THE CLASSIFICATION OF COMPLETE IMPROPER AFFINE SPHERES WITH SINGULARITIES OF LOW TOTAL CURVATURE AND NEW EXAMPLES

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Abstract. We classify complete improper affine spheres with singularities (say *improper affine fronts*) in unimodular affine three-space $\boldsymbol{R}^3$ whose total curvature is greater than or equal to -8π. We also study the asymptotic behavior of complete embedded ends of improper affine fronts. Moreover, we give new examples for this class of surfaces, including one which satisfies the equality condition of an Osserman-type inequality and is of positive genus.

1. Introduction

A locally strongly convex *improper affine sphere* is a surface in unimodular affine 3-space $\boldsymbol{R}^3$ whose affine Blaschke normal vector field is parallel and affine metric is definite (see Section 2). It is locally obtained as the graph of a smooth function $\varphi(x, y)$ on a planner domain satisfying the elliptic Monge–Ampère equation

$$\varphi_{xx}\varphi_{yy} - \varphi_{xy}^2 = 1. \tag{1.1}$$

For such surfaces, Ferrer, Martínez, and Milán established a Weierstrass-type representation formula as follows ([FMM96], [FMM99]) (see Fact 2.1 for more precise statement):

$$\psi := \left(\overline{F} + G, \frac{1}{2}(|G|^2 - |F|^2) + \operatorname{Re}\left(GF - 2\int F dG\right)\right) : \Sigma \to \boldsymbol{C} \times \boldsymbol{R} = \boldsymbol{R}^3, \tag{1.2}$$

where a pair (F, G) of holomorphic functions on a Riemann surface Σ is called *Weierstrass data*. However, as a global property, a Bernstein-type theorem for "complete" improper affine spheres is well known ([Cal58], [Cal88], [TW02], [KN12], [Kaw20], [KK24]). That is to say, any locally strongly convex affine complete (i.e., the affine metric is definite and complete) improper affine sphere is the *elliptic paraboloid* (Example 2.7). Thus, Martínez [Mar05a] introduced a concept of an *improper affine map* (Definition 2.2) (referred to as an *improper affine front* in this paper), which is defined by the same representation formula (1.2) and admits a certain kind of singularities. He also investigated a correlation between improper affine fronts and flat fronts in hyperbolic 3-space ([GMM00], [KUY04], [KRSUY05])

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in [Mar05b] and [MM14a]. In addition, Martínez, Milán, and Tenenblat [MMT15] gave a new method to transform improper affine fronts by applying the theory of Ribaucour transformations.

On the other hand, Martínez [Mar05a] introduced a completeness for improper affine fronts (Definition 2.4) like other classes of surfaces with singularities (e.g., the flat fronts in hyperbolic 3-space ([KUY04]), maximal surfaces in Lorentz–Minkowski 3-space ([UY06]), constant mean curvature 1 surfaces in de Sitter 3-space ([Fuj06])) and showed that the theory of complete improper affine fronts shares numerous global properties with the theory of complete minimal surfaces in Euclidian 3-space. As one of them, he proved a Huber–Osserman-type theorem ([Hub57], [Oss64]). Namely, the Riemann surface Σ is biholomorphic to a compact Riemann surface $\overline{\Sigma}$ minus finite points and the Weierstrass data (F, G) can be extended meromorphically to $\overline{\Sigma}$ (Fact 2.5). Moreover, he proposed a total curvature for complete improper affine fronts with respect to a certain complete Riemannian metric expressed in terms of Weierstrass data and showed an Osserman-type inequality

$$-\frac{1}{2\pi}\ \mathrm{TC}(\Sigma) \geq -\chi(\overline{\Sigma}) + 2(\text{number of ends}), \tag{1.3}$$

where $\mathrm{TC}(\Sigma)$ denotes the total curvature and $\chi(\overline{\Sigma})$ is the Euler number of $\overline{\Sigma}$ (Fact 2.6). In this paper, we study the following two topics for complete impoper affine fronts related to the minimal surface results.

Firstly, we describe an asymototic behavior of complete embedded ends of improper affine fronts in Section 3. Schoen [Sch83] proved that complete embedded ends of minimal surfaces in Euclidian 3-space with finite total curvature is asymptotic to either the plane or the catenoid. And also, Jorge and Meeks [JM83] showed a relation between embeddedness of ends and the equality of the Osserman inequality, and constructed the surface with high symmetry that attains the equality condition of the inequality. As affine correspondence to these results, we define a concept "asymptotic" for embedded ends of complete improper affine fronts and classify an asymptotic classes of embedded ends into three types (Theorem 3.2). It is associated with an equality condition of the Osserman-type inequality (Corollary 3.3). And we contstruct new examples (Examples 3.4, and 3.5) with embedded ends, which satisfy the equality condition of the Osserman-type inequality.

Secondly, we study a classification of complete improper affine fronts in terms of the total curvature in Section 4. Complete orientable minimal surfaces in Euclidian 3-space of low total curvature were classified by Osserman [Oss64] and López [Lóp92]. So this leads a natural problem "to classify complete improper affine fronts with low total curvature". The total curvature of improper affine fronts is $-2m\pi$, where m is the mapping degree of a certain holomorphic map called the *Lagrangian Gauss map* (Section 2). Here, we classify the surfaces of the total curvature greater than or equal to -8π. Namely, the main result of this paper is the following:

Theorem 1.1.
- *Complete improper affine fronts in $\boldsymbol{R}^3$ whose total curvature is greater than or equal to -6π are all genus 0 and constructed the Weierstrass data as in Theorems 4.2, 4.3, 4.5, 4.6.*
- *Genus of complete improper affine fronts with the total curvature -8π is less than or equal to 1.*
- *Complete improper affine fronts in $\boldsymbol{R}^3$ with the total curvature -8π and genus 0 are the surfaces described in Theorem 4.8.*
- *There exists a complete improper affine front in $\boldsymbol{R}^3$ with the total curvature -8π and genus 1. In particular, it is the one with the maximum total curvature and positive genus (Proposition 4.9 and Theorem 4.11).*

We show the existence in a special case for the fourth statement of Theorem 1.1. In addition, we know that there is at least one parameter family of complete improper affine fronts with the total curvature -8π and genus 1, each of which has different compex structure (Remark 4.12). The complete classification is an open problem in the genus 1 case.

Moreover, the only known example of complete improper affine fronts of genus 1 was composed by Martínez in [Mar05a, Section 4, No.6], whose total curvature is -12π. In the last of Section 4, we give a new example of genus 1 surface whose total curvature is -10π.

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2. Preliminaries

Firstly, we will briefly describe some definitions and fundamental facts about geometry of affine immersions in unimodular affine 3-space $\boldsymbol{R}^3$ (see [LSZ93] and [NS94] for datails). Let Σ be a connected and oriented 2-manifold, $\psi\colon \Sigma \to \boldsymbol{R}^3$ an immersion, and ξ a vector field of $\boldsymbol{R}^3$ along ψ which is transversal to $d\psi(T\Sigma)$. Then there uniquely exist a torsion-free affine connection ∇, a symmetric $(0,2)$-tensor h, a $(1,1)$-tensor S, and a 1-form τ on Σ, which satisfy

$$\left\{ \begin{array}{l} D_X d\psi(Y) = d\psi(\nabla_X Y) + h(X,Y)\xi, \\ D_X \xi = -d\psi(S(X)) + \tau(X)\xi, \end{array} \right. \tag{2.1}$$

where D is the canonical connection of $\boldsymbol{R}^3$, and X, Y are vector fields on Σ. Here, h is called the *affine metric* of ψ with respect to ξ. When h is definite, ψ is said to be *locally strongly convex.* Now on, we will only consider the case that ψ is locally strongly convex (for the indefinite case, see [Nak09], [Mil13], [Mil14], [MM14c], [MM15], [Mil20]). For given a locally strongly convex immersion ψ, one can uniquely choose the transversal vector field ξ which satisfies

$$\begin{cases} D_X\xi = -d\psi(S(X)), \\ \det(d\psi(X), d\psi(Y), \xi) = (h(X,X)h(Y,Y) - h(X,Y)^2)^{1/2}, \end{cases} \tag{2.2}$$

where det denotes the determinant function of $\boldsymbol{R}^3$. The transversal vector field ξ which satisfies (2.2) is called the *affine normal vector field*, and the pair (ψ, ξ) (or simply ψ) is called the *Blaschke immersion*. A Blaschke immersion ψ is said to be an *improper affine sphere* if $S = 0$ holds in (2.2). Then after equiaffine transformations of $\boldsymbol{R}^3$ ($\boldsymbol{R}^3 \ni \boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b} \in \boldsymbol{R}^3$, where A is a 3×3 matrix with $\det A = 1$, and $\boldsymbol{b} \in \boldsymbol{R}^3$), we can take the affine normal vector field ξ as $\xi = (0,0,1)$.

Next, for any improper affine sphere $\psi\colon \Sigma \to \boldsymbol{R}^3$, considering the conformal structure induced by the affine metric h, we regard Σ as a Riemann surface. In [Mar05a], Martínez introduced the following complex representation formula for improper affine spheres similar to the Weierstrass representation formula for minimal surfaces in Euclidian 3-space (see [Oss69]):

Fact 2.1. [FMM96, Theorem 4], [FMM99, Lemma 1], [Mar05a, Theorem 3] *Let Σ be a Riemann surface, and $(F, G) : \Sigma \to \boldsymbol{C}^2$ a complex regular curve (that is, F and G are holomorphic functions satisfying $(dF, dG) \neq (0,0)$) which satisfies $|dF| \neq |dG|$ and $\operatorname{Re}\int_\gamma F dG = 0$ for any closed curve γ in Σ. Then*

$$\psi := \left(\overline{F} + G, \frac{1}{2}(|G|^2 - |F|^2) + \operatorname{Re}\left(GF - 2\int F dG\right)\right) : \Sigma \to \boldsymbol{C} \times \boldsymbol{R} = \boldsymbol{R}^3 \tag{2.3}$$

gives an improper affine sphere with the affine normal vector field $\xi = (0,0,1)$. Conversely, any improper affine spheres $\psi : \Sigma \to \boldsymbol{R}^3$ with the affine normal $\xi = (0,0,1)$ are given in this way and the complex structure of the 2-manifold Σ is compatible to h.

The pair of holomorphic functions (F, G) is called the *Weierstrass data* of ψ. We find that the metric ds^2 represented as

$$ds^2 := \langle d\mathcal{X}, d\mathcal{X}\rangle = |dF|^2 + |dG|^2 + dFdG + \overline{dFdG} \tag{2.4}$$

is a non-degenerate flat metric, where $\mathcal{X} := \overline{F} + G$ is the two first coordinates in (2.3), and $\langle\ ,\ \rangle$ is the standard Euclidean inner product of $\boldsymbol{C} = \boldsymbol{R}^2$ under canonical identification. This metric ds^2 is called the *flat fundamental form*. Also, the affine metric h can be expressed as $h = |dG|^2 - |dF|^2$. When $|dG| = |dF|$ holds at a point (i.e, the affine metric h degenerates), the improper affine sphere ψ represented by (2.3) is not immersed. And the point also corresponds to the point where the flat fundamental form ds^2 degenerates. Hence, using the notations above, Martínez introduced the following concept of improper affine maps, which is a generalization of improper affine spheres in the sense of admitting singularities.

