A Coupled Optimization Framework for Correlated Equilibria in Normal-Form Games

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***Abstract*— In competitive multi-player interactions, simultaneous optimality is a key requirement for establishing strategic equilibria. This property is explicit when the game-theoretic equilibrium is the simultaneously optimal solution of coupled optimization problems. However, no such optimization problems exist for the correlated equilibrium, a strategic equilibrium where the players can correlate their actions. We address the lack of a coupled optimization framework for the correlated equilibrium by introducing an *unnormalized game*—an extension of normal-form games in which the player strategies are lifted to unnormalized measures over the joint actions. We show that the set of fully mixed generalized Nash equilibria of this unnormalized game is a subset of the correlated equilibrium of the normal-form game. Furthermore, we introduce an entropy regularization to the unnormalized game and prove that the entropy-regularized generalized Nash equilibrium is a sub-optimal correlated equilibrium of the normal form game where the degree of sub-optimality depends on the magnitude of regularization. We prove that the entropy-regularized unnormalized game has a closed-form solution, and empirically verify its computational efficacy at approximating the correlated equilibrium of normal-form games.**

I. Introduction

As autonomous and artificial intelligence-assisted technology become ubiquitous in our daily lives, game theory has emerged as an important tool for modeling and analyzing the interactions between autonomous agents. Within a game, player interactions are at an equilibrium when their strategies are *simultaneously optimal*: no player can achieve a better objective by unilaterally deviating from its current strategy. For equilibria concepts such as the Nash equilibrium and the Stackelberg equilibrium, simultaneous optimality is an explicit property: these equilibria solve coupled problems within an optimization framework. The existence of such a framework has also enabled the development of gradient-based algorithms for computing game-theoretic equilibria, in particular in autonomy and artificial intelligence [1]–[3].

The correlated equilibrium is an extension of the Nash equilibrium to the joint action space. By utilizing a *correlation device* that enables players to coordinate their actions, a correlated equilibrium is more effective than the Nash equilibrium at optimizing the social welfare, especially in competitive games with three or more players [4]. In particular, such games arise naturally in urban mobility [5], [6], robotics [7], and power markets [8]. Since correlated equilibria form a connected polytope [9], fairness and other system-level metrics can be optimized to global optimality.

Despite its advantages, the correlated equilibrium becomes exponentially more expensive to compute as the number of players and actions increase, and the lack of an optimization framework has made it difficult to apply scalable gradient-based algorithms for finding correlated equilibria. Presently, we pose and answer the following question: *can we construct a coupled optimization problem whose optimal solution is the correlated equilibrium of a normal-form game?*

Contributions. Our contribution is threefold-fold.

1) We introduce unnormalized games: an extension of normal-form games in which the player strategies are unnormalized measures over the joint action space. We prove that a strictly positive generalized Nash equilibrium of the unnormalized game is a correlated equilibrium of the normal-form game.
2) We formulate an entropy-regularized unnormalized game, and prove that its generalized Nash equilibria are sub-optimal correlated equilibria of the normal-form game. Furthermore, we compute the generalized Nash equilibrium in closed-form, and show that its degree of sub-optimality as a correlated equilibrium depends on the entropy regularization.
3) We empirically verify that the generalized Nash equilibria of entropy-regularized unnormalized games are sub-optimal correlated equilibria of normal-form games. Furthermore, we empirically derive the relationship between the degree of sub-optimality vs the magnitude of the entropy regularization of these generalized Nash equilibria.

Relevant research. First introduced in [4], the correlated equilibrium exists in both finite and infinite games, including games that possess no Nash equilibria [10]. A correlated equilibrium definition requires both a correlation device and the resulting probability distribution over the joint action space [11]. The correlated equilibrium has multiple definitions and formulations under different assumptions [2], [12]. Extensions of correlated equilibrium include constrained correlated equilibrium [13], quantal correlated equilibrium [14], extensive-form correlated equilibrium [15], and coarse correlated equilibrium [16]. A correlated equilibrium's stability properties are analyzed in [17], [18]. Learning dynamics that converge to the correlated equilibrium include uncoupled no-regret learning dynamics [19] and evolution dynamics [20]. Gradient-based learning dynamics are not well explored.

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II. EQUILIBRIA CONCEPTS IN NORMAL FORM GAMES

We consider a normal-form game with N players. Let $[A_i](A_i \in \mathbb{N})$ denote the set of actions available to player i, and let $[A] = [A_1] \times \ldots \times [A_N](A = \prod_{i\in[N]} A_i)$ denote the set of all joint actions available in the game. We denote player i's action as $a_i \in [A_i]$, the action taken by player i's opponents as a_{-i}, and every player's joint action as $a := (a_1, \ldots, a_N) \in [A]$. Under a joint action $a \in [A]$, player i incurs a cost $\ell_i(a)$, where $\ell_i : [A] \mapsto \mathbb{R}$ for all $i \in [N]$.

We denote the A_i-dimensional probability simplex over $[A_i]$ as Δ_i, the joint probability simplex as $\Delta = \Delta_1 \times \ldots \times \Delta_N$, and the A-dimensional probability simplex over $[A]$ as Δ_A. Player i's **strategy** $x_i \in \Delta_i$ is a probability distribution over the action set $[A_i]$. Under the strategy x_i, player i selects an action a_i with the probability $x_i(a_i)$ for all $a_i \in [A_i]$. The **joint strategy** $x := (x_1, \ldots, x_N) \in \prod_{i\in[N]} \Delta_i$ is the collection of all of the players' strategies. Let the opponent strategy, action space, and strategy space be respectively given by

$$x_{-i} = \prod_{j\neq i} x_i,\ [A_{-i}] = \prod_{j\neq i} [A_j],\ \Delta_{-i} = \prod_{j\neq i} \Delta_j,\ \forall i \in [N].$$

Under the joint strategy x, the expected cost for player i is given by

$$\mathbb{E}_{a\sim x}[\ell_i(a)] = \sum_{a_i\in[A_i]} x_i(a_i) \sum_{a_{-i}\in[A_{-i}]} x_{-i}(a_{-i})\ell_i(a_i, a_{-i}). \tag{1}$$

We use $\hat{\ell}_i : [A_i] \times \Delta_{-i} \mapsto \mathbb{R}$ to denote player i's expected cost for playing action a_i conditioned on the other players playing the strategy x_{-i}:

$$\hat{\ell}_i(a_i; x_{-i}) = \mathbb{E}[\ell_i(a_i, a_{-i})\ |a_j \sim x_j,\ \forall j \neq i], \forall i \in [N]. \tag{2}$$

Using the notation $\hat{\ell}_i(a_i; x_{-i})$, player i's expected cost (1) when choosing strategy x_i is given by $\mathbb{E}_{a\sim x}[\ell_i(a)] = \sum_{a_i\in[A_i]} x_i(a_i)\hat{\ell}_i(a_i; x_{-i})$ when the other players choose strategies x_{-i}.

