frac{-1}{H_{\beta+\alpha}} [i\widetilde{x}_1 \widetilde{x}_2, \widetilde{\partial}_2] \\
&\equiv \widetilde{x}_1 \widetilde{\partial}_2 + \widetilde{\partial}_2 \frac{1}{H_{\beta+\alpha}} \widetilde{x}_1 \\
&\equiv \Big(1 + \frac{1}{H_{\beta+\alpha}+1}\Big) \widetilde{\partial}_2 \widetilde{x}_1
\end{aligned}$$

Since $\widetilde{\partial}_2 \widetilde{x}_1 \equiv \widetilde{\partial}_2 P \widetilde{x}_1$ by (3.34), this proves the first relation in (1.1f).

Lastly, to prove (1.1h) and (1.1g), we use Lemma 3.3. Solving for $\widetilde{\partial}_i \widetilde{x}_i$ in (3.28) and (3.29) we get

$$\widetilde{\partial}_1 \widetilde{x}_1 + \mathrm{I\!I} = \bar{\partial}_1 \diamond \bar{x}_1 \tag{3.39}$$

$$\widetilde{\partial}_2 \widetilde{x}_2 + \mathrm{I\!I} = \bar{\partial}_2 \diamond \bar{x}_2 + \frac{1}{H_\alpha + 1} \bar{\partial}_1 \diamond \bar{x}_1 \tag{3.40}$$

Substituting (3.39) and (3.40) into the right hand side of (3.29) we get:

$$\begin{aligned}
\bar{x}_2 \diamond \bar{\partial}_2 &= -1 + \frac{H_\beta}{(H_\beta+1)(H_{\beta+\alpha}+1)} \bar{\partial}_1 \diamond \bar{x}_1 + \frac{H_\beta+2}{H_\beta+1} \Big(\bar{\partial}_2 \diamond \bar{x} + 2 + \frac{1}{H_\alpha+1} \bar{\partial}_1 \diamond \bar{x}_1 \Big) \\
&= -1 + \frac{H_\beta+2}{H_\beta+1} \bar{\partial}_2 \diamond \bar{x}_2 + \frac{H_\beta(H_\alpha+1) + (H_\beta+2)(H_{\beta+\alpha}+1)}{(H_\alpha+1)(H_\beta+1)(H_{\beta+\alpha}+1)} \bar{\partial}_1 \diamond \bar{x}_1
\end{aligned}$$

proving (1.1h) for certain f_{2j}.

Similarly, substituting (3.39) and (3.40) into (3.31), we obtain (1.1g) for some f_{1j}.

Lastly, inspecting the relations in more detail, we have proved (1.1g),(1.1h) with

$$f_{11} = 1 + \frac{(H_\alpha+1)(H_\alpha H_{\beta+\alpha} + H_{\beta+2\alpha} + 1) + (H_{\beta+2\alpha}+1)(H_\alpha - H_{\beta+\alpha} - 2)}{(H_\alpha+1)^2 (H_{\beta+\alpha}+1)(H_{\beta+2\alpha}+1)} \tag{3.41a}$$

$$f_{12} = \frac{H_\alpha - H_{\beta+\alpha} - 2}{(H_\alpha+1)(H_{\beta+\alpha}+1)} \tag{3.41b}$$

$$f_{21} = \frac{H_\beta(H_\alpha+1) + (H_\beta+2)(H_{\beta+\alpha}+1)}{(H_\alpha+1)(H_\beta+1)(H_{\beta+\alpha}+1)} \tag{3.41c}$$

$$f_{22} = 1 + \frac{1}{H_\beta + 1} \tag{3.41d}$$

One can verify that these coefficients coincide with the expressions (1.2), using that $H_{\beta+2\alpha} = H_\alpha + H_\beta$ and $H_{\beta+\alpha} = H_\alpha + 2H_\beta$. □

Corollary 3.4. (a) $D\big(\mathfrak{sp}(4)\big)$ *is a domain (i.e. there are no left or right zero-divisors).*

(b) $D\big(\mathfrak{sp}(4)\big)$ *is free as a left R-module with basis*

$$\{\bar{\partial}_1^a \bar{\partial}_2^b \bar{x}_1^c \bar{x}_2^d \mid a, b, c, d \in \mathbb{Z}_{\geq 0}\}. \tag{3.42}$$

Proof. (a) One may use part (b) to prove part (a). But we give a deformation argument using the observations made in Section 1.2. Since $D\big(\mathfrak{sp}(4)\big)$ is a subalgebra of $D^*(\mathfrak{sp}(4))$, it suffices to show the latter is a domain. Since $B = D^0(\mathfrak{sp}(4))$ is a $\mathbb{C}[\hbar]$-order in $D^*(\mathfrak{sp}(4))$ it suffices to show B is a domain. The latter has a filtration given by $B_{(k)} = B + B\hbar + \cdots + B\hbar^k$ for $k \in \mathbb{Z}_{\geq 0}$ whose associated graded algebra is $\operatorname{gr} B/\langle\hbar\rangle$. As remarked in Section 1.2, this quotient is isomorphic to $R \otimes_{\mathbb{C}} A_2(\mathbb{C})$ which is a domain. Therefore $\operatorname{gr} B$, and consequently B, is a domain.