Definition 2.2. [Mar05a, Definition 1] Let Σ be a Riemann surface and $(F, G) : \Sigma \to \boldsymbol{C}^2$ a complex regular curve satisfying the *period condition*

$$\mathrm{Re} \int_\gamma F dG = 0 \tag{2.5}$$

for any closed curve γ in Σ. Then the map $\psi \colon \Sigma \to \boldsymbol{C} \times \boldsymbol{R} = \boldsymbol{R}^3$ given by

$$\psi := \left(G + \overline{F}, \frac{1}{2}(|G|^2 - |F|^2) + \mathrm{Re}\left(GF - 2 \int F dG \right) \right) \tag{2.6}$$

is called an *improper affine map*.

The singular points of an improper affine map correspond with the points where the affine metric h degenerates, and also with the points where ds^2 degenerates. As shown in [Nak09] and [UY11], an improper affine map becomes a (wave) front. Thus in this meaning, we call the improper affine map the *improper affine front* in this paper. The differential geometry of fronts is discussed in [SUY09] and [SUY22]. We remark that the improper affine fronts are a special case of affine maximal surfaces with singularities (say affine maximal maps). The affine maximal maps are investigated in [AMM09a], [AMM09b], [AMM09c], [AMM11].

Remark 2.3. For given an improper affine front $\psi \colon \Sigma \to \boldsymbol{R}^3$ with Weierstrass data (F, G), another improper affine front constructed from $(\widetilde{F}, \widetilde{G})$ defined by

$$(\widetilde{F}, \widetilde{G}) := (\alpha F + \beta G + \mu, \overline{\beta} F + \overline{\alpha} G + \lambda) \quad (\alpha, \beta, \mu, \lambda \in \boldsymbol{C}, |\alpha|^2 - |\beta|^2 = 1) \tag{2.7}$$

gives an equiaffinely equivalent improper affine front. In particular, for any $\mu, \lambda \in \boldsymbol{C}$, $(F + \mu, G + \lambda)$ gives a Weierstrass data of parallel translation of ψ in $\boldsymbol{R}^3$. Conversely, any improper affine fronts which move to ψ by an equiaffine transformation whose differential map preserves the affine normal vector $\xi = (0, 0, 1)$ are given in this way ([Fer02]).

From now on, $\psi \colon \Sigma \to \boldsymbol{C} \times \boldsymbol{R} = \boldsymbol{R}^3$ is an improper affine front with Weierstrass data (F, G). Next, we shall review the concepts of completeness and some properties for complete improper affine fronts, shown in [Mar05a], which play important roles in this paper.

Definition 2.4. [Mar05a, Definition 2], [KUY04, Definition 3.1] An improper affine front $\psi \colon \Sigma \to \boldsymbol{C} \times \boldsymbol{R}$ is said to be *complete* if there exists a symmetric bilinear form T with a compact support such that

$$\widetilde{ds}^2 := T + ds^2 \tag{2.8}$$

is a complete Riemannian metric on Σ, where ds^2 is the flat fundamental form.

Fact 2.5. [Mar05a, Proposition 1] *Let $\psi \colon \Sigma \to \boldsymbol{C} \times \boldsymbol{R}$ be a complete improper affine front. Then Σ is biholomorphic to $\overline{\Sigma} \setminus \{p_1, \cdots, p_n\}$, where $\overline{\Sigma}$ is a compact Riemann surface, and $n \geq 1$ is an integer. Moreover, the Weierstrass data (F, G) of ψ can be extended meromorphically to $\overline{\Sigma}$. In particular, F and G have at most a pole at each p_j.*

Each puncture point p_j is called an *end*. Taking a small neighborhood U of an end such that ds^2 is non-degenerate on U (i.e., ψ is an improper affine sphere on U, which is complete at the end), we also call the puncture point the *end* of improper affine sphere. Here, an end p of ψ is said to be an *embedded end* if there is a small neighborhood U of p such that $\psi|_{U\setminus\{p\}}$ is an embedding.

Set $\Sigma = \overline{\Sigma}_g \setminus \{p_1, \cdots, p_n\}$, where $\overline{\Sigma}_g$ is a compact Riemann surface of genus g, and let $\psi\colon \Sigma \to \boldsymbol{C} \times \boldsymbol{R} = \boldsymbol{R}^3$ be a complete improper affine front. Here,

$$\rho \coloneqq \frac{dF}{dG} \tag{2.9}$$

defines a meromorphic function on Σ, and ρ is termed the *Lagrangian Gauss map*. When we set

$$\mathcal{L} \coloneqq \mathcal{X} + i\mathcal{N},$$

where $\mathcal{X} = \overline{F} + G$ and $\mathcal{N} \coloneqq \overline{F} - G$, the map $\mathcal{L}\colon \Sigma \to \boldsymbol{C}^2$ becomes the special Lagrangian immersion. The induced metric $d\tau^2$ from $\boldsymbol{C}^2$ given by

$$d\tau^2 \coloneqq \mathcal{L}^* \langle , \rangle_{\boldsymbol{C}^2} = \langle d\mathcal{X}, d\mathcal{X}\rangle + \langle d\mathcal{N}, d\mathcal{N}\rangle = 2(|dF|^2 + |dG|^2) \tag{2.10}$$

is a complete Riemannian metric and conformal to h at points where h is non-degenerate, where $\langle , \rangle_{\boldsymbol{C}^2}$ stands for the standard Euclidian inner product of $\boldsymbol{C}^2 = \boldsymbol{R}^4$ ([Mar05a], Theorem 1).

Fact 2.6. [Mar05a, Section 3] *A complere improper affine front $\psi\colon \Sigma = \overline{\Sigma}_g \setminus \{p_1, \cdots, p_n\} \to \boldsymbol{C} \times \boldsymbol{R}$ satisfies the following properties:*

- *An end p_j $(j = 1, 2, \cdots, n)$ is embedded if and only if F and G have at most a simple pole at p_j.*
- (Osserman-type inequality) *When we denote by K_τ and dA_τ the Gaussian curvature and the area element with respect to $d\tau^2$, it holds that*

$$-\frac{1}{2\pi}\int_\Sigma K_\tau dA_\tau \geq -\chi(\overline{\Sigma}_g) + 2n, \tag{2.11}$$

where $\chi(\overline{\Sigma}_g) = 2 - 2g$ is the Euler number of $\overline{\Sigma}_g$.

The integral of $K_\tau dA_\tau$ in the left hand side of (2.11) is called the *total curvature* of ψ. [Kaw13, Theorem 1.1, Corollaly 1.2] shows that the Gaussian curvature K_τ with respect to the conformal metric $d\tau^2 = 2(|dF|^2 + |dG|^2) = 2(1 + |\rho|^2)|dG|^2$ is

$$K_\tau = -\frac{1}{(1+|\rho|^2)^3}\left|\frac{d\rho}{dG}\right|^2. \tag{2.12}$$

Hence, one can varify that the total curvature satisfies

$$\int_\Sigma K_\tau dA_\tau = -2\pi \deg \rho \in -2\pi \boldsymbol{Z}_{\geq 0}, \tag{2.13}$$

and the Osserman-type inequality (2.11) can be rewritten as

$$\deg \rho \geq 2(g - 1 + n). \tag{2.14}$$

Other global results for complete improper affine fronts are also investigated in [MM14c] and [ACG07].

In the end of this section, we confirm some examples of complete improper affine fronts with only embedded ends.

Example 2.7. [Mar05a, Section 4] A complete improper affine front obtained by taking $\Sigma = \widehat{\boldsymbol{C}} \setminus \{0\}$ ($\widehat{\boldsymbol{C}} := \boldsymbol{C} \cup \{\infty\}$) and the Weierstrass data (F, G) given by

$$F = \frac{1}{z}, \quad G = \frac{b}{z} \quad (b \in \boldsymbol{C}, |b| \neq 1) \tag{2.15}$$

is called an *elliptic paraboloid* (Figure 1, (a)). Moreover for each b, we find that the elliptic paraboloid is equiaffinely equivalent to a rotational paraboloid.

Example 2.8. [Mar05a, Section 4] A complete improper affine front obtained by taking $\Sigma = \boldsymbol{C} \setminus \{0\}$ and the Weierstrass data (F, G) given by

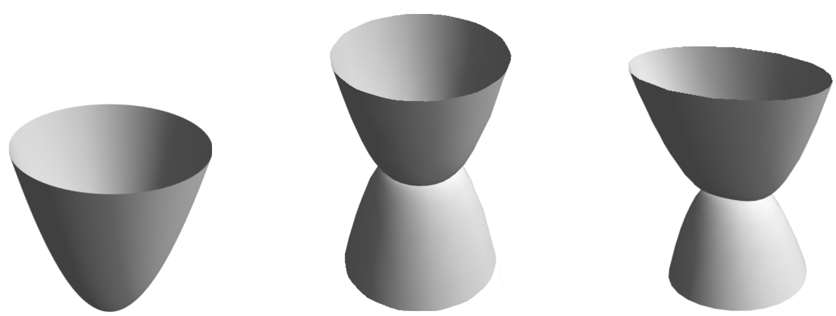
$$F = \frac{1}{z}, \quad G = az \quad (a \in \boldsymbol{R} \setminus \{0\}) \tag{2.16}$$

is called a *rotational improper affine front* (Figure 1, (b)).

Example 2.9. [Mar05a, Section 4] A complete improper affine front obtained by taking $\Sigma = \boldsymbol{C} \setminus \{0\}$ and the Weierstrass data (F, G) given by

$$F = \frac{1}{z}, \quad G = az + \frac{b}{z} \quad (a \in \boldsymbol{R} \setminus \{0\}, b \in \boldsymbol{C} \setminus \{0\}, |b| \neq 1) \tag{2.17}$$

is called a *non-rotational improper affine front* (Figure 1, (c)).



(a) Example 2.7 (b) Example 2.8 (c) Example 2.9

FIGURE 1. Complete improper affine fronts with only embedded ends

3. Asymptotic behavior of complete embedded ends

3.1. **Asymptotic behavior of the complete embedded ends.** As mentioned in Fact 2.6, the equality condition of the Osserman-type inequality (2.11) is equivalent to the condition that all ends are embedded. Using this fact and the condition for an end to be embedded, we will classify complete embedded ends in the sense of "asymptotic", into three types in a similar way to [KUY02]. Throughout this section, we take a sufficiently small local complex coordinate neighborhood centered at an end of a complete improper affine front, and consider a local expression of an improper affine front $\psi\colon \boldsymbol{D}_\varepsilon^* := \{0 < |z| < \varepsilon\} \to \boldsymbol{C} \times \boldsymbol{R}$ $(\varepsilon > 0)$ with Weierstrass data (F, G). By completeness, we may assume that $\boldsymbol{D}_\varepsilon^*$ does not contain the singular set of ψ, that is, the flat fundamental form ds^2 is non-degenerate on $\boldsymbol{D}_\varepsilon^*$. Then ψ is complete at 0 (that is, the length with respect to ds^2 of any path in $\boldsymbol{D}_\varepsilon^*$ which accumulates to 0 diverges to ∞). Hence we may also assume that ψ is an improper affine sphere on $\boldsymbol{D}_\varepsilon^*$.

