SUTURED HEEGAARD FLOER AND EMBEDDED CONTACT HOMOLOGIES ARE ISOMORPHIC

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ABSTRACT. We prove the equivalence of the sutured versions of Heegaard Floer homology, monopole Floer homology, and embedded contact homology. As applications we show that the knot versions of Heegaard Floer homology and embedded contact homology are equivalent and that product sutured 3-manifolds are characterized by the fact that they carry an adapted Reeb vector field without periodic orbits.

1. INTRODUCTION

Heegaard Floer homology, monopole Floer homology and embedded contact homology are three drastically different-looking incarnations of the same closed 3-manifold invariant. Heegaard Floer homology, introduced by Ozsváth and Szabó [OSz1, OSz2], is easily seen to be topological and admits a combinatorial description via nice Heegaard diagrams [SW]. Embedded contact homology (ECH), defined by Hutchings [Hu1, Hu2, Hu3] and Hutchings-Taubes [HT1, HT2], encodes the dynamical properties of an auxiliary Reeb vector field. Both were defined as symplectic counterparts of monopole Floer homology, defined by Kronheimer and Mrowka [KM1]. The latter was shown to be isomorphic to ECH by Taubes in [T2, T3, T4, T5, T6] and to Heegaard Floer homology by Kutluhan-Lee-Taubes in [KLT1, KLT2, KLT3, KLT4, KLT5]. Heegaard Floer homology and ECH were independently shown to be isomorphic to each other in [CGH0, CGH1, CGH2, CGH3].

All three homologies admit natural extensions to compact 3-manifolds with *sutured* boundary [Ju1, CGHH, KM2]. Sutured manifolds were introduced by Gabai [Ga] in the context of foliation theory and are now understood to be a bridge between contact geometry and its convex surface theory [Gi] on one hand and geometric decompositions of 3-manifolds/gauge-theoretic invariants on the other hand. In particular, sutured Heegaard Floer homology, developed under the impulsion of Juhász [Ju2], has striking applications to low-dimensional topology.

Baldwin and Sivek proved in [BS] that the sutured versions of monopole Floer homology and Heegaard Floer homology are isomorphic and that the isomorphism identifies the contact invariants. For what concerns the relation between the sutured

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versions of Heegaard Floer homology and ECH, we have the following conjecture, which is a slight strengthening of Conjecture 1.5 in [CGHH].

Conjecture 1.1. *If (M, Γ, ξ) is a sutured contact 3-manifold, then*

$$ECH(M, \Gamma, \xi, A) \simeq SFH(-M, -\Gamma, \mathfrak{s}_\xi + \mathrm{PD}(A))$$

as relatively graded vector spaces over $\mathbb{Z}/2\mathbb{Z}$, where $A \in H_1(M; \mathbb{Z})$, $\mathfrak{s}_\xi$ is the canonical Spin^c-structure determined by ξ, $ECH(M, \Gamma, \xi, A)$ is the sutured ECH of (M, Γ, ξ) in the homology class A, and $SFH(-M, -\Gamma, \mathfrak{s}_\xi + \mathrm{PD}(A))$ is the sutured Heegaard Floer homology of $(-M, \Gamma)$ in the Spin^c-structure $\mathfrak{s}_\xi + \mathrm{PD}(A)$. Moreover the isomorphism identifies the contact invariant of ξ in $ECH(M, \Gamma, \xi, 0)$ to that of $SFH(-M, -\Gamma, \mathfrak{s}_\xi)$.

Remark 1.2. *ECH admits a unique lifting to the integers defined by a coherent orientation of the moduli spaces defining the boundary map, while Heegaard Floer homology admits different liftings called* orientation systems*. In order to state the conjecture over the integers, one would need to identify a canonical orientation system for SFH, which has not been done yet.*

In this paper we prove part of Conjecture 1.1. In particular, we obtain the following result.

Theorem 1.3. *Let (M, Γ, ξ) be a sutured contact manifold. Then*

$$ECH(M, \Gamma, \xi) \simeq SFH(-M, -\Gamma), \tag{1.4}$$

where $ECH(M, \Gamma, \xi)$ is the sutured ECH of (M, Γ, ξ) summed over all homology classes and $SFH(-M, -\Gamma)$ is the sutured Heegaard Floer homology of $(-M, -\Gamma)$ summed over all relative Spin^c-structures.

For technical reasons we are unable to say anything about the contact invariants of ξ and only prove a partial splitting of Equation (1.4) into relative Spin^c-structures; see Theorem 5.1 for the precise statement. However, this partial splitting is sufficient to give a complete splitting into relative Spin^c-structures in the knot invariant case; see Corollary 1.7 and its stronger version Corollary 5.2.

We give two applications of Theorem 1.3. The first is the topological invariance of sutured ECH.

Corollary 1.5. *The vector spaces $ECH(M, \Gamma, \xi)$ are topological invariants of (M, Γ) (and of the canonical Spin^c-structure of ξ if we also take into account the partial decomposition in terms of relative Spin^c-structures).*

Previously it was only known that $ECH(M, \Gamma, \xi)$ is an invariant of (M, Γ, ξ) by Theorem 10.2.2 in [CGH0] and Theorem 1.2 in [KS].

As another application of Theorem 1.3, we characterize product sutured 3-manifolds by the fact that they carry compatible Reeb vector fields without periodic orbits (Theorem 6.1). This extends the proof of the Weinstein conjecture [T1] to contact 3-manifolds with sutured boundary. We also show that if (M, Γ, ξ) is a taut sutured contact 3-manifold of depth greater than $2k$ with $H_2(M) = 0$ and if an adapted Reeb vector field R_λ is nondegenerate and has no elliptic orbit, then it has at least $k+1$ hyperbolic orbits (Theorem 6.3).

