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SKdV, SmKdV flows and their supersymmetric gauge-Miura transformations

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Abstract

The construction of Integrable Hierarchies in terms of zero curvature representation provides a systematic construction for a series of integrable non-linear evolution equations (flows) which shares a common affine Lie algebraic structure. The integrable hierarchies are then classified in terms of a decomposition of the underlying affine Lie algebra $\hat{\mathcal{G}}$ into graded subspaces defined by a grading operator Q. In this paper we shall discuss explicitly the simplest case of the affine $\hat{sl}(2)$ Kac-Moody algebra within the principal gradation given rise to the KdV and mKdV hierarchies and extend to supersymmetric models.

Inspired by the dressing transformation method, we have constructed a gauge-Miura transformation mapping mKdV into KdV flows. Interesting new results concerns the negative grade sector of the mKdV hierarchy in which a double degeneracy of flows (odd and its consecutive even) of mKdV are mapped into a single odd KdV flow. These results are extended to supersymmetric hierarchies based upon the affine $\hat{sl}(2,1)$ super-algebra.

1 Introduction

Integrable field theories are very peculiar models admitting infinite number of conservation laws and soliton solutions. The KdV and mKdV equations are for instance, typical examples of such a class of models and have acted as prototypes for many new developments in the subject. In fact apart from a single equation, say mKdV (or KdV) a series of other (higher/lower grade) evolution equations of motion (flows) can be systematically derived from a zero curvature representation and a single universal operator of Lie algebraic origin (Lax operator). These evolution equations (flows) form an *Integrable Hierarchy* which are constructed in terms of a decomposition of the affine Lie algebra $\hat{\mathcal{G}}$ according to a grading operator denoted by Q. The Lax operator is further specified by a second decomposition according to a choice of a constant grade one generator E. It therefore follows that an Integrable Hierarchy is defined by three Lie algebraic ingredients, namely, i) the affine Lie algebra $\hat{\mathcal{G}}$, ii) the grading operator Q and iii) the constant grade one generator E (see for instance [1] for a review).

Both KdV and mKdV hierarchies are constructed from the decomposition of the affine $\hat{\mathcal{G}} = \hat{sl}(2)$ algebra according to the principal grading, Q. Details are explicitly displayed in the Appendix and, at this stage it should be pointed out that different decomposition generate different hierarchies. The general structure for a flow ($t_N, \ N \in \mathbb{Z}$) is given by the zero curvature representation,

$$[\partial_x + A_x, \ \partial_{t_N} + A_{t_N}] = 0 \tag{1.1}$$

in terms of a universal Lax operator $L = A_x$.

In section 2, we review the bosonic case. We first discuss the KdV and mKdV positive grade ($N > 0$) sub-hierarchies. It is shown that the graded structure of the decomposition of $\hat{\mathcal{G}} = \hat{sl}(2)$ algebra implies $N = 2n + 1$. Next, the negative sub-hierarchies are proposed and it is shown that for mKdV there are no restrictions upon the negative integers N, i.e., negative odd and negative even sub-hierarchies can be constructed consistently, [1]. For the KdV however, only a few equations were known for N negative odd (see [2, 3]). More recently, a general ansatz for constructing the entire negative odd KdV sub-hierarchy was proposed in [4]. An interesting feature of the mKdV *negative even* sub-sector requires soliton solutions build from *strictly non-zero vacuum*. These are constructed by gauge transforming the zero curvature representation (1.1) in the vacuum configuration into a non trivial configuration (dressing method) involving the construction of deformed vertex operators see [1]. Next, the gauge-Miura transformation S mapping $A_\mu^{mKdV} \to A_\mu^{KdV}$ is discussed. The mapping is shown to be one to one as far as N is odd positive. The novelty is a double degeneracy between the two negative sectors. It is verified that S maps the first negative odd and its subsequent negative flows of the mKdV hierachy into the first negative odd of the KdV hierarchy. The argument generalises to lower graded flows,

$$\begin{tikzcd}
t^{\text{mKdV}}_{-2j} \arrow[dr, "S"] & \\
& t^{\text{KdV}}_{-2j+1} \\
t^{\text{mKdV}}_{-2j+1} \arrow[ur, "S"'] &
\end{tikzcd} \tag{1.2}$$

$j = 1, \ldots$.Interesting to point out that the flows t_{-2j}^{mKdV} and t_{-2j+1}^{mKdV} require strictly non-zero and strictly zero vacuum solutions respectively, while the flow t_{-2j-1}^{KdV} admits both, see [4].

An important attempt to introduce supersymmetry in integrable models dates from the pionnering paper by Kuperschmidt [5]. Later Khovanova [6] considered a model based upon $sl(2,1)$ affine algebra and showed to be invariant under supersymmetry transformation. Moreover Miura transformation connecting the SKdV and SmKdV models was proposed. In this paper we extend those results to the entire hierarchy considering both positive and negative graded sub-hierarchies. In section 3 we discuss the construction of the supersymmetric mKdV hierarchy based upon the affine super algebra $\hat{sl}(2,1)$. We first discuss the positive hierarchy and show that the flows are labeled by odd integers and the equations admit both zero and non-zero constant vacuum solutions. Next we consider the negative sub-hierarchy and show that there is split into negative odd and negative even graded flows. The negative odd admits only zero-vacuum whilst the negative even, strictly non-zero vacuum solutions.

In section 4 we discuss the supersymmetric KdV hierarchy. It is shown that both SKdV and SmKdV possess the same algebraic structure and the SKdV flows are constructed systematically for the positive graded sub-hierarchy. The negative sub-hierarchy is then proposed by direct ansatz for the time component of two dimensional gauge potential.

In section 5 we rederive the negative SKdV flows by gauge-Miura transforming the potentials from the SmKdV. The compatibility recovers the known Miura transformation together with additional conditions involving temporal derivatives, temporal Miura transformation. These shows that the gauge-Miura mapping is one to one as far as positive sub-hierarchy is concerned and is a two to one for the negative grade sector.

Finally in section 6 we discuss the various implications about such gauge-Miura map in terms of the structure of vacuum solutions.

2 Review of bosonic case

In this section, we shall discuss the construction of the mKdV and KdV hierarchies from an algebraic formalism. Both hierarchies share the same algebraic structure and are related by the well-known Miura transformation. A crucial ingredient is the fact that the zero curvature condition is preserved under gauge transformation. We formulate the Miura transformation as gauge transformation acting on zero curvature condition (1.1). We also cover the main results related to the negative flows of both hierarchies and how they are interconnected.

The mKdV and KdV hierarchies can be constructed through gauge potentials that are elements of an affine $\hat{sl}(2)$ algebra endowed with a principal grading operator. The spatial gauge potential for the mKdV hierarchy is given by

$$A_x^{\mathrm{mKdV}} = E^{(1)} + A_0 = \begin{pmatrix} v & 1 \\ \lambda & -v \end{pmatrix}, \tag{2.1}$$

where $E^{(1)} = K_1^{(1)} = E_\alpha^{(0)} + E_{-\alpha}^{(1)} \in \mathcal{G}_1$ is a grade one constant element, and $A_0 = v\, h_1^{(0)} \in \mathcal{G}_0$ contains the field of model $v = v(x, t_N)$. The spatial gauge potential for the KdV hierarchy differs from (2.1) due to the algebraic element associated with the field of the theory, namely

$$A_x^{\mathrm{KdV}} = E^{(1)} + A_{-1} = \begin{pmatrix} 0 & 1 \\ \lambda + J & 0 \end{pmatrix}, \tag{2.2}$$

where $A_{-1} = J\, E_{-\alpha_1}^{(0)} \in \mathcal{G}_{-1}$ contains the field of model $J = J(x, t_N)$. Given the spatial and temporal gauge potentials, A_x and A_{t_N}, respectively, we can derive equations of motion associated with the temporal flow t_N from the *zero curvature condition* (1.1) where A_{t_N} is constructed from a sum of graded elements of the algebra $\mathcal{G}$. For the so-called positive sub-hierarchy of mKdV, the temporal gauge potentials are given by

$$A^{\text{mKdV}}_{t_N} = D^{(N)}_N + D^{(N-1)}_N + \cdots + D^{(0)}_N \qquad (D^{(a)}_N \in \mathcal{G}_a) \tag{2.3}$$

whereas for the KdV hierarchy, they are written as follows

$$A^{\text{KdV}}_{t_N} = \mathcal{D}^{(N)}_N + \mathcal{D}^{(N-1)}_N + \cdots + \mathcal{D}^{(0)}_N + \mathcal{D}^{(-1)}_N \qquad (\mathcal{D}^{(a)}_N \in \mathcal{G}_a) \tag{2.4}$$

The various elements $D^{(a)}$ or $\mathcal{D}^{(a)}$ are recursively determined through the graded decomposition of the zero curvature condition. It is noteworthy that for the highest grade, we obtain the following component for both mKdV and KdV hierarchies

$$\left[E^{(1)}, \mathcal{D}^{(N)}\right] = 0 \qquad \text{and} \qquad \left[E^{(1)}, D^{(N)}\right] = 0, \tag{2.5}$$

which implies that the elements $D^{(N)}$ and $\mathcal{D}^{(N)}$ belong to the Kernel $\mathcal{K}_E$ of the element $E^{(1)}$. In particular, the elements of $\mathcal{K}_E$ have odd grades for the $sl(2)$ algebra, i.e., $N = 2m+1$, with $m \in \mathbb{N}$ (see (A.5a)). Thus, the positive temporal flows of both hierarchies are constrained to be labeled by odd numbers.