Definition 3.1. An improper affine sphere $\psi\colon \boldsymbol{D}_\varepsilon^* \to \boldsymbol{C} \times \boldsymbol{R}$ complete at 0 is said to be *asymptotic* to a *type-P end* (resp. *type-R end*, *type-NR end*) if there exists a piece of an elliptic paraboloid (2.15) (resp. a rotational improper affine front (2.16), a non-rotational improper affine front (2.17))

$$\widetilde{\psi}\colon \boldsymbol{D}_\varepsilon^* \to \boldsymbol{C} \times \boldsymbol{R}$$

which is complete at 0 such that

$$|\psi(z) - \widetilde{\psi}(z)| = o(1) \tag{3.1}$$

holds, where $o(1)$ means a term tending to 0 as $z \to 0$, and $|\cdot|$ is the standard Euclidian norm. Here, we regard $\boldsymbol{C} \times \boldsymbol{R} = \boldsymbol{R}^3$ as Euclidian space, not affine space.

Theorem 3.2. *Let $\psi\colon \boldsymbol{D}_\varepsilon^* \to \boldsymbol{R}^3$ be an improper affine sphere complete at 0. Then the end 0 is embedded end if and only if ψ is asymptotic to one of the type-P end, type-R end, or type-NR end.*

Proof. Assume that the end 0 is embedded. By Fact 2.6, F and G can be expanded around $z = 0$ as

$$F = \frac{a_{-1}}{z} + \sum_{n=0}^{\infty} a_n z^n, \qquad G = \frac{b_{-1}}{z} + \sum_{n=0}^{\infty} b_n z^n, \tag{3.2}$$

and by exchanging F and G if necessary, we may assume that $a_{-1} \neq 0$. After a parallel translation of $\boldsymbol{R}^3$ and coordinate changes on $\boldsymbol{D}_\varepsilon^*$, we can suppose that $F = 1/z$ and $b_0 = 0$. In addition, the period condition (2.5) is equivalent to $b_1 \in \boldsymbol{R}$.

By (2.6), we obtain the concrete expression of ψ as,

$$\psi(z) = \left(\frac{b_{-1}}{z} + \frac{1}{\bar{z}},\ \frac{1}{2}\left(\frac{|b_{-1}|^2}{|z|^2} - \frac{1}{|z|^2} \right) - 2b_1 \log|z| \right) + o(1)$$

up to additive constant vectors of $\boldsymbol{R}^3$. We divide the situation into the following two cases.

<u>**Case 1**</u> The case of $b_{-1} = 0$.

(I) When $b_1 \neq 0$, ψ is asymptotic to the type-R end. In fact, from Example 2.8, the rotational improper affine front with the Weierstrass data $F = 1/z$, $G =$

$b_1 z$ $(b_1 \in \boldsymbol{R} \setminus \{0\})$ is expressed as

$$\widetilde{\psi}_{\mathrm{R}}(z) := \left(b_1 z + \frac{1}{\bar{z}},\ \frac{1}{2}\left(b_1^2 |z|^2 - \frac{1}{|z|^2} \right) - 2b_1 \log|z| \right) \tag{3.3}$$

up to an additive constant vector, and it holds that

$$|\psi(z) - \widetilde{\psi}_{\mathrm{R}}(z)| = \left| -\frac{1}{2} b_1^2 |z|^2 + o(1) \right| = o(1).$$

(II) When $b_1 = 0$, ψ is asymptotic to the type-P end. In fact, from Example 2.7, Weierstrass data of the elliptic paraboloid for $b = 0$, $F = 1/z$, $G = 0$ corresponds to the surface

$$\widetilde{\psi}_{\check{\mathrm{P}}}(z) := \left(\frac{1}{\bar{z}},\ -\frac{1}{2|z|^2} \right), \tag{3.4}$$

from (2.6), and one can obtain

$$|\psi(z) - \widetilde{\psi}_{\check{\mathrm{P}}}(z)| = o(1).$$

<u>**Case 2**</u> The case of $b_{-1} \neq 0$.

(I) When $b_1 \neq 0$, ψ is asymptotic to the type-NR end. In fact, from Example 2.9, Weierstrass data is given by $F = 1/z$, $G = b_1 z + b_{-1}/z$ $(b_1 \in \boldsymbol{R} \setminus \{0\}, b_{-1} \in \boldsymbol{C} \setminus \{0\}, |b_{-1}| \neq 1)$. Then this surface is expressed as

$$\widetilde{\psi}_{\mathrm{NR}}(z) := \left(b_1 z + \frac{b_{-1}}{z} + \frac{1}{\bar{z}},\ \frac{1}{2}\left(\frac{|b_{-1}|^2}{|z|^2} + |b_1|^2 |z|^2 - \frac{1}{|z|^2} \right) - 2b_1 \log|z| \right) \tag{3.5}$$

up to an additive constant vector. Hence we have

$$|\psi(z) - \widetilde{\psi}_{\mathrm{NR}}(z)| = o(1).$$

(II) When $b_1 = 0$, ψ is asymptotic to the type-P end. Indeed, from Example 2.7, Weierstrass data of elliptic paraboloid for $b = b_{-1}$ is given by $F = 1/z$, $G = b_{-1}/z$ $(|b_{-1}| \neq 1)$, and then this surface can be expressed as

$$\widetilde{\psi}_{\mathrm{P}}(z) := \left(\frac{b_{-1}}{z} + \frac{1}{\bar{z}},\ \frac{1}{2|z|^2}(|b_{-1}|^2 - 1) \right) \tag{3.6}$$

from (2.6). Therefore we find

$$|\psi(z) - \widetilde{\psi}_{\mathrm{P}}(z)| = o(1).$$

Conversely, we suppose that an improper affine sphere $\psi\colon \boldsymbol{D}_\varepsilon^* \to \boldsymbol{R}^3$ complete at 0 is asymptotic to one of those three types. Now, assume that 0 is not an embedded end. Then, from Fact 2.6, F and G can be expanded to

$$F = \sum_{n=k}^{\infty} a_n z^n, \quad G = \sum_{n=l}^{\infty} b_n z^n \quad (a_{-k}, b_{-l} \neq 0, k \leq l, k \leq -2)$$

around $z = 0$. Similarly to the first half, we may assume that $F = 1/z^k, b_0 = 0$, and the period condition is equivalent to $b_k \in \boldsymbol{R}$. Putting $\psi = (\psi_1 + i\psi_2, \psi_3) \in \boldsymbol{C} \times \boldsymbol{R}$, we can compute $\psi_3(z)$ as

$$\psi_3(z) = \frac{1}{|z|^{2k}}\left(-\frac{1}{2} + O(1)\right)$$

from (2.6), where $O(1)$ is the bounded term as $z \to 0$. If ψ is asymptotic to the type-R end, then from (3.3), we have

$$|\psi_3(z) - (\psi_{\mathrm{R}})_3(z)| = \frac{1}{|z|^{2k}}\left|-\frac{1}{2} + O(1)\right| \to \infty \quad (z \to 0).$$

This contradicts to the assumption of asymptoticity. Similarly, we can lead the contradictions for the cases of the type-P end and the type-NR end. Therefore we obtain the conclusion. □

Combining the equality condition of the Osserman-type inequality (2.11) with Theorem 3.2, one can directly show the following corollary:

Corollary 3.3. *A complete improper affine front in $\boldsymbol{C}\times\boldsymbol{R} = \boldsymbol{R}^3$ attains the equality in the Osserman-type inequality* (2.11) *if and only if each end is asymptotic to one of the type-P end, type-R end, or type-NR end.*

Symmetry, uniqueness of solutions of the exterior Plateau proplem associated to (1.1), and maximum principle at infinity for improper affine spheres are studied in [FMM96] and [FMM99].

3.2. **New examples with embedded ends.** We introduce new examples of complete improper affine fronts with embedded ends as a correspondence to the Jorge–Meeks minimal surface in Euclidian 3-space ([JM83]).

Let $n \geq 2$ be an integer and $\Sigma := \widehat{\boldsymbol{C}} \setminus \{1, \zeta, \cdots, \zeta^{n-1}, \eta, \eta\zeta, \cdots, \eta\zeta^{n-1}\}$, where $\zeta := \exp(2\pi i/n), \eta := \exp(\pi i/n)$ and $\widehat{\boldsymbol{C}} = \boldsymbol{C} \cup \{\infty\}$, and consider (F, G) given by

$$F = \sum_{j=0}^{n-1} \frac{\alpha_j}{z - \zeta^j}, \qquad G = \sum_{j=0}^{n-1} \frac{\beta_j}{z - \eta\zeta^j}, \tag{3.7}$$

where $\alpha_j, \beta_j \in \boldsymbol{C} \setminus \{0\}$, and z is the canonical complex coordinate of $\boldsymbol{C}$. We obtain the following examples by how we choose these parameters α_j, β_k.

Example 3.4. We choose α_j, β_k satisfying

$$\alpha_j \eta^{n-1}\zeta^{n-j}, \quad \beta_k \zeta^{n-k} \in \boldsymbol{R} \qquad (j, k = 0, ..., n-1).$$

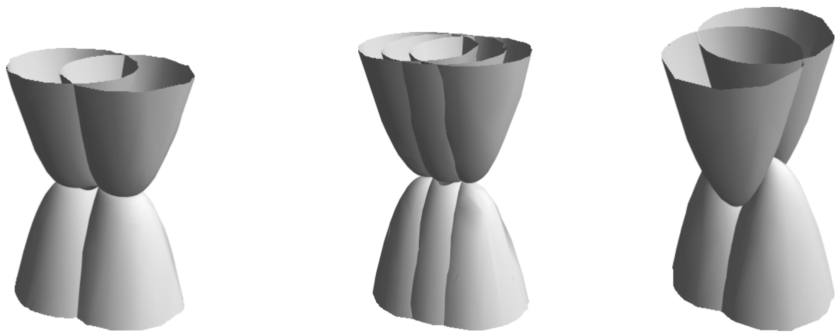
For example, we put $\alpha_j = \lambda_j \eta \zeta^j, \beta_k = \mu_k \zeta^k$, where $\lambda_j, \mu_k \in \boldsymbol{R} \setminus \{0\}$. Then (F, G) given in (3.7) induces a complete improper affine front $\psi\colon \Sigma \to \boldsymbol{R}^3$ with $2n$ embedded ends (Figure 2: (a) $n = 2, \alpha_1 = \alpha_2 = i, \beta_1 = \beta_2 = 1$, (b) $n = 3, \alpha_1 = \alpha_2 = \beta_2 = \beta_3 = 1/5, \alpha_3 = \beta_1 = -1$).

