

# GEOMETRIC SIGNALS

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**Abstract.** In signal processing, a signal is the graph of a function. We define a signal as a submanifold of a Riemannian manifold (with corners). We obtain inequalities that relate the energy of the signal and the energy of its Fourier transform. We quantify noise and filtering.

## 1. INTRODUCTION.

This paper is about a geometric generalization of the concept of signal. The goal is to build an abstract mathematical model of transmitting information via geometric objects which are, informally speaking, higher dimensional analogues of sound waveforms. We are not taking the most obvious path to defining a signal as an  $\mathbb{R}^n$ -valued function on an open subset of manifold. Instead, we are taking a more intuitive approach with cobordisms. This would allow to treat quite general geometric objects as information.

In practical applications, signal processing involves sampling, analog to digital conversion, quantization, and mathematical techniques that are needed because of the hardware used to receive and analyze the signal. We will concentrate our attention on the geometric concept of signal, rather than being concerned with digitizing such a signal or further processing the resulting data.

For background discussion, let's consider the word "dot". It can be transmitted as a finite sequence of three images, of the letters "D", "O", "T", respectively. Each of the three images can be converted into digital data (e.g. by drawing the letters on the grid paper via shading appropriate squares and entering 1 in the corresponding matrix for each shaded square, 0 for every blank square Fig. 1).

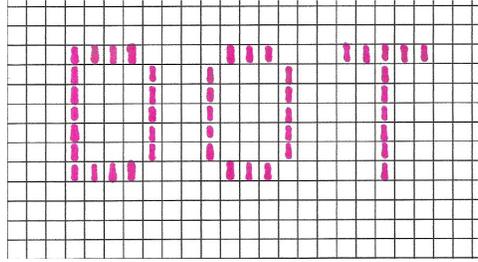


FIGURE 1. The "empty squares" are the zero entries of the rectangular matrix.

Alternatively, the three letters can be represented by the binary representations of the numbers 4, 15, 20 (their positions in the English alphabet). Or, instead, one can send the word "dot" as an audio file ( a recording of a person saying this word), or the wave soundform of this recording (which can be subdivided into three parts, each for one of the three letters of the word Fig. 2).

The mathematical aspects of these processes depend on the choice of sampling, quantization, encoding, transmission, other procedures related to speech processing, as well as the linguistics aspects such as the language and the alphabet (if the written language is based on an alphabet). For details see [1, 2].

Instead of doing all that, we can concentrate only on the geometric aspects and consider the immersed submanifolds of  $\mathbb{R}^2$  (with the standard Riemannian metric) that correspond to the data

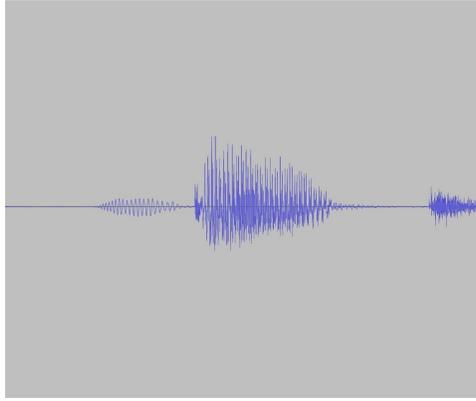


FIGURE 2. A soundform of the word "dot".

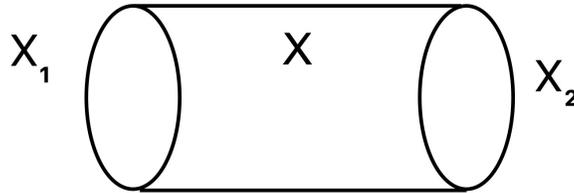
on Fig. 1, 2. This is the general point of view in this paper. In order to account, intuitively, for a "process" or "evolution" taking place, without explicitly defining a time variable, we bring in the concept of cobordism.