We also prove an isomorphism of sutured ECH with the sutured version of monopole Floer homology, denoted SHM:

Theorem 1.6. *Let (M,Γ,ξ) be a sutured contact manifold. Then*
$$ECH(M,\Gamma,\xi) \simeq SHM(-M,-\Gamma).$$

The proofs of Theorems 1.3 and 1.6 go through the construction of a contact embedding of any sutured contact manifold (M,Γ,ξ) into a closed contact manifold (Y,ξ), called the *contact closure*, on which we control the Reeb dynamics. We abuse notation by using the same name for both contact structures; this is justified by the fact that they agree where they are both defined, i.e., on M. We identify $ECH(M,\Gamma,\xi)$ as a summand in $\widehat{ECH}(Y,\xi)$ and find the analogous identification on the Heegaard Floer side, given by a result of Lekili [Le]. The isomorphism between the summands then follows from the isomorphism between $\widehat{ECH}(Y,\xi)$ and $\widehat{HF}(-Y)$, proven in the series [CGH0, CGH1, CGH2]. On the other hand, the closed 3-manifold Y is the same closure Kronheimer and Mrowka used to define sutured monopole Floer homology, and therefore Theorem 1.6 follows from the computation of $\widehat{ECH}(Y,\xi)$ and Taubes' isomorphism between monopole Floer homology and ECH proven in the series [T2]–[T6].

Juhász observed that the hat version of knot Floer homology of a knot in a 3-manifold can be interpreted as the sutured Floer homology of the knot complement with a pair of meridian sutures. Then the isomorphism between the sutured Floer homologies, in its stronger form taking into account the partial splitting according to relative Spin^c-structures proved in Theorem 5.1, can be translated into an isomorphism between knot Floer homology and ECH of a sutured manifold associated to the knot:

Corollary 1.7. *Let K be a null-homologous knot in a closed manifold M and S a Seifert surface of K. If $M(K)$ is the complement of a tubular neighborhood of K, Γ_K a pair of oppositely oriented disjoint meridians in $\partial M(K)$, and $(M(K),\Gamma_K,\xi)$ a sutured contact manifold, then, for every $d\in\mathbb{Z}$,*

(1.8)
$$\widehat{HFK}(-M,-K,[S],d) \simeq \bigoplus_{\langle c_1(\mathfrak{s}_\xi)+2\,\mathrm{PD}(A),[S]\rangle=2d} ECH(M(K),\Gamma_K,\xi,A).$$

Here $c_1(\mathfrak{s}_\xi)\in H_2(M(K),\partial M(K))$ is the relative Chern class of the canonical Spin^c-structure $\mathfrak{s}_\xi$. Spano in his thesis [Sp] gave evidence for this isomorphism by showing that the graded Euler characteristic of $SFH(M(K),\Gamma_K,\xi)$ is the Alexander polynomial.

When $K\subset M$ is a fibered knot and ξ is the Thurston-Winkelnkemper contact structure on $M(K)$, then $ECH(M(K),\Gamma_K,\xi)$ is isomorphic to a version of the periodic Floer homology of the monodromy which will be defined in Section 2.4. Thus we have the following corollary of corollary:

Corollary 1.9. *Let M be a closed manifold and K a fibered knot in M of genus g and fiber S. If $\mathfrak{h}$ is an area-preserving representative of the monodromy with zero flux, then*

$$PFH^{\sharp}(\mathfrak{h}, d) \simeq \widehat{HFK}(-M, -K, d-g). \tag{1.10}$$

Remark 1.11. *It is possible to refine Equations* (1.8) *and* (1.10) *by taking into account the splitting according to relative Spin^c-structures; the precise statement will be given in Corollary 5.2.*

When $d = 1$, periodic Floer homology reduces to the usual symplectic Floer homology of a surface automorphism, and therefore Corollary 1.9 generalizes previous results of Ni [Ni] and Ghiggini–Spano [GS]. The proof here is similar in spirit to that of [Ni], which goes from the knot to a (different) closed manifold and uses the isomorphism between monopole Floer homology and periodic Floer homology due to Lee-Taubes [LT], followed by the isomorphism of [KLT1]–[KLT5]. On the other hand, the proof in [GS] is almost completely independent of the isomorphisms as it uses only the (simpler) open-closed map of [CGH1] and "standard" symplectic geometry.

In [KM2] Kronheimer and Mrowka defined knot monopole Floer homology groups $HKM(M, K, [S], d)$, where M is a closed manifold, $K \subset M$ a null-homologous knot, S a Seifert surface for K, and $d \in \mathbb{Z}$, as the monopole Floer homology of the sutured manifold $(M(K), \Gamma_K)$. The same argument proving Corollary 1.7 also proves the following corollary:

Corollary 1.12. *Let K be a null-homologous knot in a closed manifold M and S a Seifert surface of K. Then, for every $d \in \mathbb{Z}$,*

$$HKM(-M, -K, [S], d) \simeq \bigoplus_{\langle c_1(\mathfrak{s}_\xi) + 2\,\mathrm{PD}(A), [S]\rangle = 2d} ECH(M(K), \Gamma_K, \xi, A). \tag{1.13}$$

Remark 1.14. *The reason why sutured monopole Floer homology does not have a decomposition into relative Spin^c summands but knot monopole Floer homology does have an Alexander grading is the same reason why we could not get a full Spin^c-decomposition in Theorem 1.3 but we could prove that the isomorphism in Corollary 1.7 preserves the Alexander grading.*

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2. SUTURED MANIFOLDS AND THEIR FLOER HOMOLOGIES

In this section we review some ingredients from [CGHH], [CGH0] and [Ju1]. All the Floer-type homology groups will be defined over the ground field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

2.1. Balanced sutured manifolds.

The various sutured invariants mentioned in the introduction are defined for *balanced* sutured manifolds, a restricted class of sutured manifolds introduced by Juhász in [Ju1]. Here we present the definition in a slightly modified form because it is convenient for us to present M as a manifold with corners and include the choice of a tubular neighborhood of the suture in the definition.