Concerning the negative temporal flows, t_{-N}, we can obtain them through the temporal gauge potentials defining the so-called negative sub-hierarchy of mKdV, given by

$$A^{\text{mKdV}}_{t_{-N}} = D^{(-N)}_{-N} + D^{(-N+1)}_{-N} + \cdots + D^{(-1)}_{-N} \qquad (D^{(a)}_{-N} \in \mathcal{G}_a) \tag{2.6}$$

similar to the positive temporal flows, we can decompose the zero curvature condition according to its graded structure and determine its various components $D^{(-a)}_{-N}$. We therefore obtain equations of motion which are in general non-local. From the the lowest grade of the decomposition, we find the non local equation for $D^{(-N)}_{-N}$,

$$\partial_x D^{(-N)}_{-N} + \left[A_0, D^{(-N)}_{-N}\right] = 0 \tag{2.7}$$

and henceforth find no restrictions for the negative temporal flows of mKdV hierarchy.

Now, concerning the KdV negative sub-hierarchy, consider the temporal gauge potential given by

$$A^{\text{KdV}}_{t_{-N}} = \mathcal{D}^{(-N-2)}_{-N} + \mathcal{D}^{(-N-1)}_{-N} + \cdots + \mathcal{D}^{(-1)}_{-N} \qquad (\mathcal{D}^{(a)}_{-N} \in \mathcal{G}_a), \tag{2.8}$$

which can also be determined by the decomposition of the zero curvature condition. Unlike the mKdV case (2.7), we obtain the following equation from the lowest grade,

$$\left[A_{-1}, \mathcal{D}^{(-N-2)}_{-N}\right] = 0. \tag{2.9}$$

Since $A_{-1} = J(x, t_{-N}) E^{(0)}_{-\alpha_1}$, the above relation will be only satisfied if the element $\mathcal{D}^{(-N-2)}_{-N}$ is proportional to $E^{(-m)}_{-\alpha_1}$. This condition constraints the negative temporal flows to be $N = 2m+1$, i.e., *the negative sub-hierarchy of KdV admits only odd flows.*

For the **mKdV hierarchy**, we have the following equations of motion for the few first temporal flows:

- **for** $N = 1$**:**

$$\partial_{t_1} v = \partial_x v, \tag{2.10}$$

which is just the wave equation.

- **for** $N = 3$**:**

$$4\partial_{t_3} v = \partial_x^3 v - 6v^2 \partial_x v \tag{2.11}$$

we obtain the well-known mKdV equation, which names the whole hierarchy.

- **for** $N = -1$**:**

$$\partial_{t_{-1}} \partial_x \phi = 2\sinh(2\phi), \tag{2.12}$$

we get the well-known sinh-Gordon equation. Here, we have introduced a convenient reparametrization for the field, $v(x, t_{-N}) = \partial_x \phi(x, t_{-N})$.

- **for** $N = -2$**:**

$$\partial_{t_{-2}} \partial_x \phi = -2 \left(e^{-2\phi} \partial_x^{-1} e^{2\phi} + e^{2\phi} \partial_x^{-1} e^{-2\phi} \right), \tag{2.13}$$

where the anti-derivative operator is defined by $\partial_x^{-1} f = \int^x f(y) dy$.

For the **KdV hierarchy**, we have the following equations of motion for the first temporal flows:

- **for** $N = 1$**:**

$$\partial_{t_1} J = \partial_x J, \tag{2.14}$$

corresponds to the wave equation.

- **for** $N = 3$**:**

$$4\partial_{t_3} J = \partial_x^3 J - 6J \partial_x J, \tag{2.15}$$

gives the well-known KdV equation, which names the hierarchy.

- **for** $N = -1$**:**

$$\partial_{t_{-1}} \partial_x^3 \eta - 4\partial_x \eta \partial_{t_{-1}} \partial_x \eta - 2\partial_x^2 \eta \partial_{t_{-1}} \eta = 0, \tag{2.16}$$

where we have defined $J(x, t_{-N}) = \partial_x \eta(x, t_{-N})$. This equation is the counterpart of the sinh-Gordon equation in the KdV hierarchy.

3 Gauge-Miura transformations

The mKdV and KdV equations can be related through the well-known *Miura transformation*, initially proposed in [7]. Other formulations have been proposed to perform this procedure, such as [8, 9], including mappings not only between individual equations but also between each flow of both hierarchies. For the case of $sl(2)$ algebra, this was achieved for the temporal flows in [12], and generalized to $sl(n+1)$ in [13]. More recently, the relation between negative temporal flows has been addressed using an approach in which Miura transformations are performed as gauge transformations that links the gauge potentials of both hierarchies, as discussed in [17, 4].

Our proposal to establish a connection between the two hierarchies involves relating the spatial gauge potentials (2.1) and (2.2) by a gauge transformation, which we have dubbed the *gauge-Miura transformation*,

$$A_x^{\mathrm{KdV}} = S_\pm A_x^{\mathrm{mKdV}} S_\pm^{-1} + S_\pm \partial_x S_\pm^{-1}, \tag{3.1}$$

where

$$S_+ = \begin{pmatrix} 1 & 0 \\ v & 0 \end{pmatrix} \qquad \text{and} \qquad S_- = \begin{pmatrix} 0 & 1/\lambda \\ 1 & -v/\lambda \end{pmatrix}. \tag{3.2}$$

The spatial gauge potentials (2.1) and (2.2) satisfy the relation (3.1) provided that the mKdV and KdV fields are related as follows,

$$J = v^2 \mp \partial_x v, \tag{3.3}$$

which are precisely the well-known *Miura transformations*, depending on the respective transformation $S_\pm$. These gauge-Miura transformations also act on all temporal potentials of both hierarchies,

$$A_{t_N}^{\mathrm{KdV}} = S_\pm A_{t_N}^{\mathrm{mKdV}} S_\pm^{-1} + S_\pm \partial_{t_N} S_\pm^{-1}. \tag{3.4}$$

It was shown in [4, 17], that the *positive flows* between the mKdV and KdV hierarchies can be related in a *one-to-one* correspondence,

$$t_N^{\mathrm{mKdV}} \xrightarrow{S} t_N^{\mathrm{KdV}}, \quad N = 1, 3, \dots \tag{3.5}$$

On the other hand, the negative flows satisfy a quite peculiar two-to-one correspondence, namely

$$\begin{tikzcd} t_{-N}^{\mathrm{mKdV}} \arrow[dr, "S"] & \\ & t_{-N}^{\mathrm{KdV}} \\ t_{-N-1}^{\mathrm{mKdV}} \arrow[ur, "S"'] & \end{tikzcd} \tag{3.6}$$

$N = 1, 3, \dots$. In the case of negative flows an additional relation is required involving time derivatives, *temporal Miura* (see [4]). For example, eqn. (3.4) for $N = -1$ is valid provided the following condition is satisfied,

$$\partial_{t_{-1}} \eta = 2\, e^{-2\phi(x, t_{-1})}. \tag{3.7}$$

On the other hand, if the map occurs between t_{-2}^{mKdV} and t_{-1}^{KdV} the relation is completely different

$$\partial_{t_{-1}}\eta = 4\, e^{-2\phi(x,t_{-2})}\partial_x^{-1}(e^{2\phi(x,t_{-2})}), \tag{3.8}$$

and similarly for lower flows. Notice that (3.7) involves solution of mKdV according to flow t_{-1} while (3.8) involves solution according to t_{-2}.

These correspondences are particularly useful when analyzing the solutions of these equations. For the positive flows, for each mKdV solution, we can obtain two distinct solutions for the corresponding KdV equation through (3.3). In contrast, for the negative flows, there is a greater degeneracy of solutions for KdV due to the two-to-one correspondence between the flows. That is, for each pair of negative mKdV flows related to their negative KdV counterpart, there will be four distinct solutions, see [18].