Each player minimizes its expected cost $\mathbb{E}_{a\sim x}[\ell_i(a)]$ through unilateral changes in its own strategy $x_i \in \Delta_i$. At the joint strategy $x = (x_1, \ldots, x_N)$ and for each $i \in [N]$, if x_i minimizes $\sum_{a_i\in[A_i]} x_i(a_i)\hat{\ell}_i(a_i; x_{-i})$ simultaneously, x is a Nash equilibrium.

Definition 1 (**Nash equilibrium**)**.** *The joint strategy $x^\star = (x_1^\star, \ldots, x_N^\star) \in \Delta$ is a Nash equilibrium if for each $i \in [N]$, $x_i^\star$ satisfies*

$$\sum_{a_i\in[A_i]} x_i^\star(a_i)\hat{\ell}_i(a_i; x_{-i}^\star) \leq \sum_{a_i\in[A_i]} x_i(a_i)\hat{\ell}_i(a_i; x_{-i}^\star),\ \forall\, x_i \in \Delta_i. \tag{3}$$

The set of Nash equilibria is equivalent to the set of KKT points of the following coupled linear program for all $i \in [N]$. In general, the set is disconnected [9].

$$\begin{aligned} \min_{x_i\in\Delta_i} & \sum_{a_i\in[A_i]} x_i(a_i)\hat{\ell}_i(a_i; x_{-i}), \\ \text{s.t.} & \sum_{a_i\in[A_i]} x_i(a_i) = 1, x_i(a_i) \geq 0,\ \forall a_i \in [A_i], \end{aligned} \tag{4}$$

The concept of Nash equilibrium extends the notion of single-player optimality to *simultaneous optimality* under unilateral deviations in the players' strategies [21]. A Nash equilibrium strategy $(x_1, \ldots, x_N)$ ensures that, within player i's own strategy space Δ_i, the strategy x_i minimizes player i's expected cost when the other players play strategy x_{-i}.

Independent decision-making induces inequity. The concept of Nash equilibrium implicitly assumes that the players make decisions independently—i.e., x_i, x_j are independent probability distributions for all $i, j \in [N]$, $j \neq i$. While this assumption holds for game-theoretic models such as the Prisoner's Dilemma [22], it fails to take advantage of the additional coordination structure that exists in large-scale cyber-physical systems. Furthermore, independent decision-making often induces inequity among players.

Example 1 (Vehicle standoff)**.** *Consider a single-lane road with bi-directional traffic and an unexpected pothole on its right side. Vehicles can choose to veer left or right to pass each other. Two pure Nash equilibria are (left, right) and (right, left), but the traffic direction that chooses the pothole side will constantly be at a disadvantage. A mixed Nash equilibrium can ensure that both traffic directions are equally likely to encounter unexpected potholes, but it also means that with positive probability, both directions' vehicles will choose the same roadside and stall traffic.*

In Example 1, a more robust solution is to *coordinate* both traffic directions to alternate between the two Nash equilibria (left, right) and (right, left), without choosing the joint action pairs (right, right) or (left, left). By doing so, the vehicles are choosing to *correlate* their strategies.

Definition 2 (Correlated strategy)**.** *The A-dimensional probability distribution $y \in \Delta_A$ is a correlated strategy if $y(a) \geq 0$ denotes the probability of the joint action $a = (a_1, \ldots, a_N)$ occurring, for all $a \in [A]$ and $\sum_{a\in[A]} y(a) = 1$ [9], [23].*

To employ correlated strategies, the players must have the incentive and the means to coordinate. As illustrated in Example 1, one possible incentive may be to ensure greater equity among players, and a possible coordination method is a traffic operator.

Correlated strategies require a *correlation device* [11] that coordinate actions among players in order to be implementable. Presently, we assume that such a correlation device exists for every correlated strategy satisfying Definition 2, so that the players can accurately coordinate and realize every joint action $a \in [A]$ [9].

Every joint strategy induces a correlated strategy, but not every correlated strategy can be reduced to a joint strategy. Furthermore, all correlated strategies induced by joint strategies are rank one in their tensor form.

Example 2 (Rank of correlated strategy tensors)**.** *Consider a two-player normal form game with finite action sets $[U]$ and $[V]$. We will cast the correlated strategy $y \in \Delta_{UV}$ to a matrix $Y \in \mathbb{R}^{U\times V}$. For a joint strategy $(x_U, x_V) \in \Delta_U \times \Delta_V$, the* *corresponding correlated strategy Y is given by*

$$Y = x_U x_V^\top.$$

Thus, all the joint strategies $x = (x_U, x_V)$ produce rank one correlated strategies in its matrix form.

On the other hand, let Y_0 be any feasible correlated strategy, then the complete set of correlated strategies is given by $Y_0 + \mathcal{N}$ where $\mathcal{N}$ is defined as

$$\mathcal{N} = \{Y \in \mathbb{R}_+^{U\times V} | \sum_{u\in[U]}\sum_{v\in[V]} Y(u,v) = \mathbb{1}^\top Y \mathbb{1} = 0\},$$

From the constraint $\mathbb{1}^\top Y \mathbb{1} = 0$, matrices in $\mathcal{N}$ have a maximum rank of $\min\{U, V\} - 1$.