Moreover, we consider the following example in the special case of $n = 2$. Set $\alpha_j = \beta_j = 1$ $(j = 0, 1)$ in (3.7). In this case, $\Sigma = \widehat{\boldsymbol{C}} \setminus \{\pm 1, \pm i\}$.

Example 3.5. Let

$$F = \frac{1}{z-1} + \frac{1}{z+1}, \quad G = \frac{1}{z-i} + \frac{1}{z+i}. \tag{3.8}$$

Since the residues of FdG at each ends are $\mathrm{Res}(FdG, \pm 1) = \mathrm{Res}(FdG, \pm i) = 0$, the period condition (2.5) is satisfied. Hence (3.8) induces a complete improper affine front with four embedded ends. This surface is not equiaffinely equivalent to the surface in Example 3.4, $n = 2$. (Figure 2, (c))



(a) Example 3.4, $n = 2$ (b) Example 3.4, $n = 3$ (c) Example 3.5

FIGURE 2. New examples with $2n$ embedded ends

4. Classification of complete improper affine fronts of total curvature $-2m\pi$

In the minimal surface theory, Osserman [Oss64] and López [Lóp92] classified complete minimal surfaces in the Euclidian 3-space whose total curvature is $-4m\pi$ ($0 \le m \le 2$) (that is, the mapping degree m of the Gauss map satisfies $0 \le m \le 2$). In this section, we will classify complete improper affine fronts, up to transformations in Remark 2.7, whose total curvatures are $0, -2\pi, -4\pi, -6\pi$, and -8π.

Let $\psi\colon \Sigma = \overline{\Sigma}_g \setminus \{p_1, \cdots, p_n\} \to \boldsymbol{C} \times \boldsymbol{R} = \boldsymbol{R}^3$ be a complete improper affine front with Weierstrass data (F, G). At first, we start from describing three facts which was showed in [Mar05a] to classify surfaces.

Fact 4.1. [Mar05a, Theorems 5, 6, 7], [AMM11, Theorems 12, 15]

- *A complete improper affine front is the elliptic paraboloid if and only if its Lagrangian Gauss map ρ in* (2.9) *is constant.*
- *A complete improper affine front with only one end, which is embedded, is the elliptic paraboloid referred to in Example 2.7.*
- *If a complete improper affine front with exact two ends, which are embedded, then it is either the rotational improper affine front or the non-rotational improper affine front described in Examples 2.8 and 2.9.*

The first assertion of Fact 4.1 yields the following theorem directly:

Theorem 4.2. *A complete improper affine front with the total curvature* 0 *is the elliptic paraboloid.*

From now on, without loss of generality, we may assume that $\rho(p) = \infty$ at one end $p \in \overline{\Sigma}$ (if necessary, exchange F with G and conduct a transformation (2.7)). In addition, we remark the following:

- (Residue condition) The residues of F' and G' necessarily have to vanish, where z is a local complex coordinate and $' := d/dz$, by the uniqueness of Laurent expansion.
- If ρ has a pole at except ends of order $-k$, then it is a zero point of G' of order k, and $F' \neq 0$ holds there because (F, G) satisfies the relation

$$dF = \rho dG, \tag{4.1}$$

  and $(dF, dG) \neq (0, 0)$ on Σ (cf., Definition 2.2). In particular, $dG \neq 0$ on $\Sigma \setminus \rho^{-1}(\{\infty\})$.
- Sum of orders of a meromorphic function on a compact Riemann surface is zero ([Mir95]).
- The number of ends n satisfies $n \geq 1$. In fact, if $n = 0$, then F and G become holomorphic functions on the compact Riemann surace $\overline{\Sigma}_g$, so they have to be constant. It contradicts $(dF, dG) \neq (0, 0)$.

4.1. **The case of total curvature -2π.**

Theorem 4.3. *A complete improper affine front with the total curvature* -2π *is obtained from the Weierstrass deta*

$$F = az^2, \quad G = z \qquad (a > 0) \tag{4.2}$$

defined on $\Sigma = \boldsymbol{C}$ *(Figure 3).*

Proof. From (2.14), (g, n) satisfies $g+n \leq 3/2$. Then we see that $(g, n) = (0, 1)$. We may assume that $\Sigma = \widehat{\boldsymbol{C}} \setminus \{\infty\} = \boldsymbol{C}$ and F, G are both polynomials. Moreover, ρ is a Möbius transformation because of $\deg \rho = 1$. By (4.1), it holds that $G' = c$ $(\neq 0)$. Hence we may set $G = z$. Since the Möbius transformation ρ satisfies $\rho(\infty) = \infty$, ρ can be written as $\rho = az + b$ $(a \neq 0)$. Then we obtain $F = az^2 + bz$. Applying a coordinate change and equiaffine transformations (2.7), we get (4.2). $\square$

Remark 4.4. The [KRSUY05] criteria of singularities for improper affine fronts ([Kod21]) shows that the improper affine front in Theorem 4.3 has three swallowtails. A relation between this surface and a flat front in hyperbolic 3-space with three swallowtails is refferred to in [MM14b] (see also [MM14c]).

4.2. **The case of total curvature -4π.**

FIGURE 3. Comlete improper affine front with total curvature -2π : (4.2)

Theorem 4.5. *Complete improper affine fronts with the total curvature -4π are the rotational improper affine front, the non-rotational improper affine front, and the surfaces constructed by the Weierstrass data*

$$F = az^3 + bz, \quad G = z \qquad (a > 0, b \in \boldsymbol{C}), \tag{4.3}$$

$$F = az^3 + bz^2 + cz, \quad G = z^2 \qquad (a > 0, c \in \boldsymbol{C} \setminus \{0\}) \tag{4.4}$$

defined on $\Sigma = \boldsymbol{C}$ (Figure 4).

Proof. It follows from (2.14) that $g + n \leq 2$, and the pairs of (g, n) are $(g, n) = (0, 2), (1, 1), (0, 1)$. Recalling Fact 4.1, we find that if $(g, n) = (0, 2)$, then the surface is either the rotational improper affine front (2.16) or the non-rotational improper affine front (2.17). The case of $(g, n) = (1, 1)$ can not happen by Fact 4.1. So we only have to investigate the case of $(g, n) = (0, 1)$. As the case of $\deg \rho = 1$, we may assume that $\Sigma = \boldsymbol{C}$. Then F and G are polynomials.

Here, we will consider the following two cases.

(I) The case $\operatorname{ord}_\infty \rho = -2$.

We can set $G = z$ and $\rho = a_2 z^2 + a_1 z + a_0$ $(a_2 \neq 0, a_1, a_0 \in \boldsymbol{C})$ by the same reason as in the proof of Theorem 4.3. Thus F can be computed, and we obtain (4.3).

(II) The case where there uniquely exists $p \in \boldsymbol{C}$ which is a pole of ρ and satisies $\operatorname{ord}_\infty \rho = \operatorname{ord}_p \rho = -1$.

Without loss of generality, we may assume $p = 0$. Then we have $G' = az$ $(a \neq 0)$. Hence ρ and G can be written as $G = z^2$, $\rho = a_1 z + a_{-1}/z + a_0$ $(a_1, a_{-1} \in \boldsymbol{C} \setminus \{0\}, a_0 \in \boldsymbol{C})$, Thus rewriting the parameters of ρ and changing coordinate, we have the Weierstrass data (4.4).

Therefore, the proof is completed. $\square$

4.3. **The case of total curvature -6π.**

Theorem 4.6. *Complete improper affine fronts with the total curvature -6π are constructed by the Weierstrass data*

$$F = az^4 + bz^2 + cz, \quad G = z \quad (a > 0), \tag{4.5}$$

(a) (4.3) (b) (4.4)

FIGURE 4. Comlete improper affine front with total curvature -4π

$$F = az^4 + bz^3 + cz^2 + dz, \quad G = z^2 \quad (a > 0, c \neq 0), \tag{4.6}$$

$$F = az^4 + bz^3 + cz^2 + dz, \quad G = z^3 \quad (a > 0, d \neq 0), \tag{4.7}$$

$$F = az^4 + bz^3 + cz^2 + dz, \quad G = \alpha(2z^3 - 3z^2) \quad (a > 0,\ d, \alpha \neq 0), \tag{4.8}$$

which are defined on $\Sigma = \boldsymbol{C}$ (Figure 5),

$$F = az^2 + bz + \frac{c}{z}, \quad G = \frac{1}{z} \quad (a > 0, b \in \boldsymbol{R}, |c| \neq 1), \tag{4.9}$$

$$F = az + \frac{b}{z} + \frac{c}{z^2}, \quad G = \frac{1}{z^2} \quad (a > 0, |c| \neq 1), \tag{4.10}$$

$$F = \frac{a}{z^2} + \frac{b}{z} + cz, \quad G = \frac{1}{z} \quad (a > 0, c \in \boldsymbol{R} \setminus \{0\}), \tag{4.11}$$

$$F = az + \frac{b}{z} + \frac{c}{z^2}, \quad G = \alpha\left(\frac{1}{2z^2} - \frac{1}{z}\right) \quad (a > 0, \alpha \in \boldsymbol{C} \setminus \{0\}, 2|c| \neq |\alpha|), \tag{4.12}$$

$$F = az^2 + bz + \frac{c}{z}, \quad G = \alpha\left(z + \frac{1}{z}\right) \quad (a > 0,\ b, \alpha \neq 0, c - a \in \boldsymbol{R}, |c| \neq |\alpha|), \tag{4.13}$$

which are defined on $\boldsymbol{C} \setminus \{0\}$ (Figure 5, 6 and 7).

proof of Theorem 4.6. Seeing (2.14), we find $g + n \leq 5/2$, and hence $(g, n) = (0, 2), (1, 1), (0, 1)$. These cases are the same (g, n) as the case of the total curvature -4π , but note that at least one end is not embedded in the case $(g, n) = (0, 2)$.

<u>Case 1</u> $(g, n) = (0, 1)$ $(\Sigma = \boldsymbol{C})$.

We further divide **Case 1** into the following **(I)-(III)**.

(I) The case $\mathrm{ord}_\infty \rho = -3$.

In this case, as with **(I)** in the case of -4π and $(g, n) = (0, 1)$, we obtain (4.5).

(II) The case where there uniquely exists $p \in \boldsymbol{C}$ which is a pole of ρ.

We have to consider the two more cases:

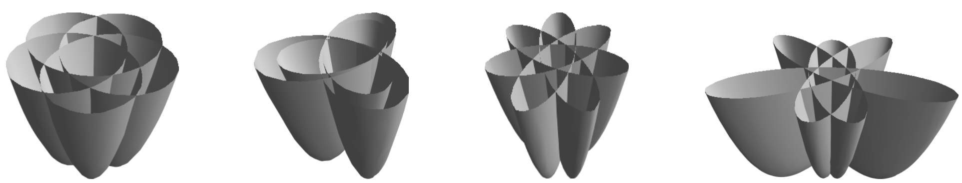
$$\text{(II-a)}\ \mathrm{ord}_p \rho = -1, \quad \mathrm{ord}_0 \rho = -2, \quad \text{(II-b)}\ \mathrm{ord}_p \rho = -2, \quad \mathrm{ord}_0 \rho = -1.$$

(II-a) We can set $G' = \alpha(z - p)$, and then after some transformations, we get $G = z^2$. Since ρ is expressed as $\rho = a_2 z^2 + a_1 z + a_{-1}/z + a_0$ $(a_2, a_{-1} \neq 0)$, computing F from the relation (4.1), and conducting some transformations, we have Weierstrass data (4.6).