Definition 2.1. *A* balanced *sutured* 3*-manifold is a triple* $(M, \Gamma, U(\Gamma))$*, where* M *is a compact* 3*-manifold with boundary and corners,* Γ *is an oriented* 1*-manifold in* ∂M *called the* suture*, and* $U(\Gamma) \simeq [-1, 0] \times \Gamma \times [-1, 1]$ *is a neighborhood of* $\Gamma \simeq \{0\} \times \Gamma \times \{0\}$ *in* M *with coordinates* $(\tau, t) \in [-1, 0] \times [-1, 1]$*, such that the following hold:*

- M *has no closed components;*
- $U(\Gamma) \cap \partial M \simeq (\{0\} \times \Gamma \times [-1, 1]) \cup ([-1, 0] \times \Gamma \times \{-1\}) \cup ([-1, 0] \times \Gamma \times \{1\})$*;*
- $\partial M \setminus (\{0\} \times \Gamma \times (-1, 1))$ *is the disjoint union of two submanifolds which we call* $R_-(\Gamma)$ *and* $R_+(\Gamma)$*, where the orientation of* ∂M *agrees with that of* $R_+(\Gamma)$ *and is opposite that of* $R_-(\Gamma)$*, and the orientation of* Γ *agrees with the boundary orientation of* $R_\pm(\Gamma)$*;*
- *the corners of* M *are precisely* $\{0\} \times \Gamma \times \{\pm 1\}$*;*
- $R_\pm(\Gamma)$ *have no closed components and* $\chi(R_-(\Gamma)) = \chi(R_+(\Gamma))$.

Definition 2.2. *If* $(M, \Gamma, U(\Gamma))$ *is a sutured* 3*-manifold,* $(M, \Gamma, U(\Gamma), \xi)$ *is a* sutured contact manifold *if there exists a contact form* λ *for* ξ *with Reeb vector field* R_λ *such that:*

(C1) R_λ *is positively transverse to* $R_+(\Gamma)$ *and negatively transverse to* $R_-(\gamma)$*;*
(C2) $\lambda = C dt + \beta$ *on* $U(\Gamma)$ *for some constant* $C > 0$*, where* β *is independent of* t*. In particular,* $R_\lambda = \frac{1}{C}\partial_t$ *on* $U(\Gamma)$.

A contact form λ *satisfying (C1) and (C2), and the contact structure* $\xi = \ker \lambda$*, are said to be* adapted to $(M, \Gamma, U(\Gamma))$.

From now on, to simplify notation, we will always omit the neighborhood $U(\Gamma)$ in the data associated to a sutured contact manifold. Sometimes we will even regard M as a manifold with (smooth) boundary and Γ as a closed codimension-one submanifold with boundary; in such a case it is understood that we introduce convex corners along the boundary of a neighborhood of Γ.

2.2. **Sutured Floer homology and knot Floer homology.** The sutured Heegaard Floer homology $SFH(M,\Gamma)$ of a balanced sutured 3-manifold (M,Γ) is a topological invariant of (M,Γ). It decomposes according to relative Spin^c-structures:

$$SFH(M,\Gamma)=\bigoplus_{\mathfrak{s}\in \mathrm{Spin}^c(M,\Gamma)} SFH(M,\Gamma,\mathfrak{s}).$$

We refer to the original paper [Ju1] for the definition.

If M is a closed manifold and $B\subset M$ is a closed ball, we define the balanced sutured manifold (M_B,Γ_B), where $M_B=M\setminus \mathrm{int}(B)$ and Γ_B is a connected, embedded closed curve in $\partial M_B\simeq S^2$. (In [Ju1] the sutured manifold (M_B,Γ_B) is denoted by $M(1)$.) By [Ju1] there is a tautological isomorphism

$$\widehat{HF}(M)\simeq SFH(M_B,\Gamma_B). \tag{2.3}$$

When K is a knot in a 3-manifold M, one can form the sutured manifold

$$(M(K),\Gamma_K)=(M\setminus \mathrm{int}(N(K)),\Gamma_K),$$

where $N(K)$ is a tubular neighborhood of K in M and Γ_K consists of two disjoint curves parallel to the meridian of K in $\partial N(K)$. Let $\widehat{HFK}(M,K)$ be the hat version of knot Floer homology defined in [OSz3]. Then by [Ju1] there is a (tautological) isomorphism

$$\widehat{HFK}(M,K)\simeq SFH(M(K),\Gamma_K). \tag{2.4}$$

Assume now that K bounds an oriented embedded surface $\Sigma\subset M$. Let $M_0(K)$ be the 3-manifold obtained by zero-surgery on M along K, where the surgery coefficient is computed with respect to the framing induced by Σ. Then the knot Floer homology group decomposes according to Spin^c-structures on $M_0(K)$:

$$\widehat{HFK}(M,K)=\bigoplus_{\underline{\mathfrak{s}}\in \mathrm{Spin}^c(M_0(K))} \widehat{HFK}(M,K,\underline{\mathfrak{s}}).$$

Let $\widehat{\Sigma}\subset M_0(K)$ be the closed surface obtained by capping off Σ. Every relative Spin^c-structure $\mathfrak{s}\in \mathrm{Spin}^c(M(K),\Gamma_K)$ extends uniquely to a Spin^c-structure $\underline{\mathfrak{s}}\in \mathrm{Spin}^c(M_0(K))$ such that

$$\langle c_1(\mathfrak{s}),[\Sigma]\rangle=\langle c_1(\underline{\mathfrak{s}}),[\widehat{\Sigma}]\rangle$$

and Equation (2.4) can be refined to

$$\widehat{HFK}(M,K,\underline{\mathfrak{s}})\simeq SFH(M(K),\Gamma_K,\mathfrak{s}). \tag{2.5}$$

Finally we recall that one defines, for $d\in\mathbb{Z}$,

$$\widehat{HFK}(M,K,[\Sigma],d)=\bigoplus_{\substack{\underline{\mathfrak{s}}\in \mathrm{Spin}^c(M_0(K))\\ \langle c_1(\underline{\mathfrak{s}}),[\widehat{\Sigma}]\rangle=2d}} \widehat{HFK}(M,K,\underline{\mathfrak{s}}).$$

The integer d is called the *Alexander grading*.