Example 2's tensor formulation of correlated strategies is extendable to the N-player setting: every joint strategy $(x_1, \ldots, x_N)$, where $x_i \in \Delta_i$ for all $i \in [N]$, induces a correlated strategy $\hat{y}$ given by

$$\hat{y}(a_1, \ldots, a_N) = \prod_{i\in[N]} x_i(a_i),\ \forall (a_1, \ldots, a_N) \in [A]. \quad (5)$$

If we cast $\hat{y} \in \Delta_A$ to an N-dimensional tensor $Y \in \mathbb{R}^{A_1\times\ldots\times A_N}$, we observe that $\hat{y}$ is again a rank one tensor.

Comparison of solution spaces Δ and Δ_A. The joint strategy's and correlated strategy's solution spaces differ significantly in size. A joint strategy is given by N independent probability distributions $x_i \in \Delta_i$, and its overall dimension is $\sum_i A_i$. On the other hand, a correlated strategy has dimension $A = \prod_{i\in[N]} A_i$. When the number of players or the number of player actions increases, the joint strategy space Δ's dimension scales linearly, while the correlated strategy space Δ_A's dimension scales *exponentially*.

Player optimality in correlated strategy space Δ_A. If a correlated strategy is optimal for player i, the joint action (a_i, a_{-i}) is played only when no other action $\hat{a}_i \in [A_i]$ can be played with a_{-i} in place of a_i to improve player i's expected cost ℓ_i. This notion of optimality is the same as the Nash equilibrium (3). However, unlike the Nash equilibrium, defining the optimality of an independent strategy is no longer sufficiently descriptive. We formally define correlated equilibrium as below.

Definition 3 (**Correlated equilibrium [4]**). *The correlated strategy $y \in \Delta_A$ (Definition 2) is a correlated equilibrium if for all $i \in [N]$ and $a_i, \hat{a}_i \in [A_i]$,*

$$\sum_{a_{-i}\in[A_{-i}]} \Big(\ell_i(a_i, a_{-i}) - \ell_i(\hat{a}_i, a_{-i})\Big) y(a_i, a_{-i}) \le 0. \quad (6)$$

Intuitively, condition (6) implies that player i cannot independently swap action a_i for $\hat{a}_i$ while the other players play a_{-i} and achieve a lower expected cost. On the set of correlated strategies induced by joint strategies, the correlated equilibrium condition (6) is equivalent to the Nash equilibrium condition (3).

Lemma 1. *Over the set of correlated strategies induced by joint strategies as in* (5)*, the correlated equilibrium condition* (6) *is equivalent to the Nash equilibrium condition* (3).

Proof. Over the subset of correlated strategies induced by a joint strategy (5), the correlated equilibria condition (6) is equivalent to

$$\begin{aligned}
\sum_{a_{-i}\in[A_{-i}]} \Big(\ell_i(a_i, a_{-i}) - \ell_i(\hat{a}_i, a_{-i})\Big) \prod_{j\in[N]} x_j(a_j) \le 0 \\
x_i(a_i) \sum_{a_{-i}\in[A_{-i}]} \Big(\ell_i(a_i, a_{-i}) - \ell_i(\hat{a}_i, a_{-i})\Big) \prod_{j\neq i} x_j(a_j) \le 0 \\
x_i(a_i)\Big(\hat{\ell}_i(a_i; x_{-i}) - \hat{\ell}_i(\hat{a}_i; x_{-i})\Big) \le 0,
\end{aligned} \quad (7)$$

for all $(a_i, \hat{a}_i) \in [A_i]$, $i \in [N]$. When $x_i(a_i) > 0$, (7) implies that $\hat{\ell}_i(a_i; x_{-i}) = \min_{\hat{a}_i\in[A_i]} \hat{\ell}(\hat{a}_i; x_{-i})$. When $x_i(a_i) = 0$, the probability simplex constraints on x_i enforce the existence of $a_i' \in [A_i]$, such that $x_i(a_i') > 0$. Applying (7) to a_i', $\hat{\ell}_i(a_i'; x_{-i}) = \min_{\hat{a}_i\in[A_i]} \hat{\ell}_i(\hat{a}_i; x_{-i})$, such that the cost $\hat{\ell}_i$ at the original action a_i must have a cost greater than or equal to $\hat{\ell}_i(a_i'; x_{-i})$: $\hat{\ell}_i(a_i'; x_{-i}) \le \hat{\ell}_i(a_i; x_{-i})$. In summary, the following holds for all $a_i \in [A_i],\ i \in [N]$:

$$\begin{cases} \hat{\ell}_i(a_i; x_{-i}) = \min_{\hat{a}_i\in[A_i]} \hat{\ell}_i(\hat{a}_i; x_{-i}) & x_i(a_i) > 0 \\ \min_{\hat{a}_i\in[A_i]} \hat{\ell}_i(\hat{a}_i; x_{-i}) \le \hat{\ell}_i(a_i; x_{-i}) & x_i(a_i) = 0 \end{cases}.$$

We define $\lambda_i = \min_{a_i\in[A_i]} \hat{\ell}_i(a_i; x_{-i})$ and $\mu_i(a_i) = \hat{\ell}_i(a_i; x_{-i}) - \lambda_i$ for each player $i \in [N]$. By construction, (x_i, λ_i, μ_i) satisfy the KKT conditions of (4) when the other players use strategies x_{-i} for all $i \in [N]$, given by

$$\begin{aligned}
\hat{\ell}_i(a_i; x_{-i}) - \lambda_i - \mu_i(a_i) = 0, \quad \forall a_i \in [A_i] \\
x_i(a_i) \ge 0,\ \mu_i(a_i) \ge 0,\ \mu_i(a_i)x_i(a_i) = 0, \quad \forall a_i \in [A_i], \\
\sum_{a_i\in[A_i]} x_i(a_i) = 1.
\end{aligned} \quad (8)$$

Since the optimization problem (4) is a linear program in x_i, the KKT conditions fully characterize optimality and $(x_1, \ldots, x_N)$ is a Nash equilibrium.