(b) The the given set spans $D\big(\mathfrak{sp}(4)\big)$ as a left R-module follows from Theorem 1.1. That the set is independent follows from the fact that $\mathscr{A} \cong R \otimes_{\mathbb{C}} U(\mathfrak{n}_-) \otimes_{\mathbb{C}} A_2(\mathbb{C}) \otimes_{\mathbb{C}} U(\mathfrak{n}_+)$ as left R-modules. In particular, the canonical projection $\mathscr{A} \to \mathscr{A}/\mathrm{II} = D\big(\mathfrak{sp}(4)\big)$ has a section ϕ which is a left R-module map. Explicitly, ϕ sends an ordered monomial in the barred variables to the corresponding variables with tilde. □

4. Generalized Weyl Algebras

Differential operators are ubiquitous in representation theory and mathematical physics, with the previous sections as examples. In the 1990s, Bavula [3] described a class of algebras that generalizes the nth Weyl algebra A_n. These so-called generalized Weyl algebras (GWAs) include important examples from representation theory and ring theory, such as $U(\mathfrak{sl}(2))$, $U_q(\mathfrak{sl}(2))$, down-up algebras [4], skew Laurent-polynomial rings, to name a few. The class is closed under taking tensor products and certain skew polynomial extensions. See [8] for a recent survey on GWAs.

In [33], GWA presentations are given for certain subalgebras of Mickelsson step algebras $S(\mathfrak{g}_{n+1}, \mathfrak{g}_n)$ where $\mathfrak{g}_n = \mathfrak{sl}(n)$ or $\mathfrak{so}(n)$. In [25] certain reduction algebras were shown to be twisted GWAs.

In this section, we recall the definition of GWAs and state two theorems about their center and irreducible weight representations. We also observe that $D\big(\mathfrak{gl}(2)\big)$ and $D\big(\mathfrak{sp}(4)\big)$ are examples of an interesting subclass of GWAs that we term *skew-affine* GWAs.

4.1. Definition of Generalized Weyl Algebras.

Definition 4.1. Let $\Bbbk$ be a field. Let B be an associative $\Bbbk$-algebra, n be a positive integer, $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \operatorname{Aut}_{\Bbbk}(B)^n$ an n-tuple of commuting $\Bbbk$-algebra automorphisms of B, $t = (t_1, t_2, \ldots, t_n) \in Z(B)^n$ an n-tuple of elements of the center of B. The associated *generalized Weyl algebra (GWA) of rank (or degree) n*, denoted $B(\sigma, t)$ is the B-ring generated by $X_1, Y_1, \ldots, X_n, Y_n$ subject to the following relations, for $i, j = 1, 2 \ldots, n$ and all $b \in B$:

$$X_i b = \sigma_i(b) X_i, \quad bY_i = Y_i \sigma_i(b), \tag{4.1a}$$

$$X_i X_j = X_j X_i, \quad Y_i Y_j = Y_j Y_i, \quad X_i Y_j = Y_j X_i \text{ if } i \neq j, \tag{4.1b}$$

$$Y_i X_i = t_i, \quad X_i Y_i = \sigma_i(t_i). \tag{4.1c}$$

Taking $\Bbbk = \mathbb{C}$, $B = \mathbb{C}[u_1, u_2, \ldots, u_n]$, σ_i defined by $\sigma_i(u_j) = u_j - \delta_{ij}$, and $t_i = u_i$, the GWA $B(\sigma, t)$ is isomorphic to the nth Weyl algbera $A_n(\mathbb{C})$ via $X_i \mapsto x_i$, $Y_i \mapsto \partial_i$.

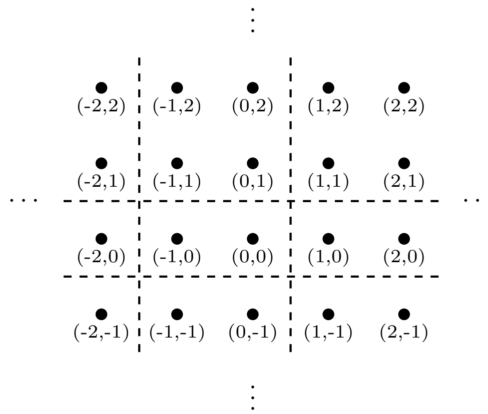

FIGURE 2. Example of a $\mathbb{Z}^2$-orbit $\mathcal{O}$ in $\operatorname{MaxSpec}(B)$. A point $(a,b)\in\mathbb{Z}^2$ corresponds to the maximal ideal $\sigma_1^a\sigma_2^b(\mathfrak{m}_0)$. The dashed break lines divide $\mathcal{O}$ into 9 equivalence classes ω_i, only one of which is finite. For example, the vertical break line $x_1=-1.5$ corresponds to the statement that $t_1\in\sigma_1^{-2}\sigma_2^b(\mathfrak{m}_0)$ for all $b\in\mathbb{Z}$.

4.2. **Center.** To describe the center of a GWA, first note that since the automorphisms σ_i commute, they define an action of $\mathbb{Z}^n$ on B, by the formula $(k_1,k_2,\ldots,k_n).b=\sigma_1^{k_1}\sigma_2^{k_2}\cdots\sigma_n^{k_n}(b)$.