2.3. **Sutured ECH.** Let λ be a nondegenerate contact form adapted to (M, Γ) and J a *tailored* almost complex structure from [CGHH, Section 3.1]. Since such a J prevents families of holomorphic curves in the symplectization of M from exiting along its boundary [CGHH, Proposition 5.20], Hutchings' definition of ECH extends in a straightforward manner to (M, Γ, λ, J). Just recall here that the sutured ECH chain complex $ECC(M, \Gamma, \lambda, J)$ is generated over $\mathbb{F}$ by orbit sets $\boldsymbol{\gamma} = \{(\gamma_i, m_i) \mid i = 1, \ldots, k; k \in \mathbb{Z}_{\geq 0}\}$ — this includes the empty set — where γ_i is a simple orbit of the Reeb vector field R_λ, $m_i \in \mathbb{Z}_{>0}$, and $m_i = 1$ whenever γ_i is a hyperbolic orbit. We will sometimes write the orbit set $\boldsymbol{\gamma}$ multiplicatively as $\prod_i \gamma_i^{m_i}$. We call m_i the *multiplicity* of γ_i in $\boldsymbol{\gamma}$.

Convention 2.6. *In this paper, when we write "orbit" we mean "closed/periodic orbit".*

The coefficient $\langle \partial \boldsymbol{\gamma}, \boldsymbol{\gamma}' \rangle$ in the differential counts ECH index $I = 1$ J-holomorphic curves in the symplectization of (M, λ) that are asymptotic to the orbit sets $\boldsymbol{\gamma}$ at $+\infty$ and $\boldsymbol{\gamma}'$ at $-\infty$; see [Hu1]. The ECH index 1 property implies strong restrictions on the asymptotic behavior of a curve approaching an orbit, called *partition conditions*, for which we refer to [Hu2, Definitions 4.13 and 4.14 and Theorem 4.15]. Relying on the analogous result for closed manifolds, we proved in [CGH0, Theorem 10.2.2] (see also [KS]) that sutured ECH, denoted by $ECH(M, \Gamma, \xi)$, is an invariant of the sutured contact 3-manifold (M, Γ, ξ). As in the closed case, there exists a direct sum decomposition into homology classes $A \in H_1(M; \mathbb{Z})$ of orbit sets as follows:

$$ECH(M, \Gamma, \xi) = \bigoplus_{A \in H_1(M;\mathbb{Z})} ECH(M, \Gamma, \xi, A).$$

If M is a closed manifold, $B \subset M$ a closed ball, ξ is a contact structure that is adapted to (M_B, Γ_B) and $A \in H_1(M; \mathbb{Z}) \simeq H_1(M_B; \mathbb{Z})$, then we define

$$\widehat{ECH}(M, \xi, A) = ECH(M_B, \Gamma_B, \xi, A).$$

The hat version of ECH was originally defined as the mapping cone of a U-map, and its equivalence with a sutured ECH was proved in [CGH0, Theorem 10.3.1].

2.4. **Periodic Floer homology and sutured ECH.** When K is a fibered knot in M, the sutured ECH of $(M(K), \Gamma_K)$ can be interpreted as a version of the periodic Floer homology of a special representative of the monodromy of K.

Let S be a fiber of K and let (ρ, θ) be coordinates on a collar neighborhood $[-1, 0] \times S^1 \subset \partial S$ such that $\partial S = \{0\} \times S^1$. There exist a 1-form λ and a representative $\mathfrak{h}\colon S \xrightarrow{\sim} S$ of the monodromy such that:

- $d\lambda$ is an area form on S and $\lambda = e^\rho d\theta$ near ∂S;
- $\mathfrak{h}^*\lambda - \lambda$ is exact;
- the periodic points of $\mathfrak{h}$ in $\mathrm{int}(S)$ are nondegenerate; and
- $\mathfrak{h}|_{\partial S} = \mathrm{id}_{\partial S}$ and the linearized first return map at every point of ∂S is of the form $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$ with $a < 0$ (i.e., ∂S is a negative Morse-Bott circle of fixed points).

The existence of λ and $\mathfrak{h}$ follows from [CGH0, Lemma 9.3.2] and a standard genericity argument for nondegenerate periodic points.

The mapping torus $N_{(S,\mathfrak{h})}$ of $(S,\mathfrak{h})$ carries a suspension flow which is transverse to the fibers and whose first return map is $\mathfrak{h}$. The boundary of $N_{(S,\mathfrak{h})}$ admits an S^1-family of simple orbits of the suspension flow and we choose one orbit that we call h. As in the definition of ECH, the *periodic Floer homology* chain complex $PFC^\sharp(\mathfrak{h})$ is generated, as a vector space over $\mathbb{F}$, by orbit sets $\boldsymbol{\gamma}=\{(\gamma_i,m_i)\mid i=1,\dots,k; k\in\mathbb{Z}_{\geq 0}\}$ (including the empty set), where γ_i is a simple orbit of the suspension flow in $\operatorname{int}(N_{(S,\mathfrak{h})})$ or the orbit h on the boundary, $m_i\in\mathbb{Z}_{>0}$, and $m_i=1$ whenever γ_i is a hyperbolic orbit or h (i.e., h is treated as a hyperbolic orbit, hence the symbol h). The name "periodic Floer homology" is due to the fact that closed orbits of the suspension flow are in bijection with orbits of periodic points of $\mathfrak{h}$.