To show that a joint strategy $x = (x_1, \ldots, x_N)$ produces a correlated equilibrium y (5) if x is a Nash equilibrium, we first note that x is a Nash equilibrium if and only if there exist Lagrange multipliers λ_i and μ_i for all $i \in [N]$ such that (x_i, λ_i, μ_i) satisfies the KKT conditions (8) against opponent strategy x_{-i}. Furthermore, the KKT condition implies that

$$\lambda_i = \min_{\hat{a}_i\in[A_i]} \hat{\ell}_i(\hat{a}_i; x_{-i}),\ \forall i \in [N]. \quad (9)$$

When $x_i(a_i) > 0$, $\hat{\ell}_i(a_i; x_{-i}) \le \hat{\ell}_i(\hat{a}_i; x_{-i})$ for all $\hat{a}_i \in [A_i]$. When $x_i(a_i) = 0$, $x_i(a_i)\Big(\hat{\ell}_i(a_i; x_{-i}) - \hat{\ell}_i(a_i; x_{-i})\Big) = 0$. We can conclude that (7) holds for all $i \in [N]$, such that the correlated strategy y constructed via (5) from the Nash equilibrium $(x_1, \ldots, x_N)$ satisfies (6). ☐

Lemma 1 provides an optimization-based proof for previously known equivalence between Nash equilibria and correlated equilibria [9], [24], [25].

Correlated equilibrium polytope. In the original [4] formulation of correlated equilibrium, the set of correlated equilibria is shown to be equivalent to the following linear polytope on the joint action space.

$$\mathcal{P}_{CE} := \Big\{ y \in \Delta \mid \mathbb{1}^\top y = 0, y \geq 0 \\ \sum_{a_{-i} \in [A_{-i}]} y(a_i, a_{-i}) \Big(\ell_i(a_i, a_{-i}) - \ell_i(\hat{a}_i, a_{-i}) \Big) \leq 0, \\ \forall a_i, \hat{a}_i \in [A_i],\ i \in [N] \Big\}. \quad (10)$$

In [9], the authors showed that in addition to being a connected polytope, $\mathcal{P}_{CE}$'s boundary set $\partial \mathcal{P}_{CE}$ contains the correlated strategies induced by Nash equilibria. However, computing the $\mathcal{P}_{CE}$ suffers from the curse of dimensionality both due to the dimension of Δ being exponential in the N and the number of $\mathcal{P}_{CE}$'s constraints, $\sum_{i \in [N]} \binom{A_i}{2}$, being exponential in A_i.

III. Lifting correlated equilibrium to generalized Nash equilibrium

While a correlated equilibrium has the interpretation of being a 'simultaneously optimal' strategy in literature, this interpretation lacks an explicit optimization formulation like the one that exists for Nash equilibrium in the form of (4). In this section, we formulate a novel game in which the player strategy spaces are lifted from the probability measure space over $[A_i]$ to the unnormalized measure space over $[A] = \prod_{i \in [N]} [A_i]$. We show that a fully mixed generalized Nash equilibrium of the lifted game over unnormalized measures corresponds to a fully mixed correlated equilibrium of the normal form game.

A. Unnormalized measures

We first consider a relaxation of probability distributions to unnormalized measures with finite mass [26], [27].

Definition 4 (Unnormalized measure)**.** *Given an action set $[A]$ and a corresponding probability simplex Δ_A, the vector $\alpha \in \mathbb{R}^A_+/\{0\}$ is an unnormalized measure over the sample space $[A]$ if $\alpha(a) \geq 0$ for all $a \in [A]$.*

Given two unnormalized measures α_1, α_2 over $[A]$, we denote their **element-wise product** by $\alpha_1 \circ \alpha_2$, such that

$$(\alpha_1 \circ \alpha_2) \in \mathbb{R}^A_+, (\alpha_1 \circ \alpha_2)(a) = \alpha_1(a)\alpha_2(a),\ \forall a \in [A].$$

We consider the decomposition of a correlated strategy y (Definition 2) into N unnormalized measures.

Definition 5 (Normalized Decomposition)**.** *Given a correlated strategy $y \in \Delta_A$, we say that $(\alpha_1, \ldots, \alpha_N)$ is a normalized decomposition of y and y is a product of $(\alpha_1, \ldots \alpha_N)$ if*

$$y = \alpha_1 \circ \ldots \circ \alpha_N,\ \alpha_i \in \mathbb{R}^A_+,\ \forall i \in [N]. \quad (11)$$

The mapping $(\alpha_1, \ldots, \alpha_N) \mapsto y$ is surjective but not injective. Every correlated strategy has at least one valid decomposition, but multiple sets of unnormalized measures can produce the same correlated strategy.

Lemma 2. *Every correlated strategy $y \in \Delta_A$ has an infinite number of decompositions in the form of* (11)*. Furthermore, if $(\alpha_1, \ldots, \alpha_N)$ satisfies*

$$\mathbb{1}^\top (\alpha_1 \circ \ldots \circ \alpha_N) = 1, \alpha_i \geq 0,\ \forall i \in [N], \quad (12)$$

then $y = \alpha_1 \circ \ldots \circ \alpha_N$ is a correlated strategy.

Proof. Consider the unnormalized measures $(\alpha_1, \ldots, \alpha_N)$ that satisfy (12) and $y = \alpha_1 \circ \ldots \circ \alpha_N$, then y is a correlated strategy as defined in Definition 2. To show that every correlated strategy y has an infinite number of decomposition's, we first show there exist at least one feasible decomposition: $\alpha_1 = y$, $\alpha_j = \mathbb{1}$ for all $j \neq 1$. We then select $d \in \mathbb{R}$, $d > 0$ and define the measures $\hat{\alpha}_1 = d\alpha_1$, $\hat{\alpha}_2 = \frac{1}{d}\alpha_2$, and $\hat{\alpha}_j = \alpha_j$ for all $j \notin \{1, 2\}$, then the product of $(\hat{\alpha}_1, \hat{\alpha}_2, \ldots \circ \hat{\alpha}_N)$ satisfies $y = \hat{\alpha}_1 \circ \hat{\alpha}_2 \ldots, \hat{\alpha}_N$. Since we can select arbitrary positive real number d, there exists an infinite number of decompositions. □