## 2. COBORDISMS AS SIGNALS

Let  $k \geq 0$  be an integer. Let  $(X; X_1, X_2)$  be a  $(k + 1)$ -dimensional (oriented) cobordism, i.e. a compact  $(k + 1)$ -dimensional oriented manifold  $X$  with boundary

$$\partial X = X_1 \sqcup X_2$$

where  $X_1$  and  $X_2$  are closed  $k$ -dimensional oriented manifolds.

FIGURE 3. A cobordism  $(X; X_1, X_2)$ .

Let  $(Y; Y_1, Y_2)$  be a  $(k + 1)$ -dimensional cobordism. Let  $(M; X, Y, \Sigma; \partial X, \partial Y)$  be a  $(k + 2)$ -dimensional cobordism of manifolds with boundary in the sense of [3], i.e.  $(\Sigma, \partial X, \partial Y)$  is a  $(k + 1)$ -dimensional cobordism, and

$$\partial M = X \cup \Sigma \cup Y.$$

Assume, moreover, that

$$\Sigma = A \sqcup B$$

and  $(A; X_1, Y_1)$ ,  $(B; X_2, Y_2)$  are  $(k + 1)$ -dimensional cobordisms.  $M$  is a manifold with corners [5]. Figure 4 shows an example where  $k = 0$ , the boundary of  $X$  consists of two points and the boundary of  $Y$  consists of two points.

*Example 2.1.* For applications, a simplest typical example would be  $k = 1$ ,  $X$  is a compact smooth surface with the boundary which is the disjoint union of  $n_1 + n_2$  circles ( $n_1, n_2 \in \mathbb{N}$ ),  $X_1$  is a disjoint union of  $n_1$  circles,  $X_2$  is a disjoint union of  $n_2$  circles,  $Y = X$ ,  $M = X \times [0, 1]$ . In Figure 5,  $X$  has values  $n_1 = 2$  and  $n_2 = 3$ .

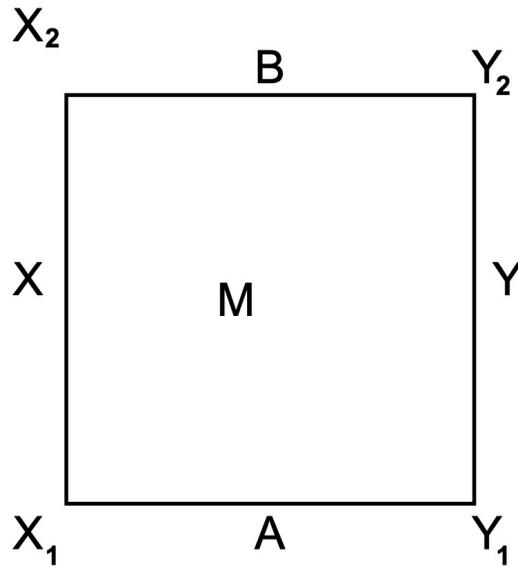


FIGURE 4. A cobordism  $(M; X, Y, \Sigma; \partial X, \partial Y)$ ,  $k = 0$ .

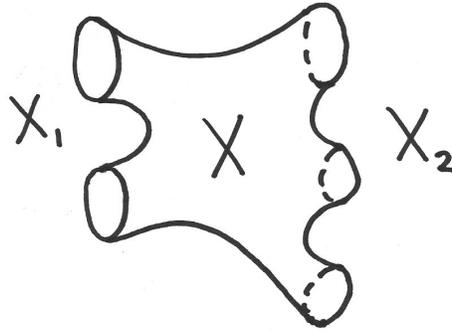


FIGURE 5. A cobordism  $(X; X_1, X_2)$ ,  $n_1 = 2$ ,  $n_2 = 3$ .