The manifold $N_{(S,\mathfrak{h})}$ carries a natural stable Hamiltonian structure (α_0,ω) induced by $d\lambda$ (see [CGH1, Section 3.1]). Let J be a generic almost complex structure on $\mathbb{R}\times N_{(S,\mathfrak{h})}$ which is adapted to (α_0,ω) in the sense of Definition [CGH1, Definition 3.2.1]. The analytical foundations of ECH go through for stable Hamiltonian structures on mapping tori (see [Hu1] and [LT]) and therefore we define the boundary operator on $PFH^\sharp(\mathfrak{h})$ by counting $I=1$ J-holomorphic maps in $\mathbb{R}\times N_{(S,\mathfrak{h})}$ asymptotic to orbit sets at the positive and negative ends. Here the situation is less standard than the one considered in [LT] due to the presence of the orbit h belonging to a Morse-Bott family. This situation was treated in detail in [CGH0, Section 7], where a similar chain complex $ECC^\sharp(N_{(S,\mathfrak{h})},\alpha)$ is defined for a contact form α on $N_{(S,\mathfrak{h})}$, and the argument goes through unchanged for periodic Floer homology.

Periodic Floer homology splits as a direct sum over homology classes

$$PFH^\sharp(\mathfrak{h})=\bigoplus_{A\in H_1(N_{(S,\mathfrak{h})})}PFH^\sharp(\mathfrak{h},A),$$

as usual. We also define, for $d\in\mathbb{Z}$,

$$PFH^\sharp(\mathfrak{h},d)=\bigoplus_{A\cdot[S]=d}PFH^\sharp(\mathfrak{h},A),$$

where $[S]$ is the class of a fiber and $A\cdot[S]$ is the algebraic intersection number.

Note that $N_{(S,\mathfrak{h})}\simeq M(K)$. We have the following isomorphism.

Lemma 2.7. *Let ξ be a contact structure on $(M(K),\Gamma_K)$ obtained by a small perturbation of the tangent planes of the fibers. Then, for every $A\in H_1(M(K))$,*

$$ECH(M(K),\Gamma_K,\xi,A)\simeq PFH^\sharp(\mathfrak{h},A).$$

Proof. The lemma follows from [CGH0, Theorem 10.3.2] and the arguments of [CGH1, Section 3.6]. □

3. Proofs of Theorems 1.3 and 1.6

3.1. **Reduction to connected sutures.** In this subsection we show that we may assume without loss of generality that the suture of (M,Γ,ξ) is connected.

Lemma 3.1. *If Theorems 1.3 and 1.6 hold for sutured contact manifolds with connected sutures, then they hold for all sutured contact manifolds.*

Proof. Let (M, Γ, ξ) be a sutured contact manifold with disconnected suture. We glue sutured contact product 1-handles $(H \times [-1,1], \ker(dt+\beta))$ to (M, Γ), where t is the coordinate of $[-1,1]$ and β is a Liouville form on H, i.e., we take an *interval-fibered extension*, to obtain a sutured contact manifold (M', Γ', ξ') with a connected suture Γ'. From [CGHH, Section 9] we obtain the isomorphism

$$ECH(M', \Gamma', \lambda') \simeq ECH(M, \Gamma, \lambda).$$

From [Ju1, Lemma 9.13] we obtain the isomorphism

$$SFH(-M', -\Gamma') \simeq SFH(-M, -\Gamma),$$

since $(-M, -\Gamma)$ is obtained from $(-M', -\Gamma')$ by a sequence of product disk decompositions along the cocores of $H \times [-1,1]$. Finally from [KM2, Lemma 4.6, Proposition 6.5 and Proposition 6.7] we obtain the isomorphism

$$SHM(-M', -\Gamma') \simeq SHM(-M, -\Gamma),$$

since there is a product annulus splitting $(-M', -\Gamma')$ into the disjoint union of $(-M, -\Gamma)$ and a product sutured manifold. $\square$

3.2. **The contact closure.** Let (M, Γ, ξ) be a sutured contact 3-manifold with connected suture. We pick a compact, oriented surface S of genus $g \geq 3$ with connected boundary, together with a $[-1,1]$-invariant contact structure ξ[1] on $S \times [-1,1]_t$ such that:

- the dividing set of $S \times \{\pm 1\}$ consists of a single circle in $\mathrm{int}(S) \times \{\pm 1\}$ bounding a disk $D \times \{\pm 1\}$;
- $D \times \{+1\}$ is the negative region of $S \times \{+1\}$ and $D \times \{-1\}$ is the positive region of $S \times \{-1\}$; and
- the characteristic foliation, oriented in the usual way, enters S along ∂S.

We then glue the product $(S \times [-1,1], \xi)$ to (M, Γ, ξ) along $\partial S \times [-1,1] \simeq \Gamma \times [-1,1]$. We obtain a contact 3-manifold (Y_Σ, ξ) with boundary components

$$\Sigma_+ = R_+ \cup_{\partial R_+ \simeq \partial S \times \{1\}} (S \times \{1\}),$$
$$\Sigma_- = R_- \cup_{\partial R_- \simeq \partial S \times \{-1\}} (S \times \{-1\}).$$

Lastly we consider the closed contact 3-manifold (Y, ξ) obtained by identifying Σ_+ and Σ_- by a ξ-compatible diffeomorphism

$$\widetilde{\psi} \colon \Sigma_+ \to \Sigma_-$$

that is the identity between $S \times \{1\}$ and $S \times \{-1\}$. We denote

$$\psi = \widetilde{\psi}|_{R_+(\Gamma)} \colon R_+(\Gamma) \to R_-(\Gamma)$$

and assume that ψ is the identity between $U(\Gamma) \cap R_+(\Gamma)$ and $U(\Gamma) \cap R_-(\Gamma)$.