Example 3 (Normalized decompositions)**.** *In a two-player, finite action game where $A_1 = A_2 \in \mathbb{N}$. We can represent the unnormalized measures by $A_1 \times A_2$-dimensional matrices, $\alpha \in \mathbb{R}^{A_1 \times A_2}$, such that any element-wise product $\alpha_i \circ \alpha_j$ is equivalent to the Hadarmard product between their matrix counterparts. The following are all valid normalized decompositions and their correlated strategy product.*

$$\Bigg\{ \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{\alpha_1}, \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{\alpha_2} \Bigg\}, \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{y_\alpha}, \quad (13)$$

$$\Bigg\{ \underbrace{\frac{1}{45}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{\beta_1}, \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}}_{\beta_2} \Bigg\}, \underbrace{\frac{1}{45}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}}_{y_\beta}, \quad (14)$$

$$\Bigg\{ \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{\gamma_1}, \underbrace{\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}}_{\gamma_2} \Bigg\}, \underbrace{\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{y_\gamma}. \quad (15)$$

B. Unnormalized game

We define an **unnormalized game** as the following extension of the normal-form game: instead of choosing probability distributions supported on each player's individual action space, each player i chooses an unnormalized measure α_i over the joint action space $[A]$ as defined in Definition 4, constrained by the condition that the product $y = \alpha_1 \circ \ldots \alpha_N$ is a correlated strategy. The player objectives remain the expected cost incurred by each player (2), which is a multi-linear function of the unnormalized measures through (11) and (2). Each player's optimization problem is given by

$$\begin{aligned} \min_{\alpha_i \in \mathbb{R}_+^A} & \sum_{a\in[A]} \ell_i(a)\alpha_1(a)\dots\alpha_N(a), \\ \text{s.t.} & \sum_{a\in[A]} (\alpha_1 \circ \dots \circ \alpha_N)(a) = 1. \end{aligned} \tag{16}$$

In the unnormalized game, each player's strategy α_i has the same dimension as the correlated strategy of the original finite game. Given the other players' strategies α_{-i}, player i uses their strategy α_i to optimize the expected cost $\sum_{a\in[A]} \ell_i(a)\alpha_1(a)\dots\alpha_N(a)$, constrained by a mass constraint: $\sum_{a\in[A]}(\alpha_1 \circ \dots \circ \alpha_N)(a) = 1$. The optimal solution of this game is a generalized Nash equilibrium, where, in addition to minimizing their expected cost, each player's strategy must be feasible with respect to the other players' strategies.

Definition 6 (Generalized Nash equilibrium)**.** *A joint strategy $(\alpha_1^\star, \dots, \alpha_N^\star)$ is a generalized Nash equilibrium if for all $i \in [N]$, $\alpha_i^\star$ is the optimal solution to* (16)*.*

A generalized Nash equilibrium extends the standard Nash equilibrium (Definition 3) to games where each player's strategy feasibility depends on the other players' strategies. In the unnormalized game, all players share the strategy constraint given by (12). Next, we restrict our analysis to fully mixed unnormalized measures and show that when a set of fully mixed unnormalized measures forms a generalized Nash equilibrium, their product is a correlated equilibrium.

Assumption 1 (Fully mixed measures)**.** *A measure $\alpha \in \mathbb{R}_+^A$ is fully mixed if for all $a \in [A]$, $\alpha(a) > 0$.*

When a correlated strategy is fully mixed, all of its normalized decompositions $(\alpha_1, \dots, \alpha_N)$ satisfy Assumption 1. Games with certain player cost structures, such as zero-sum games and games with non-dominant strategies, tend to have fully mixed Nash and correlated equilibria [28], [29].

C. Equivalence between generalized Nash equilibrium and correlated equilibrium

For fully mixed correlated strategies y with a normalized decomposition $(\alpha_1, \dots, \alpha_N)$, we can show that y is a correlated equilibrium of the normal form game if and only if $(\alpha_1, \dots, \alpha_N)$ is a generalized Nash equilibrium of the unnormalized game.

Proposition 1. *If $(\alpha_1, \dots, \alpha_N)$ is a generalized Nash equilibrium of the unnormalized game* (16)*, and the α_i's satisfy Assumption 1, then the product $y = \alpha_1 \circ \dots \circ \alpha_N$ is a correlated equilibria of the normal form game* (6)*.*

Proof. We prove this proposition by showing that Assumption 1 and the coupled KKT conditions of (16) together imply the correlated equilibrium condition in (6). From [30, Thm.3.3], the coupled KKT conditions of (16) are necessary and sufficient for $(\alpha_1, \dots, \alpha_N)$ to be a generalized Nash equilibrium of the unnormalized game. Therefore, we show that (6) holds for $y = \alpha_1 \circ \dots \circ \alpha_N$ for all the KKT points $(\alpha_1, \dots \alpha_N)$ of (16).

From the unnormalized game (16) for player i, we assign the Lagrange multipliers $\sigma_i \in \mathbb{R}$ for the constraint $\sum_{a\in[A]}(\alpha_1 \circ \dots \circ \alpha_N)(a) = 1$ and $\mu_i(a)$ for the constraints $\alpha_i(a) \geq 0$. The first-order gradient condition and the complementarity condition of the KKT are given by

$$\begin{aligned} &\ell_i(a)\alpha_{-i}(a) - \sigma_i\alpha_{-i}(a) - \mu_i(a) = 0, \\ &\mu_i(a) = \begin{cases} \geq 0 & \alpha_i(a) = 0 \\ = 0 & \alpha_i(a) > 0 \end{cases}, \forall (i, a) \in [N] \times [A]. \end{aligned} \tag{17}$$

When $\alpha_{-i}(a) > 0$, the KKT conditions above imply that

$$\ell_i(a) \begin{cases} = \sigma_i, & \text{if } \alpha_i(a) > 0 \\ \geq \sigma_i, & \alpha_i(a) = 0 \end{cases}, \ \forall (i, a) \in [N] \times [A]. \tag{18}$$

From Assumption 1, $\alpha_{-i}(a) > 0$ and $\alpha_i(a) > 0$ for all $a \in [A]$. Therefore, $\mu_i(a) = 0$, $\sigma_i = \ell_i(a)$ for all $a \in [A]$. In particular, $\ell_i(a_i, a_{-i}) = \ell_i(\hat{a}_i, a_{-i})$ for all $a_i, \hat{a}_i \in [A_i]$. The correlated equilibrium condition (6) $(\ell_i(a_i, a_{-i}) - \ell_i(\hat{a}_i, a_{-i}))y(a_i, a_{-i})$ will then evaluate strictly to 0 for all $\hat{a}_i \in [A_i]$ and $i \in [N]$. We conclude that $y = \alpha_1 \circ \dots \circ \alpha_N$ is a correlated equilibrium. ☐

