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Cost of excursions until first return for random walks and Lévy flights: an exact general formula

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We consider a discrete-time random walk where a cost is incurred at each jump. We obtain an exact analytical formula for the distribution of the total cost of a trajectory until the process returns to the origin for the first time. The formula is valid for arbitrary jump distribution and cost function (heavy- and light-tailed alike), provided they are symmetric and continuous. The tail behavior of the distribution is universal and independent of the details of the model. Applications are given to the motion of an active particle in one dimension and extensions to multiple cost variables are considered. The analytical results are in perfect agreement with numerical simulations.

Introduction - In a variety of disciplines, key events occur when a stochastic process reaches a predefined target state for the first time [1–3]. For instance, in finance, limit orders are employed to execute trades when a specified price level is reached. Similarly, in the study of foraging behavior among animals [4] it is important to estimate the typical duration of an exploration phase, until the animal goes back to its home for the first time.

This type of problems have been thoroughly characterized within the theory of first passage events. Consider the paradigmatic case of a discrete-time random walk

$$x_k = x_{k-1} + \eta_k \,, \tag{1}$$

with $x_0 = 0$ and where the jumps η_k are independently drawn from a symmetric and continuous distribution $f(\eta)$. A central quantity to characterize first-passage events is the probability F_n that the walker crosses the origin for the first time at step n. A cornerstone of first-passage theory is the *Sparre Andersen theorem* [5], which states that F_n is completely universal, i.e., independent of the jump distribution $f(\eta)$, as long as $f(\eta)$ is continuous and symmetric. Crucially, the universal behavior is valid for any finite n and even for Lévy flights, corresponding to fat-tailed $f(\eta) \sim 1/|\eta|^{\mu+1}$, with $0 < \mu < 2$. This universality extends to many quantities, e.g., the number of records [6], and processes, including active particles [7, 8] and resetting systems [9].

However, the classical treatment of the first-return problems usually ignores a very important variable: there are often costs or rewards, e.g., in terms of monetary fees or energy consumption, associated with the change of state of the process. For instance, in the example of animal foraging, there is an *energy cost* associated to a roaming trajectory – as well as a potential *energy gain* if food is actually found along the way. *Markov reward models* [10–12], where a Markov process drives an auxiliary cost/reward dynamics have proven useful in wireless communications [13, 14], biochemistry [15], insurance models [16] and software development [17]. However, a general result that quantifies the cost associated with first-passage events (in the spirit of the *Sparre Andersen theorem*) is, to the best of our knowledge, still missing.

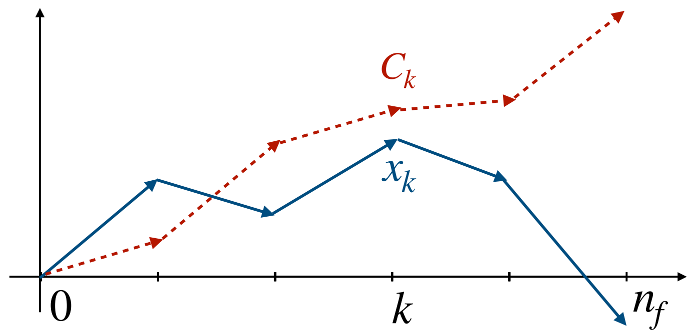

FIG. 1. Typical trajectory of a random walk x_k on a line starting at $x_0 = 0$ and evolving in discrete time via the jump process (1). The process stops at step n_f when the walker crosses the origin for the first time. A cost C_k is associated with the trajectory of the random walk up to step k.

In this Letter, we derive a closed formula for the distribution of the total cost until first return, which is surprisingly general and valid for *any* (continuous and symmetric) jump distribution and cost function per jump. Analytical results of such breadth and generality are exceedingly rare. We provide a direct and natural application to the run-and-tumble particle (RTP) model of an active particle in one dimension. Moreover, our results extend naturally to describe several cost variables, with interesting applications to prey-predators models [18] and excursions in environments with feedback-coupling [19].

Setting and main result - To describe the cost associated to the process, we couple to the random walk in Eq. (1) a cost variable C_k, evolving according to

$$C_k = C_{k-1} + h(\eta_k) \,, \tag{2}$$

with $C_0 = 0$. The cost function $h(\eta) > 0$ is assumed to be continuous and symmetric, i.e., $h(\eta) = h(-\eta)$, but is otherwise arbitrary. The function $h(\eta)$ can be interpreted as the energy spent or the cost incurred by the walker in making a single jump η. Cost dynamics of the type (2) with a non-linear function $h(\eta)$ have been used to model the fare structure of taxi rides, static friction under random applied forces, and depinning transition in spatially inhomogeneous media [20–28]. Additionally, recent works have investigated the cost associated to resetting processes [29–31].

We focus on first-passage processes of the random walk variable x_n. The process stops when the walker, starting from $x_0 = 0$, crosses the origin for the first time at step n_f (see Fig. (1)). Note that n_f itself is a random variable that fluctuates from one trajectory to another. Given $f(\eta)$ and $h(\eta)$, we are interested in computing the distribution $Q(C)$ of the total cost $C = \sum_{k=1}^{n_f} h(\eta_k)$ till the first-passage time to the origin.

Beyond its natural interpretation as a cost function, the variable C can describe a broad class of *first-passage functionals* of discrete-time random walks. Indeed, depending on the choice of $h(\eta)$, C can describe different observables of a first-passage trajectory. For instance: with $h(\eta) = 1$ the variable C simply coincides with the first passage time n_f, with $h(\eta) = |\eta|$, the variable C describes the total distance traveled by the walker, with $h(\eta) = \theta(|\eta| - \eta_c)$, where $\theta(z)$ is the Heaviside step function, C describes the number of steps longer than η_c. In the continuous-time setting, first-passage functionals of stochastic processes have been widely studied in the literature with many applications [32–37] (for a review see Ref. [38]). Note, however, that most of these works have considered functionals depending on the state of the process x_k, instead of the step size η_k, i.e., of the form $C = \sum_{k=1}^{n_f} h(x_k)$. On the other hand, for discrete-time random walks and functionals depending on the step size, there are, to the best of our knowledge, no general results.

In this letter, we derive an explicit exact formula for the Laplace transform of the cost distribution $Q(C)$ that reads

$$\int_0^\infty e^{-p\,C}\, Q(C)\, dC = 1 - \sqrt{1 - 2A(p)}\ , \tag{3}$$

where

$$A(p) = \int_0^\infty e^{-p\,h(\eta)}\, f(\eta)\, d\eta\ . \tag{4}$$

This result is valid for arbitrary jump distribution $f(\eta)$ and arbitrary cost function $h(\eta) > 0$, as long as both functions are symmetric and continuous. Eqs. (3) and (4) constitute our main result. In the rest of the letter, we will discuss several applications and then provide an intuitive derivation of this formula. Several cases where the Laplace transform can be explicitly inverted will be discussed in the Supp. Mat. [39].