Theorem 4.2. *Let $B(\sigma,t)$ be a GWA where B is commutative. Assume that the action of $\mathbb{Z}^n$ on B is faithful. Then the center of $B(\sigma,t)$ is equal to the set of all $b\in B$ such that $\sigma_i(b)=b$ for all $i=1,2\ldots,n$.*

4.3. **Irreducible Weight Modules.** A left module V over a GWA $B(\sigma,t)$ is a *weight module* if

$$V=\bigoplus_{\mathfrak{m}}V_{\mathfrak{m}},\quad V_{\mathfrak{m}}=\{v\in V\mid \mathfrak{m}v=0\}$$

where the sum runs over the set $\operatorname{MaxSpec}(B)$ of all maximal ideals $\mathfrak{m}$ of B. The *support* of a weight module V is defined to be $\operatorname{Supp}(V)=\{\mathfrak{m}\in\operatorname{MaxSpec}(B)\mid V_{\mathfrak{m}}\neq 0\}$.

Since σ_i are automorphisms, the image $\sigma_i(\mathfrak{m})$ of a maximal ideal is a maximal ideal. Thus we have an action of $\mathbb{Z}^n$ on the set $\operatorname{MaxSpec}(B)$ of all maximal ideals of B. Furthermore, by (4.1a),

$$X_iV_{\mathfrak{m}}\subset V_{\sigma_i(\mathfrak{m})},\quad Y_iV_{\sigma(\mathfrak{m})}\subseteq V_{\mathfrak{m}},\quad \forall i=1,2,\ldots,n;\ \forall\mathfrak{m}\in\operatorname{MaxSpec}(B). \tag{4.2}$$

For simplicity, we assume the action of $\mathbb{Z}^n$ on MaxSpec(B) is faithful. Let $\sim$ be the smallest equivalence relation on $\operatorname{MaxSpec}(B)$ such that $t_i\notin\mathfrak{m}\Rightarrow\mathfrak{m}\sim\sigma_i(\mathfrak{m})$, and let $\Omega=\operatorname{MaxSpec}(B)/\sim$ be the set of equivalence classes. Note that each equivalence class $\omega\in\Omega$ is contained in some $\mathbb{Z}^n$-orbit in $\operatorname{MaxSpec}(B)$. In more detail, if we fix $\mathfrak{m}_0\in\operatorname{MaxSpec}(B)$, the map $\mathbb{Z}^n\to\mathcal{O}=\mathbb{Z}^n.\mathfrak{m}_0$, $k=(k_1,k_2,\ldots,k_n)\mapsto\sigma^k(\mathfrak{m}_0):=\sigma_1^{k_1}\sigma_2^{k_2}\cdots\sigma_n^{k_n}(\mathfrak{m}_0)$ is bijective. Then, whenever $t_i\in\mathfrak{m}=\sigma^k(\mathfrak{m}_0)$, the hyperplane $\{x\in\mathbb{R}^n\mid x_i=k_i+\frac{1}{2}\}$ breaks $\mathcal{O}$ into two pieces. Thus any equivalence class $\omega\in\Omega$ can be thought of as a rectangular parallelotope (box) in $\mathbb{Z}^n$ bounded by a set of these break hyperplanes, see Figure 2

Theorem 4.3. *Let $A=B(\sigma,t)$ be a GWA, where B is commutative.*

(i) *If B is a finitely generated $\mathbb{C}$-algebra, then any finite-dimensional irreducible representation of A is a weight module.*

(ii) *Assume that the action of $\mathbb{Z}^n$ on* $\operatorname{MaxSpec}(B)$ *is faithful. Then the map* $[V] \mapsto \operatorname{Supp}(V)$ *is a bijection from the set of isomorphism classes of irreducible weight modules over A and the set Ω. Furthermore,* $\dim V = |\operatorname{Supp}(V)|$ *for any irreducible weight module V.*

4.4. **The $\mathfrak{gl}(2)$ GWA.** We state here the connection between $D\big(\mathfrak{gl}(2)\big)$ and GWAs which follows immediately from Theorem 2.4 and Corollary 2.5.

Corollary 4.4. *$D\big(\mathfrak{gl}(2)\big)$ is isomorphic to the GWA $B(\sigma, t)$ with*

$$B = R[t_1, t_2], \qquad R = \mathbb{C}[H, I][(H-n)^{-1} \mid n \in \mathbb{Z}]$$

and

$$\sigma_1(H) = H - 1, \qquad \sigma_1(I) = I - 1 \tag{4.3a}$$
$$\sigma_2(H) = H + 1, \qquad \sigma_2(I) = I - 1 \tag{4.3b}$$

and $\sigma_i(t_j)$ for $j \neq i$ and

$$\sigma_i(t_i) = \widehat{c}_i + \widehat{f}_{i1} t_1 + \widehat{f}_{i2} t_2 \tag{4.4}$$

with

$$\widehat{c}_1 = -H, \qquad \widehat{f}_{11} = \frac{H}{H+1}, \qquad \widehat{f}_{12} = \frac{-1}{H+1}, \tag{4.5a}$$
$$\widehat{c}_2 = -H - 2, \qquad \widehat{f}_{21} = \frac{1}{H+1}, \qquad \widehat{f}_{22} = \frac{H+2}{H+1}. \tag{4.5b}$$

Note that $\widehat{f}_{11}$ and $\widehat{f}_{22}$ are invertible in R (in fact, $\widehat{f}_{12}$ and $\widehat{f}_{21}$ are too, but this is not in general required in order for σ_i to be automorphisms). Furthermore, one can directly verify that Equations (4.12a)–(4.12f) hold. This follows independently from the compatibility between the relation $\bar{x}_1 \bar{x}_2 = \bar{x}_2 \bar{x}_1$ and $\bar{x}_i a = \sigma_i(a) \bar{x}_i$ for all $a \in B$, and the fact that $D(\mathfrak{gl}(2))$ is a domain.