Let  $(\tilde{M}, \tilde{g})$  be a Riemannian manifold such that  $M \subset \tilde{M}$  (a subset of  $\tilde{M}$  which is an embedded submanifold via the identity map). We will write  $M_g$  for  $M$  equipped with the Riemannian metric  $g$  induced by  $\tilde{g}$ . Unless explicitly stated otherwise, we will also assume that the Riemannian metric on every submanifold of  $M$  is the one induced by  $g$ . In Figure 4,  $M \subset \mathbb{R}^2 = \tilde{M}$ . Figure 6 shows a cobordism as an Example 2.1, with  $k = 1$ ,  $n_1 = n_2 = 1$ , and  $M \subset \mathbb{R}^3 = \tilde{M}$ .

Denote by  $dV_g$  the volume form of  $g$  and by  $\rho_g$  the Riemannian distance with respect to  $g$ . Define

$$f_X : M \rightarrow \mathbb{R}$$

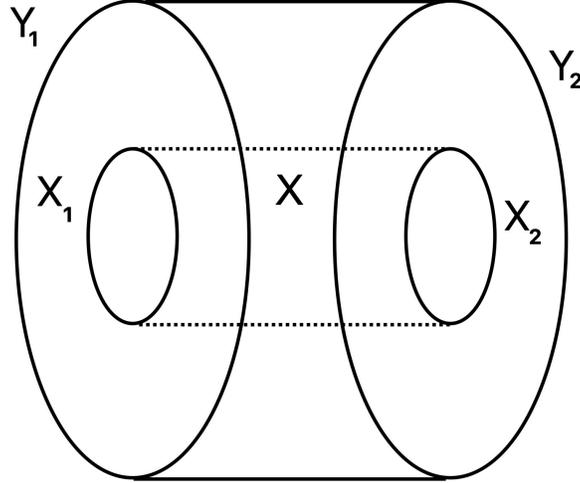


FIGURE 6. A cobordism  $(M; X, Y, \Sigma; \partial X, \partial Y)$ .

by  $f_X(x) = \rho_g(x, X)$ . Since  $X$  is compact,  $f_X$  is well defined and continuous on  $M$ . Similarly, define

$$f_A : M \rightarrow \mathbb{R}$$

by  $f_A(x) = \rho_g(x, A)$ .

*Remark 2.2.* Regularity of the distance function to a submanifold is treated in [4]. In [4], the discussion is for  $\mathbb{R}^n$  and it is noted that the proofs are similar for submanifolds of Riemannian manifolds.

*Remark 2.3.* The reason to keep track of orientation is for calculation of volumes.

*Remark 2.4.* We are able to complete the proofs of the theorems below with the standard techniques of Riemannian geometry. For further work, the methods developed by R. Melrose would be useful (*b*-calculus and analysis on manifolds with corners).

Denote  $W = X \sqcup Y$ . Define

- the energy of the signal  $M_g$

$$(1) \quad E(M_g) = \int_M f_A(x) dV_g(x)$$

- the Fourier transform  $F$  of the signal  $M_g$  as the map of cobordisms of manifolds with boundary

$$(M; X, Y, \Sigma; \partial X, \partial Y) \rightarrow (M; A, B, W; \partial X, \partial Y)$$

induced by the identity map  $M \rightarrow M$

- the energy of the Fourier transform of the signal  $F(M_g)$

$$E(F(M_g)) = \int_M f_X(x) dV_g(x).$$

- a noise  $(U, h)$  where  $U \subset M$  is an open set and  $h$  is a Riemannian metric on  $M$  such that  $h = g$  on  $M - U$ . Then

$$E(M_h) = \int_M \rho_h(x, A) dV_h(x)$$

$$E(F(M_h)) = \int_M \rho_h(x, X) dV_h(x).$$

- a *filter* a signal  $M' = (M'; X', Y', \Sigma'; \partial X', \partial Y')$ , such that  $\partial X' = X'_1 \sqcup X'_2$ ,  $\partial Y' = Y'_1 \sqcup Y'_2$ ,  $\Sigma' = A' \sqcup B'$ ,  $M' \subset M$ ,  $X' \subset X$  and  $Y' \subset Y$ .
- composition of two signals  $M = (M; X, Y, \Sigma = A \sqcup B; \partial X, \partial Y)$ ,  $M' = (M'; X', Y', \Sigma' = A' \sqcup B'; \partial X', \partial Y')$  such that  $Y = X'$  and  $M \cap M' = X'$  is the signal