Applications - Active particles consume energy directly from the environment and move via self-propelled motions. For example, *E. Coli* bacteria typically move in space by alternating phases of 'run' and 'tumble'. Consider for simplicity the standard RTP model in one dimension. The particle starts at the origin, say with a positive velocity v_0 and runs ballistically with speed v_0 during a random time τ chosen from the exponential distribution $p(\tau) = \gamma\, e^{-\gamma\,\tau}$ where $1/\gamma$ is the persistence time. After this initial run, the particle 'tumbles' instantaneously, i.e., it chooses a new velocity $\pm v_0$ with equal probability and then moves ballistically again with this chosen velocity during another random period drawn from $p(\tau)$. Then it tumbles again and the dynamics continues via the alternating 'run' and 'tumble' phases. A natural question is how much energy does the RTP spend till the first return to its starting position? This model can be clearly mapped to the Markov jump process in Eq. (1) where x_n denotes the position of the RTP after the n-th run – this is also known as a 'persistent random walk' [40]. This mapping has been recently used successfully to derive interesting universal results for the survival probability of a generalized RTP model with arbitrary velocity distribution after each tumbling [7, 8]. Via this mapping, the run length ℓ_n of the n-th run in the standard RTP model (with $\pm v_0$ velocities) is precisely the jump length η_n of the random walk model in Eq. (1). Since, $\ell_n = v_0\,\tau_n$ where τ_n is the exponentially distributed run time of the n-th run, it follows that this RTP model then corresponds precisely to the jump distribution

$$f(\eta) = \frac{\gamma}{2\,v_0}\, e^{-\frac{\gamma}{v_0}\,|\eta|}\,, \tag{5}$$

in the random walk model in Eq. (1). One can then associate a cost function $h(\eta)$ denoting the energy spent during each run. Hence, $Q(C)$ is precisely the distribution of the total energy spent by the particle till its first return to its starting point. For the double-exponential jump distribution in Eq. (5) and the cost function $h(\eta) = |\eta|$ corresponding to energy being consumed at a constant rate, the Laplace transform (3) can be explicitly inverted (see [39] for details) to yield the distribution of the cost of first return to the origin

$$Q(C) = \frac{1}{2}\, e^{-C/2}\left[I_0\left(\frac{C}{2}\right) - I_1\left(\frac{C}{2}\right)\right]\,, \tag{6}$$

where $I_0(z)$ and $I_1(z)$ are modified Bessel functions (see Fig. 2). Note that, since $h(\eta) = |\eta|$, the variable C also describes the total distance traveled by the particle until first passage to the origin.

The distribution has asymptotic behaviors

$$Q(C) \approx \begin{cases} \frac{1}{2} - \frac{3}{8}\,C + \frac{5}{32}\,C^2 & \text{as} \quad C \to 0 \\ \\ \frac{1}{2\sqrt{\pi}}\,\frac{1}{C^{3/2}} & \text{as} \quad C \to \infty\,. \end{cases} \tag{7}$$

The asymptotic power law decay $\sim C^{-3/2}$ for large cost is universal for sufficiently light-tailed jump dis-

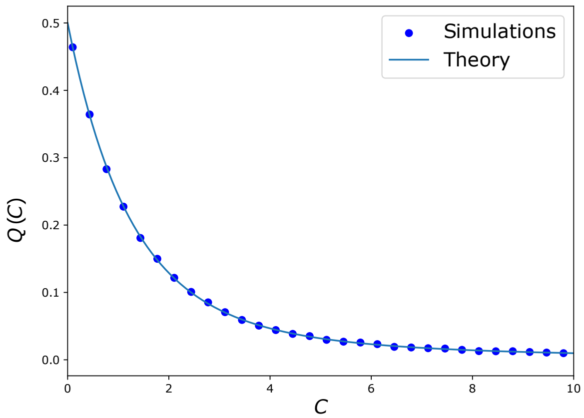

FIG. 2. Distribution $Q(C)$ of the cost C until first return for jump distribution $f(\eta) = e^{-|\eta|}/2$ and cost function $h(\eta) = |\eta|$. The continuous blue line corresponds to the exact result in Eq. (6) while the dots are obtained from numerical simulations with 10^6 samples.

tributions (see below for details), and can be understood by a heuristic scaling argument. The distribution $P(n_f)$ of the first-passage time n_f for random walks with symmetric and continuous jump distributions is universal and given by the Sparre Andersen law [5] (see also Refs. [1, 2, 41, 42]). For large n_f, we have $P(n_f) \approx (1/2\sqrt{\pi})n_f^{-3/2}$. Now, since the cost C in n_f steps scale typically as $C \sim n_f$, it follows that the distribution of C for large C also has exactly the same power law tail. The fact that also the pre-factors match is just a coincidence in this special case. In general, the pre-factor is different and nontrivial and we compute it exactly later.

Sketch of the derivation - We provide here the main ideas behind the derivation of our main result in Eq. (3) (see [39] for more detailed steps). To begin, it is useful to define $Q(x_0, C)$ as the probability density function of the total cost C until first-passage to the origin with initial position x_0. Note that by definition $Q(C) = Q(x_0 = 0, C)$. We first derive an exact integral equation satisfied by $Q(x_0, C)$ for arbitrary choices of $f(\eta)$ and $h(\eta)$. It is natural to use a backward Fokker-Planck like approach where we examine what happens after the first jump and then follow the rest of the trajectory till the first-passage time. We can then directly write down the following backward integral equation using the Markov property of the process

$$Q(x_0, C) = \int_0^\infty dC' \int_0^\infty dx_1\, Q(x_1, C')\, f(x_1 - x_0)\delta\left(C' + h(x_1 - x_0) - C\right) + \int_{-\infty}^0 \delta\left(C - h(x_1 - x_0)\right)\, f(x_1 - x_0)\, dx_1 \,. \tag{8}$$

The two terms on the right-hand side of (8) correspond to the two possibilities: (i) after the first jump, occurring with density $f(x_1 - x_0)$, the new position remains non-negative, i.e, $x_1 = x_0 + \eta_1 \geq 0$ and then the process continues to the next step. The total cost C is the sum of the cost $h(x_1 - x_0)$ of the first jump, and the cost C' of the rest of the trajectory, whose distribution until first passage is $Q(x_1, C')$. (ii) the walker crosses the origin after the first jump – whose cost is $h(x_1 - x_0)$ – to a position $x_1 < 0$, and the process stops. Finally, one needs to integrate over x_1 in both cases, and over $C' \geq 0$ in the first contribution. Defining

$$\tilde{Q}_p(x_0) = \int_0^\infty e^{-p\,C}\, Q(x_0, C)\, dC \,, \tag{9}$$

and taking the Laplace transform with respect to C on both sides of Eq. (8) we obtain

$$\tilde{Q}_p(x_0) = \int_0^\infty dx_1\, \tilde{Q}_p(x_1)\, e^{-p\,h(x_1 - x_0)}\, f(x_1 - x_0) + \int_{-\infty}^{-x_0} e^{-p\,h(\eta_1)}\, f(\eta_1)\, d\eta_1 \,. \tag{10}$$