Remark 1. *Proposition 1 suggests that a correlated equilibrium is fully mixed only if $\ell_i(a)$ evaluates to the same value for all $a \in [A]$. While this may seem restrictive, we use entropy regularizations in Section IV to produce games in which the regularized costs are all equal for each opponent action a_{-i}. We can show that the generalized Nash equilibrium under regularization will approximate the correlated equilibrium of the normal-form game even if no fully mixed correlated equilibrium exists.*

The reverse of Proposition 1 is not true: if y is a fully mixed correlated equilibrium, then it may not have a normalized decomposition that is a generalized Nash equilibrium of the unnormalized game.

Example 4 (Correlated equilibria not captured by gNE)**.** *Consider a 2×2 matrix game where player one chooses the row and player two chooses the column. The player costs are given by matrices A and B, respectively, defined as*

$$P_1 = \begin{bmatrix} 3 & 3 \\ 2 & 4 \end{bmatrix}, \ P_2 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}.$$

We vectorize the joint action space as $[A] = \{a_1, a_2, a_3, a_4\}$, corresponding to the counterclockwise sequence of joint actions in matrix P_i starting from the top left. For $y \in \Delta_4$ to be a correlated equilibrium as defined in (6)*, it must satisfy*

$$\begin{aligned} 3y(a_1) + 3y(a_4) - 2y(a_2) - 4y(a_3) \leq 0 \\ 1y(a_1) + 1y(a_2) - 0y(a_3) - 2y(a_4) \leq 0 \end{aligned} \tag{19}$$

We can verify that $y_{CE} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$ satisfies (19)*. The unnormalized game played is given by*

$$\begin{aligned} \min_{\alpha_1} & \frac{1}{2}\Big(3\alpha_1(a_1) + 2\alpha_1(a_2) + 4\alpha_1(a_3) + 4\alpha_1(a_4)\Big) \\ \textit{s.t. } & \alpha_1(a_1) + \alpha_1(a_4) + \alpha_1(a_2) + \alpha_1(a_3) = 2 \\ & \alpha_1(a_i) \geq 0, \ \forall i \in [4], \end{aligned} \tag{20}$$

where the mass constraint simplifies since player two's strategy is $\alpha_2(a_j) = \frac{1}{2}$ for all $j \in [4]$. Consider the decomposition $\alpha_1 = \alpha_2 = \begin{bmatrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{bmatrix}$, we can verify that α_1 does not minimize (20). Specifically, $\hat{\alpha}_1 = \begin{bmatrix}0 & 1 & \frac{1}{2} & \frac{1}{2}\end{bmatrix}$ can achieve a lower objective than α_1 against $\alpha_2 = \begin{bmatrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{bmatrix}$.

From Propositions 1, we can say that if the generalized Nash equilibrium of (16) is strictly positive, then their product y is a fully mixed correlated equilibrium of the original normal-form game. A natural follow-up question remains: when do strictly positive correlated strategies exist? We explore this in the following section using entropy regularizations.

IV. Entropy-Regularized Correlated Equilibria

A coupled optimization formulation for the correlated equilibrium expands the set of analysis techniques applicable to it. In this section, we demonstrate how entropy regularization can be applied to the unnormalized game to find ϵ-correlated equilibria of the original normal-form game.

We consider the entropy-regularized counterpart of the unnormalized game (16), where each player solves the optimization problem given by

$$\begin{aligned}\min_{\alpha_i \in \mathbb{R}^A_+} & \sum_{a\in[A]} \left(\ell_i(a) + \frac{1}{\lambda_i}\Big(\log\big(\alpha_i(a)\big) - 1\Big)\right)\alpha_i(a)\alpha_{-i}(a)\\ \text{s.t.} & \sum_{a\in[A]} (\alpha_1 \circ \ldots \circ \alpha_N)(a) = 1.\end{aligned} \tag{21}$$

Here, $\lambda_i \geq 0$, $\lambda_i \in \mathbb{R}$ denotes the magnitude of the entropy regularization. The total entropy of the correlated strategy $y = \alpha_1 \circ \ldots \circ \alpha_N$ is given by $\sum_{a\in[A]} y(a)\log(y(a))$, such that $\sum_{a\in[A]} y(a)\log(\alpha_i)$ is equivalent to player i's contribution to the total entropy. For two unnormalized measures that achieve equal costs $\sum_{a\in[A]} \ell_i(a)\alpha_i(a)\alpha_{-i}(a)$, (21) will favor the measure with the greater entropy and thus achieving a lower cost as defined by (16).

Remark 2. *In applications of game-theoretic equilibrium, player costs and transitions are often obtained from noisy and imperfect data. When faced with such modeling inaccuracy data, it is in the player's best interest to seek strategies that not only optimize their expected cost but also maximize the entropy over the action space. Entropy regularization has been used in single-agent reinforcement learning and Nash equilibrium to find optimal policies that are robust to modeling inaccuracies and 'unforeseen changes in the environment' [31], [32]. The Nash equilibrium of entropy regularized games is also known as* logit quantal response equilibrium *[33] and is an important tool for finding ϵ-Nash equilibria in policy gradient-based reinforcement learning [34]. Furthermore, the logit quantal response equilibrium is shown to be a more robust equilibrium model than the Nash equilibrium for games models involving human players [35].*

We can show that the entropy-regularized unnormalized distribution game has the following closed-form solution.