$$M'' = (M''; X, Y', \Sigma''; \partial X, \partial Y')$$

where

$$M'' = M \cup M'$$

$$A'' = A \cup A'$$

$$B'' = B \cup B'$$

$$\Sigma'' = \Sigma \cup \Sigma' = A'' \sqcup B''$$

*Remark 2.5.* If there is no ambiguity about the metric and the metric is presumed to be the one induced by  $g$ , then we will sometimes omit  $g$  from notation and write  $E(M) = E(M_g)$ , similarly for volume, diameter etc.

*Remark 2.6.* In signal processing, a discrete signal is represented by a finite sequence of real numbers  $(x_n)$ . These come (via sampling) from the sound waveform and should be understood as the numbers that represent the air pressure at time  $n$ . The signal is characterized by its *energy*

$$\sum_n |x_n|^2$$

and *magnitude*  $\max\{|x_n|\}$ . The energy of a continuous one-dimensional signal is

$$(2) \quad \int |x(t)|^2 dt.$$

Our definition of energy (1) is the integral of the distance function. Intuitively, this makes sense, as the integrand is a second order quantity as in (2).

*Remark 2.7.* To give an example of noise, one can consider a local diffeomorphism  $\varphi$  of a small ball that is the identity map on the boundary of this ball, and the metric  $h = \varphi^*g$ . The unit ball in  $\mathbb{R}^n$  admits a diffeomorphism that is identity on its boundary and extends smoothly (by the identity map) to a neighborhood of the boundary.

**Theorem 2.8.** *Let  $M = (M; X, Y, \Sigma = A \sqcup B; \partial X, \partial Y)$  be a signal.*

(i) *There is a constant  $r > 0$  such that*

$$\frac{1}{1 + \frac{3 \operatorname{diam}(M)\operatorname{vol}(M)}{r \operatorname{vol}(X)}} \leq \frac{E(F(M))}{E(M)} \leq 1 + \frac{3 \operatorname{diam}(M)\operatorname{vol}(M)}{r \operatorname{vol}(A)}.$$

(ii) *Let  $(U, h)$  be noise, where*

$$U = \{x \in M \mid \rho_g(x, p) < \delta\}$$

*for some  $p \in M$  and  $\delta > 0$ . Let  $0 < \delta_0 < \delta$  and let  $0 < \varepsilon < 1$ .*

*Then there is a smooth function  $a_\varepsilon : M \rightarrow \mathbb{R}$  such that*

$$(3) \quad a_\varepsilon(x) = \begin{cases} \varepsilon, & \text{if } x \in \{x \in M \mid \rho_g(x, p) \leq \delta_0\} \\ 1, & \text{if } x \in \{x \in M \mid \rho_g(x, p) \geq \delta\} \end{cases}$$

$$(4) \quad 0 < a_\varepsilon(x) < 1 \text{ for all } x \in \{x \in M \mid \delta_0 < \rho_g(x, p) < \delta\},$$

and as  $\varepsilon \rightarrow 0$

$$\frac{E(F(M_{ha_\varepsilon}))}{E(M_{ha_\varepsilon})} = \frac{\beta}{\gamma} \left( 1 + C\varepsilon^{\frac{k+2}{2}} + O(\varepsilon^{k+2}) \right)$$

where

$$\begin{aligned} \beta &= \int_{\{x \in M | \rho_g(p, x) > \delta_0\}} \rho_{ha_\varepsilon}(x, X) a_\varepsilon(x)^{\frac{k+2}{2}} dV_h(x) \\ \gamma &= \int_{\{x \in M | \rho_g(p, x) > \delta_0\}} \rho_{ha_\varepsilon}(x, A) a_\varepsilon(x)^{\frac{k+2}{2}} dV_h(x) \\ C &= \frac{\int_{\{x \in M | \rho_g(p, x) < \delta_0\}} \rho_{ha_\varepsilon}(x, X) dV_h(x)}{\beta} - \frac{\int_{\{x \in M | \rho_g(p, x) < \delta_0\}} \rho_{ha_\varepsilon}(x, A) dV_h(x)}{\gamma}. \end{aligned}$$