This equation is valid for arbitrary $f(\eta)$ and $h(\eta)$. This backward equation is of the Wiener-Hopf type that are known to be notoriously difficult to solve. However, it turns that when both $f(\eta)$ and $h(\eta)$ are continuous and symmetric functions of η, we can obtain an explicit and general formula for $\tilde{Q}_p(x_0)$ by an adaptation of the Pollaczek-Spitzer result for the first-passage probability of a continuous and symmetric random walk [43–45] (for a pedagogical introduction see the review [46]).

To solve the integral equation (10), we first notice its similarity with a well-known integral equation for the first-passage probability. Let $F(x_0, n)$ be the probability that the random walk x_k, with $x_0 = 0$, crosses the origin for the first time at step n. By defining the generating function $\tilde{F}(x_0, s) = \sum_{n=1}^\infty F(x_0, n)\, s^n$, one can show that [39]

$$\tilde{F}(x_0, s) = s \int_0^\infty \tilde{F}(x_1, s)\, f(x_1 - x_0)\, dx_1 + s \int_{-\infty}^{-x_0} f(\eta_1)\, d\eta_1 \,. \tag{11}$$

A central result of first-passage theory is the Pollaczek-Spitzer formula for the first-passage probability of a random walk [43, 45]. This formula states that the generating function of the first-passage probability reads [**?**]

$$\int_0^\infty \tilde{F}(x_0, s)\, e^{-\lambda\, x_0}\, dx_0 = \frac{1}{\lambda}\left[1 - \sqrt{1 - s}\, \phi(\lambda, s)\right] \,, \tag{12}$$

where

$$\phi(\lambda, s) = \exp\left[-\frac{\lambda}{\pi}\int_0^\infty \frac{\ln\left(1 - s\,\hat{f}(k)\right)}{\lambda^2 + k^2}\,dk\right] \ . \tag{13}$$

In this formula, the jump distribution enters through its Fourier transform $\hat{f}(k) = \int_{-\infty}^{\infty} f(\eta)\, e^{\mathrm{i}\,k\,\eta}\, d\eta$. This explicit formula has been employed in many different contexts, see, e.g., Refs. [46–52].

Note however that the exact solution in Eq. (12) requires the step function $f(\eta)$ to be continuous, symmetric, and normalized to unity $\int_{-\infty}^{\infty} f(\eta)\, d\eta = 1$. Therefore, the Pollaczek-Spitzer formula cannot be immediately applied to solve the integral equation (10) for $\tilde{Q}_p(x_0)$. We first need to cast the integral equation (10) into the form (11) by defining the auxiliary kernel

$$g_p(\eta) = \frac{e^{-p\,h(\eta)}\, f(\eta)}{2\int_0^\infty e^{-p\,h(\eta)}\, f(\eta)\, d\eta} \ , \tag{14}$$

parametrized by p, which is symmetric, continuous and by construction, is normalized to unity $\int_{-\infty}^{\infty} g_p(\eta)\, d\eta = 1$. This allows us to rewrite (10) in the form

$$\tilde{Q}_p(x_0) = 2A(p)\int_0^\infty dx_1\, \tilde{Q}_p(x_1)\, g_p(x_1 - x_0) + 2A(p)\int_{-\infty}^{-x_0} g_p(\eta_1)\, d\eta_1 \ , \tag{15}$$

with $A(p) = \int_0^\infty e^{-p\,h(\eta)}\, f(\eta)\, d\eta$. Comparing (15) and (11), we can directly use the exact result (12) after the replacements: $\tilde{F}(x_0, s) \to \tilde{Q}_p(x_0)$, $s \to 2A(p)$ and $f(\eta) \to g_p(\eta)$. This then provides the exact solution of our integral equation (10), namely

$$\int_0^\infty \tilde{Q}_p(x_0)\, e^{-\lambda\, x_0}\, dx_0 = \frac{1 - \sqrt{1 - 2A(p)}\,\phi\left(\lambda, 2A(p)\right)}{\lambda} \ , \tag{16}$$

with $A(p)$ and $\phi(\lambda, s)$ defined above. Taking the $\lambda \to \infty$ limit on both sides after the change of variables $\lambda x_0 = y$ in the integral leads via drastic simplifications to the Laplace transform of the special case $\tilde{Q}_p(x_0 = 0)$, whose final expression reads as in (3) - see [39] for details.

Asymptotics - Let us first consider the small p limit of Eq. (3). By expanding in powers of p, one finds that to leading order in small p

$$\tilde{Q}_p(0) = 1 - a\sqrt{p} + \mathcal{O}(p) \ , \quad a = \sqrt{2\int_0^\infty h(\eta)\, f(\eta)\, d\eta} \, . \tag{17}$$

This is quite generic as long as the constant $a = \sqrt{\mu_1}$ above exists, where $\mu_1 = \int_{-\infty}^{\infty} h(\eta)\, f(\eta)\, d\eta$ is just the mean cost per jump. In that case, the small p expansion in (17) indicates, by the Tauberian theorem, that $Q(0, C)$ has a universal power law tail for large C,

$$Q(0, C) \simeq \frac{a}{2\sqrt{\pi}\, C^{3/2}} \quad \text{as } C \to \infty \ , \tag{18}$$

as previously noted for the special case in Eq. (7).