Another relevant fact about both hierarchies is their behavior concerning vacuum solutions, i.e., trivial solutions. For the mKdV hierarchy, the positive flows admit both zero, $v = 0$ and non-zero constant vacuum solutions $v = v_0$. However, the odd negative flows of the mKdV hierarchies admit only zero vacuum solutions, while the even negative flows admits strictly non-zero vacuum solution. In [4], we discuss that such classification in terms of vacuum orbits and define two different hierarchies: mKdV-I and mKdV-II.The mKdV-I contains positive and negative odd flows and is defined in the orbit of a zero vacuum. The mKdV-II, in turn contains positive odd and negative even flows and is defined in the orbit of strictly non-zero vacuum. This analysis can be extended to the KdV hierarchy through Miura transformations as shown in diagrams (3.5) and (3.6) how the two different vacuum structures are couched within the two, KdV and mKdV hierarchies.

In the next sections, we will present the main elements for constructing the supersymmetric versions of the mKdV and KdV hierarchies. We derive the equations of motion for the first temporal flows, and show how to derive the supersymmetric version of the gauge-Miura transformations, along with their implications for the correspondence between the flows of both hierarchies.

4 Super mKdV hierarchy

In this section we extend previous results to supersymmetric KdV/mKdV hierarchies. The algebraic structure is based on the super Kac-Moody algebra $sl(2,1)$, endowed with the principal grading operator Q, and the grade one constant element $\mathcal{E}^{(1)}$, which decompose the algebra into graded subspaces $\mathcal{G}_n$, of grade n. All the details can be found in appendix B (see also [10], [11]).

Consider the following spatial Lax operator [17]:

$$A_x^{\text{SmKdV}} = \mathcal{E}^{(1)} + A_0 + A_{\frac{1}{2}}, \tag{4.1}$$

where $\mathcal{E}^{(1)} = K_1^{(1)} + K_2^{(1)} \in \mathcal{G}_1$ is the grade one constant element, and $A_0 = v\; h_1^{(0)} \in \mathcal{G}_0$ and $A_{\frac{1}{2}} = \bar{\psi}\; G_2^{(\frac{1}{2})} \in \mathcal{G}_{1/2}$ contain the bosonic and fermionic fields of the model, $v(x,t_N)$ and $\bar{\psi}(x,t_N)$, respectively. Concerning the smKdV positive sub-hierarchy, the temporal gauge potential is defined as follows,

$$A_{t_N}^{\text{SmKdV}} = D_N^{(N)} + D_N^{(N-1/2)} + \cdots + D_N^{(1/2)} + D_N^{(0)} \qquad (D_N^{(a)} \in \mathcal{G}_a). \tag{4.2}$$

By decomposing the zero curvature condition (1.1), we find the following set of equations

$$\left[\mathcal{E}^{(1)}, D_N^{(N)}\right] = 0, \tag{4.3a}$$

$$\left[\mathcal{E}^{(1)}, D_N^{(N-\frac{1}{2})}\right] + \left[A_{\frac{1}{2}}, D_N^{(N)}\right] = 0, \tag{4.3b}$$

$$\left[\mathcal{E}^{(1)}, D_N^{(N-1)}\right] + \left[A_{\frac{1}{2}}, D_N^{(N-\frac{1}{2})}\right] + \left[A_0, D_N^{(N)}\right] + \partial_x D_N^{(N)} = 0, \tag{4.3c}$$

$$\vdots$$

$$\left[A_{\frac{1}{2}}, D_N^{(0)}\right] + \left[A_0, D_N^{(\frac{1}{2})}\right] + \partial_x D_N^{(\frac{1}{2})} - \partial_{t_N} A_{\frac{1}{2}} = 0, \tag{4.3d}$$

$$\left[A_0, D_N^{(0)}\right] + \partial_x D_N^{(0)} - \partial_{t_N} A_0 = 0 \tag{4.3e}$$

Since our model is based on a superalgebra, the ansatz for the temporal gauge potential (4.2) contains elements with semi-integer grades. From the highest grade (4.3a), we find that $D_N^{(N)}$ must belong to the kernel of $\mathcal{E}^{(1)}$, implying that $N = 2m + 1$, with $m \in \mathbb{N}$ (see details in (B.4)). Thus, the positive flows of the super mKdV hierarchy are constrained to be odd. The first non-trivial positive temporal flow occurs when $N = 3$, which yields the following equations of motion,

$$4\partial_{t_3} v = \partial_x^3 v - 6v^2 \partial_x v - 3\bar{\psi}\partial_x \left(v\partial_x \bar{\psi}\right), \tag{4.4a}$$

$$4\partial_{t_3} \bar{\psi} = \partial_x^3 \bar{\psi} - 3v\partial_x \left(v\bar{\psi}\right), \tag{4.4b}$$

which are the well-known *Super mKdV equations.* Another relevant flow is for $N = 1/2$, which provides us with supersymmetry transformations relating the bosonic and fermionic fields of our model, as follows

$$\partial_{t_{\frac{1}{2}}} v = \xi\, \partial_x \bar{\psi}, \tag{4.5a}$$

$$\partial_{t_{\frac{1}{2}}} \bar{\psi} = \xi\, v, \tag{4.5b}$$

where ξ is a fermionic constant. It can be verified that equations (4.4a) and (4.4b) are invariant under these supersymmetric transformations.

To complete the construction of our hierarchy, let us propose the gauge potential for the negative sub-hierarchy of super mKdV as

$$A_{t_{-N}}^{\mathrm{SmKdV}} = D_{-N}^{(-N)} + D_{-N}^{(-N+\frac{1}{2})} + \cdots + D_{-N}^{(-\frac{1}{2})} \qquad (D_{-N}^{(a)} \in \mathcal{G}_a). \tag{4.6}$$

It can be solved from the zero curvature equation (1.1) recursively from the following equations,

$$\left[A_0, D_{-N}^{(-N)}\right] + \partial_x D_{-N}^{(-N)} = 0, \tag{4.7a}$$

$$\left[A_{\frac{1}{2}}, D_{-N}^{(-N)}\right] + \left[A_0, D_{-N}^{(-N+\frac{1}{2})}\right] + \partial_x D_{-N}^{(-N+\frac{1}{2})} = 0, \tag{4.7b}$$

$$\vdots$$

$$\left[\mathcal{E}^{(1)}, D_{-N}^{(-1)}\right] + \left[A_0, D_{-N}^{(-\frac{1}{2})}\right] - \partial_{t_{-N}} A_0 = 0, \tag{4.7c}$$

$$\left[\mathcal{E}^{(1)}, D_{-N}^{(-\frac{1}{2})}\right] - \partial_{t_{-N}} A_{\frac{1}{2}} = 0. \tag{4.7d}$$

Unlike the positive flows, in this case we do not get any restriction upon the values of N. The $N=-1$ temporal flow provide us with the following temporal Lax

$$A_{t_{-1}}^{\mathrm{SinhG}} = \cosh 2\phi\; K_1^{(-1)} + K_2^{(-1)} - \sinh 2\phi\; M_2^{(-1)} - \psi \sinh\phi\; F_2^{(-\frac{1}{2})} - \psi\cosh\phi\; G_1^{(-\frac{1}{2})} \tag{4.8}$$

associated with the super sinh-Gordon equation,

$$\partial_x \partial_{t_{-1}} \phi = 2\sinh 2\phi - 2\bar{\psi}\psi \sinh\phi, \tag{4.9a}$$

$$\partial_{t_{-1}} \bar{\psi} = 2\psi \cosh\phi, \tag{4.9b}$$

$$\partial_x \psi = 2\bar{\psi}\cosh\phi, \tag{4.9c}$$

where $v(x,t_N) = \partial_x \phi(x,t_N)$. For $N=-2$, we obtain,

$$\begin{aligned} A_{t_{-2}}^{\mathrm{mKdV}} &= M_1^{(-2)} - \frac{e^{\phi}\psi_-}{2}\left(F_1^{(-\frac{3}{2})} + G_2^{(-\frac{3}{2})}\right) - \frac{e^{-\phi}\psi_+}{2}\left(F_1^{(-\frac{3}{2})} - G_2^{(-\frac{3}{2})}\right) \\ &\quad + a_-\left(K_1^{(-1)} + M_2^{(-1)}\right) - a_+\left(K_1^{(-1)} - M_2^{(-1)}\right) + \left(1+\psi_-\psi_+\right) K_2^{(-1)} \\ &\quad + \Omega_+\left(F_2^{(-\frac{1}{2})} + G_1^{(-\frac{1}{2})}\right) + \Omega_-\left(F_2^{(-\frac{1}{2})} - G_1^{(-\frac{1}{2})}\right) \end{aligned} \tag{4.10}$$

yielding the pair of equations

$$\partial_{t_{-2}} \partial_x \phi = -2\left(a_- + a_+\right) + 2\bar{\psi}\left(\Omega_- + \Omega_+\right), \tag{4.11a}$$

$$\partial_{t_{-2}} \bar{\psi} = -2\left(\Omega_+ - \Omega_-\right), \tag{4.11b}$$

where

$$\psi_\pm = \partial_x^{-1}\left(e^{\pm\phi}\bar{\psi}\right), \tag{4.12a}$$

$$a_\pm = e^{\pm 2\phi}\partial_x^{-1}\left[e^{\mp 2\phi}\left(1 + \psi_\mp \partial_x \psi_\pm\right)\right], \tag{4.12b}$$

and

$$\Omega_\pm = \frac{e^{\pm\phi}}{2}\partial_x^{-1}\left[e^{\mp 2\phi}\psi_\pm - \bar{\psi}_\mp \mp \partial_x \psi_\mp \partial_x^{-1}\left(\mp 2a_\pm + \bar{\psi}_- \bar{\psi}_+ + 1\right)\right]. \tag{4.13}$$

In the following section, we will systematize the supersymmetric hierarchy of KdV.