Proposition 2. *In the entropy-regularized unnormalized game where each player solves (21), there exists a generalized Nash equilibrium $(\alpha_1, \ldots, \alpha_N)$, where α_i is given by*

$$\alpha_i(a) = \frac{\exp\big(-\lambda_i\ell_i(a)\big)}{\Big(\sum_a \exp\big(-\sum_j \lambda_j\ell_j(a)\big)\Big)^{\lambda_i/\sum_j\lambda_j}}, \quad \forall i, a \in [N]\times[A]. \tag{22}$$

Proof. Since (21) has strongly convex objectives, convex independent constraints, and a shared constraint that is convex in each individual α_i, the generalized Nash equilibrium of the game is equivalent to the coupled KKT points of (21) [30, Thm.3.3]. We assign the Lagrange multipliers $\sigma_i \in \mathbb{R}$ to the constraint $\sum_{a\in[A]}(\alpha_1 \circ \ldots \circ \alpha_N)(a) = 1$ and $\mu_i(a) \in \mathbb{R}_+$ to the constraints $\alpha_i(a) \geq 0$, for all $a \in [A]$. The KKT conditions of (21) are given by

$$\begin{aligned} 0 =& \Big(\ell_i(a) + \frac{1}{\lambda_i}\log\big(\alpha_i(a)\big) + \sigma_i\Big)\alpha_{-i}(a) - \mu_i(a),\ \forall a \in [A],\\ 1 =& \sum_{a\in[A]} (\alpha_1 \circ \ldots \circ \alpha_N)(a),\\ 0 \leq& \alpha_i(a), 0 \leq \mu_i(a), 0 = \alpha_i(a)\mu_i(a),\ \forall a \in [A]. \end{aligned} \tag{23}$$

When $\alpha_{-i}(a) > 0$ and $\alpha_i(a) > 0$, the first KKT condition reduces to $0 = \ell_i(a) + \frac{1}{\lambda_i}\log\big(\alpha_i(a)\big) + \sigma_i$. We can solve for $\alpha_i(a)$ as

$$\alpha_i(a) = \exp\big(-\lambda_i(\sigma_i + \ell_i(a)\big). \tag{24}$$

For each joint action $a \in [A]$, the product $(\alpha_1 \circ \ldots \circ \alpha_N)(a)$ is given by

$$(\alpha_1 \circ \ldots \circ \alpha_N)(a) = \exp\Big(-\sum_i \lambda_i\sigma_i - \sum_i \lambda_i\ell_i(a)\Big). \tag{25}$$

From primal feasibility of the KKT conditions, $1 = \sum_{a\in[A]}(\alpha_1 \circ \ldots \circ \alpha_N)(a)$. We combine this with (25) to derive

$$\exp\big(-\sum_i \lambda_i\sigma_i\big)\sum_a \exp\big(-\sum_i \lambda_i\ell_i(a)\big) = 1. \tag{26}$$

Let $\lambda_N = \sum_i \lambda_i$ and $\sigma_i = \sigma$ for all $i \in [N]$. We can then solve for $\exp(\sigma)$ in (26) as $\Big(\sum_a \exp\big(-\sum_i \lambda_i\ell_i(a)\big)\Big)^{-1/\lambda_N}$. Let $\beta_i(a) = \exp\big(-\lambda_i\ell_i(a)\big)$, each $\alpha_i(a)$ (24) is given by

$$\alpha_i(a) = \frac{\beta_i(a)}{\Big(\sum_a \prod_j \beta_j(a)\Big)^{\lambda_i/\lambda_N}}. \tag{27}$$

When $\lambda_i, \ell_i(a)$ are finite, $\alpha_i(a) > 0$ for all $i, a \in [N]\times[A]$. Therefore $\alpha_1, \ldots, \alpha_N$ satisfy the KKT conditions and are the optimal solutions to the unnormalized game. □

The correlated equilibrium corresponding to (22) is

$$y(a) = \frac{\exp\big(-\sum_j \lambda_j\ell_j(a)\big)}{\sum_a \exp\big(-\sum_j \lambda_j\ell_j(a)\big)},\ \forall a \in [A]. \tag{28}$$

The resulting correlated equilibrium is a softmax function over the regularized and weighted sum of individual player costs, where the level of entropy introduced is controlled by λ_i: the smaller the λ_i is, the closer the resulting correlated equilibrium is to the completed mixed correlated strategy, $y'(a) = 1/A$ for all $a \in [A]$. We note that while Proposition 2 provides one possible solution for the entropy-regularized unnormalized game (21), other generalized Nash equilibria exist. In particular, each strictly positive generalized Nash equilibrium is an ϵ-correlated equilibrium of the original normal form game (16), even if the original normal form game does not have any strictly positive correlated equilibrium.

Corollary 1 (ϵ-correlated equilibrium)**.** *If the entropy-regularized generalized Nash equilibrium $(\alpha_1, \ldots, \alpha_N)$ satisfies Assumption 1 for each $i \in [N]$, their product y (28) is an ϵ-correlated equilibria—i.e., for all $i \in [N]$,*

$$\sum_{a_{-i}} y(a_i, a_{-i})\Big(\ell_i(a_i, a_{-i}) - \ell_i(\hat{a}_i, a_{-i})\Big) \leq \frac{\epsilon_i}{\lambda_i}, \ \forall a_i, \hat{a}_i \in [A_i], \tag{29}$$

where $\epsilon_i = \max_{a, \hat{a} \in [A]} \log\Big(\alpha_i(a)/\alpha_i(\hat{a})\Big)$ and $\epsilon = \max_i \epsilon_i$.

Proof. From (23), we derived the coupled KKT conditions of the entropy-regularized unnormalized game. When $\alpha_i(a) > 0$ for all $(i, a) \in [N] \times [A_i]$, the following holds:

$$\ell_i(a) + \frac{1}{\lambda_i}\log(\alpha_i(a)) + \sigma_i = 0, \forall a \in [A]. \tag{30}$$

Then for any $a, a' \in [A]$, their cost difference is given by

$$\ell_i(a_i, a_{-i}) - \ell_i(a'_i, a_{-i}) = \frac{1}{\lambda_i}\log(\frac{\alpha_i(a_i, a_{-i})}{\alpha_i(a'_i, a_{-i})}). \tag{31}$$