If the mean cost per jump $\mu_1 = \int_{-\infty}^{\infty} h(\eta)\, f(\eta)\, d\eta$ is divergent then the analysis above does not hold. Consider, for instance, the case where $h(\eta) = |\eta|$ (linear cost) and the jump distribution has a power law tail $f(\eta) \sim 1/|\eta|^{\mu+1}$ for large $|\eta|$ with the exponent $0 < \mu \leq 2$ corresponding to Lévy flights. In this case, clearly μ_1 is divergent for $0 < \mu < 1$ and convergent for $\mu > 1$. For $\mu > 1$, one will again find the $C^{-3/2}$ decay of the cost distribution for large C. However, for $0 < \mu < 1$, one finds that

$$A(p) = \int_0^\infty e^{-p\,h(\eta)}\, f(\eta)\, d\eta \sim \frac{1}{2} - b_\mu\, p^\mu \quad \text{as} \quad p \to 0 \ , \tag{19}$$

where b_μ is an unimportant positive constant. Consequently, from Eq. (3), we find that to leading order for small p

$$\tilde{Q}_p(0) \approx 1 - \sqrt{2\, b_\mu}\; p^{\mu/2} \ . \tag{20}$$

This indicates, again via the Tauberian theorem, that for large C, the cost distribution has a power law tail: $Q(0, C) \sim C^{-(1+\mu/2)}$. Thus, summarizing for the Lévy jump distributions with Lévy index $0 < \mu \leq 2$, the cost distibution $Q(0, C) \sim C^{-\theta}$ for large C where the exponent θ is given by

$$\theta = \begin{cases} \frac{\mu}{2} + 1 & \text{for} \quad 0 < \mu < 1 \\ \\ \frac{3}{2} & \text{for} \quad 1 < \mu \leq 2 \ . \end{cases} \tag{21}$$

Thus the exponent θ increases linearly with μ for $\mu \in [0, 1]$ and then freezes at the value $3/2$ for $\mu > 1$, with a logarithmic correction exactly at $\mu = 1$ (for example for the Cauchy jump distribution $f(\eta) = 1/[\pi(\eta^2 + 1)]$). We work out an explicit example of a Lévy flight corresponding to $f(\eta) = \exp(-1/(4|\eta|))/2\sqrt{4\pi|\eta|^3}$ and $h(\eta) = |\eta|$ in [39].

It turns out that unlike the universal behavior of $Q(0, C)$ for large C as discussed above, the small C behavior is rather non-universal and it depends on the choices of $f(\eta)$ and $h(\eta)$. But even these non-universal behaviors can be extracted from our exact result (3).

Another interesting application of our formula in Eq. (4) concerns the cost function $h(\eta) = \theta(|\eta| - \eta_c)$, where $\theta(z)$ is the Heaviside step function. In this case, the variable C describes the number of steps of length larger than η_c until the first passage time. In [39], we derive an exact formula for the distribution of C, valid for arbitrary continuous and symmetric $f(\eta)$.

Extensions - Our main result Eq. (3) generalizes to the case of N cost variables $C^i = \sum_{k=1}^{n_f} h_i(\eta_k)$ following Eq. (2) each with its own $h_i(\eta)$, and correlated through the noise term of the random walk in Eq. (1). The N-fold Laplace transform of the joint distribution

$Q(0, C^1, \ldots, C^N)$ of the cost variables until first return of the master walk to the origin satisfies

$$\left\langle e^{-\sum_{i=1}^{N} p_i C^i} \right\rangle = 1 - \sqrt{1 - 2A(p_1, p_2, \ldots, p_N)} \qquad (22)$$

where

$$A(p_1, p_2, \ldots, p_N) = \int_0^\infty e^{-\sum_{i=1}^{N} p_i h_i(\eta)} f(\eta) d\eta \ . \qquad (23)$$

A thorough analysis of the consequences of the generalized formula (22) is deferred to a separate publication.

Conclusions - We have derived a general formula Eq. (3) for the distribution of the total cost incurred by a one-dimensional random walker until its first return to the origin, provided the jump distribution $f(\eta)$ and the cost function $h(\eta) > 0$ are symmetric and continuous. Several examples can be worked out exactly (see [39]), whose asymptotic behavior for large cost is found to be universal. Ours is one of the few exact and general first-passage results for discrete-time random walks, and can be generalized to the case of N cost variables that are correlated via the noise term of a main process.

It would be interesting to consider extensions of our setting to higher dimensions and different velocity distributions for the RTP model. Nonlinear cost functions are also particularly interesting to investigate in this context [20, 21], as well as modified cost processes subject to an independent noise term as well. Additionally, it would be interesting to extend our framework to resetting processes.

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SUPPLEMENTAL MATERIAL

EXTENDED DERIVATION OF MAIN RESULT

In this section, we derive an exact solution to the integral equation

$$\tilde{Q}_p(x_0) = \int_0^\infty dx_1\, \tilde{Q}_p(x_1)\, e^{-p\, h(x_1-x_0)}\, f(x_1-x_0) + \int_{-\infty}^{-x_0} e^{-p\, h(\eta_1)}\, f(\eta_1)\, d\eta_1\,, \tag{24}$$

where the step distribution $f(\eta)$ and the cost function $h(\eta)$ are continuous and symmetric.

Consider our random walk in Eq. 1 of the main text, starting at $x_0 \geq 0$, with a continuous and symmetric jump distribution $f(\eta)$. Let $S(x_0, n)$ denote the survival probability of the walk up to step n, i.e., the probability that the walker does not cross the origin to the negative side up to n steps, given that it starts at $x_0 \geq 0$. More precisely,

$$S(x_0, n) = \text{Prob.}\left[x_1 \geq 0,\, x_2 \geq 0, \ldots, x_n \geq 0 | x_0\right]\ . \tag{25}$$

It is again easy to write the backward Fokker-Planck like integral equation for $S(x_0, n)$ by examining what happens after the first jump. One simply gets [46]

$$S(x_0, n) = \int_0^\infty dx_1\, S(x_1, n-1)\, f(x_1 - x_0)\ , \tag{26}$$

starting from the initial condition $S(x_0, 0) = 1$ for all $x_0 \geq 0$. The solution of this Wiener-Hopf integral equation for arbitrary kernel $f(\eta)$ is very hard. However, when $f(\eta)$ is continuous, symmetric and normalized to unity $\int_{-\infty}^{\infty} f(\eta)\, d\eta = 1$, there is an explicit formula known as the Pollaczek-Spitzer formula that reads [43, 45]

$$\sum_{n=0}^{\infty} s^n \int_0^\infty dx_0\, S(x_0, n)\, e^{-\lambda\, x_0} = \frac{\phi(\lambda, s)}{\lambda\sqrt{1-s}}\ , \tag{27}$$

where $\phi(\lambda, s)$ is given by

$$\phi(\lambda, s) = \exp\left[-\frac{\lambda}{\pi}\int_0^\infty \frac{\ln\left(1 - s\,\hat{f}(k)\right)}{\lambda^2 + k^2}\, dk\right]\ . \tag{28}$$

In this formula, the jump distribution enters through its Fourier transform

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(\eta)\, e^{\mathrm{i}\, k\, \eta}\, d\eta\ . \tag{29}$$

This formula, though explicit, is not very user-friendly and to extract the asymptotic behavior of $S(x_0, n)$ for large or small arguments is far from trivial. It has been analyzed in many different contexts, see e.g. Refs. [46–52].