(II-b) Similarly, as (II-a), we can set $G = z^3$, and ρ is written by the formation $\rho = a_{-2}/z^2 + a_{-1}/z + a_0 + a_1 z$ $(a_{-2}, a_1 \neq 0)$. Then we obtain (4.7) by (4.1).

(III) The case where there exist distinct points $p, q \in \boldsymbol{C}$ which are poles of ρ of each order -1.

Without loss of generality, we can set $p = 0, q = 1$. Then G' can be written as $G' = \alpha z(z-1)$ $(\alpha \neq 0)$, and by retaking the parameter α, we have $G = \alpha(2z^3 - 3z^2)$. Therefore by (4.1), we get (4.8).



(a) (4.5) (b) (4.6) (c) (4.7) (d) (4.8)

FIGURE 5. Comlete improper affine front with total curvature -6π, one end

Case 2 $(g, n) = (0, 2)$ $(\Sigma = \boldsymbol{C} \setminus \{0\})$.

Note that at least one of F and G must not be a polynomial.

Considering this case, we must remark the period condition and completeness. We divide this case into the following **(I)-(III)**.

(I) The case $\operatorname{ord}_\infty \rho = -3$.

We can set $\rho = a_3 z^3 + a_2 z^2 + a_1 z + a_0$ $(a_3 \neq 0)$, and G, G' must have a pole at $z = 0$. And considering (4.1), we can set $G' = \alpha/z^k$ $(\alpha \neq 0, k = 2, 3)$, and G is calculated as $G = 1/z^{k-1}$. Hence after some transformation, F is calculated, and we have (4.9) and (4.10).

(a) (4.9) (b) (4.10)

FIGURE 6. Complete improper affine fronts with total curvature -6π, and two ends, No.1

(II) The case where there uniquely exists $p \in \boldsymbol{C}$ which is a pole of ρ.

We have to consider the following two cases:

$$\text{(II-a)}\ \operatorname{ord}_p \rho = -1, \quad \operatorname{ord}_\infty \rho = -2, \quad \text{(II-b)}\ \operatorname{ord}_p \rho = -2, \quad \operatorname{ord}_\infty \rho = -1.$$

(II-a) In this case, ρ can be expressed as $\rho(z) = a_{-1}/(z-p) + a_0 + a_1 z + a_2 z^2$ $(a_{-1}, a_2 \neq 0)$, and $G' \neq 0$ on $\boldsymbol{C} \setminus \{0, p\}$. Moreover, we divide (II-a) into two more cases.
(II-a-1) When $p = 0$, we can set $G' = \alpha/z^2$, so we obtain $G = 1/z$. Thus, we have (4.11).

(II-a-2) When $p \neq 0$ (we may assume $p = 1$), we can set $G' = \alpha(z-1)/z^3$. Then $a_1 - a_2 = 0$ must hold because of the residue condition for F'. Therefore, we get (4.12).

(II-b) ρ is expressed by $\rho(z) = a_{-2}/(z-p)^2 + a_{-1}/(z-p) + a_0 + a_1 z$ $(a_{-2}, a_1 \in \boldsymbol{C} \setminus \{0\})$. Moreover, we divide (II-b) into two more cases.

(II-b-1) When $p = 0$, since $G' = \alpha$ $(\alpha \neq 0)$, we can set $G = z$. Given the residue condition for F', we have $F = a/z + bz + cz^2$. However, by changing coordinates, we find that this data is the same as (4.11).

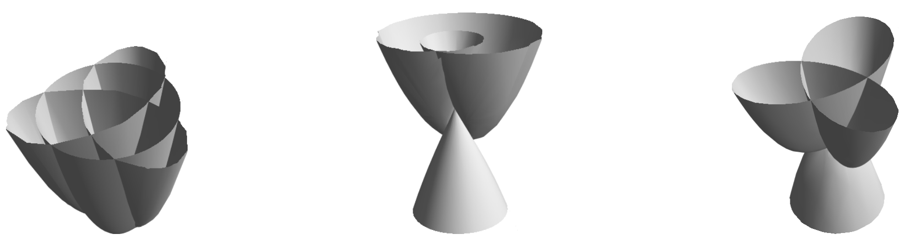
(II-b-2) When $p \neq 0$, we can set $G' = \alpha(z-p)^2/z^k$ $(\alpha \neq 0)$, and k satisfies $k \geq 4$. Then one can varify that F does not have a pole at ∞. Thus, this is impossible.

(III) The case where there exist distinct points $p, q \in \boldsymbol{C}$ which are poles of ρ of orders -1.

ρ is expressed as $\rho = a_{-1}/(z-p) + b_{-1}/(z-q) + a_0 + a_1 z$ $(a_{-1}, b_{-1}, a_1 \neq 0)$. We will consider the following two cases:

(III-a) When $p = 0$, $G'(q) = 0$ of order 1. Then we can set $G' = \alpha(z-q)/z^k$ $(\alpha \neq 0, k \geq 3)$. However $\operatorname{ord}_\infty F' = \operatorname{ord}_\infty \rho + \operatorname{ord}_\infty G' = -1 + (k-1) = k-2 \geq 1$, and this is impossible because both F and G do not have a pole at ∞.

(III-b) When $p, q \neq 0$ (we can set $p = 1$), G' can be written as $G' = \alpha(z-1)(z-q)/z^k$ $(\alpha \neq 0)$, and k satisfies $k = 2$ if $1 + q = 0$, or $k \geq 4$. However, the latter is impossible for the same reason as (III-a). If $k = 2$, then we obtain $G = \alpha(z + 1/z)$, $\rho = a_{-1}/(z-1) + b_{-1}/(z+1) + a_1 z + a_0$ $(a_{-1}, b_{-1}, a_1 \neq 0)$. Retaking parameters, we obtain (4.13).



(a) (4.11) (b) (4.12) (c) (4.13)

Figure 7. Complete improper affine fronts with total curvature -6π, and two ends, No.2

<u>**Case 3**</u> $(g, n) = (1, 1)$.

Let τ be a complex number satisfying $\operatorname{Im} \tau > 0$, and set

$$T_\tau \coloneqq \boldsymbol{C}/[1, \tau], \tag{4.14}$$

where $[1,\tau]$ is a lattice defined by $[1,\tau] := \{m+n\tau\ ;\ m,n \in \boldsymbol{Z}\}$. In this case, we may assume that $\Sigma = T_\tau \setminus \{[0]\}$ ($[x]$ stands for an equivalence class where x belongs). Since F,G, and ρ are meromorphic functions on T_τ, we can identify them with the elliptic functions on $\boldsymbol{C}$ associated with $[1,\tau]$. We will use general theory of the elliptic functions when we consider the case of genus 1 (see [HC44] for datails). Let $\Pi_0 := \{x+y\tau\ ;\ 0 \le x, y < 1\}$ be a fundamental period parallelogram (FPP, in short). Here, the Weierstrass $\wp$-function associated to $[1,\tau]$ is defined by

$$\wp(z) := \frac{1}{z^2} + \sum_{\omega\in[1,\tau],\omega\neq 0}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right),$$

and the Laurent expansion of $\wp$ around 0 is

$$\wp(z) = \frac{1}{z^2} + 3G_4 z^2 + 5G_6 z^4 + \cdots,$$

where $G_k := \sum_{\omega\in[1,\tau]\setminus\{0\}}(1/\omega^k)$ $(k = 4, 5, \cdots)$.

Fact 4.7 ([HC44]**).** (1) *Set* $e_j := \wp(\omega_j)$ $(j = 1,2,3)$, *where* $\omega_1 := 1/2, \omega_2 := (1+\tau)/2, \omega_3 := \tau/2, g_2 := 60G_4$, *and* $g_3 := 140G_6$. *Then* $\wp$ *satisfies*

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3,$$

$$g_2 = -4(e_1e_2 + e_2e_3 + e_3e_1), \quad g_3 = 4e_1e_2e_3. \tag{4.15}$$

(2) *An elliptic function has at least one pole in the FPP. In particular, an elliptic function which is holomorphic on the FPP is constant.*
(3) *The sum of residues in the FPP of an elliptic function is* 0*.*
(4) *The number of zero points in the FPP of an elliptic function is equal to the number of its poles in the FPP. Here, the number of zero points (resp., poles) is the sum of the order at each zero point (resp., pole).*

So we divide **Case 3** into the following **(I)-(III)**.

(I) The case $\mathrm{ord}_0\,\rho = -3$.

Here, $G' \neq 0$ holds on Π_0 by (4.1). Since G' is holomorphic in Π_0, $G'(z) = c\ (\neq 0)$ holds, and then $G(z) = cz$, but this is not an elliptic function. Thus **(I)** does not happen.

(II) The case where there uniquely exists $p \in \Pi_0 \setminus \{0\}$ which is a pole of ρ.

We have to consider two more cases :

$$\text{(II-a)}\ \mathrm{ord}_p\,\rho = -1, \quad \mathrm{ord}_0\,\rho = -2, \quad \text{(II-b)}\ \mathrm{ord}_p\,\rho = -2, \quad \mathrm{ord}_0\,\rho = -1.$$

(II-a) In this case, $G'(p) = 0$ of order 1 and $G' \neq 0$ otherwise. Then $z = 0$ is the only pole of G' of order -1, and thus $\mathrm{Res}(G', z = 0) \neq 0$. This is impossible.

(II-b) In this case, $z = 0$ is the only pole of G' of order -2, and then $z = 0$ is the only pole of G of order -1. This is impossible for the same reason as (II-a).

(III) The case where there exist distinct points $p, q \in \Pi_0 \setminus \{0\}$ which are poles of ρ of orders -1.

Then since $G'(p) = G'(q) = 0$ of order 1, G' has the unique pole at $z = 0$ of order -2. Hence this is impossible for the same reason as (II-b).

Summing up the arguments above, we find that there does not exist complete improper affine fronts of genus 1 with total curvature -6π. $\square$

4.4. **The case of total curvature -8π.** In this case, from Osserman-type inequality (2.14), we obtain $g+n \leq 3$ and find $(g, n) = (0, 1), (0, 2), (1, 1), (1, 2), (2, 1), (0, 3)$. The cases $(g, n) = (1, 2), (2, 1)$ can not happen by Fact 4.1.