[1]Since the contact structures will be glued, the contact structures will all be denoted by ξ in this subsection.

We let Σ be the glued $\Sigma_+ = \Sigma_-$ in Y, oriented as Σ_+. Let $e(\xi)$ be the Euler class of ξ. Then

$$\langle e(\xi), [\Sigma] \rangle = \chi(\Sigma) - 2.$$

The topological part of such a construction — turning a sutured manifold into a closed one — was first considered by Kronheimer and Mrowka in the context of monopole Floer homology [KM2].

The key technical result of this article is the following isomorphism:

Theorem 3.2. *Let (M, Γ, ξ) be a sutured contact 3-manifold with connected suture and (Y, ξ) its contact closure. Then*

$$\bigoplus_{A \cdot [\Sigma]=1} \widehat{ECH}(Y, \xi, A) \simeq ECH(M, \Gamma, \xi) \oplus ECH(M, \Gamma, \xi)[1]. \tag{3.3}$$

The proof of this theorem will occupy Section 4.

3.3. **Proofs of Theorems 1.3 and 1.6 assuming Theorem 3.2.** We introduce the notation

$$\widehat{HF}(-Y|\Sigma) = \bigoplus_{\langle c_1(\mathfrak{s}), [\Sigma] \rangle = \chi(\Sigma)} \widehat{HF}(-Y, \mathfrak{s}),$$

$$\widehat{ECH}(Y, \xi|\Sigma) = \bigoplus_{A \cdot [\Sigma]=1} \widehat{ECH}(Y, \xi, A).$$

Similar notation will be used also for HF^+ and monopole Floer homology.

Lemma 3.4. $\widehat{HF}(-Y|\Sigma) \simeq \widehat{ECH}(Y, \xi|\Sigma)$.

Proof. Let $\mathfrak{s}_\xi$ be the canonical Spin^c-structure determined by ξ. Since $\langle e(\xi), [\Sigma] \rangle = \chi(\Sigma) - 2$, the map

$$A \mapsto \mathfrak{s}_\xi + PD(A)$$

gives a bijection between the homology classes satisfying $A \cdot [\Sigma] = 1$ and the Spin^c-structures satisfying $\langle c_1(\mathfrak{s}), [\Sigma] \rangle = \chi(\Sigma)$. Finally, by [CGH1, Theorem 1.2.1] there is an isomorphism

$$\widehat{ECH}(Y, \xi, A) \simeq \widehat{HF}(-Y, \mathfrak{s}_\xi + PD(A)).$$ □

Theorem 3.2 provides a link between $ECH(M, \Gamma, \xi)$ and $\widehat{ECH}(Y, \xi|\Sigma)$. In order to prove Theorem 1.3 assuming Theorem 3.2, it remains to relate $\widehat{HF}(-Y|\Sigma)$ to $SFH(-M, -\Gamma)$.

Lemma 3.5. *If $R_+(\Gamma)$ and $R_-(\Gamma)$ have minimal genus in their relative homology class in $H_2(M, \Gamma)$, then Σ has minimal genus in its homology class.*

Proof. In this lemma we use Gabai's original definition of sutured manifolds, i.e., we allow empty boundary and empty sutures. Suppose that $R_\pm(\Gamma)$ is genus-minimizing in its class in $H_2(M, \Gamma)$. Writing $M = M' \# N$ where $\partial N = \emptyset$ and M' is irreducible, $R_\pm(\Gamma)$ is still genus-minimizing in its class in $H_2(M', \Gamma)$, and therefore (M', Γ') is a taut sutured manifold; see [Ga, Definition 2.10].

We have connected sum decompositions $Y_\Sigma = Y'_\Sigma \# N$ and $Y = Y' \# N$. Since (M', Γ) is obtained from Y'_Σ by a sequence of product annulus and disk decompositions, Y'_Σ, seen as a sutured manifold with empty suture, is taut by Lemma [Ga, Lemma 3.12]. Then there is a taut foliation on Y' with Σ as a closed leaf (see [Ga, Section 5]), and therefore Σ minimizes the genus in its class in $H_2(Y')$ by the genus-minimizing property of closed leaves in taut foliations; see Corollary 2 of Section 3 of [Th]. Finally Σ also minimizes the genus in Y because any minimal genus surface $\widetilde{\Sigma}$ in the homology class of Σ can be made disjoint from the connected sum sphere by an isotopy because it is incompressible. □

Lemma 3.6. $\widehat{HF}(-Y|\Sigma)) \simeq SFH(-M, -\Gamma) \oplus SFH(-M, -\Gamma)[1]$.

Proof. If Σ is not genus-minimizing, then $\widehat{HF}(-Y|\Sigma) = 0$ by the adjunction inequality [OSz2, Theorem 1.6], together with [OSz2, Proposition 2.1] and [OSz2, Theorem 2.4]. On the other hand, if Σ is not genus-minimizing, then $R_\pm(\Gamma)$ are not genus-minimizing either by Lemma 3.5. Then $SFH(-M, -\Gamma) = 0$ by [Ju1, Proposition 9.18] and [Ju1, Proposition 9.15]. This proves the lemma in the trivial case when Σ is not genus-minimizing.