Now, let $F(x_0, n)$ denote the first-passage probability, i.e., the probability that starting at $x_0 \geq 0$, the walker crosses the origin for the first time at step n. Then $F(x_0, n)$, for $n \geq 1$, satisfies the following backward equation

$$F(x_0, n) = \int_0^\infty F(x_1, n-1)\, f(x_1 - x_0)\, dx_1 + \delta_{n,1}\int_{-\infty}^0 f(x_1 - x_0)\, dx_1\ , \tag{30}$$

starting from $F(x_0, 0) = 0$. This equation can be understood again by investigating what happens after the first jump. There are two possibilities: (i) either the walker jumps to a positive position $x_1 \geq 0$ and the walk continues, explaining the first term on the r.h.s of Eq. (30) or (ii) the walker jumps to the negative side in the first step which gives rise to the second term in Eq. (30). Let us now define the generating function

$$\tilde{F}(x_0, s) = \sum_{n=1}^{\infty} F(x_0, n)\, s^n\ . \tag{31}$$

Taking the generating function with respect to n on both sides of Eq. (30) then gives

$$\tilde{F}(x_0, s) = s \int_0^{\infty} \tilde{F}(x_1, s)\, f(x_1 - x_0)\, dx_1 + s \int_{-\infty}^{-x_0} f(\eta_1)\, d\eta_1 \ , \tag{32}$$

where we made a change of variable $x_1 = x_0 + \eta_1$ in the second term in Eq. (30).

Can we find the solution of this integral equation (32) for arbitrary s and arbitrary $f(\eta)$ that is continuous, symmetric and normalized to unity? The answer is yes as we now show. The solution for $\tilde{F}(x_0, s)$ can be derived from the Pollaczek-Spitzer formula by noting that the first-passage probability $F(x_0, n)$ and the survival probability $S(x_0, n)$ satisfy the simple relation

$$F(x_0, n) = S(x_0, n-1) - S(x_0, n) \ . \tag{33}$$

This simply follows from the fact that the trajectories that survive up to step $n-1$ may either survive till step n or jump to the negative side at step n. Now, taking the generating function with respect to n on both sides of (33) and using $S(x_0, 0) = 1$ for all $x_0 \geq 0$ gives

$$\tilde{F}(x_0, s) = 1 - (1-s)\, \tilde{S}(x_0, s) \quad \text{where} \quad \tilde{S}(x_0, s) = \sum_{n=0}^{\infty} S(x_0, n)\, s^n \ . \tag{34}$$

Now, taking the Laplace transform with respect to x_0 and using the Pollaczek-Spitzer formula (27) gives the exact result

$$\int_0^{\infty} \tilde{F}(x_0, s)\, e^{-\lambda\, x_0}\, dx_0 = \frac{1}{\lambda} \left[1 - \sqrt{1-s}\, \phi(\lambda, s) \right] \ , \tag{35}$$

where $\phi(\lambda, s)$ is defined in Eq. (28).

Hence, the point of this derivation is that, thanks to the Pollaczek-Spitzer formula for the survival probability and the exact relation (33), we managed to find the Laplace transform of the exact solution of the integral equation (32) for arbitrary positive $s < 1$ and arbitrary normalized jump distribution $f(\eta)$ (symmetric and continuous) and it is given by Eq. (35). Armed with this information, we can now go back to our actual integral equation (24) and seek its exact solution.

The two integral equations, namely (24) and (32), look indeed superficially similar to each other. But there is an important caveat. One of the important conditions for (35) to be an exact solution of (32) is that the kernel $f(x_1 - x_0)$ that appears in Eq. (32) is not only continuous and symmetric, but also is normalized to unity, i.e., $\int_{-\infty}^{\infty} f(\eta)\, d\eta = 1$. However, the kernel $e^{-p\, h(x_1 - x_0)}\, f(x_1 - x_0)$ that appears in Eq. (24) is not normalized to unity. So, we can not directly lift the solution (35). To make the kernel normalized to unity we divide and multiply by the normalizing factor, i.e., we use

$$e^{-p\, h(x_1 - x_0)} f(x_1 - x_0) = 2A(p)\, \frac{e^{-p\, h(x_1 - x_0)}\, f(x_1 - x_0)}{2A(p)} \quad \text{where} \quad A(p) = \int_0^{\infty} e^{-p\, h(\eta)}\, f(\eta)\, d\eta \ . \tag{36}$$

Consequently the new kernel

$$g_p(\eta) = \frac{e^{-p\, h(\eta)}\, f(\eta)}{2 \int_0^{\infty} e^{-p\, h(\eta)}\, f(\eta)\, d\eta} \ , \tag{37}$$

parametrized by p, is symmetric, continuous and by construction, is normalized to unity

$$\int_{-\infty}^{\infty} g_p(\eta)\, d\eta = 1 \ . \tag{38}$$

Substituting (36) in Eq. (24) we rewrite it as

$$\tilde{Q}_p(x_0) = 2A(p) \int_0^{\infty} dx_1\, \tilde{Q}_p(x_1)\, g_p(x_1 - x_0) + 2A(p) \int_{-\infty}^{-x_0} g_p(\eta_1)\, d\eta_1 \ , \tag{39}$$

with $A(p)$ defined in Eq. (36).

We are now ready to compare the two integral equations (32) and (39). We see that these two equations are identical if we replace $\tilde{F}(x_0, s)$ in (32) by $\tilde{Q}_p(x_0)$, the constant parameter s in (32) by $2A(p)$ and the jump distribution $f(\eta)$ (symmetric, continuous and normalized to unity) in Eq. (32) by the effective jump distribution $g_p(\eta)$ defined in Eq. (37) which also happens to be symmetric, continuous and normalized to unity. Now, we know the exact solution of Eq. (32) is given by (35), and is valid for arbitrary positive parameter s and arbitrary (symmetric, continuous and normalized to unity) jump distribution $f(\eta)$. Hence, we can use the exact result (35) after the replacements: $\tilde{F}(x_0, s) \to \tilde{Q}_p(x_0)$, $s \to 2A(p)$ and $f(\eta) \to g_p(\eta)$. This then provides the exact solution of our integral equation (24), namely

$$\int_0^\infty \tilde{Q}_p(x_0)\, e^{-\lambda\, x_0}\, dx_0 = \frac{1}{\lambda}\left[1 - \sqrt{1 - 2A(p)}\, \phi\left(\lambda, 2A(p)\right)\right] \,, \tag{40}$$

with $A(p)$ given in (36) and $\phi(\lambda, s)$ defined in (28).

In principle, by inverting this Laplace transform, one can obtain the full distribution $\tilde{Q}_p(x_0)$ for arbitrary x_0. For specific choices of $f(\eta)$ and $h(\eta)$, this can be done. For instance, it can be checked that for the choice $f(\eta) = (1/2)e^{-|\eta|}$ and $h(\eta) = |\eta|$, we do recover, from the general solution (40), the explicit result found in Eq. (49). However, for other choices of $f(\eta)$ and $h(\eta)$, this Laplace inversion of (40) looks rather difficult.