4.5. $D\big(\mathfrak{sp}(4)\big)$ **is a Generalized Weyl Algebra.** Let $B = R[t_1, t_2]$. Define $\mathbb{C}$-algebra automorphisms σ_i of B as follows:

$$\sigma_1(H_\alpha) = H_\alpha - 1, \qquad \sigma_2(H_\alpha) = H_\alpha + 1 \tag{4.6a}$$
$$\sigma_1(H_\beta) = H_\beta, \qquad \sigma_2(H_\beta) = H_\beta - 1 \tag{4.6b}$$
$$\sigma_1(t_1) = \widehat{c}_1 + \widehat{f}_{11} t_1 + \widehat{f}_{12} t_2, \qquad \sigma_2(t_1) = t_1 \tag{4.6c}$$
$$\sigma_1(t_2) = t_2, \qquad \sigma_2(t_2) = \widehat{c}_2 + \widehat{f}_{21} t_1 + \widehat{f}_{22} t_2, \tag{4.6d}$$

where

$$\widehat{c}_1 = -H_\alpha(H_{\beta+\alpha} + 1), \qquad \widehat{c}_2 = -(H_\alpha + 2)(H_{\beta+\alpha} + 1), \tag{4.7}$$
$$\widehat{f}_{i1} = f_{i1} \frac{(H_\alpha + i)(H_{\beta+\alpha} + 1)}{(H_\alpha + 2)(H_{\beta+\alpha} + 2)}, \qquad \widehat{f}_{i2} = f_{i2} \frac{(H_\alpha + i)(H_{\beta+\alpha} + 1)}{(H_\alpha + 1)(H_{\beta+\alpha} + 2)}, \quad i = 1, 2. \tag{4.8}$$

and $f_{ij} = f_{ij}(H_\alpha, H_\beta)$ were defined in (1.2).

It is far from obvious that the automorphisms σ_1 and σ_2 actually commute. However, we will show this indirectly during the proof of Theorem 4.5. This is not a problem because the definition of GWA goes through even without the assumption that the σ_i commute (the algebra could just be somewhat degenerate, possibly trivial).

Note that using the generators a_{ii} in place of H_α, H_β and using $H_\alpha = a_{11} - a_{22}$, $H_\beta = a_{22} + \frac{1}{2}$, (4.6a)-(4.6b) may be replaced by $\sigma_i(a_{jj}) = a_{jj} - \delta_{ij}$.

Theorem 4.5. *There is a $\mathbb{C}$-algebra isomorphism*

$$\phi : D\big(\mathfrak{sp}(4)\big) \longrightarrow B(\sigma, t) \tag{4.9}$$

satisfying

$$\begin{aligned}
\phi(H_\alpha) &= H_\alpha,\\
\phi(H_\beta) &= H_\beta,\\
\phi(X_1) &= \widehat{x}_1 = \bar{x}_1,\\
\phi(X_2) &= \widehat{x}_2 = (H_\alpha + 2)\bar{x}_2,\\
\phi(Y_1) &= \widehat{\partial}_1 = \bar{\partial}_1(H_\alpha + 1)(H_{\beta+\alpha} + 1),\\
\phi(Y_2) &= \widehat{\partial}_2 = \bar{\partial}_2(H_{\beta+\alpha} + 1).
\end{aligned}$$

Proof. Let $\phi : B \to D\big(\mathfrak{sp}(4)\big)$ be the $\mathbb{C}$-algebra homomorphism which is the identity on R and $\phi(t_i) = \widehat{\partial}_i \diamond \widehat{x}_i$ for $i = 1, 2$. Extend ϕ to $B \cup \{X_1, Y_1, X_2, Y_2\}$ by $\phi(X_i) = \widehat{x}_i$, $\phi(Y_i) = \widehat{\partial}_i$. We must show that the GWA relations (4.1) are satisfied by the images of ϕ. First we prove that relations (4.1b) are preserved.