Theorem 4.8. *Complete improper affine fronts with genus* 0 *whose total curvature is* -8π *are constructed by the Weierstrass data*

$$F = az^5 + bz^3 + cz^2 + dz, \quad G = z \quad (a > 0), \tag{4.16}$$

$$F = az^5 + bz^4 + cz^3 + dz^2 + ez, \quad G = z^2 \quad (a > 0, e \in \boldsymbol{C} \setminus \{0\}), \tag{4.17}$$

$$F = az^5 + bz^4 + cz^3 + dz^2 + ez, \quad G = z^3 \quad (a > 0, e \in \boldsymbol{C} \setminus \{0\}), \tag{4.18}$$

$$F = az^5 + bz^4 + cz^3 + dz^2 + ez, \quad G = z^4 \quad (a > 0, e \in \boldsymbol{C} \setminus \{0\}), \tag{4.19}$$

$$F = az^5 + bz^4 + cz^3 + dz^2 + ez, \quad G = \alpha(2z^3 - 3z^2) \quad (a > 0, \alpha, e, F'(1) \neq 0), \tag{4.20}$$

$$F = az^5 + bz^4 + cz^3 + dz^2 + ez, \quad G = \alpha(3z^4 - 4z^3) \quad (a > 0, \alpha, e, F'(1) \neq 0), \tag{4.21}$$

$$F = az^5 + bz^4 + cz^3 + dz^2 + ez, \quad G = \alpha(3z^4 - 4(1+r)z^3 + 6rz^2) \tag{4.22}$$

$(a > 0, r \notin \{0, 1\}, \alpha, e, F'(1), F'(r) \neq 0)$*, defined on* $\Sigma = \boldsymbol{C}$*,*

$$F = az^3 + bz^2 + cz + \frac{d}{z}, \quad G = \frac{1}{z} \quad (a > 0, c \in \boldsymbol{R}, |d| \neq 1), \tag{4.23}$$

$$F = az^2 + bz + \frac{c}{z} + \frac{d}{z^2}, \quad G = \frac{1}{z^2} \quad (a > 0, |d| \neq 1), \tag{4.24}$$

$$F = az^2 + bz + \frac{c}{z} + \frac{d}{z^2}, \quad G = \frac{1}{z} \quad (a > 0, d \neq 0), \tag{4.25}$$

$$F = az + \frac{b}{z} + \frac{c}{z^2} + \frac{d}{z^3}, \quad G = \frac{1}{z^2} \quad (a > 0, d \neq 0), \tag{4.26}$$

$$F = az^2 + bz + \frac{c}{z} + \frac{d}{z^2}, \quad G = \alpha\left(-\frac{1}{z} + \frac{1}{2z^2}\right) \tag{4.27}$$

$(a > 0, b \in \boldsymbol{R}, \alpha, F'(1) \neq 0, 2|d| \neq |\alpha|)$,

$$F = az + \frac{b}{z} + \frac{c}{z^2} + \frac{d}{z^3}, \quad G = \alpha\left(-\frac{1}{2z^2} + \frac{1}{3z^3}\right) \tag{4.28}$$

$(a > 0, F'(1) \neq 0, 3|d| \neq |\alpha|)$,

$$F = az + \frac{b}{z} + \frac{c}{z^2} + \frac{d}{z^3}, \quad G = \frac{1}{z} \quad (a > 0, d \neq 0), \tag{4.29}$$

$$F = az + \frac{b}{z} + \frac{c}{z^2} + \frac{d}{z^3}, \quad G = \alpha\left(-\frac{1}{z} + \frac{1}{z^2} - \frac{1}{3z^3}\right) \tag{4.30}$$

$(a>0, F'(1)\neq 0, 3|d|\neq|\alpha|)$,

$$F = az+\frac{b}{z}+\frac{c}{z^2}+\frac{d}{z^3}\quad G=\alpha\left(\frac{1}{2z^2}-\frac{1}{z}\right)\quad (a>0,\alpha,d\neq 0), \tag{4.31}$$

$$F = az^3+bz^2+cz+\frac{d}{z},\quad G=\alpha\left(z+\frac{1}{z}\right) \tag{4.32}$$

$(a>0,\alpha,F'(\pm 1)\neq 0, -c+d\in\boldsymbol{R}, |d|\neq|\alpha|)$,

$$F = az+\frac{b}{z}+\frac{c}{z^2}+\frac{d}{z^3}\quad G=\alpha\left(-\frac{1}{z}+\frac{p+1}{2z^2}-\frac{p}{3z^3}\right) \tag{4.33}$$

$(a>0,\alpha\in\boldsymbol{R}, p\in\boldsymbol{C}\setminus\{0,1\},\alpha,F'(1),F'(p)\neq 0, 3|d|\neq|\alpha||p|)$,

$$F = az^3+bz^2+cz+\frac{d}{z},\quad G=\alpha\left(z^2-6z+\frac{8}{z}\right) \tag{4.34}$$

$(a>0,\alpha(3d+4c)\in\boldsymbol{R},\alpha,F'(1),F'(-2)\neq 0,|d|\neq 8|\alpha|)$,

$$F = az^2+bz+\frac{c}{z}+\frac{d}{z^2},\quad G=\alpha\left(z+\frac{3}{4z}+\frac{1}{8z^2}\right) \tag{4.35}$$

$(a>0,\alpha(a+3b-4c)\in\boldsymbol{R},F'(1),F'(-1/2),\alpha\notin\boldsymbol{C}\setminus\{0\},2|d|\neq|\alpha|)$,

$$F = az^2+bz+\frac{c}{z}+\frac{d}{z^2},\quad G=\alpha\left(z+\frac{1}{z}\right) \tag{4.36}$$

$(a>0,\alpha,d,F'(\pm 1)\notin\boldsymbol{C}\setminus\{0\},\alpha(b-c)\in\boldsymbol{R})$,

$$F = az^3+bz^2+cz+\frac{d}{z},\quad G=\alpha\left(z^2+2(pq-1)z+\frac{2pq}{z}\right) \tag{4.37}$$

$(a>0,p\neq q,p,q\neq 0,1,p+q=-qr,(d-c)pq-d\in\boldsymbol{R},|d|\neq 2|pq|,F'(p),F'(q),$
$F'(1)\notin\boldsymbol{C}\setminus\{0\})$,

$$F = az^2+bz+\frac{c}{z}+\frac{d}{z^2},\quad G=\alpha\left(z+\frac{q^2+q+1}{z}-\frac{q(q+1)}{2z^2}\right) \tag{4.38}$$

$(a>0,\alpha\neq 0,q\neq 0,\pm 1,\alpha(4c-4b(q^2+q+1)+aq(q+1))\in\boldsymbol{R},4|d|\neq|\alpha q(q+1)|,F'(1),F'(q),F'(-1-q)\notin\boldsymbol{C}\setminus\{0\})$, *defined on* $\boldsymbol{C}\setminus\{0\}$, *and*

$$F = az+\frac{b}{z-1}+\frac{c}{z},\quad G=\frac{\alpha}{z-1}\quad (a>0,c,\alpha\in\boldsymbol{R}\setminus\{0\},|b|\neq\alpha) \tag{4.39}$$

$$F = az+\frac{b}{z-1}+\frac{c}{z},\quad G=\alpha\left(\frac{1}{z}-\frac{1}{z-1}\right), \tag{4.40}$$

$(a>0,\mathrm{Im}(b+c)=0,\alpha\in\boldsymbol{C}\setminus\{0\},|b|,|c|\neq|\alpha|)$,

$$F = az+\frac{b}{z-1}+\frac{c}{z},\quad G=\alpha\left(\frac{pq-1}{z-1}-\frac{pq}{z}\right), \tag{4.41}$$

$(a>0,\alpha\in\widehat{\boldsymbol{C}}\setminus\{0\},p,q\notin\{0,1\},pq\neq 1,p+q=2pq,\mathrm{Im}(1-2pq)=0,\mathrm{Im}(2(b-c)pq+2c-a)=0,|c|\neq|\alpha||pq|,|b|\neq|\alpha||pq-1|)$, *defined on* $\Sigma=\boldsymbol{C}\setminus\{0,1\}$ *(Figure 8).*

Proof. In the exact same way as before cases of $(g,n)=(0,1),(0,2)$, we get the above Weierstrass data.

Hence we only need to consider the case of $(g,n)=(0,3)$.

In this case, the three ends are all embedded. We may assume that $\Sigma = \boldsymbol{C}\setminus\{0,1\}$ and $\rho(\infty) = \infty$. Firstly, we shall divide this case into the following **(I)-(IV)**.

(I) The case $\operatorname{ord}_\infty \rho = -4$.

By (4.1), we find $G' \neq 0$ on $\boldsymbol{C}$, and observe that $(\operatorname{ord}_\infty G', \operatorname{ord}_0 G', \operatorname{ord}_1 G') = (4, -2, -2)$. Hence, G' can be expressed as $G' = \alpha/z^2(z-1)^2$ $(\alpha \neq 0)$. However, since $\operatorname{Res}(G', 0) = 2\alpha \neq 0$, this case does not happen.

(II) The case where there uniquely exists $p \in \boldsymbol{C}$ which is a pole of ρ.

In addition, we must investigate two cases (II-a) $p \in \{0, 1\}$ and (II-b) otherwise.

(II-a) The case $p \in \{0, 1\}$.

It is sufficient to consider the case of $p = 0$. Moreover, this case has to be divided into the following two cases by the orders of ρ:

(II-a-1) The case $(\operatorname{ord}_\infty \rho, \operatorname{ord}_0 \rho) = (-1, -3)$.

We observe $(\operatorname{ord}_\infty G', \operatorname{ord}_0 G', \operatorname{ord}_1 G') = (4, -2, -2),\ (2, 0, -2),\ (0, 2, -2)$. The first case is the same as **(I)**, so impossible. The second is also impossible by $\operatorname{ord}_\infty F' = \operatorname{ord}_\infty \rho + \operatorname{ord}_\infty G' = -1 + 2 = 1$. For the last case, G' can be written as $G' = \alpha z^2/(z-1)^2$ $(\alpha \neq 0)$. Then we find $\operatorname{Res}(G', 1) = 2\alpha \neq 0$, and this is impossible.

(II-a-2) The case $(\operatorname{ord}_\infty \rho, \operatorname{ord}_0 \rho) = (-2, -2)$.

One can find $(\operatorname{ord}_\infty G', \operatorname{ord}_0 G', \operatorname{ord}_1 G') = (4, -2, -2), (2, 0, -2), (0, 2, -2)$. We find $\operatorname{Res}(G', 1) \neq 0$ in the first and second cases, so they are not possible. In the third case, we obtain

$$G = \frac{\alpha}{z-1} \quad (\alpha \in \boldsymbol{C} \setminus \{0\}).$$

From (4.1), we get the Weierstrass deta (4.39).

(II-b) The case $p \notin \{0, 1\}$.

Furthermore, this case is divided into the following three cases:

(II-b-1) The case $(\operatorname{ord}_\infty \rho, \operatorname{ord}_p \rho) = (-3, -1)$.