Fortuitously, however, a simplification occurs if we choose the starting position $x_0 = 0$, i.e., for the first return problem. To extract this limiting result from the general expression (40) we proceed as follows. We first make the change of variable $\lambda x_0 = y$ on the left hand side (l.h.s) of Eq. (40) and then take the large λ limit. Assuming $\tilde{Q}_p(0)$ exists, the l.h.s. behaves for large λ as

$$\int_0^\infty \tilde{Q}_p(x_0)\, e^{-\lambda\, x_0}\, dx_0 \to \frac{1}{\lambda}\, Q_p(0) \,. \tag{41}$$

Now we take the $\lambda \to \infty$ limit on the r.h.s. of (40). Now, from the definition of $\phi(\lambda, s)$ in (28), it is easy to see that $\phi(\lambda, s) \to 1$ as $\lambda \to \infty$. Comparing the l.h.s. in (41) we then obtain the simplified result mentioned in Eq. (3) in the main text, namely,

$$\tilde{Q}_p(0) = \int_0^\infty e^{-p\, C}\, Q(0, C)\, dC = 1 - \sqrt{1 - 2\, A(p)} = 1 - \sqrt{1 - 2\int_0^\infty e^{-p\, h(\eta)}\, f(\eta)\, d\eta} \,. \tag{42}$$

This completes the extended derivation of our main result.

EXPLICIT SOLUTION OF THE BACKWARD INTEGRAL EQUATION FOR A SPECIAL CASE

In this section, we show how to solve the integral equation (24) of the main text for the special choice

$$f(\eta) = \frac{1}{2}\, e^{-|\eta|}\,; \quad \text{and} \quad h(\eta) = |\eta| \,. \tag{43}$$

Substituting the choice (43) in the integral equation (24) one gets

$$\tilde{Q}_p(x_0) = \frac{1}{2}\int_0^\infty dx_1\, \tilde{Q}_p(x_1)\, e^{-(p+1)\,|x_1 - x_0|} + \frac{1}{2}\int_{x_0}^\infty e^{-(p+1)\,\eta}\, d\eta \,, \tag{44}$$

where we used the symmetry to change the limits of the integral in the second term of the r.h.s in Eq. (24). This integral equation can be reduced to a differential equation following the trick used in numerous contexts before, see e.g. Ref. [47, 48] (including an application to black hole physics as in the recent paper [53]). This trick uses the following identity

$$\frac{d^2}{dx^2}\left[e^{-a\,|x-b|}\right] = -2\, a\, \delta(x - b) + a^2\, e^{-a\,|x-b|} \,. \tag{45}$$

Differentiating Eq. (44) twice with respect to x_0 and using the identity (45) simply gives, for $x_0 \geq 0$,

$$\frac{d^2 \tilde{Q}_p(x_0)}{dx_0^2} = p(p+1)\, \tilde{Q}_p(x_0) \,. \tag{46}$$

The general solution, for $x_0 \geq 0$, is trivially

$$\tilde{Q}_p(x_0) = A\, e^{-\sqrt{p(p+1)}\, x_0} + B\, e^{\sqrt{p(p+1)}\, x_0} \ , \tag{47}$$

where A and B are arbitrary constants. First we notice that when $x_0 \to \infty$, the solution $\tilde{Q}_p(x_0)$ can not diverge exponentially. This immediately fixes $B = 0$. However, finding the constant A is more tricky since we do not have any available boundary condition on $\tilde{Q}_p(x_0)$. The important point is that in arriving at the differential equation from the integral equation, we took two derivatives and thus we lost some information, including constant and linear terms in x_0. Hence, we need to ensure that the solution of the differential equation $\tilde{Q}_p(x_0) = A\, e^{-\sqrt{p(p+1)}\, x_0}$ actually satisfies also the integral equation (46). Indeed, substituting back this solution into the integral equation (46) we find that this is indeed the solution of the integral equation as well, provided

$$A = 1 - \sqrt{\frac{p}{p+1}} \ . \tag{48}$$

This then fixes the solution uniquely as

$$\tilde{Q}_p(x_0) = \left[1 - \sqrt{\frac{p}{p+1}}\right] e^{-\sqrt{p(p+1)}\, x_0} \ . \tag{49}$$

Setting, in particular, $x_0 = 0$ for simplicity, we get

$$\tilde{Q}_p(0) = \int_0^\infty Q(0, C)\, e^{-p\, C}\, dC = 1 - \sqrt{\frac{p}{p+1}} \ . \tag{50}$$

It turns out that a direct inversion of this Laplace transform is slightly difficult. To circumvent this problem, we first differentiate both sides with respect to p to obtain

$$\int_0^\infty e^{-p\, C}\, C\, Q(0, C)\, dC = \frac{1}{2\sqrt{p}(1+p)^{3/2}} \ . \tag{51}$$

Now this Laplace transform can be inverted using Mathematica and we explicitly obtain the distribution of the cost of first return to the origin (see Eq. 6 of the main text)

$$Q(0, C) = \frac{1}{2}\, e^{-C/2} \left[I_0\left(\frac{C}{2}\right) - I_1\left(\frac{C}{2}\right)\right] \ , \tag{52}$$

where $I_0(z)$ and $I_1(z)$ are modified Bessel functions. The distribution has asymptotic behaviors

$$Q(0, C) \approx \begin{cases} \frac{1}{2} - \frac{3}{8}\, C + \frac{5}{32}\, C^2 & \text{as} \quad C \to 0 \\ \\ \frac{1}{2\sqrt{\pi}}\, \frac{1}{C^{3/2}} & \text{as} \quad C \to \infty \ . \end{cases} \tag{53}$$

The result for $Q(0, C)$ in Eq. (6) of the main text can of course be recovered from the main result in Eq. (3) of the main text. For the special choice: $f(\eta) = (1/2)e^{-|\eta|}$ and $h(\eta) = |\eta|$, we get

$$A(p) = \int_0^\infty e^{-p\, h(\eta)}\, f(\eta)\, d\eta = \frac{1}{2(p+1)} \ . \tag{54}$$

Consequently, the general result in Eq. (3) of the main text gives

$$\tilde{Q}_p(0) = 1 - \sqrt{\frac{p}{p+1}} \ , \tag{55}$$

which precisely coincides with the result in Eq. (50) obtained previously by a different method.

EXPLICIT $Q(C)$ FOR THREE OTHER EXAMPLES.