Using $\bar{x}_1 H_\alpha = (H_\alpha - 1)\bar{x}_1$ (and omitting $\diamond$ for brevity),

$$\widehat{x}_1\widehat{x}_2 = \bar{x}_1(H_\alpha + 2)\bar{x}_2 = (H_\alpha + 1)\bar{x}_1\bar{x}_2 = (H_\alpha + 1)\frac{H_\alpha + 2}{H_\alpha + 1}\bar{x}_2\bar{x}_1 = \widehat{x}_2\widehat{x}_1$$

Using $H_{\beta+\alpha}\bar{\partial}_1 = \bar{\partial}_1(H_{\beta+\alpha} - 1)$, $\bar{\partial}_2 H_\alpha = (H_\alpha - 1)\bar{\partial}_2$, and $\bar{\partial}_2 H_{\beta+\alpha} = (H_{\beta+\alpha} + 1)\bar{\partial}_2$ we get:

$$\begin{aligned}
\widehat{\partial}_2\widehat{\partial}_1 &= \bar{\partial}_2(H_{\beta+\alpha} + 1)\bar{\partial}_1(H_\alpha + 1)(H_{\beta+\alpha} + 1) = \bar{\partial}_2\bar{\partial}_1(H_\alpha + 1)H_{\beta+\alpha}(H_{\beta+\alpha} + 1)\\
&= \bar{\partial}_1\bar{\partial}_2\frac{H_\alpha + 2}{H_\alpha + 1}(H_\alpha + 1)H_{\beta+\alpha}(H_{\beta+\alpha} + 1) = \bar{\partial}_1\bar{\partial}_2(H_\alpha + 2)H_{\beta+\alpha}(H_{\beta+\alpha} + 1)\\
&= \bar{\partial}_1(H_\alpha + 1)(H_{\beta+\alpha} + 1)\bar{\partial}_2(H_{\beta+\alpha} + 1) = \widehat{\partial}_1\widehat{\partial}_2.
\end{aligned}$$

Next, since $H_{\beta+\alpha}$ commutes with $\bar{x}_1\bar{\partial}_2$, and $H_{\beta+\alpha}\bar{\partial}_2 = \bar{\partial}_2(H_{\beta+\alpha} - 1)$, we have

$$\begin{aligned}
\widehat{x}_1\widehat{\partial}_2 &= \bar{x}_1\bar{\partial}_2(H_{\beta+\alpha} + 1) = (H_{\beta+\alpha} + 1)\bar{x}_1\bar{\partial}_2 = (H_{\beta+\alpha} + 1)\frac{H_{\beta+\alpha} + 2}{H_{\beta+\alpha} + 1}\bar{\partial}_2\bar{x}_1 = (H_{\beta+\alpha} + 2)\bar{\partial}_2\bar{x}_1\\
&= \bar{\partial}_2(H_{\beta+\alpha} + 1)\bar{x}_1 = \widehat{\partial}_2\widehat{x}_1
\end{aligned}$$

We have $H_\alpha\bar{x}_2 = \bar{x}_2(H_\alpha - 1)$, $H_\alpha\bar{\partial}_1 = \bar{\partial}_1(H_\alpha - 1)$, and $\bar{x}_2 H_{\beta+\alpha} = (H_{\beta+\alpha} - 1)\bar{x}_2$, so

$$\widehat{x}_2\widehat{\partial}_1 = (H_\alpha + 2)\bar{x}_2\bar{\partial}_1(H_\alpha + 1)(H_{\beta+\alpha} + 1) = \bar{x}_2\bar{\partial}_1 H_\alpha(H_\alpha + 1)(H_{\beta+\alpha} + 1)$$

$$
\begin{aligned}
&= \bar{\partial}_1 \bar{x}_2 \frac{H_{\beta+\alpha}+2}{H_{\beta+\alpha}+1} H_\alpha (H_\alpha+1)(H_{\beta+\alpha}+1) = \bar{\partial}_1 \bar{x}_2 H_\alpha (H_\alpha+1)(H_{\beta+\alpha}+2) \\
&= \bar{\partial}_1 (H_\alpha+1)(H_\alpha+2)(H_{\beta+\alpha}+1)\bar{x}_2 = \widehat{\partial}_1 \widehat{x}_2.
\end{aligned}
$$

This proves that the four relations in (4.1b) are preserved by ϕ.

Next, to show that (4.1a) holds for all $b \in B$, it suffices to show it for $b \in \{H_\alpha, H_\beta, t_1, t_2\}$ since σ_i are $\mathbb{C}$-algebra homomorphisms. For $b = H_\alpha$ and $b = H_\beta$, it follows directly from (1.1a)–(1.1d) and the definition of σ_i.

For $b = t_i$ the identities are immediately verified once we prove that $\phi(t_i) = \phi(Y_i)\phi(X_i)$ and $\phi\big(\sigma_i(t_i)\big) = \phi(X_i)\phi(Y_i)$, because then:

$$
\phi(X_i)\phi(t_i) = \phi(X_i)\phi(Y_i)\phi(X_i) = \phi\big(\sigma_i(t_i)\big)\phi(X_i)
$$

and for $j \neq i$

$$
\phi(X_i)\phi(t_j) = \phi(X_i)\phi(Y_j)\phi(X_j) = \widehat{x}_i \widehat{\partial}_j \widehat{x}_j = \widehat{x}_j \widehat{\partial}_j \widehat{x}_i = \phi\big(\sigma_i(t_j)\big)\phi(X_i).
$$

Similarly for the case of Y_i.