We multiply (31) by $y(a_i, a_{-i})$ and sum over $a_{-i} \in [A_{-i}]$ to obtain

$$\sum_{a_{-i} \in [A_{-i}]} y(a_i, a_{-i})\Big(\ell_i(a_i, a_{-i}) - \ell_i(a'_i, a_{-i})\Big) = \frac{1}{\lambda_i}\sum_{a_{-i} \in [A_{-i}]} y(a_i, a_{-i})\log(\frac{\alpha_i(a_i, a_{-i})}{\alpha_i(a'_i, a_{-i})}). \tag{32}$$

Let $\max_{a, a' \in [A]} \log(\frac{\alpha_i(a)}{\alpha_i(a')}) = \epsilon_i$. It follows that

$$\sum_{a_{-i} \in [A_{-i}]} y(a_i, a_{-i})\log(\frac{\alpha_i(a_i, a_{-i})}{\alpha_i(a'_i, a_{-i})}) \leq \sum_{a \in [A]} y(a)\epsilon_i \leq \epsilon_i. \tag{33}$$

We can then conclude the proposed statement (29). □

V. Computing ϵ-Correlated Equilibrium

We apply the results of Section IV to compute the generalized Nash equilibrium of the unnormalized game (16) and evaluate its feasibility as an ϵ-correlated equilibrium of the original normal-form game.

We simulate normal-form games (4) with $N = \{2, 3\}$ players and individual action spaces of size $A = \{2, 5, 10\}$.

Over $K = 1000$ randomly generated normal-form games, we compute the entropy-regularized generalized Nash equilibrium as (22). We plot the empirical violation with a 5% standard deviation range of the correlated equilibrium condition (6) under ϵ(empirical) and the theoretical bound $\max_i \epsilon_i/\lambda_i$ (29) under ϵ(bound) for the regularization values $\lambda = \{0.1, 10, 30, 100, 1000, 1e4\}$. We assume that all players use the same entropy regularization, $\lambda_i = \lambda, \ \forall i \in [N]$.

For each game, we compute its entropy-regularized generalized Nash equilibrium $y^\star$ via (22) and evaluate $y^\star$'s empirical sub-optimality $\epsilon_{ce} = \epsilon$ (empirical) as

$$\max_{\substack{i \in [N] \\ a_i, a'_i \in [A_i]}} \sum_{a_{-i} \in [A_{-i}]} y^\star(a_i, a_{-i})\Big(\ell_i(a_i, a_{-i}) - \ell_i(a'_i, a_{-i})\Big). \tag{34}$$

We note that ϵ_{ce} is equivalent to the distance between y_{ce} and the correlated polytope in ∞ vector norm.

Finally, we note that a key challenge in applying correlated equilibrium for autonomous interactions is its poor scalability in the number of agents and actions. To this end, (22) provides an approximation that significantly reduces the computation complexity. We observe this in Figure 2.

As shown in Figure 2, the computation time still scales poorly in the number of actions and players. However, the overall computation time is significantly lower than solving for a feasible point of the correlated equilibrium polytope via linear programming. For comparison, it takes approximately 1.87 seconds to use CVXPY to compute a correlated equilibrium for the game with $N = 3$ players each with $A_i = 3$, whereas approximating it using the entropy-regularized generalized Nash equilibrium only takes $4e - 3$ seconds.

VI. Conclusion

We introduced an extension of finite player normal-form games to coupled strategies on unnormalized measures over the joint action space and showed that for fully mixed unnormalized measures, the set of generalized Nash equilibria of the unnormalized measure game produces fully mixed correlated equilibria in the original normal-form game. Leveraging the optimization structure this imposes on the correlated equilibria, we introduce an entropy-regularized version of the unnormalized game, and show that its generalized Nash equilibrium is within ϵ distance of the correlated equilibrium polytope, where ϵ is dependent on the entropy of each player's unnormalized measure as well as the entropy regularization. Our optimization framework is the first step in connecting correlated equilibrium to the wider literature on gradient-based multi-agent learning algorithms.

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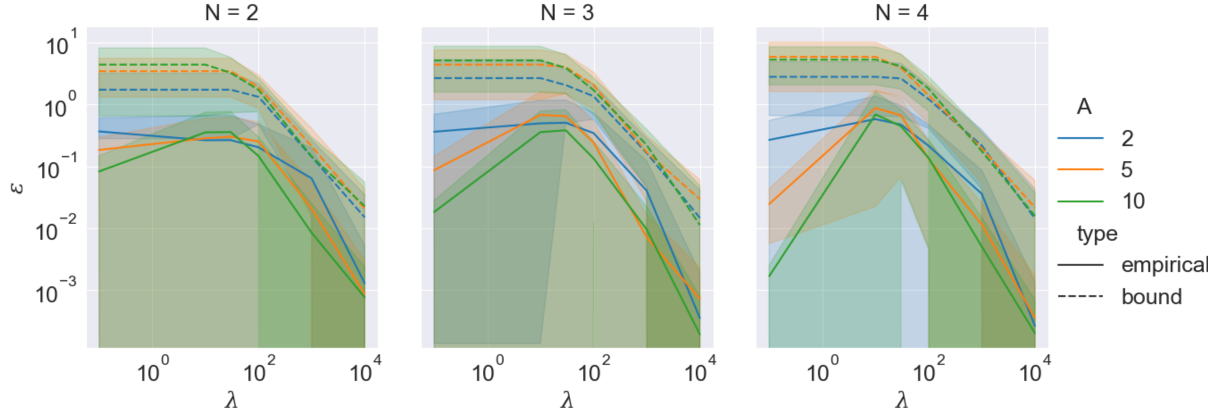

Fig. 1: Empirical vs theoretical sub-optimality of the entropy-regularized generalized Nash equilibrium (28) as a correlated equilibrium.

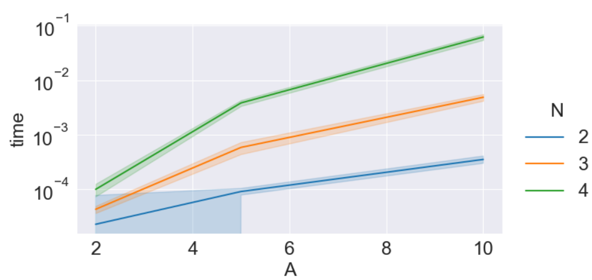

Fig. 2: Computation time (seconds) of the ϵ-correlated equilibrium for different numbers of players and actions.

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