Example 1 - Gaussian steps

Let us consider another choice for the jump distribution $f(\eta)$ and the cost function $h(\eta)$ different from the example in Eq. (43). We consider the case when

$$f(\eta) = \frac{1}{\sqrt{\pi}}\, e^{-\eta^2} \,, \quad \text{and} \quad h(\eta) = \eta^2 \;. \tag{56}$$

This corresponds to a random walk with Gaussian jump distribution and a quadratic cost function, which is also a rather natural choice. In this case, our exact formula (42) gives

$$\tilde{Q}_p(0) = \int_0^\infty Q(0,C)\, e^{-p\,C}\, dC = 1 - \sqrt{1 - \frac{1}{\sqrt{1+p}}} \;. \tag{57}$$

One needs some tricks to invert this Laplace transform. Let us first define

$$Q(0,C) = e^{-C}\, G(C) \;. \tag{58}$$

Substituting (58) in (57) and denoting $s = p + 1$, we get

$$\int_0^\infty G(C)\, e^{-s\,C}\, dC = 1 - \sqrt{1 - \frac{1}{\sqrt{s}}} \;. \tag{59}$$

We now expand the r.h.s. of (59) in a power series in $1/\sqrt{s}$ and invert each term by using the identity

$$\mathcal{L}^{-1}_{s\to C}\left[s^{-n/2}\right] = \frac{C^{n/2-1}}{\Gamma(n/2)} \;. \tag{60}$$

This leads to a power series for $G(C)$

$$G(C) = \frac{2}{C}\sum_{n=1}^{\infty} \frac{\Gamma(2n-1)}{\Gamma(n)\,\Gamma(n+1)\,\Gamma(n/2)\,4^n}\, C^{n/2} \;. \tag{61}$$

Amazingly, Mathematica was able to resum this series and express it in terms of hypergeometric functions. Finally, using the relation (58), we get a nice and explicit expression for $Q(0,C)$

$$Q(0,C) = \frac{e^{-C}}{2\sqrt{\pi\, C}}\, {}_2F_2\left[1/4,\, 3/4;\, 1/2,\, 3/2; C\right] + \frac{e^{-C}}{8}\, {}_2F_2\left[3/4,\, 5/4;\, 3/2,\, 2; C\right] \;. \tag{62}$$

One can easily plot this function (see Fig. 3) and it has the asymptotic behaviors

$$Q(0,C) \approx \begin{cases} \frac{1}{2\sqrt{\pi\, C}} & \text{as} \quad C \to 0 \\ \\ \frac{1}{2\,\sqrt{2}\,\pi\, C^{3/2}} & \text{as} \quad C \to \infty \;. \end{cases} \tag{63}$$

The large C decay $C^{-3/2}$ is in accordance with the general result in Eq. 18 of the main text, while it also diverges as $C \to 0$ (though still integrable). Comparing to the result for the RTP case in Eq. (7) of the main text, we indeed see that the small C behavior is rather nonuniversal, while the large C result is a universal $C^{-3/2}$ law, as long as the mean cost per jump μ_1 is finite.

Example 2 - Lévy flight

Let us consider another example where the Laplace transform in Eq. (3) of the main text, namely,

$$\tilde{Q}_p(0) = \int_0^\infty e^{-p\,C}\, Q(0,C)\, dC = 1 - \sqrt{1 - 2\int_0^\infty e^{-p h(\eta)}\, f(\eta)\, d\eta} \tag{64}$$

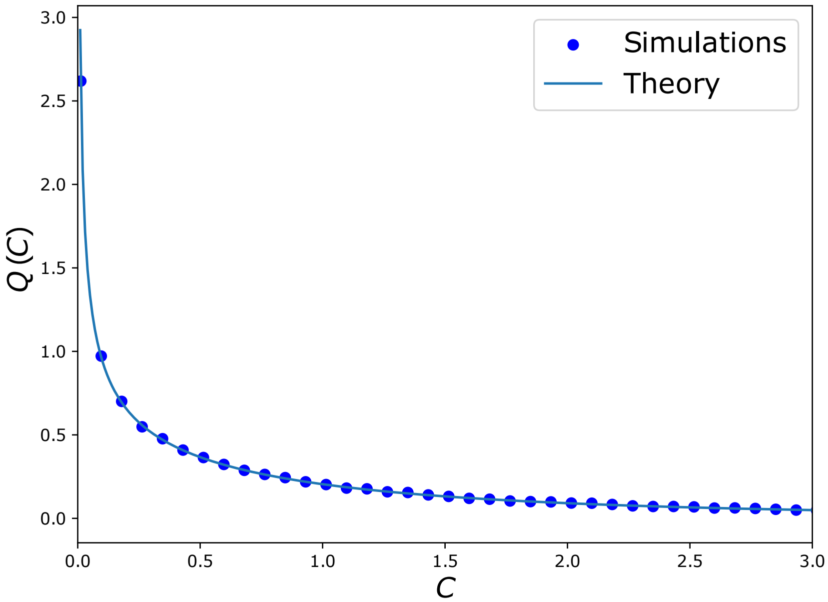

FIG. 3. Distribution $Q(C)$ of the cost C until first return for jump distribution $f(\eta) = e^{-\eta^2}/\sqrt{\pi}$ and cost function $h(\eta) = \eta^2$. The continuous blue line corresponds to the exact result in Eq. (62) while the dots are obtained from numerical simulations with 10^6 samples.

can be explicitly inverted. This example corresponds to the choice

$$f(\eta) = \frac{1}{2\sqrt{4\,\pi\,|\eta|^3}}\, e^{-1/(4|\eta|)}\,; \qquad \text{and} \quad h(\eta) = |\eta|\ . \tag{65}$$

This is thus an example of a Lévy flight where the jump distribution $f(\eta) \sim 1/|\eta|^{\mu+1}$ for large $|\eta|$, with a Lévy exponent $\mu = 1/2$. We now use the following identity

$$\int_0^\infty \frac{a}{4\pi t^3}\, e^{-a^2/4t}\, e^{-p\,t} dt = e^{-a\,\sqrt{p}}\ , \tag{66}$$

valid for all $a > 0$. Setting $t = \eta$ and $a = 1$ in this identity, we then get for the choice in Eq. (65)

$$A(p) = 2\int_0^\infty e^{-ph(\eta)}\, f(\eta)\, d\eta = e^{-\sqrt{p}}\,. \tag{67}$$

Hence, Eq. (64) gives

$$\tilde{Q}_p(0) = \int_0^\infty e^{-p\,C}\, Q(0,C)\, dC = 1 - \sqrt{1 - e^{-\sqrt{p}}}\ . \tag{68}$$

We now use the following power series expansion

$$(1-x)^{1/2} = 1 - \sum_{m=1}^{\infty} \frac{(2m-2)!}{m!\,(m-1)!}\, \frac{x^m}{2^{2m-1}}\ , \tag{69}$$

to write Eq. (68) as

$$\tilde{Q}_p(0) = \int_0^\infty e^{-p\,C}\, Q(0,C)\, dC = \tilde{Q}_p(0) = \sum_{m=1}^{\infty} \frac{(2m-2)!}{m!\,(m-1)!\,2^{2m-1}}\, e^{-m\,\sqrt{p}}\ . \tag{70}$$