Actually, that $\phi(t_i) = \phi(Y_i)\phi(X_i)$ is immediate by definition of $\phi(t_i)$. So, it only remains to prove that

$$
\phi\big(\sigma_i(t_i)\big) = \phi(X_i)\phi(Y_i) \tag{4.10}
$$

for $i = 1, 2$. The left hand side of (4.10) equals

$$
\begin{aligned}
\phi(\sigma_i(t_i)) &= \phi(c_i + \widehat{f}_{i1} t_1 + \widehat{f}_{i2} t_2) = c_i + \widehat{f}_{i1} \widehat{\partial}_1 \widehat{x}_1 + \widehat{f}_{i2} \widehat{\partial}_2 \widehat{x}_2 \\
&= c_i + \widehat{f}_{i1} \cdot (H_\alpha+2)(H_{\beta+\alpha}+2)\bar{\partial}_1 \bar{x}_1 + \widehat{f}_{i2} \cdot (H_\alpha+1)(H_{\beta+\alpha}+2)\bar{\partial}_2 \bar{x}_2 \\
&= c_i + (H_\alpha+i)(H_{\beta+\alpha}+1)\Big(f_{i1}\bar{\partial}_1 \bar{x}_1 + f_{i2}\bar{\partial}_2 \bar{x}_2\Big). \qquad (4.11)
\end{aligned}
$$

The right hand side of (4.10) equals

$$
\widehat{x}_i \widehat{\partial}_i = (H_\alpha + i)(H_{\beta+\alpha}+1)\bar{x}_i \bar{\partial}_i
$$

which equals (4.11), by (1.1g).

At this point we can establish that σ_1 and σ_2 actually commute. We have for any $b \in B$,

$$
0 = (\widehat{x}_1 \widehat{x}_2 - \widehat{x}_2 \widehat{x}_1) b = \big(\sigma_1(\sigma_2(b)) - \sigma_2(\sigma_1(b))\big)\widehat{x}_1 \widehat{x}_2.
$$

By Corollary 3.4, it follows that σ_1 and σ_2 commute.

We have shown all GWA relations (4.1) are preserved by ϕ. Therefore there exists a well-defined $\mathbb{C}$-algebra homomorphism $\phi : B(\sigma, t) \to D\big(\mathfrak{sp}(4)\big)$ as in the statement of the theorem. Moreover, ϕ is a homomorphism of left R-modules. By a known result about GWAs of the type where $B = R[t_1, t_2, \ldots, t_n]$ for some subring R, the monomials $Y_1^a Y_2^b X_1^c X_2^d$ $(a, b, c, d \in \mathbb{Z}_{\geq 0})$ form a basis for $B(\sigma, t)$ as a left R-module. These are mapped under ϕ to the monomials $\widehat{\partial}_1^a \widehat{\partial}_2^b \widehat{x}_1^c \widehat{x}_2^d$ which form a left R-basis for $D\big(\mathfrak{sp}(4)\big)$ by Corollary 3.4(b). Therefore ϕ is bijective. $\square$

4.6. **Rank Two Skew-affine GWAs.** We consider a class of GWAs we call *skew-affine*. We restrict to the case of rank two. The reduction algebras considered in this paper turn out to be examples of such GWAs.

Suppose R is a finitely generated commutative $\mathbb{C}$-algebra and let $B = R[t_1, t_2]$ be the polynomial algebra over R in two indeterminates t_i. Consider the following Ansatz for $\mathbb{C}$-algebra automorphisms σ_i of B:

$$\begin{aligned} \sigma_i(t_i) &= c_i + f_{i1}t_1 + f_{i2}t_2, \quad i = 1, 2; \\ \sigma_i(t_j) &= t_j, \quad i \neq j; \\ \sigma_i|_R &\in \mathrm{Aut}_{\mathbb{C}}(R). \end{aligned}$$

where c_i and f_{ij} are some fixed elements of R. The requirement that the σ_i are invertible and commute impose some conditions on the coefficients:

Lemma 4.6.

(a) *σ_i is invertible if and only iff f_{ii} is invertible in R.*

(b) *σ_1 and σ_2 commute if and only if $\sigma_1|_R$ and $\sigma_2|_R$ commute, and the following equations hold:*

$$c_1 = \sigma_2(c_1) + \sigma_2(f_{12})c_2, \tag{4.12a}$$
$$f_{11} = \sigma_2(f_{11}) + \sigma_2(f_{12})f_{21}, \tag{4.12b}$$
$$f_{12} = \sigma_2(f_{12})f_{22}, \tag{4.12c}$$
$$c_2 = \sigma_1(c_2) + \sigma_1(f_{21})c_1, \tag{4.12d}$$
$$f_{21} = \sigma_1(f_{21})f_{11}, \tag{4.12e}$$
$$f_{22} = \sigma_1(f_{21})f_{12} + \sigma_1(f_{22}). \tag{4.12f}$$