In this time, we observe $\operatorname{ord}_p G' = 1, \operatorname{ord}_0 G' = \operatorname{ord}_1 G' = -2$, and $\operatorname{ord}_\infty G' = 3$. Hence we can write G' as

$$G' = \frac{\alpha(z-p)}{z^2(z-1)^2} \quad (\alpha \in \boldsymbol{C} \setminus \{0\}),$$

and by $\operatorname{Res}(G', 0) = \alpha(1-2p), \operatorname{Res}(G', 1) = \alpha(2p-1)$, p must be $p = 1/2$. Thus we obtain

$$G = \alpha\left(\frac{1}{z} - \frac{1}{z-1}\right) \quad (\alpha \neq 0).$$

We rewrite $\alpha/2$ as the same α. Then F can be calculated by (4.1), and we get the Weierstrass data (4.40).

(II-b-2) The case $(\operatorname{ord}_\infty \rho, \operatorname{ord}_p \rho) = (-2, -2)$.

In this case, the orders of G' satisfies $(\operatorname{ord}_\infty G', \operatorname{ord}_0 G', \operatorname{ord}_1 G', \operatorname{ord}_p G') = (2, -2, -2, 2)$. Hense G' can be expressed as $G' = \alpha(z-p)^2/z^2(z-1)^2$ $(\alpha \neq 0)$. Then it hold that $\operatorname{Res}(G', 0) = 2\alpha p(p-1)$. Thus $p \neq 0, 1$ yields a contradiction.

(II-b-3) The case $(\operatorname{ord}_\infty \rho, \operatorname{ord}_p \rho) = (-1, -3)$.

Then $(\operatorname{ord}_\infty G', \operatorname{ord}_0 G', \operatorname{ord}_1 G', \operatorname{ord}_p G') = (1, -2, -2, 3)$, and we can express G' as $G' = \alpha(z-p)^3/z^2(z-1)^2$ $(\alpha \neq 0)$. It is necessary to hold $\operatorname{Res}(G', 0) = \alpha p^2(2p+3) = 0$ and $\operatorname{Res}(G', 1) = \alpha(p-1)^2(2p+1) = 0$, but it is impossible for $p \neq 0, 1$.

(III) The case where there exists distinct points $p, q \in \boldsymbol{C}$ which are poles of ρ.

We will check the three cases (III-a) $p = 0, q = 1$, (III-b) $q = 0$, $p \notin \{0, 1\}$, and (III-c) $p, q \notin \{0, 1\}$. The first case (III-a) is impossible for the same reason as (II-a).

(III-b) The case $q = 0$, $p \notin \{0, 1\}$.

In this case, the orders of ρ are

$$(\operatorname{ord}_\infty \rho, \operatorname{ord}_0 \rho, \operatorname{ord}_p \rho) =^{(1)} (-2, -1, -1),^{(2)} (-1, -2, -1),^{(3)} (-1, -1, -2),$$

and $\operatorname{ord}_1 G' = -2$ must hold.

(III-b-1) The case $(\operatorname{ord}_\infty \rho, \operatorname{ord}_0 \rho, \operatorname{ord}_p \rho) = (-2, -1, -1)$.

Since $\operatorname{ord}_p G' = 1, \operatorname{ord}_0 G' \geq 1$, $\operatorname{ord}_\infty G' \geq 2$, and $\operatorname{ord}_1 G' = -2$, we know that $\sum_{x \in \widehat{\boldsymbol{C}}} \operatorname{ord}_x G' \geq 1 + 1 + 2 - 2 = 2$. This is impossible.

(III-b-2) The case $(\operatorname{ord}_\infty \rho, \operatorname{ord}_0 \rho, \operatorname{ord}_p \rho) = (-1, -2, -1)$.

We know that $(\operatorname{ord}_\infty G', \operatorname{ord}_0 G', \operatorname{ord}_1 G', \operatorname{ord}_p G') = (1, -2, 0, 1),\ (1, 0, -2, 1)$. In the first case, we find $\operatorname{Res}(G', 0) \neq 0$. In the second case, we also find $\operatorname{Res}(G', 1) \neq 0$. Hence these cases are impossible.

(III-b-3) The case $(\operatorname{ord}_\infty \rho, \operatorname{ord}_0 \rho, \operatorname{ord}_p \rho) = (-1, -1, -2)$.

Given $\operatorname{ord}_p G' = 2, \operatorname{ord}_\infty G' = 1$, and $\operatorname{ord}_1 G' = -2$, we know $\operatorname{ord}_0 G' = -1$. However, this is impossible.

(III-c) The case $p, q \notin \{0, 1\}$.

Furthermore, this case is divided into the following two cases:

$$(\operatorname{ord}_\infty \rho, \operatorname{ord}_p \rho, \operatorname{ord}_q \rho) =^{(1)} (-2, -1, -1),^{(2)} (-1, -2, -1).$$

(III-c-1) The case $(\operatorname{ord}_\infty \rho, \operatorname{ord}_p \rho, \operatorname{ord}_q \rho) = (-2, -1, -1)$.

We observe $(\operatorname{ord}_\infty G', \operatorname{ord}_0 G', \operatorname{ord}_1 G', \operatorname{ord}_p G', \operatorname{ord}_q G') = (2, -2, -2, 1, 1)$ and can set

$$G' = \frac{\alpha(z-p)(z-q)}{z^2(z-1)^2} = \alpha\left(\frac{pq-p-q+1}{(z-1)^2} + \frac{pq}{z^2} + \frac{p+q-2pq}{z-1} + \frac{2pq-p-q}{z}\right).$$

Then $p + q = 2pq, pq \neq 0, 1$ must hold. Hence we obtain

$$G = \alpha\left(\frac{pq-1}{z-1} - \frac{pq}{z}\right),$$

and by (4.1), we get the Werstrass data (4.41).

(III-c-2) The case $(\operatorname{ord}_\infty \rho, \operatorname{ord}_p \rho, \operatorname{ord}_q \rho) = (-1, -2, -1)$.

In the same way of the case of (III-c-1), we have $(\operatorname{ord}_\infty G', \operatorname{ord}_0 G', \operatorname{ord}_1 G', \operatorname{ord}_p G', \operatorname{ord}_q G') = (1, -2, -2, 2, 1)$. Then we obtain

$$G' = \alpha\left(-\frac{(q-1)(p-1)^2}{(z-1)^2} - \frac{p^2 q}{z^2} - \frac{(p-1)(2pq-p-1)}{z-1} + \frac{p(p+2q-2pq)}{z}\right)$$

and $2pq - p - 1 = 0$ and $p + 2q - 2pq = 0$ must hold, but these do not coincide.

(IV) The case where there exist distinct points $p, q, r \in \boldsymbol{C}$ which are poles of ρ of each order -1.

And we investigate three more cases:

(IV-a) The case $p = 0, q = 1$, and $r \notin \{0, 1\}$.

Then $\operatorname{ord}_r G' = 1, \operatorname{ord}_0 G', \operatorname{ord}_1 G', \operatorname{ord}_\infty G' \geq 1$ hold. Hence we have $\sum_{x \in \widehat{\boldsymbol{C}}} \operatorname{ord}_x G' \geq 4$, and it does not happen.

(IV-b) The case $p = 0$, and $q, r \notin \{0, 1\}$.

It holds that $\operatorname{ord}_1 G' = -2, \operatorname{ord}_q G' = \operatorname{ord}_r G' = 1, \operatorname{ord}_0 G' \geq 0$ and $\operatorname{ord}_\infty G' \geq 1$. Thus $\sum_{x \in \widehat{\boldsymbol{C}}} \operatorname{ord}_x G' \geq 1$ holds, and this is a contradiction.

(IV-c) The case $p, q, r \notin \{0, 1\}$.

Then we find that $(\operatorname{ord}_\infty G', \operatorname{ord}_0 G', \operatorname{ord}_1 G', \operatorname{ord}_p G', \operatorname{ord}_q G', \operatorname{ord}_r G') = (1, -2, -2, 1, 1, 1)$. Hence

$$G' = \alpha \left(\frac{a}{(z-1)^2} + \frac{2pqr - pq - qr - rp + 1}{z - 1} - \frac{b}{z^2} + \frac{pq + qr + rp - 2pqr}{z} \right)$$

$(a, b \in \boldsymbol{C}, \alpha \neq 0)$ holds, but this does not happen. In fact, $2pqr - pq - qr - rp + 1 = 0$ and $pq + qr + rp - 2pqr = 0$ must hold, but they do not coincide.

Therefore, the proof is finished. $\square$

(a) (4.39) (b) (4.40) (c) (4.41)

FIGURE 8. Complete improper affine front with total curvature -8π, genus 0, and three embedded ends

We will investigate the rest case $(g, n) = (1, 1)$. We may assume that $\Sigma = T_\tau \setminus \{[0]\}$, where T_τ is in (4.14), and $\rho([0]) = \infty$, and identify ρ, F, and G with the elliptic functions on $\boldsymbol{C}$. Since $\deg \rho = 4$, ρ may have other poles except 0. We shall divide this case into the following **(I)-(IV)**.

(I) The case $\operatorname{ord}_0 \rho = -4$

By (4.1), $G' \neq 0$ on $\Pi_0 \setminus \{0\}$ and $G'(0) \neq \infty$. In particular, G' is holomorphic on Π_0, so G' is a non-zero constant. Hence G is not elliptic.

(II) The case where there exists $p \neq 0$ which is a pole of ρ.

$$(\operatorname{ord}_0 \rho, \operatorname{ord}_p \rho) =^{(a)} (-1, -3),^{(b)} (-2, -2),^{(c)} (-3, -1).$$

(II-a) The case $(\operatorname{ord}_0 \rho, \operatorname{ord}_p \rho) = (-1, -3)$.

In this case, since $\operatorname{ord}_p G' = 3$ and $G' \neq 0$ on $\Pi_0 \setminus \{0, p\}$, $\operatorname{ord}_0 G' = -3$, and then $\operatorname{ord}_0 G = -2$. Hence, G is given by $G = c\wp$ $(c \neq 0)$. Also, $\operatorname{ord}_0 F' = -3$ holds.

Thus Weierstrass data is given by

$$F = a\wp' + b\wp, \quad G = c\wp \quad (a > 0, c \neq 0).$$

(II-b) The case $(\operatorname{ord}_0 \rho, \operatorname{ord}_p \rho) = (-2, -2)$.

Since $\operatorname{ord}_p G' = 2$, we know $\operatorname{ord}_0 G' = -2$ and $\operatorname{ord}_0 G = -1$. This is impossible.

(II-c) The case $(\operatorname{ord}_0 \rho, \operatorname{ord}_p \rho) = (-3, -1)$

Since $\operatorname{ord}_p G' = 1$, we find that $\operatorname{ord}_0 G' = -1$. This is also impossible.

(III) The case where there are $p, q \in \Pi_0 \setminus \{0\}$ $(p \neq q)$ which are poles of ρ.