5 Super KdV Hierarchy

The super KdV hierarchy shares the same algebraic structure as the super mKdV hierarchy, based on the superalgebra $sl(2,1)$ detailed in Appendix B, and is characterized by the spatial gauge potential,

$$A_x^{\text{SKdV}} = \mathcal{E}^{(1)} + A_{-1} + A_{-\frac{1}{2}}, \tag{5.1}$$

where $\mathcal{E}^{(1)}$ is the same given in eqn. (4.1) for the SmKdV, $A_{-1} = J\ E_{\alpha_1}^{(0)} \in \mathcal{G}_{-1}$ contains the bosonic field, $J(x,t_N)$ and $A_{-\frac{1}{2}} = \bar{\chi}\ (E_{\alpha_2}^{(-\frac{1}{2})} - E_{-\alpha_2}^{(-\frac{1}{2})}) \in \mathcal{G}_{-1/2}$ the fermionic field $\bar{\chi}(x,t_N)$ of the theory.

The positive sub-hierarchy of sKdV is given by the temporal gauge potential defined as follows,

$$A_{t_N}^{\text{SKdV}} = \mathcal{D}_N^{(N)} + \mathcal{D}_N^{(N-1/2)} + \mathcal{D}_N^{(N-1)} + \cdots + \mathcal{D}_N^{(0)} + \mathcal{D}_N^{(-1/2)} + \mathcal{D}_N^{(-1)} \qquad (D_N^{(n)} \in \mathcal{G}_n). \tag{5.2}$$

By decomposing the zero curvature condition (1.1), we obtain,

$$\left[\mathcal{E}^{(1)}, \mathcal{D}_N^{(N)}\right] = 0, \tag{5.3a}$$

$$\left[\mathcal{E}^{(1)}, \mathcal{D}_N^{(N-\frac{1}{2})}\right] = 0, \tag{5.3b}$$

$$\partial_x \mathcal{D}_N^{(N)} + \left[\mathcal{E}^{(1)}, \mathcal{D}_N^{(N-1)}\right] = 0, \tag{5.3c}$$

$$\vdots$$

$$\partial_x \mathcal{D}_N^{(-\frac{1}{2})} + \left[A_{-1}, \mathcal{D}_N^{(\frac{1}{2})}\right] + \left[A_{-\frac{1}{2}}, \mathcal{D}_N^{(0)}\right] - \partial_{t_N} A_{-\frac{1}{2}} = 0, \tag{5.3d}$$

$$\partial_x \mathcal{D}_N^{(-1)} + \left[A_{-1}, \mathcal{D}_N^{(0)}\right] + \left[A_{-\frac{1}{2}}, \mathcal{D}_N^{(-\frac{1}{2})}\right] - \partial_{t_N} A_{-1} = 0. \tag{5.3e}$$

The highest grade eqn. (5.3a) implies that $\mathcal{D}_N^{(N)}$ must belong to the kernel of $\mathcal{E}^{(1)}$, and hence $N = 2m+1$, $m \in \mathbb{N}$ (see details in (B.4)). Thus, the positive flows of the super KdV hierarchy are odd. The first non-trivial positive temporal flow occurs for $N = 3$ yielding the *Super KdV equation*,

$$4\partial_{t_3} J = \partial_x^3 J - 6J\partial_x J - 3\bar{\chi}\partial_x^2 \bar{\chi}, \tag{5.4a}$$

$$4\partial_{t_3} \bar{\chi} = \partial_x^3 \bar{\chi} - 3\partial_x \left(J\bar{\chi}\right), \tag{5.4b}$$

As before, the positive temporal flow for $N = 1/2$ provides us with supersymmetry transformations namely,

$$\partial_{t_{\frac{1}{2}}} J = \xi\ \partial_x \bar{\chi}, \tag{5.5a}$$

$$\partial_{t_{\frac{1}{2}}} \bar{\chi} = \xi\ J, \tag{5.5b}$$

where ξ is a fermionic constant. The equations of motion (5.4) remains invariant under supersymmetry transformations (5.5). Now, to complete the construction of the sKdV hierarchy, let us consider the temporal gauge potential associated to negative grades,

$$A_{t_{-N}}^{\rm SKdV} = \mathcal{D}_{-N}^{(-N-2)} + \mathcal{D}_{-N}^{(-N-\frac{3}{2})} + \mathcal{D}_{-N}^{(-N-1)} + \cdots + D_{-N}^{(-1)} + D_{-N}^{(-\frac{1}{2})} \qquad (D_{-N}^{(n)} \in \mathcal{G}_n) \tag{5.6}$$

which by the decomposition of (1.1) leads to the following equations,

$$\left[A_{-1}, \mathcal{D}_{-N}^{(-N-2)}\right] = 0, \tag{5.7a}$$

$$\left[A_{-\frac{1}{2}}, \mathcal{D}_{-N}^{(-N-2)}\right] + \left[A_{-1}, \mathcal{D}_{-N}^{(-N-\frac{3}{2})}\right] = 0, \tag{5.7b}$$

$$\partial_x \mathcal{D}_{-N}^{(-N-2)} + \left[A_{-1}, \mathcal{D}_{-N}^{(-N-1)}\right] + \left[A_{-\frac{1}{2}}, \mathcal{D}_{-N}^{(-N-\frac{3}{2})}\right] = 0, \tag{5.7c}$$

$$\vdots$$

$$\partial_x \mathcal{D}_{-N}^{(-1)} + \left[A_{-\frac{1}{2}}, \mathcal{D}_{-N}^{(-\frac{1}{2})}\right] + \left[\mathcal{E}^{(1)}, \mathcal{D}_{-N}^{(-2)}\right] - \partial_{t_{-N}} A_{-1} = 0, \tag{5.7d}$$

$$\partial_x \mathcal{D}_{-N}^{(-\frac{1}{2})} + \left[\mathcal{E}^{(1)}, \mathcal{D}_{-N}^{(-\frac{3}{2})}\right] - \partial_{t_{-N}} A_{-\frac{1}{2}} = 0, \tag{5.7e}$$

$$\left[\mathcal{E}^{(1)}, \mathcal{D}_{-N}^{(-1)}\right] = 0, \tag{5.7f}$$

$$\left[\mathcal{E}^{(1)}, \mathcal{D}_{-N}^{(-\frac{1}{2})}\right] = 0. \tag{5.7g}$$

From the lowest-grade equation (5.7a), we deduce that the $\mathcal{D}_{-N}^{(-N-2)}$ must be proportional to $E_{-\alpha_1}^{(m)} \in \mathcal{G}_{2m+1}$, thus the temporal flows are always odd, i.e., $N = 2m+1$. Considering the first negative temporal flow, we find for the temporal gauge potential:

$$A_{t_{-1}}^{\rm SKdV} = \mathcal{D}_{-1}^{(-3)} + \mathcal{D}_{-1}^{(-\frac{5}{2})} + \mathcal{D}_{-1}^{(-2)} + \mathcal{D}_{-1}^{(-\frac{3}{2})} + \mathcal{D}_{-1}^{(-1)} + \mathcal{D}_{-1}^{(-\frac{1}{2})} \tag{5.8}$$

with

$$\mathcal{D}_{-1}^{(-3)} = -\frac{1}{8}\left\{\partial_x(\partial_{t_{-1}}\partial_x\eta + \partial_x\gamma) - 2\partial_x\eta\left(\partial_{t_{-1}}\eta + \gamma\right)\right.$$
$$\left. +\partial_x\bar{\eta}\left(\bar{\nu}_- - \bar{\nu}_+ + 2\bar{\gamma} - \partial_{t_{-1}}\partial_x\bar{\gamma}\right)\right\}\left(K_1^{(-3)} - M_2^{(-3)}\right), \tag{5.9a}$$

$$\mathcal{D}_{-1}^{(-\frac{5}{2})} = \frac{1}{8}\left(\partial_x(\bar{\nu}_+ + \bar{\nu}_-) - \partial_{t_{-1}}\partial_x^2\bar{\gamma} + 2\partial_x\bar{\gamma}\,\gamma + 2\partial_x\bar{\gamma}\right)\left(F_2^{(-\frac{5}{2})} + G_1^{(-\frac{5}{2})}\right), \tag{5.9b}$$

$$\mathcal{D}_{-1}^{(-2)} = \frac{1}{4}\left(\partial_{t_{-1}}\partial_x\eta + \partial_x\gamma\right)\; M_1^{(-2)}, \tag{5.9c}$$

$$\mathcal{D}_{-1}^{(-\frac{3}{2})} = \frac{1}{4}\left(\bar{\nu}_+ - \bar{\nu}_- - 2\bar{\gamma}\right)\; F_1^{(-\frac{3}{2})} - \frac{1}{4}\partial_{t_{-1}}\partial_x\bar{\gamma}\; G_2^{(-\frac{3}{2})}, \tag{5.9d}$$

$$\mathcal{D}_{-1}^{(-1)} = \frac{1}{2}\gamma\left(K_1^{(-1)} + K_2^{(-1)}\right) + \frac{1}{2}\partial_{t_{-1}}\eta\; K_1^{(-1)} + K_2^{(-1)}, \tag{5.9e}$$

$$\mathcal{D}_{-1}^{(-\frac{1}{2})} = \frac{1}{2}\partial_{t_{-1}}\bar{\gamma}\; F_2^{(-\frac{1}{2})} \tag{5.9f}$$