(iii) Let  $(U, h)$  be noise as in (ii) and let  $M'$  be a filter such that  $X'_1 = X_1$ ,  $Y'_1 = Y_1$ ,  $A' = A$ ,  $U \subset M - M'$ . Then

$$E(M'_g) \leq E(M_g)$$

$$E(M'_h) \leq E(M_h).$$

*Remark 2.9.* In Theorem 2.8 (ii) the function  $a_\varepsilon$ , in a sense, modulates the noisy signal, from  $M_h$  to  $M_{ha_\varepsilon}$ . In (iii),  $M'$  filters out the noise.

**Theorem 2.10.** Let  $M''$  be the composition of signals  $M$  and  $M'$ . Then

$$(5) \quad E(M'') \leq E(M) + E(M')$$

$$(6) \quad E(F(M'')) \geq E(F(M)).$$

### 3. PROOFS

**Proof of Theorem 2.8.** By the mean value theorem, there is  $a_0 \in M$  such that

$$E(M) = \int_M f_A(x) dV_g(x) = f_A(a_0) \text{vol}_g(M)$$

and there is  $x_0 \in M$  such that

$$E(F(M)) = \int_M f_X(x) dV_g(x) = f_X(x_0) \text{vol}_g(M).$$

Since  $A$  and  $X$  are compact, there are  $a_1 \in A$  and  $x_1 \in X$  such that  $\rho_g(a_0, a_1) = \rho_g(a_0, A)$  and  $\rho_g(x_0, x_1) = \rho_g(x_0, X)$ . Let  $b \in A \cap X$ . Then

$$\begin{aligned} \frac{E(F(M))}{E(M)} &= \frac{f_X(x_0)}{f_A(a_0)} = \frac{\rho_g(x_0, X)}{f_A(a_0)} \leq \frac{\rho_g(x_0, b) + \rho_g(x_1, b)}{f_A(a_0)} \leq \\ &\frac{\rho_g(a_0, x_0) + \rho_g(a_0, b) + \rho_g(x_1, b)}{f_A(a_0)} \leq \frac{\rho_g(a_0, x_0) + \rho_g(a_0, a_1) + \rho_g(a_1, b) + \rho_g(x_1, b)}{f_A(a_0)} \leq \\ (7) \quad &1 + \frac{\text{diam}(M) + \text{diam}(A) + \text{diam}(X)}{\rho_g(a_0, A)}. \end{aligned}$$

By the collar neighborhood theorem, there is  $\varepsilon > 0$ , an open neighborhood  $A_\varepsilon$  of  $A$  in  $M$  and a diffeomorphism

$$\alpha : A_\varepsilon \rightarrow A \times [0, 1).$$

Let  $\mathcal{U} = \{U_1, \dots, U_m\}$  be an open cover of  $A$  by manifold charts  $\{U_i, \varphi_i\}$  and let  $\{\psi_1, \dots, \psi_m\}$  be a smooth partition of unity subordinate to  $\mathcal{U}$ . Then

$$f_A(a_0) = \frac{\int_M f_A(x) dV_g(x)}{\text{vol}(M)} \geq \frac{1}{\text{vol}(M)} \int_{A_\varepsilon} f_A(x) dV_g(x) = \frac{1}{\text{vol}(M)} \int_{A_\varepsilon} f_A(x) \sum_{j=1}^m \psi_j(x) dV_g(x) =$$