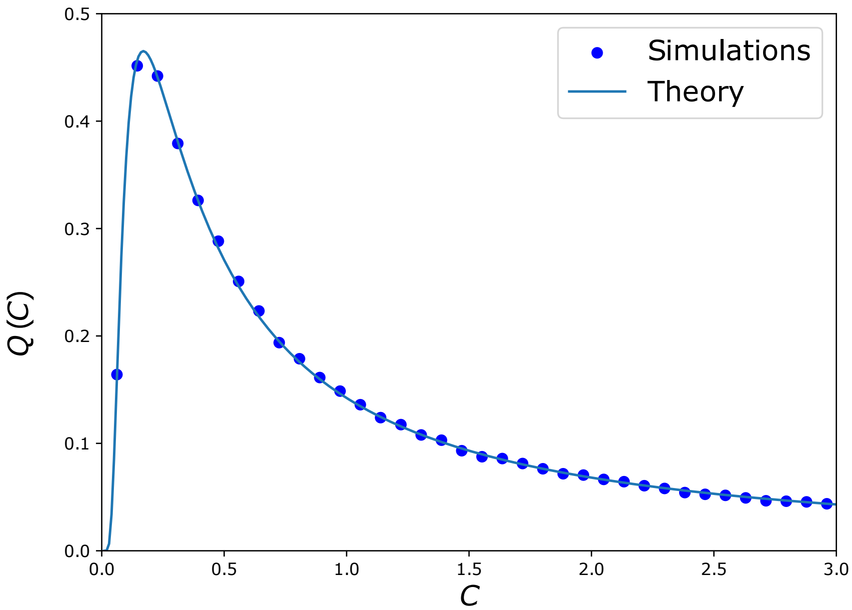

FIG. 4. Distribution $Q(C)$ of the cost C until first return for jump distribution $f(\eta) = \frac{1}{2\sqrt{4\pi|\eta|^3}} e^{-1/(4|\eta|)}$ and cost function $h(\eta) = |\eta|$. The continuous blue line corresponds to the exact result in Eq. (71) while the dots are obtained from numerical simulations with 10^7 samples.

We can now invert this Laplace transform with respect to p term by term using the same identity as in Eq. (66) by setting $a = m$ for the inversion of the m-th term. This then gives an explicit result

$$Q(0,C) = \frac{1}{\sqrt{4\pi C^3}} \sum_{m=1}^{\infty} \frac{(2m-2)!}{(m-1)!\,(m-1)!\,2^{2m-1}}\, e^{-m^2/4C} \,. \tag{71}$$

Unfortunately Mathematica could not resum this series, but it can easily evaluate it numerically and plot it as a function of C, as shown in Fig. 4. The asymptotic behavior of $Q(0,C)$ for small and large C can be easily derived from Eq. (71). For example, for small C, the $m=1$ term dominates giving, $Q(0,C) \simeq e^{-1/(4C)}/\sqrt{16\pi C^3}$. On the other hand, for large C the dominant contribution to the sum in Eq. (71) comes from large m, where we can use Stirling's formula to approximate the factorials and then the sum can be converted to an integral which can be explicitly evaluated. Summarizing, we get the following asymptotic behaviors

$$Q(0,C) \approx \begin{cases} \frac{1}{2\sqrt{4\pi C^3}}\, e^{-1/(4C)} & \text{as} \quad C \to 0 \\ \\ \frac{1}{4\Gamma(3/4)}\, \frac{1}{C^{5/4}} & \text{as} \quad C \to \infty \,. \end{cases} \tag{72}$$

The large C power law decay $C^{-5/4}$ is consistent with the general result $\mu/2+1$ for $\mu = 1/2 < 1$, as stated in the main text.

Example 3 - Number of steps longer than a threshold

Let us consider the cost function $h(\eta) = \theta(|\eta| - \eta_c)$, where $\theta(z)$ is the Heaviside step function and $\eta_c \geq 0$ is fixed. Using this cost function, the variable C is the number of steps of length greater than η_c until the first passage event. Note that when $\eta_c = 0$, we have $q = 0$ and C reduces to the first passage time. We find

$$A(p) = \frac{1}{2}\left[q + e^{-p}(1-q)\right] \,, \tag{73}$$

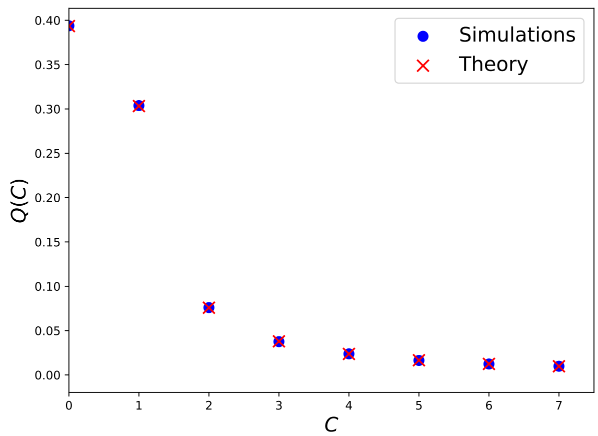

FIG. 5. Probability distribution of the number C of steps of length $|\eta|$ larger than $\eta_c = 1$. The simulations are obtained from 10^6 samples with step function $f(\eta) = e^{-|\eta|}/2$.

where

$$q = \int_{-\eta_c}^{\eta_c} f(\eta) d\eta \,, \tag{74}$$

is the probability that a single step is shorter than η_c. Plugging this expression for $A(p)$ and inverting the Laplace transform, we get

$$Q(0) = 1 - \sqrt{1-q} \,, \tag{75}$$

and

$$Q(C) = \sqrt{1-q} \, \frac{(2(C-1))!}{C!(C-1)!} 2^{-2C+1} \tag{76}$$

for $C \geq 1$. This result in Eqs. (75) and (76) is in perfect agreement with numerical simulations (see Fig. 5).

Interestingly, for $C \geq 1$, the probability of C can be written as

$$Q(C) = \sqrt{1-q} \; F_C \,, \tag{77}$$

where

$$F_C = \frac{(2(C-1))!}{C!(C-1)!} 2^{-2C+1} \tag{78}$$

is the first passage probability to the origin after C steps. From Eq. (75), we immediately find that $\sqrt{1-q}$ is the probability that there is at least a step of length greater than η_c up to the first passage time. As a consequence, we find that the probability of C conditioned on the event $C \geq 1$ is exactly the same as the probability distribution of the first passage time

$$Q(C|C \geq 1) = F_C \,. \tag{79}$$

Remarkably, this last rest result is completely universal, as F_C is independent of the jump distribution $f(\eta)$.