Here we have used the following relations,

$$J(x,t_N) = \partial_x\eta(x,t_N), \qquad \bar{\chi}(x,t_N) = \partial_x\bar{\gamma}(x,t_N), \qquad \partial_x\gamma = \partial_x\bar{\gamma}\partial_{t_{-1}}\bar{\gamma}, \tag{5.10a}$$

$$\partial_x\bar{\nu}_+ = \partial_{t_{-1}}\eta\partial_x\bar{\gamma}, \qquad \partial_x\bar{\nu}_- = \partial_x\eta\partial_{t_{-1}}\bar{\gamma}. \tag{5.10b}$$

yielding the following pair of equations of motion,

$$\partial_x^3\partial_{t_{-1}}\eta + \partial_x^2\gamma - 2\partial_x^2\eta\left(\partial_{t_{-1}}\eta + \gamma\right) - 4\partial_x\eta\partial\partial_x\eta + \partial_x\bar{\gamma}\left(\partial_{t_{-1}}\bar{\gamma}\partial_x\eta - \partial_{t_{-1}}\partial_x\bar{\gamma} - \partial\partial_x^2\bar{\gamma}\right)$$
$$+ \partial_x^2\bar{\gamma}\left(\bar{\nu}_- - \bar{\nu}_+ + 2\bar{\gamma} - \partial_{t_{-1}}\partial_x\bar{\gamma}\right) - 2\partial_x\eta\partial_x\bar{\gamma}\partial_{t_{-1}}\bar{\gamma} = 0, \tag{5.11a}$$

$$\partial_x\left(\partial_{t_{-1}}\eta\partial_x\bar{\gamma} + \partial_x\eta\partial_{t_{-1}}\bar{\gamma} - \partial_{t_{-1}}\partial_x^2\bar{\gamma} + 2\partial_x\bar{\gamma}\right) + \partial_x\eta\left(\bar{\nu}_+ - \bar{\nu}_- - 2\bar{\gamma} + 2\partial_{t_{-1}}\partial_x\bar{\gamma}\right)$$
$$+ \partial_x\bar{\gamma}\left(\partial_{t_{-1}}\partial_x\eta + \partial_x\gamma\right) = 0. \tag{5.11b}$$

A natural step now is to determine the connection between the *SmKdV* and *SKdV* hierarchies using a gauge-Miura transformation.

6 Gauge Super Miura transformation

Now that we have established the structure for both SmKdV and SKdV Hierarchies, we are able to proceed in determining the Super Miura transformation via gauge transformation, in such a way that it will be possible to map not only the SmKdV equation into SKdV equation, but also the entire hierarchy, including the negative flows. The unifying element employed here is the mapping the spatial Lax operators which in the matrix form can be written as

$$A_x^{\mathrm{SKdV}} = \begin{pmatrix} \sqrt{\lambda} & 1 & 0 \\ J+\lambda & \sqrt{\lambda} & \bar{\chi} \\ -\bar{\chi} & 0 & 2\sqrt{\lambda} \end{pmatrix}, \qquad A_x^{\mathrm{SmKdV}} = \begin{pmatrix} \sqrt{\lambda}+v & 1 & -\bar{\psi} \\ \lambda & \sqrt{\lambda}-v & \sqrt{\lambda}\bar{\psi} \\ \sqrt{\lambda}\bar{\psi} & -\bar{\psi} & 2\sqrt{\lambda} \end{pmatrix}. \tag{6.1}$$

We therefore search for a gauge transformation $\mathcal{S}$ such that,

$$A_x^{\mathrm{SKdV}} = \mathcal{S}A_x^{\mathrm{SmKdV}}\mathcal{S}^{-1} + \mathcal{S}\partial_x\mathcal{S}^{-1}. \tag{6.2}$$

From the experience gathered with the bosonic case [13] we propose the following ansatz,

$$\mathcal{S} = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} + \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{\lambda}}a_{13} \\ 0 & 0 & a_{23} \\ a_{31} & \frac{1}{\sqrt{\lambda}}a_{32} & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{\lambda}}b_{11} & \frac{1}{\lambda}a_{12} & 0 \\ a_{21} & \frac{1}{\sqrt{\lambda}}b_{22} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\lambda}}b_{33} \end{pmatrix} \tag{6.3}$$

which leads to two different solutions,

$$\mathcal{S}_\pm = \begin{pmatrix} 1 & 0 & 0 \\ v & 1 & -\bar{\psi} \\ \pm\bar{\psi} & 0 & \pm 1 \end{pmatrix}. \tag{6.4}$$

These, in turn leads to the *Super Miura transformation* relating SKdV and SmKdV field variables [1] [2]

$$\begin{aligned} J^{(\pm)} &= v^2 - \partial_x v + \bar{\psi}\partial_x\bar{\psi}, \\ \bar{\chi}^{(\pm)} &= \mp v\bar{\psi} \pm \partial_x\bar{\psi}. \end{aligned} \tag{6.5}$$

We notice that if we consider the inversion matrix L_- acting upon the fermionic subspace, i.e.,

$$L_- = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \tag{6.6}$$

the following relations holds

$$\mathcal{S}_- = L_-\,\mathcal{S}_+.$$

Thus, we can proceed assuming $\mathcal{S} = \mathcal{S}_+$ without loss of generalization. Such fact is very convenient, as is possible to write the gauge-Miura $\mathcal{S}_+$ in the exponential form

$$\mathcal{S} \equiv \mathcal{S}_+ = e^{\frac{v}{2}\left(K_1^{(-1)} - M_2^{(-1)}\right) - \frac{\bar{\psi}}{2}\left(F_2^{(-\frac{1}{2})} + G_1^{(-\frac{1}{2})}\right)}. \tag{6.7}$$

As expected, we have verified after tedious but straightforward calculation that the Super gauge Miura transformation indeed maps the temporal Lax $A_{t_3}^{\mathrm{SmKdV}}$ into $A_{t_3}^{\mathrm{SKdV}}$ connecting equations (4.4) and (5.4). In fact, for higher positive flows it is possible to show using

[1]It also possible to obtain a second pair of Super gauge Miura transformation using a second matrix with a different ansatz, given by $\mathcal{S}_{2,\pm} = \begin{pmatrix} 0 & \frac{1}{\lambda} & 0 \\ 1 & -\frac{v}{\lambda} & \bar{\psi} \\ 0 & \mp\bar{\psi} & \frac{\pm 1}{\sqrt{\lambda}} \end{pmatrix}$ associated with the two pairs of super Miura transformation: $J^{(2,\pm)} = v^2 + \partial_x v + \bar{\psi}\partial_x\bar{\psi}$ and $\bar{\chi}^{(2,\pm)} = \mp v\bar{\psi} \mp \partial_x\bar{\psi}$. This choice leads to the Mathieu's Super Miura transformation [19] under $\bar{\chi} \to i\bar{\chi}$ and $\bar{\psi} \to i\bar{\psi}$ transformation.