$$(8) \quad \frac{1}{\text{vol}(M)} \sum_{j=1}^m \int_{U_j \times [0,1]} f_A(\alpha^{-1}(a, t)) \psi_j(\alpha^{-1}(a, t)) dV_g(\alpha^{-1}(a, t)),$$

where  $a \in A, t \in [0, 1)$ . Denote by  $a_1^{(i)}, \dots, a_{k+1}^{(i)}$  the coordinates in the  $i$ -th chart. Using the Fubini theorem, we get that (8) equals

$$\frac{1}{\text{vol}(M)} \int_{[0,1]} \sum_{i=1}^m \int_{\varphi_i(U_i)} f_A(\alpha^{-1}(\varphi_i^{-1}(a_1, \dots, a_{k+1}), t)) \psi_i(\alpha^{-1}(\varphi_i^{-1}(a_1, \dots, a_{k+1}), t))$$

$$\sqrt{\det g(a_1, \dots, a_{k+1}, t)} da_1 \dots da_{k+1} dt.$$

By the mean value theorem, there is  $0 < t_0 < 1$  such that

$$\int_{[0,1]} \sum_{i=1}^m \int_{\varphi_i(U_i)} f_A(\alpha^{-1}(\varphi_i^{-1}(a_1, \dots, a_{k+1}), t)) \psi_i(\alpha^{-1}(\varphi_i^{-1}(a_1, \dots, a_{k+1}), t))$$

$$\sqrt{\det g(a_1, \dots, a_{k+1}, t)} da_1 \dots da_{k+1} dt =$$

$$(9) \quad \sum_{i=1}^m \int_{\varphi_i(U_i)} f_A(\alpha^{-1}(\varphi_i^{-1}(a_1, \dots, a_{k+1}), t_0)) \psi_i(\alpha^{-1}(\varphi_i^{-1}(a_1, \dots, a_{k+1}), t_0))$$

$$\sqrt{\det g(a_1, \dots, a_{k+1}, t_0)} da_1 \dots da_{k+1}.$$

Then

$$f_A(a_0) \geq \frac{1}{\text{vol}(M)} \sum_{i=1}^m \int_{\varphi_i(U_i)} f_A(\alpha^{-1}(\varphi_i^{-1}(a_1, \dots, a_{k+1}), t_0)) \psi_i(\alpha^{-1}(\varphi_i^{-1}(a_1, \dots, a_{k+1}), t_0))$$

$$\sqrt{\det g(a_1, \dots, a_{k+1}, t_0)} da_1 \dots da_{k+1} =$$

$$\frac{1}{\text{vol}(M)} \sum_{i=1}^m \int_{U_i} f_A(\alpha^{-1}(a, t_0)) \psi_i(\alpha^{-1}(a, t_0)) d\mu_A(a) =$$

$$\frac{1}{\text{vol}(M)} \sum_{i=1}^m \int_A f_A(\alpha^{-1}(a, t_0)) \psi_i(\alpha^{-1}(a, t_0)) d\mu_A(a) = \frac{1}{\text{vol}(M)} \int_A f_A(\alpha^{-1}(a, t_0)) d\mu_A(a)$$

where  $d\mu_A$  is the measure on  $A$  induced by the Riemannian metric. Hence there is  $r_1 > 0$  such that

$$\rho_g(a_0, A) = f_A(a_0) \geq r_1 \frac{\text{vol}(A)}{\text{vol}(M)}.$$

Then, with (7),

$$\frac{E(F(M))}{E(M)} \leq 1 + \frac{3 \text{diam}(M) \text{vol}(M)}{r_1 \text{vol}(A)}.$$

Repeating the argument for  $\frac{E(M)}{E(F(M))}$ , we get: there is  $r_2 > 0$  such that

$$\frac{E(M)}{E(F(M))} \leq 1 + \frac{3 \text{diam}(M) \text{vol}(M)}{r_2 \text{vol}(X)}.$$

Set  $r = \min\{r_1, r_2\}$ . Since  $\text{diam}(A) \leq \text{diam}(M)$  and  $\text{diam}(X) \leq \text{diam}(M)$ , (i) follows.