When Σ is genus-minimizing, [Le, Theorem 24] shows that

$$HF^+(-Y|\Sigma) \simeq SFH(-M, -\Gamma), \tag{3.7}$$

and moreover by [Le, Corollary 20]

$$\widehat{HF}(-Y|\Sigma) \simeq HF^+(-Y|\Sigma) \oplus HF^+(-Y|\Sigma)[1], \tag{3.8}$$

because the U-map is zero when restricted to Spin^c-structures $\mathfrak{s}$ such that $\langle c_1(\mathfrak{s}), [\Sigma]\rangle = \chi(\Sigma)$. □

Proof of Theorem 1.3. Theorem 1.3 follows from Lemma 3.4, Equations (3.7) and (3.8), and Theorem 3.2. □

In order to prove Theorem 1.6, we prove the analogue of [Le, Corollary 20], i.e., the vanishing of the U-map in the relevant Spin^c-structures, for monopole Floer homology. The proof of the following lemma was suggested to us by Francesco Lin.

Lemma 3.9. *Let Y be a closed, connected and oriented 3-manifold and $\Sigma \subset Y$ an embedded closed, connected, oriented surface of genus at least 2. Then for every Spin^c-structure $\mathfrak{s}$ such that $\langle c_1(\mathfrak{s}), [\Sigma]\rangle = \chi(\Sigma)$, the map*

$$U\colon \widecheck{HM}_\bullet(Y, \mathfrak{s}) \to \widecheck{HM}_\bullet(Y, \mathfrak{s})$$

is trivial.

Proof. Since $\mathfrak{s}$ is nontorsion, there is an isomorphism

$$\widecheck{HM}_\bullet(Y, \mathfrak{s}) \simeq \widehat{HM}_\bullet(Y, \mathfrak{s}),$$

because $\overline{HM}_\bullet(Y, \mathfrak{s}) = 0$, as its definition only involves reducible solutions. Then it suffices to prove that the map

$$U\colon \widehat{HM}_\bullet(Y, \mathfrak{s}) \to \widehat{HM}_\bullet(Y, \mathfrak{s})$$

is trivial.

First we consider the case $Y = S^1 \times \Sigma$. By [KM2, Lemma 2.2],

$$\widehat{HM}(S^1 \times \Sigma|\Sigma) \simeq \mathbb{F},$$

and therefore U is trivial on $\widehat{HM}_\bullet(S^1 \times \Sigma, \mathfrak{s})$ for every $\mathfrak{s}$ such that $\langle c_1(\mathfrak{s}), [\Sigma]\rangle = \chi(\Sigma)$. We treat the general case by a cobordism argument. Take $W = [-1, 1] \times Y$. Then the map

$$\widehat{HM}(W) \colon \widehat{HM}_\bullet(Y, \mathfrak{s}) \to \widehat{HM}_\bullet(Y, \mathfrak{s})$$

is the identity. Now let $W_0 \subset W$ be the union of (i) a closed tubular neighborhood of $\{0\} \times \Sigma$ contained in $[-\frac{1}{3}, \frac{1}{3}] \times Y$; (ii) $[-1, -\frac{2}{3}] \times Y$; and (iii) a tube (i.e., a neighborhood of an arc) connecting them. Then W_0 is a cobordism from Y to $Y\#(S^1 \times \Sigma)$ and $W_1 := W \setminus \mathrm{int}(W_0)$ is a cobordism from $Y\#(S^1 \times \Sigma)$ to Y, and moreover $\widehat{HM}(W_1) \circ \widehat{HM}(W_0) = \widehat{HM}(W) = \mathrm{Id}$. Since U commutes with the cobordism maps, to prove the lemma it suffices to prove that U vanishes on $\widehat{HM}(Y\#(S^1 \times \Sigma), \mathfrak{s}_\#)$, where $\mathfrak{s}_\#$ is the restriction to $Y\#(S^1 \times \Sigma)$ of the Spin^c-structure on W induced by $\mathfrak{s}$. If the connected sum is performed along balls that do not intersect $\Sigma' = \{\theta\} \times \Sigma \subset S^1 \times \Sigma$, then $\langle c_1(\mathfrak{s}_\#), [\Sigma']\rangle = \chi(\Sigma')$, and the vanishing of U on $\widehat{HM}(Y\#(S^1 \times \Sigma), \mathfrak{s}_\#)$ follows from Bloom, Mrowka and Ozsváth's connected sum formula (see [Lin, Theorem 5]) and the vanishing of U on $\widehat{HM}_\bullet(S^1 \times \Sigma, \mathfrak{s}_\#|_{S^1\times\Sigma})$. □

Proof of Theorem 1.6. By the definition of sutured monopole Floer homology [KM2, Definition 4.3],

$$SHM(-M, -\Gamma) \simeq \widecheck{HM}_\bullet(-Y|\Sigma). \tag{3.10}$$

Let $\widetilde{HM}_\bullet(-Y|\Sigma)$ be the cone of U-map in $\widecheck{HM}_\bullet(-Y|\Sigma)$. Then by Equation (3.10) and Lemma 3.9,

$$\widetilde{HM}_\bullet(-Y|\Sigma) \simeq SHM(-M, -\Gamma) \oplus SHM(-M, -\Gamma)[1].$$

By [T2] there is an isomorphism between $\widecheck{HM}_\bullet(-Y|\Sigma)$ and $ECH(Y, \xi|\Sigma)$ which commutes with the U-maps, and therefore induces an isomorphism $\widetilde{HM}_\bullet(-Y|\Sigma) \simeq \widehat{ECH}(Y, \xi|\Sigma)$. The theorem then follows from Theorem 3.2. □

4. ECH OF THE CONTACT CLOSURE

4.1. **Construction of a Reeb vector field.** The proof of Theorem 3.2 relies in large part on a careful construction of a Reeb vector field, which is given in this subsection. We decompose $Y = Y' \cup Y''$, where

$$Y'' = S \times S^1 = S \times ([-1, 1]/-1 \sim 1) \quad \text{and} \quad Y' = M/(R_+ \overset{\psi}{\sim} R_-).$$

The submanifolds Y' and Y'' are glued along their torus boundary. The Reeb vector field is constructed in three steps: first we modify the contact form on Y' near the boundary to introduce a "buffer zone" which will restrict holomorphic curves from going between Y' and Y'', then we construct a contact form on Y'' whose Reeb vector field is Morse-Bott and easy to understand, and finally we perturb the Reeb vector field to make the relevant Reeb orbits nondegenerate.