[2]The fact that exists four different Super Miura transformation is a manifestation of the symmetry of Super mKdV equation under the parity transformation $v \to -v$ and $\bar{\psi} \to -\bar{\psi}$.

only the exponential form (6.7) together with (6.5) that each positive *SmKdV* flow can be mapped into its corresponding *SKdV* flow:

$$t_N^{\text{SmKdV}} \xrightarrow{\mathcal{S}} t_N^{\text{SKdV}}. \tag{6.8}$$

Let us now extend our analysis to the negative sector using the exponential form of $\mathcal{S}$ (6.7). Consider first a generic negative *odd* flow $A_{t_{-2n+1}}^{\text{SmKdV}}$ under the gauge transformation induced by $\mathcal{S}$. The gauge transformation (6.7) yields the following graded structure

$$\begin{aligned} A_{t_{-2n+1}}^{\text{SKdV}} &\equiv \mathcal{S}\left(D_{-2n+1}^{(-2n+1)} + D_{-2n+1}^{(-2n+\frac{3}{2})} + \cdots + D_{-2n+1}^{(-\frac{1}{2})}\right)\mathcal{S}^{-1} + \mathcal{S}\partial_{t_{-2n+1}}\mathcal{S}^{-1} \\ &= \mathcal{D}_{-2n+1}^{(-2n-1)} + \mathcal{D}_{-2n+1}^{(-2n+\frac{1}{2})} + \mathcal{D}_{-2n+1}^{(-2n)} + \cdots + \mathcal{D}_{-2n+1}^{(-1)} + \mathcal{D}_{-2n+1}^{(-\frac{1}{2})}. \end{aligned} \tag{6.9}$$

On the other hand, if we now consider the subsequent negative *even* flow $A_{t_{-2n}}^{\text{SmKdV}}$, its lower operator is now proportional to $D_{-2n}^{(-2n)} \sim M_1^{(-2n)}$, such that the final transformation presents the same algebraic structure, i.e.,

$$\begin{aligned} \widetilde{A}_{t_{-2n+1}}^{\text{SKdV}} &\equiv \mathcal{S}\left(D_{-2n}^{(-2n)} + D_{-2n}^{(-2n+\frac{1}{2})} + \cdots + D_{-2n}^{(-\frac{1}{2})}\right)\mathcal{S}^{-1} + \mathcal{S}\partial_{t_{-2n}}\mathcal{S}^{-1} \\ &= \widetilde{\mathcal{D}}_{-2n+1}^{(-2n-1)} + \widetilde{\mathcal{D}}_{-2n+1}^{(-2n+\frac{1}{2})} + \widetilde{\mathcal{D}}_{-2n+1}^{(-2n)} + \cdots + \widetilde{\mathcal{D}}_{-2n+1}^{(-1)} + \widetilde{\mathcal{D}}_{-2n+1}^{(-\frac{1}{2})}. \end{aligned} \tag{6.10}$$

Since the potentials A_x^{SKdV} and A_x^{SmKdV} are universal within the hierarchies, the zero curvature condition for (6.9) and (6.10) must yield the same operator, i.e.,

$$A_{t_{-2n+1}}^{\text{SKdV}} = \widetilde{A}_{t_{-2n+1}}^{\text{SKdV}}, \tag{6.11}$$

and the two gauge potentials provide the same evolution equations. We therefore conclude that, as in the pure bosonic case, *subsequent negative integer odd and even SmKdV flows collapse into the same negative odd SKdV flow*, which is consistent with the fact that there is no *even* negative flow within the SKdV hierarchy. This can be illustrated by

$$\begin{array}{ccc} t_{-N}^{\text{SmKdV}} & & \\ & \overset{\mathcal{S}}{\searrow} & \\ & & t_{-N}^{\text{SKdV}} \\ & \underset{\mathcal{S}}{\nearrow} & \\ t_{-N-1}^{\text{SmKdV}} & & \end{array} \tag{6.12}$$

for $N = 2n-1$, $n = 1, 2, \ldots$.

In order to illustrate such phenomena, we consider an explicit example of $A_{t_{-1}}^{\text{SmKdV}}$ and $A_{t_{-2}}^{\text{SmKdV}}$ given by (4.8) and (4.10). Indeed the gauge transformation (6.2) results in the same temporal Lax $A_{t_{-1}}^{\text{SKdV}}$ (5.8).

Nevertheless, to obtain such result, it is necessary to introduce an additional information concerning *time derivatives*. If the mapping occurs between t_{-1}^{SmKdV} and t_{-1}^{SKdV}

$$A_{t_{-1}}^{\text{SKdV}} = \mathcal{S}A_{t_{-1}}^{\text{SmKdV}}\mathcal{S}^{-1} + \mathcal{S}\partial_x\mathcal{S}^{-1}, \tag{6.13}$$

the KdV fields $(\eta, \bar{\gamma})$ must obey the following relations

$$\eta_{t_{-1}} = 2\left(e^{-2\phi} + e^{-\phi}\psi\bar{\psi}\right), \tag{6.14}$$

$$\bar{\gamma}_{t_{-1}} = 2e^{-\phi}\psi, \tag{6.15}$$

where the mKdV fields in the r.h.s are solutions of t_{-1} eqns. (4.9) .

However, if we are mapping t_{-2}^{SmKdV} into t_{-1}^{SKdV}

$$A_{t_{-1}}^{\mathrm{SKdV}} = \mathcal{S}A_{t_{-2}}^{\mathrm{SmKdV}}\mathcal{S}^{-1} + \mathcal{S}\partial_x\mathcal{S}^{-1}, \tag{6.16}$$

this relation is completely different, namely

$$\eta_{t_{-1}} = 4\left(a_- + \Omega_-\bar{\psi} + \frac{1}{2}\bar{\psi}_+\bar{\psi}_-\right), \tag{6.17}$$

$$\bar{\gamma}_{t_{-1}} = 4\Omega_-, \tag{6.18}$$

where $\bar{\psi}_\pm$, $a_\pm$ and $\Omega_\pm$ are given by (4.12a), (4.12b) and (4.13) and satisfy eqns. (4.12).

Notice that such set of relations involving KdV variables, $(\eta, \bar{\gamma})$ define a distinct set of *solutions* for equation (5.11). One class of solutions must respect relations (6.5) together with the pair (6.14) and (6.15), and the second one must obeys (6.5), (6.14) and (6.15). This allows us to determine a larger range of solutions for the negative flows within the $SKdV$ Hierarchy.

7 Discussion and further developments

In this paper, we have discussed the Miura mapping between the mKdV and KdV flows, extending the approach already used in the pure bosonic case to the supersymmetric case based upon the $sl(2,1)$ affine algebra and the zero curvature representation. The approach employed here involves a gauge transformation acting upon the zero curvature condition. Such a framework has the virtue of relating the entire two hierarchies and henceforth is dubbed the Gauge Super Miura transformation. Using such an approach, we are able to recover well-known results such as the Super Miura transformation [19], and also, to discover new ones, such as the coalescence of two subsequent negative flows of SmKdV hierarchy into a single flow of the SKdV hierarchy. We also provide a complete algebraic formulation for the $SKdV$ hierarchy and extend the $SmKdV$ hierarchy [14] to the negative *even flows.* This result represents a generalization of the bosonic case proposed in [4] and demonstrates how an approach focused on algebraic structure together with the formulation of zero curvature enables a general structure that allows the discover of new results.

On the other hand, it is important to make some comments on such coalescence feature, in particular on the vacuum structure of the equations of motion involved. In order to construct solutions (or a family of solutions) for an integrable, model such as SmKdV, it is necessary to define a vacuum orbit, i.e., a simple solution which leads to more general ones. In such a scenario, soliton solutions can be obtained by gauge-transforming the gauge potentials in the vacuum, $A_\mu^{vac} = A_\mu(\phi_0)$, into a nontrivial configuration $A_\mu(\phi)$. Such a framework is the basis of the *dressing method* [15]. The fact that the flows share the same vacuum orbit is crucial to their involution, and guarantees the existence of the hierarchy [20].

It has been shown in several works [1, 16, 21] that not only is possible to have a zero vacuum orbit $(v, \bar{\psi}) = (0, 0)$, but it is also possible to use a non-zero vacuum orbit $(v, \bar{\psi}) = (v_0, \bar{\psi}_0)$. Now, it turns out that for the SmKdV system, we can verify that equation (4.4) admits, besides the zero and non-zero vacuum solutions, intermediary states as $(0, \bar{\psi}_0)$ and $(v_0, 0)$. However, this is not true for the negative flows. For instance, in the case of $N = -1$ (Super sinh-Gordon), only a vacuum solution is possible; and for $N = -2$, only non-zero bosonic vacuum is possible, $(v_0, \bar{\psi}_0)$ with $v_0 \neq 0$. In fact, this leads to two different SmKdV hierarchies:

- The SmKdV-I hierarchy has *negative odd* and *positive odd flows*, and is defined in the orbit of a *bosonic* and *fermionic zero vacuum*;
- The SmKdV-II hierarchy has *negative even* and *positive odd flows*, and is defined in the orbit of a *nonzero bosonic vacuum* and both *zero* or *nonzero fermionic vacuum*.

For the SKdV Hierarchy, this picture is completely different. As one might notice analyzing equation (5.4) and (5.11), all the equations shared the same vacuum orbit, either $(J, \bar{\chi}) = (0, 0)$, $(J, \bar{\chi}) = (J_0, \bar{\chi}_0)$, $(J, \bar{\chi}) = (0, \bar{\chi}_0)$ or $(J, \bar{\chi}) = (J_0, 0)$, are valid for all the equations[3]. In such a case,

- Each integrable model within the positive and negative part of the SKdV hierarchy admits both zero as well as nonzero vacuum solutions.