Proof of (ii). Let  $\{\varphi, \psi\}$  be a smooth partition of unity subordinate to the open cover

$$\{\{x \in M \mid \rho_g(p, x) < \delta\}, \{x \in M \mid \rho_g(p, x) > \delta_0\}\}.$$

Then the function  $a_\varepsilon$  defined by

$$a_\varepsilon(x) = \varepsilon\varphi(x) + \psi(x)$$

satisfies (3), (4). We have:

$$\begin{aligned} \frac{E(F(M_{ha_\varepsilon}))}{E(M_{ha_\varepsilon})} &= \frac{\int_M \rho_{ha_\varepsilon}(x, X) dV_{ha_\varepsilon}(x)}{\int_M \rho_{ha_\varepsilon}(x, A) dV_{ha_\varepsilon}(x)} = \frac{\int_M \rho_{ha_\varepsilon}(x, X) a_\varepsilon(x)^{\frac{k+2}{2}} dV_h(x)}{\int_M \rho_{ha_\varepsilon}(x, A) a_\varepsilon(x)^{\frac{k+2}{2}} dV_h(x)} = \\ &= \frac{\int_{\{x \in M \mid \rho_g(x, p) > \delta_0\}} \rho_{ha_\varepsilon}(x, X) a_\varepsilon(x)^{\frac{k+2}{2}} dV_h(x) + \varepsilon^{\frac{k+2}{2}} \int_{\{x \in M \mid \rho_g(x, p) \leq \delta_0\}} \rho_{ha_\varepsilon}(x, X) dV_h(x)}{\int_{\{x \in M \mid \rho_g(x, p) > \delta_0\}} \rho_{ha_\varepsilon}(x, A) a_\varepsilon(x)^{\frac{k+2}{2}} dV_h(x) + \varepsilon^{\frac{k+2}{2}} \int_{\{x \in M \mid \rho_g(x, p) \leq \delta_0\}} \rho_{ha_\varepsilon}(x, A) dV_h(x)} = \\ &= \frac{\beta \mathbf{1} + \varepsilon^{\frac{k+2}{2}} \frac{1}{\beta} \int_{\{x \in M \mid \rho_g(p, x) < \delta_0\}} \rho_{ha_\varepsilon}(x, X) dV_h(x)}{\gamma \mathbf{1} + \varepsilon^{\frac{k+2}{2}} \frac{1}{\gamma} \int_{\{x \in M \mid \rho_g(p, x) < \delta_0\}} \rho_{ha_\varepsilon}(x, A) dV_h(x)}. \end{aligned}$$

The Maclaurin series for

$$\frac{1}{1 + \varepsilon^{\frac{k+2}{2}} \frac{1}{\gamma} \int_{\{x \in M \mid \rho_g(p, x) < \delta_0\}} \rho_{ha_\varepsilon}(x, A) dV_h(x)}$$

yields the desired statement.

Proof of (iii).

$$E(M'_g) = \int_{M'} \rho_g(x, A') dV_g(x).$$

Since  $A' = A$ ,  $M' \subset M$ , and  $U \cap M' = \emptyset$ , both inequalities follow.  $\square$

**Proof of Theorem 2.10.**

$$E(M'') = \int_{M''} \rho_g(x, A'') dV_g(x) = \int_M \rho_g(x, A'') dV_g(x) + \int_{M'} \rho_g(x, A'') dV_g(x)$$

Since for every  $x$ ,  $\rho_g(x, A'') \leq \rho_g(x, A)$  and  $\rho_g(x, A'') \leq \rho_g(x, A')$ , the inequality (5) follows.

$$E(F(M'')) = \int_{M''} \rho_g(x, X) dV_g(x) = \int_M \rho_g(x, X) dV_g(x) + \int_{M'} \rho_g(x, X) dV_g(x)$$

The inequality (6) follows.  $\square$

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