If $\operatorname{ord}_p \rho = \operatorname{ord}_q \rho = -1, \operatorname{ord}_0 \rho = -2$, then $\operatorname{ord}_p G' = \operatorname{ord}_q G' = 1$, and $\operatorname{ord}_0 G' = -2$, so $\operatorname{ord}_0 G = -1$. This does not happen. On the other hand, we assume that $\operatorname{ord}_p \rho = \operatorname{ord}_0 \rho = -1$, $\operatorname{ord}_q \rho = -2$. Then we obtain $\operatorname{ord}_p G' = 1, \operatorname{ord}_q G' = 2$ and then $\operatorname{ord}_0 G' = -3$. From this, we find that $\operatorname{ord}_0 G = -2$. Hence G can be written as $G = c\wp$ $(c \neq 0)$. Thus for the same reason as (II-a), we have Weierstrass data

$$F = a\wp' + b\wp, \quad G = c\wp \quad (a > 0, c \neq 0).$$

(IV) The case where there are distinct points $p, q, r \in \Pi_0 \setminus \{0\}$ which are poles of ρ of each order -1.

Since $\operatorname{ord}_0 G = -2$, we obtain

$$F = a\wp' + b\wp, \quad G = c\wp \quad (a > 0, c \neq 0).$$

Also, in each case, since $G'(p), G'(q), G'(r) = 0$, the points p, q, and r are half periods of $[1, \tau]$ because of $G' = c\wp'$ and Fact 4.7. Thus the case (II-a) and **(III)** do not happen. Indeed, for the case (II-a), $3p \equiv 0 \mod [1, \tau]$ ([HC44]). So p is not a half period, which is impossible. **(III)** is also impossible for the same reason as (II-a). Therefore we only have to consider the case of **(IV)** and may assume $p = 1/2, q = (1 + \tau)/2, r = \tau/2$.

Now, we shall consider the period condition (2.5). Direct computations give that

$$\int F dG = c\left(\frac{a}{30}\wp'''(z) + \frac{2}{5}ag_2\zeta(z) - \frac{3}{5}ag_3 z + \frac{1}{2}b\wp(z)^2\right)$$

up to additive constant, where $\zeta(z)$ is the Weierstrass ζ-function which satisfies $\zeta'(z) = -\wp(z)$ and $\lim_{z \to 0}(\zeta(z) - 1/z) = 0$, and g_2, g_3 are in (4.15). In addition, consider two curves $\gamma_1(t) := 1/4 + \tau t, \gamma_2(t) := t + \tau/4$ $(t \in [0, 1])$, which generete the fundamental group of T, and a loop γ around the origin. Then it holds that

$$\int_{\gamma_1} F dG = \frac{1}{5}ac\left(2g_2\eta_2 - 3g_3\tau\right), \int_{\gamma_2} F dG = \frac{1}{5}ac\left(2g_2\eta_1 - 3g_3\right), \int_{\gamma} F dG = 0,$$

where η_1, η_2 are constant numbers, which are determined by the lattice $[1, \tau]$, satisfying $\eta_1 = \zeta(z + 1) - \zeta(z), \eta_2 = \zeta(z + \tau) - \zeta(z)$ for all $z \in \boldsymbol{C}$. Hence the period conditions (2.5) are equivalent to

$$\text{(4.42)} \qquad \begin{cases} c\left(2g_2\eta_2 - 3g_3\tau\right) \in i\boldsymbol{R}, \\ c\left(2g_2\eta_1 - 3g_3\right) \in i\boldsymbol{R}. \end{cases}$$

Proposition 4.9. *Complete improper affine fronts with genus* 1 *whose total curvature is* -8π *are constructed by the Weierstrass data*

$$F = a\wp' + b\wp, \quad G = c\wp \quad (a > 0, c \neq 0), \tag{4.43}$$

defined on $\boldsymbol{C}/[1,\tau] \setminus \{[0]\}$ *and satisfying the period condition* (4.42).

Remark 4.10. Nore that Proposition 4.9 still does not show the existence of the surface because we need to determine the modulus τ of the torus and choose $c \in \boldsymbol{C} \setminus \{0\}$ that satisfy the period condition (4.42). If $\tau = i$ (i.e., T_τ is a square torus and this case corresponds with the Chen–Gackstatter [CG82]minimal surface case) or $\tau = e^{(2\pi i)/3}$ (i.e., T_τ is a equilateral triangle torus), then one can observe that these cases are impossible. In fact, when $\tau = i$, it holds that $g_2 > 0, g_3 = 0$ and $\eta_1 = -i\pi$ ([AS64, Section 18]) and then (4.42) yields $c = 0$. When $\tau = e^{(2\pi i)/3}$, it holds that $g_2 = 0, g_3 > 0$ and $\eta_1 = 2\pi/\sqrt{3}$. Then (4.42) implies $c = 0$.

From now on, we will show existence of the surface in the special case where $\tau = e^{i\alpha}$ ($\alpha \in (0,\pi)$) in Proposition 4.9. If the period condition (4.42) holds, then one can see that

$$\mathrm{Im}\left((\overline{2g_2\eta_1 - 3g_3})(2g_2\eta_2 - 3g_3\tau)\right) = 0.$$

Since the inveriants of the $\wp$-function and the ζ-function, namely g_2, g_3, η_1, and η_2 are continuous functions of τ, we put $g_2 = g_2(\tau), g_3 = g_3(\tau), \eta_1 = \eta_1(\tau)$, and $\eta_2 = \eta_2(\tau)$. We then set

$$P(\alpha) := \mathrm{Im}\left(\overline{p_1(\alpha)}p_2(\alpha)\right), \tag{4.44}$$

where $p_1(\alpha) := 2g_2(e^{i\alpha})\eta_1(e^{i\alpha}) - 3g_3(e^{i\alpha}),\ p_2(\alpha) := 2g_2(e^{i\alpha})\eta_2(e^{i\alpha}) - 3g_3(e^{i\alpha})e^{i\alpha}$.

Theorem 4.11. *There exists* $\alpha_0 \in (\pi/3, \pi/2)$ *such that* $P(\alpha_0) = 0$. *In particular, there exists a complete improper affine front* $\psi : \boldsymbol{C}/[1, e^{i\alpha_0}] \setminus \{[0]\} \to \boldsymbol{R}^3$ *of genus* 1 *whose total curvature is* -8π *(Figure 9).*

Proof. [AS64, Section 18] shows that the concrete values of g_2, g_3, η_1 and η_2 are

$$g_2\left(\frac{\pi}{3}i\right) = 0,\ g_3\left(\frac{\pi}{3}i\right) > 0,\ \eta_1\left(\frac{\pi}{3}i\right) = \frac{2\pi}{\sqrt{3}},\ \eta_2\left(\frac{\pi}{3}i\right) = \frac{2\pi}{\sqrt{3}}e^{-\frac{i\pi}{3}},$$
$$g_2(i) > 0,\ g_3(i) = 0,\ \eta_1(i) = \pi,\ \eta_2(i) = -i\pi.$$

Then, direct computations give that $P(\pi/3) = (9\sqrt{3}g_3^2)/2 > 0,\ P(\pi/2) = -4g_2^2\pi^2 < 0$. Since the function $P(\alpha)$ is continuous on $(0,\pi)$, the intermidiate value theorem yields that there exists $\alpha_0 \in (\pi/3, \pi/2)$ such that $P(\alpha_0) = 0$.

Here, either $p_1(\alpha)$ or $p_2(\alpha)$ does not vanish for any $\alpha \in (0,\pi)$. In fact, if $p_1(\alpha) = p_2(\alpha) = 0$ for some α, then it holds that

$$2g_2(e^{i\alpha})(\eta_1(e^{i\alpha})e^{i\alpha} - \eta_2(e^{i\alpha})) = 0.$$

By the Legendre's identity $\eta_1(e^{i\alpha})e^{i\alpha} - \eta_2(e^{i\alpha}) = 2\pi i$, one can observe that $g_2(e^{i\alpha}) = 0$ and then the torus is a equilateral torus. From Remark 4.10, it is impossible.

Thus, we choose as complex number c in (4.43) the non-zero either of

$$c = i\overline{p_1(\alpha_0)} \quad \text{and} \quad c = i\overline{p_2(\alpha_0)}$$

and hence one can observe that the period conditions (4.42) is satisfied. Therefore, we complete the proof. □

FIGURE 9. Complete improper affine front of genus 1 with total curvature -8π when $c = i\overline{p_1(\alpha_0)}$. The values of α_0 and c can be estimated as $\alpha_0 \approx 1.37048$, $c = i\overline{p_1(\alpha_0)} \approx 1265.89 + 370.33i$ by using the Mathematica software.

Theorem 4.11 shows that there is a complete improper affine front with the maximum total curvature and positive genus.

Remark 4.12. Now we consider a function

$$\widetilde{P}(\tau) := \operatorname{Im}\left(\overline{\widetilde{p_1}(\tau)}\widetilde{p_2}(\tau)\right), \tag{4.45}$$

where $\widetilde{p_1}(\tau) := 2g_2(\tau)\eta_1(\tau) - 3g_3(\tau)$, $\widetilde{p_2}(\tau) := 2g_2(\tau)\eta_2(\tau) - 3g_3(\tau)\tau$ are defined on the upper half plane $\boldsymbol{H} := \{\tau \in \boldsymbol{C}; \operatorname{Im}\tau > 0\}$. Theorem 4.11 shows that $\widetilde{P}(e^{i\alpha_0}) = P(\alpha_0) = 0$. On the other hand, the invariants $g_2(\tau), g_3(\tau), \eta_1(\tau)$, and $\eta_2(\tau)$ have an expression by the Weierstrass θ-function. The Mathematica software computes

$$\left.\frac{d\widetilde{P}}{d\alpha}(e^{i\alpha})\right|_{\alpha=\alpha_0} \approx -7.74116 \times 10^6 \neq 0.$$

Thus, from the implicit function theorem, there exists an interval I ($\ni 0$) and a smooth curve $\phi\colon I \to \boldsymbol{C}$ such that

$$\phi(0) = e^{i\alpha_0}, \quad \widetilde{P}(\phi(t)) = 0 \ (t \in I).$$

Hence, when we set $W := \left\{\tau = \phi(t) \in \boldsymbol{H}; \widetilde{P}(\phi(t)) = 0 \ (t \in I)\right\}$ and choose as c the non-zero either of $i\overline{\widetilde{p_1}(\tau)}$ and $i\overline{\widetilde{p_2}(\tau)}$ for each $\tau \in W$, the period condition (4.42) holds. Therefore, it implies the existence of an real one parameter family with respect to the modulus τ of complete improper affine fronts of genus 1 and the total curvature -8π.

Finally, we give a new example of a complete improper affine front of genus 1 (for known examples of genus 1 whose total curvature is -12π, see [Mar05a, Section 4, No.6]).

Example 4.13. Let $\Sigma = \boldsymbol{C}/(\boldsymbol{Z} \oplus i\boldsymbol{Z}) \setminus \{[0]\}$ be the square torus minus one point and define F, G by

$$F = \wp'' + \frac{5g_2}{7\pi}\wp, \quad G = \wp'. \tag{4.46}$$

One can verify that these F, G satisfy the period condition (2.5). Therefore, (F, G) induces a complete improper affine front $\psi\colon \Sigma \to \boldsymbol{R}^3$ of genus 1 with the total curvature -10π (Figure 10).

FIGURE 10. Complete improper affine front of genus 1 with total curvature -10π

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