This feature certainly explains why the two flows of the negative part of the SmKdV model collapses into one SKdV time flow, since each one posses the vacuum configuration that is necessary for the negative SKdV time flow.

It would be interesting to construct *soliton solutions* for the *SmKdV* and the SKdV hierarchies, by implementing both the *dressing method*, and the Super gauge Miura transformation developed in this work. These issues also currently under investigations and will be reported elsewhere.

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[3] To verify this for the negative flows of SKdV, it is crucial to use the temporal Super Miura relations (6.14), (6.15) or (6.17), (6.18) due the existence of pure temporal derivatives in the equation of motion

A Algebra $sl(2)$

Consider the $\mathcal{G} = sl(2)$ centerless Kac-Moody algebra generated by[4]

$$\mathcal{G} = sl(2) = \left\{ h_1^{(m)} = \lambda^m h_1,\ E_{\alpha_1}^{(m)} = \lambda^m E_{\alpha_1},\ E_{-\alpha_1}^{(m)} = \lambda^m E_{\alpha_{-1}} \right\} \tag{A.1}$$

where α_1 is a simple root. The *principal grading operator* is defined by

$$Q = 2\hat{d} + \frac{1}{2} h_1, \tag{A.2}$$

where $\hat{d}$ is a derivation operator that satisfies

$$\left[\hat{d}, T_a^{(m)}\right] = m\, T_a^{(m)}, \qquad\qquad T_a^{(m)} \in \mathcal{G}$$

The *principal grading operator* decomposes the algebra in graded subspaces, $\mathcal{G} = \bigoplus_i \mathcal{G}_i$, where

$$[Q, \mathcal{G}_a] = a\, \mathcal{G}_a, \qquad [\mathcal{G}_a, \mathcal{G}_b] \in \mathcal{G}_{a+b},$$

for $a, b \in \mathbb{Z}$. For our purposes, the subspaces to consider are:

$$\begin{aligned} \mathcal{G}_{2m} &= \left\{ h_1^{(m)} \right\}, \\ \mathcal{G}_{2m+1} &= \left\{ E_{\alpha_1}^{(m)},\ E_{-\alpha_1}^{(m+1)} \right\}. \end{aligned} \tag{A.3}$$

Another key ingredient to construct our models is a grade one constant element,

$$E^{(1)} = E_{\alpha_1}^{(0)} + E_{-\alpha_1}^{(1)} = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \tag{A.4}$$

that decomposes the algebra into $\mathcal{G} = \mathcal{K} \bigoplus \mathcal{M}$ where $\mathcal{K}_E = \{x \in \mathcal{G} \,|\, [E, x] = 0\}$ and $\mathcal{M}_E$ its complement,

$$\mathcal{K}_E = \left\{ K_1^{(2m+1)} = E_{\alpha_1}^{(m)} + E_{-\alpha_1}^{(m+1)} \right\}, \tag{A.5a}$$

$$\mathcal{M}_E = \left\{ M_1^{(2m)} = h_1^{(m)},\ M_2^{(2m+1)} = E_{\alpha_1}^{(m)} + E_{-\alpha_1}^{(m+1)} \right\}. \tag{A.5b}$$

From (A.2) and (A.4), we can reorganize the graded subspaces (A.3) in terms of decomposition kernel-image as follows

$$\begin{aligned} \mathcal{G}_{2m} &= \left\{ M_1^{(2m)} \right\}, \\ \mathcal{G}_{2m+1} &= \left\{ K_1^{(2m+1)},\ M_2^{(2m+1)} \right\}. \end{aligned} \tag{A.6}$$

The commutation relations of the algebra are given by

$$\begin{aligned} \left[K_1^{(2m+1)}, K_1^{(2n+1)}\right] &= 0, & \left[M_1^{(2m)}, M_1^{(2n)}\right] &= 0, \\ \left[K_1^{(2m+1)}, M_1^{(2n)}\right] &= -2M_2^{(2m+2n+1)}, & \left[M_1^{(2m)}, M_2^{(2n+1)}\right] &= 2K_1^{(2m+2n+1)}, \\ \left[K_1^{(2m+1)}, M_2^{(2n+1)}\right] &= -2M_1^{(2(m+n+1))}, & \left[M_2^{(2m+1)}, M_2^{(2n+1)}\right] &= 0. \end{aligned}$$

[4]We employ the following representation for generators:
$h_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_{\alpha_1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{-\alpha_1} = E_{\alpha_1}^\dagger.$

B Superalgebra $sl(2,1)$

In this section we employ the algebraic formalism to construct an integrable hierarchies with supersymmetry. Consider the $\mathcal{G} = sl(2,1)$ centerless super Kac-Moody algebra generated by[5]

$$L_0 = \left\{ h_1^{(m)} = \lambda^m h_1,\ h_2^{(m)} = \lambda^m h_2,\ E_{\pm\alpha_1}^{(m)} = \lambda^m E_{\pm\alpha_1} \right\},$$

$$L_1 = \left\{ E_{\pm\alpha_2}^{(m)} = \lambda^m E_{\pm\alpha_2},\ E_{\pm(\alpha_1+\alpha_2)}^{(m)} = \lambda^m E_{\pm(\alpha_1+\alpha_2)} \right\},$$

where $\lambda \in \mathbb{C}$ and $m \in \mathbb{N}$. The L_0 and L_1 are called the bosonic and fermionic parts of algebra, respectively, that satisfying the following relations

$$[L_0, L_0] \subset L_0, \qquad [L_0, L_1] \subset L_1, \qquad [L_1, L_1] \subset L_0.$$

The *principal grading operator* is defined by

$$Q = 2\hat{d} + \frac{1}{2} h_1 \tag{B.1}$$

where $\hat{d}$ is a derivation operator that satisfies

$$\left[\hat{d}, T_a^{(m)}\right] = m\, T_a^{(m)}, \qquad T_a^{(m)} \in \mathcal{G}$$

The *principal grading operator* decomposes the algebra in graded subspaces, $\mathcal{G} = \bigoplus_i \mathcal{G}_i$, where

$$[Q, \mathcal{G}_a] = a\, \mathcal{G}_a, \qquad [\mathcal{G}_a, \mathcal{G}_b] \in \mathcal{G}_{a+b},$$

for $a, b \in \mathbb{Z}$. For our purposes, the subspaces to consider are:

$$\begin{aligned}
\mathcal{G}_{2m} &= \left\{ h_1^{(m)} \right\}, \\
\mathcal{G}_{2m+\frac{1}{2}} &= \left\{ E_{\alpha_2}^{(m+1/2)},\ E_{\alpha_1+\alpha_2}^{(m)},\ E_{-\alpha_2}^{(m)},\ E_{-\alpha_1-\alpha_2}^{(m+1/2)} \right\}, \\
\mathcal{G}_{2m+1} &= \left\{ E_{\alpha_1}^{(m)},\ E_{-\alpha_1}^{(m+1)},\ h_2^{(m+1/2)} \right\}, \\
\mathcal{G}_{2m+\frac{3}{2}} &= \left\{ E_{\alpha_2}^{(m+1/2)},\ E_{\alpha_1+\alpha_2}^{(m+1/2)},\ E_{-\alpha_2}^{(m+1/2)},\ E_{-\alpha_1-\alpha_2}^{(m+1)} \right\}.
\end{aligned} \tag{B.2}$$

Another key ingredient to construct our models is a grade one constant element,

$$\mathcal{E}^{(1)} = E_{\alpha_1}^{(0)} + E_{-\alpha_1}^{(1)} + h_1^{(1/2)} + 2h_2^{(1/2)} = \begin{pmatrix} \sqrt{\lambda} & 1 & 0 \\ \lambda & \sqrt{\lambda} & 0 \\ 0 & 0 & 2\sqrt{\lambda} \end{pmatrix}, \tag{B.3}$$

[5] We employ the following representation for generators:
$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $E_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$,
$E_{\alpha_1+\alpha_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $E_{-\alpha_1} = E_{\alpha_1}^\dagger$, $E_{-\alpha_2} = E_{\alpha_2}^\dagger$, $E_{-(\alpha_1+\alpha_2)} = E_{(\alpha_1+\alpha_2)}^\dagger$.