Proof. (a) Say $i = 1$. Suppose ψ_1 is the inverse of σ_1. Then $\psi_1|_R = (\sigma_1|_R)^{-1}$ and $\psi_1(t_2) = \sigma_1^{-1}(t_2) = t_2$ and $\psi_1(t_1) = a(t_2) + g(t_2)t_1 + p(t_1, t_2)t_1^2$ for some $a, g \in R[t_2]$, $p \in B$. Now $t_1 = \psi_1(\sigma_1(t_1)) = \psi_1(c_1) + \psi_1(f_{11})\psi_1(t_1) + \psi_1(f_{12})t_2$. Substituting $\psi_1(t_1)$ we see that $\psi_1(f_{11})g(t_2) = 1$. Thus f_{11} is a unit in $R[t_2]$, hence a unit in R. Conversely, if f_{11} is a unit, an inverse ψ_1 can be found of skew-affine form $\psi_1(t_1) = a_1 + g_{11}t_1 + g_{12}t_2$. Indeed,

$$\begin{aligned} \psi_1(\sigma_1(t_1)) &= \psi_1(c_1) + \psi_1(f_{11})\big(a_1 + g_{11}t_1 + g_{12}t_2\big) + \psi_1(f_{12})t_2 \\ &= \Big(\psi_1(c_1) + \psi_1(f_{11})a_1\Big) + \Big(\psi_1(f_{11})g_{11}\Big)t_1 + \Big(\psi_1(f_{11})g_{12} + \psi_1(f_{12})\Big)t_2 \end{aligned}$$

which equals t_1 iff $a_1 = -\psi_1(c_1/f_{11})$, $g_{11} = \psi_1(1/f_{11})$, $g_{12} = -\psi_1(f_{12}/f_{11})$. With these choices, $\sigma_1(\psi_1(t_1)) = t_1$ too.

(b) By definition of σ_i, the automorphisms commute if and only if they commute on R and $\sigma_1\sigma_2(t_i) = \sigma_2\sigma_1(t_i)$ for $i = 1, 2$. For $i = 1$ we have

$$\sigma_1\sigma_2(t_1) = \sigma_1(t_1) = c_1 + f_{11}t_1 + f_{12}t_2 \tag{4.13}$$

and

$$\sigma_2\sigma_1(t_1) = \sigma_2(c_1 + f_{11}t_1 + f_{12}t_2) =$$

$$
\begin{aligned}
&= \sigma_2(c_1) + \sigma_2(f_{11})t_1 + \sigma_2(f_{12})\Big(c_2 + f_{21}t_1 + f_{22}t_2)\Big) \\
&= \Big(\sigma_2(c_1) + \sigma_2(f_{12})c_2\Big) + \Big(\sigma_2(f_{11}) + \sigma_2(f_{12})f_{21}\Big)t_1 + \Big(\sigma_2(f_{12})f_{22}\Big)t_2
\end{aligned} \tag{4.14}
$$

Since B is a polynomial algebra in t_1, t_2 over R, identifying coefficients of (4.13) and (4.14), we see that $\sigma_1\sigma_2(t_1) = \sigma_2\sigma_1(t_1)$ if and only if (4.12a)–(4.12c) hold. Similarly, $\sigma_1\sigma_2(t_2) = \sigma_2\sigma_1(t_2)$ holds if and only if (4.12d)–(4.12f) hold. □

As explained in Section 4.3, to understand weight modules, it becomes important to know when t_i belongs to a maximal ideal of B. The maximal ideals of B have the form $B\mathfrak{m}_R + (t_1 - \lambda_1, t_2 - \lambda_2)$ where $\mathfrak{m}_R$ is a maximal ideal of R. In particular, the following calculation is useful:

Lemma 4.7. *Assume that σ_1 and σ_2 commute. For $i = 1, 2$ and any non-negative integer n:*

$$\sigma_i^n(t_i) = c_i^{(n)} + f_{i1}^{(n)}t_1 + f_{i2}^{(n)}t_2, \tag{4.15}$$

where

$$
\begin{aligned}
c_1^{(n)} &= \sum_{k=0}^{n-1} \frac{f_{11}^{(n)}}{f_{11}^{(k+1)}}\sigma_1^k(c_1), \quad & f_{11}^{(n)} &= f_{11}\sigma_1(f_{11})\cdots\sigma_1^{n-1}(f_{11}), \quad & f_{12}^{(n)} &= \sum_{k=0}^{n-1} \frac{f_{11}^{(n)}}{f_{11}^{(k+1)}}\sigma_1^k(f_{12}), \\
c_2^{(n)} &= \sum_{k=0}^{n-1} \frac{f_{22}^{(n)}}{f_{22}^{(k+1)}}\sigma_2^k(c_2), \quad & f_{21}^{(n)} &= \sum_{k=0}^{n-1} \frac{f_{22}^{(n)}}{f_{22}^{(k+1)}}\sigma_2^k(f_{21}), \quad & f_{22}^{(n)} &= f_{22}\sigma_2(f_{22})\cdots\sigma_2^{n-1}(f_{22}),
\end{aligned}
$$

where $f_{ii}^{(n)}/f_{ii}^{(k+1)}$ is shorthand for $\sigma_i^{k+1}(f_{ii})\sigma_i^{k+2}(f_{ii})\cdots\sigma_i^{n-1}(f_{ii}) = \sigma_i^{k+1}(f_{ii}^{(n-1-k)})$.

Proof. Follows by definition of σ_i using induction on $n \geq 0$. □

5. Open Problems and Conjectures

Problem 5.1. *Classify all rank two skew-affine GWAs by solving the equations in Lemma 4.6.*

Problem 5.2. *Classify simple weight modules over such skew-affine GWAs (perhaps using the formulas in Lemma 4.7).*

Problem 5.3. *Generalize the previous two problems to higher rank.*

Conjecture 5.4. *The differential reduction algebra of $\mathfrak{sp}(2n)$ is an example of a skew-affine GWA of rank n.*

It is known that there is no need to define "super GWAs" as the class of twisted GWAs is already general enough to include such examples, see [9]. Therefore we make the following conjecture.

Conjecture 5.5. *The differential reduction algebra of $\mathfrak{osp}(1|2n)$ is an example of a skew-affine twisted GWA of rank n.*

A. Alternative Computation of One Identity

See Lemma 3.3. By the implication of (3.26g), we have $\widetilde{x}_2 P\widetilde{\partial}_2 = \widetilde{x}_2 P_\beta P_{\beta+\alpha} P_{\beta+2\alpha} P_\alpha \widetilde{\partial}_2 \equiv \widetilde{x}_2 P_\beta P_{\beta+\alpha}\widetilde{\partial}_2$. Continuing:

$$\begin{aligned}
\widetilde{x}_2 P\widetilde{\partial}_2 &\equiv \widetilde{x}_2 P_\beta P_{\beta+\alpha}\widetilde{\partial}_2 \\
&\equiv \Big(\widetilde{x}_2 + [\widetilde{x}_2, F_\beta]\frac{-1}{H_\beta}E_\beta\Big)\Big(\widetilde{\partial}_2 + F_{\beta+\alpha}\frac{-1}{H_{\beta+\alpha}}[E_{\beta+\alpha}, \widetilde{\partial}_2]\Big) \\
&\equiv \Big(\widetilde{x}_2 + (-i\widetilde{\partial}_2)\frac{-1}{H_\beta}E_\beta\Big)\Big(\widetilde{\partial}_2 + F_{\beta+\alpha}\frac{-1}{H_{\beta+\alpha}}(-i\widetilde{x}_1)\Big) \\
&\equiv \Big(\widetilde{x}_2 + \frac{i}{H_\beta+1}\widetilde{\partial}_2 E_\beta\Big)\Big(\widetilde{\partial}_2 + \frac{i}{H_{\beta+\alpha}+2}F_{\beta+\alpha}\widetilde{x}_1\Big) \\
&\equiv \widetilde{x}_2\widetilde{\partial}_2 + \frac{i}{H_\beta+1}\widetilde{\partial}_2[E_\beta, \widetilde{\partial}_2] + \widetilde{x}_2\frac{i}{H_{\beta+\alpha}+2}F_{\beta+\alpha}\widetilde{x}_1 - \frac{1}{H_\beta+1}\widetilde{\partial}_2 E_\beta\frac{1}{H_{\beta+\alpha}+2}F_{\beta+\alpha}\widetilde{x}_1 \\
&\equiv \widetilde{x}_2\widetilde{\partial}_2 + \frac{1}{H_\beta+1}\widetilde{\partial}_2\widetilde{x}_2 + \widetilde{x}_2\frac{i}{H_{\beta+\alpha}+2}F_{\beta+\alpha}\widetilde{x}_1 - \frac{1}{H_\beta+1}\widetilde{\partial}_2\frac{1}{H_{\beta+\alpha}}E_\beta F_{\beta+\alpha}\widetilde{x}_1 \\
&\equiv \widetilde{x}_2\widetilde{\partial}_2 + \frac{1}{H_\beta+1}\widetilde{\partial}_2\widetilde{x}_2 + \frac{i}{H_{\beta+\alpha}+1}[\widetilde{x}_2, F_{\beta+\alpha}]\widetilde{x}_1 - \frac{1}{(H_\beta+1)(H_{\beta+\alpha}+1)}\widetilde{\partial}_2[E_\beta, F_{\beta+\alpha}]\widetilde{x}_1 \\
&\equiv \widetilde{x}_2\widetilde{\partial}_2 + \frac{1}{H_\beta+1}\widetilde{\partial}_2\widetilde{x}_2 + \frac{i}{H_{\beta+\alpha}+1}(-i\widetilde{\partial}_1)\widetilde{x}_1 - \frac{1}{(H_\beta+1)(H_{\beta+\alpha}+1)}[\widetilde{\partial}_2, F_\alpha]\widetilde{x}_1 \\
&\equiv \widetilde{x}_2\widetilde{\partial}_2 + \frac{1}{H_\beta+1}\widetilde{\partial}_2\widetilde{x}_2 + \frac{1}{H_{\beta+\alpha}+1}\widetilde{\partial}_1\widetilde{x}_1 - \frac{1}{(H_\beta+1)(H_{\beta+\alpha}+1)}\widetilde{\partial}_1\widetilde{x}_1 \\
&\equiv \widetilde{x}_2\widetilde{\partial}_2 + \frac{H_\beta}{(H_\beta+1)(H_{\beta+\alpha}+1)}\widetilde{\partial}_1\widetilde{x}_1 + \frac{1}{H_\beta+1}\widetilde{\partial}_2\widetilde{x}_2 \\
&\equiv \widetilde{\partial}_2\widetilde{x}_2 + \frac{H_\beta}{(H_\beta+1)(H_{\beta+\alpha}+1)}\widetilde{\partial}_1\widetilde{x}_1 + \frac{H_\beta+2}{H_\beta+1}\widetilde{\partial}_2\widetilde{x}_2
\end{aligned}$$

This coincides with (3.30).

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