that decomposes the algebra into $\mathcal{G} = \mathcal{K} \bigoplus \mathcal{M}$ where $\mathcal{K}_{\mathcal{E}} = \{x \in \mathcal{G} \,|\, [E, x] = 0\}$ and $\mathcal{M}_{\mathcal{E}}$ its complement,

$$
\begin{aligned}
\mathcal{K}_{\text{Bose}} &= \mathcal{K}_{\mathcal{E}} \cap L_0 = \left\{K_1^{(2m+1}, \; K_2^{(2m+1)}\right\}, & \mathcal{M}_{\text{Bose}} &= \mathcal{M}_{\mathcal{E}} \cap L_0 = \left\{M_1^{(2m)}\right\}, \\
\mathcal{K}_{\text{Fermi}} &= \mathcal{K}_{\mathcal{E}} \cap L_1 = \left\{F_1^{(2m+1/2)}, \; F_2^{(2m+3/2)}\right\}, & \mathcal{M}_{\text{Fermi}} &= \mathcal{M}_{\mathcal{E}} \cap L_1 = \left\{G_1^{(2m+3/2)}, \; G_2^{(2m+1/2)}\right\},
\end{aligned}
\tag{B.4}
$$

where the bosonic generators are be defined as

$$
\begin{aligned}
K_1^{(2m+1)} &= E_{\alpha_1}^{(m)} + E_{-\alpha_1}^{(m+1)}, & M_1^{(2m)} &= h_1^{(m)}, \\
K_2^{(2m+1)} &= h_1^{(m+1/2)} + 2h_2^{(m+1/2)}, & M_2^{(2m+1)} &= E_{\alpha_1}^{(m)} - E_{-\alpha_1}^{(m+1)},
\end{aligned}
$$

and the fermionic generators are

$$
\begin{aligned}
F_1^{(2m+\frac{1}{2})} &= \left(E_{\alpha_2}^{(m+\frac{1}{2})} + E_{\alpha_1+\alpha_2}^{(m)}\right) + \left(E_{-\alpha_2}^{(m)} + E_{-(\alpha_1+\alpha_2)}^{(m+\frac{1}{2})}\right), \\
F_2^{(2m+\frac{3}{2})} &= \left(E_{\alpha_2}^{(m+1)} + E_{\alpha_1+\alpha_2}^{(m+\frac{1}{2})}\right) - \left(E_{-\alpha_2}^{(m+\frac{1}{2})} + E_{-(\alpha_1+\alpha_2)}^{(m+1)}\right), \\
G_1^{(2m+\frac{3}{2})} &= \left(E_{\alpha_2}^{(m+1)} - E_{\alpha_1+\alpha_2}^{(m+\frac{1}{2})}\right) + \left(E_{-\alpha_2}^{(m+\frac{1}{2})} - E_{-(\alpha_1+\alpha_2)}^{(m+1)}\right), \\
G_2^{(2m+\frac{1}{2})} &= \left(E_{\alpha_2}^{(m+\frac{1}{2})} - E_{\alpha_1+\alpha_2}^{(m)}\right) - \left(E_{-\alpha_2}^{(m)} - E_{-(\alpha_1+\alpha_2)}^{(m+\frac{1}{2})}\right).
\end{aligned}
$$

From (A.2) and (B.3), we can reorganize the graded subspaces (B.2) in terms of decomposition kernel-image as follows

$$
\begin{aligned}
\mathcal{G}_{2m} &= \left\{M_1^{(2m)}\right\}, \\
\mathcal{G}_{2m+\frac{1}{2}} &= \left\{F_1^{(2m+\frac{1}{2})}, \; G_2^{(2m+\frac{1}{2})}\right\}, \\
\mathcal{G}_{2m+1} &= \left\{K_1^{(2m+1)}, \; K_2^{(2m+1)}, \; M_2^{(2m+1)}\right\}, \\
\mathcal{G}_{2m+\frac{3}{2}} &= \left\{F_2^{(2m+\frac{3}{2})}, \; G_1^{(2m+\frac{3}{2})}\right\}.
\end{aligned}
\tag{B.6}
$$

The commutation relations of the algebra are then given by

$$
\begin{aligned}
\left[K_1^{(2m+1)}, K_1^{(2n+1)}\right] &= 0, \\
\left[K_1^{(2m+1)}, K_2^{(2n+1)}\right] &= 0, \\
\left[K_1^{(2m+1)}, M_1^{(2n)}\right] &= -2M_2^{(2m+2n+1)}, \\
\left[K_1^{(2m+1)}, M_2^{(2n+1)}\right] &= -2M_1^{(2(m+n+1))}, \\
\left[K_2^{(2m+1)}, K_2^{(2n+1)}\right] &= 0, \\
\left[K_1^{(2m+1)}, F_1^{(2n+1/2)}\right] &= F_2^{(2(m+n)+3/2)}, \\
\left[K_1^{(2m+1)}, F_2^{(2n+3/2)}\right] &= F_1^{(2(m+n+1)+1/2)}, \\
\left[K_1^{(2m+1)}, G_1^{(2n+3/2)}\right] &= -G_2^{(2(m+n+1)+1/2)}, \\
\left[K_1^{(2m+1)}, G_2^{(2n+1/2)}\right] &= -G_1^{(2(m+n)+3/2)}, \\
\left[K_2^{(2m+1)}, F_1^{(2n+1/2)}\right] &= -F_2^{(2(m+n)+3/2)}, \\
\left[K_2^{(2m+1)}, F_2^{(2n+3/2)}\right] &= -F_1^{(2(m+n+1)+1/2)}, \\
\left[K_2^{(2m+1)}, G_1^{(2n+3/2)}\right] &= -G_2^{(2(m+n+1)+1/2)}, \\
\left[K_2^{(2m+1)}, G_2^{(2n+1/2)}\right] &= -G_1^{(2(m+n)+3/2)},
\end{aligned}
\qquad
\begin{aligned}
\left[K_2^{(2m+1)}, M_1^{(2n)}\right] &= 0, \\
\left[K_2^{(2m+1)}, M_2^{(2n+1)}\right] &= 0, \\
\left[M_1^{(2m)}, M_1^{(2n)}\right] &= 0, \\
\left[M_1^{(2m)}, M_2^{(2n+1)}\right] &= 2K_1^{(2m+2n+1)}, \\
\left[M_2^{(2m+1)}, M_2^{(2n+1)}\right] &= 0. \\
\left[M_1^{(2m)}, F_1^{(2n+1/2)}\right] &= -G_2^{(2(m+2)+1/2)}, \\
\left[M_1^{(2m)}, F_2^{(2n+3/2)}\right] &= -G_1^{(2(m+2)+3/2)}, \\
\left[M_1^{(2m)}, G_1^{(2n+3/2)}\right] &= -F_2^{(2(m+2)+3/2)}, \\
\left[M_1^{(2m+1)}, G_2^{(2n+1/2)}\right] &= -F_1^{(2(m+2)+1/2)}, \\
\left[M_2^{(2m+1)}, F_1^{(2n+1/2)}\right] &= -G_1^{(2(m+n)+3/2)}, \\
\left[M_2^{(2m+1)}, F_2^{(2n+3/2)}\right] &= -G_2^{(2(m+n+1)+1/2)}, \\
\left[M_2^{(2m+1)}, G_1^{(2n+3/2)}\right] &= F_1^{(2(m+n+1)+1/2)}, \\
\left[M_2^{(2m+1)}, G_2^{(2n+1/2)}\right] &= F_2^{(2(m+n)+3/2)}.
\end{aligned}
$$

and the anti-commutations relations, given by

$$
\begin{aligned}
\left\{F_1^{(2m+1/2)}, F_1^{(2n+1/2)}\right\} &= 2(K_1^{(2m+2n+1)} + K_2^{(2m+2n+1)}), \\
\left\{F_1^{(2m+1/2)}, F_2^{(2n+3/2)}\right\} &= 0, \\
\left\{F_1^{(2m+1/2)}, G_1^{(2n+3/2)}\right\} &= -2M_1^{(2(m+n+1))}, \\
\left\{F_1^{(2m+1/2)}, G_2^{(2n+1/2)}\right\} &= -2M_2^{(2m+2n+1)}, \\
\left\{F_2^{(2m+3/2)}, F_2^{(2n+3/2)}\right\} &= -(K_1^{(2(m+n+1)+1)} + K_2^{(2(m+n+1)+1)}), \\
\left\{F_2^{(2m+3/2)}, G_1^{(2n+3/2)}\right\} &= 2M_2^{(2(m+n+1)+1)}, \\
\left\{F_2^{(2m+3/2)}, G_2^{(2n+1/2)}\right\} &= 2M_1^{(2(m+n+1))}, \\
\left\{G_1^{(2m+3/2)}, G_1^{(2n+3/2)}\right\} &= -2(K_1^{(2(m+n+1)+1)} - K_2^{(2(m+n+1)+1)}), \\
\left\{G_1^{(2m+3/2)}, G_2^{(2n+1/2)}\right\} &= 0, \\
\left\{G_2^{(2m+1/2)}, G_2^{(2n+1/2)}\right\} &= 2(K_1^{(2m+2n+1)} + K_2^{(2m+2n+1)}).
\end{aligned}
$$

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