4.1.1. *The buffer zone.* Let λ be a contact form on Y', obtained by gluing a contact form adapted to (M,Γ). The goal of Section 4.1.1 is to make a particular modification to (Y',λ) on a collar neighborhood $N \subset Y'$ of $\partial Y'$, which we refer to as "installing a buffer zone".

Let $N := [-1,1]_s \times T^2_{\phi,t} \subset Y'$ be a collar neighborhood of $\partial Y' = \{s = 1\}$ such that ϕ is the coordinate in the Γ-direction and t is still the coordinate in the fiber direction. Without loss of generality, we may assume that $\lambda|_N = e^s d\phi + dt$. Choose $\epsilon > 0$ small. On N we consider a contact form $\lambda_0|_N$ of the form

$$\lambda_0|_N = a(s,t)d\phi + b(s)dt,$$

whose Reeb vector field $R_{\lambda_0|_N}$ is parallel to $-\frac{\partial b}{\partial s}\partial_\phi + \frac{\partial a}{\partial s}\partial_t - \frac{\partial a}{\partial t}\partial_s$. Here a and b are chosen such that:

- (C1) The contact condition $b\frac{\partial a}{\partial s} - a\frac{\partial b}{\partial s} > 0$ holds. Geometrically this means that along the curve $(a(s,t), b(s))$ for fixed t, $((\frac{\partial a}{\partial s}, \frac{\partial b}{\partial s}), (a,b))$ is an oriented basis.
- (C2) $a(s,t) = e^s$ for s near ± 1 and a does not depend on t when $s \notin [-\epsilon, \epsilon]$.
- (C3) $b(s) = 1$ for s near 1 and $b(s) = 1 + \delta$ for s near -1 and $\delta > 0$ small.
- (C4) On $s \in [-1, -\epsilon]$ (resp. $s \in [\epsilon, 1]$), as s increases, $(\frac{\partial a}{\partial s}, \frac{\partial b}{\partial s})$ rotates in the clockwise (resp. counterclockwise) direction from horizontal to nearly vertical (resp. nearly vertical to horizontal). See Figure 1.

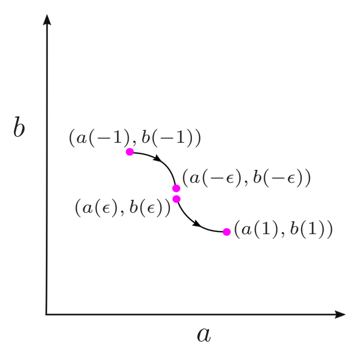

FIGURE 1. The curve $(a(s), b(s))$ on $[-1, -\epsilon]$ and $[\epsilon, 1]$.

- (C5) On $[-\epsilon, \epsilon] \times S^1_t$, a is a Morse function C^1-close to 1, with two index one critical points $h_{+,0}$ and $h_{-,0}$, a local maximum $e_{+,0}$, and a local minimum $e_{-,0}$, and whose level sets are drawn in Figure 2, and b satisfies $\frac{\partial b}{\partial s} < 0$.

We define the contact form λ_0 on Y' such that $\lambda_0|_{Y' \setminus N} = (1+\delta)\lambda|_{Y' \setminus N}$ and on N agrees with $\lambda_0|_N$ constructed above. We will refer to $(N, \lambda_0|_N)$ as the *buffer zone*.

Next we describe the dynamics of the Reeb vector field $R_{\lambda_0|_N}$ on the buffer zone. Write $Y = \frac{\partial a}{\partial s}\partial_t - \frac{\partial a}{\partial t}\partial_s$ for the st-component of $R_{\lambda_0|_N}$.

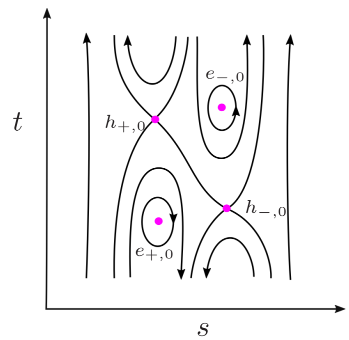

FIGURE 2. The buffer zone for $s \in [-\epsilon, \epsilon]$. The level sets of the Morse function a are oriented by the projection of the Reeb vector field to the (t, s)-annulus.

Remark 4.1. *Since $da(R_{\lambda_0|_N}) = da(Y) = 0$, $R_{\lambda_0|_N}$ is tangent to the level sets of a.*

On $[-\epsilon, \epsilon] \times S^1_t$, the function a has a minimum, a maximum, and two saddle points. By (C5) and Figure 2, each saddle has a homoclinic connection that gives a closed level line which is positively transverse to $\{t = const\}$ away from the zero of Y, and there are two heteroclinic connections between the two saddle points that together form a closed level line which is negatively transverse to $\{t = const\}$ away from the zeros of Y. Thus there are 4 horizontal orbits $e_{-,0}$, $h_{-,0}$, $h_{+,0}$, and $e_{+,0}$ corresponding to the critical points of a, where $e_{-,0}$ (resp. $e_{+,0}$) corresponds to the minimum (resp. maximum) of a, and no orbit has negative algebraic intersection with $\{t = const\}$.

Convention 4.2. *Given a torus parallel to $T^2_{\phi,t}$ with induced (ϕ, t)-coordinates, we define the slope of a curve tangent to $q\partial_\phi + p\partial_t$ (or isotopic to such a curve) to be (q, p).*

Lemma 4.3. *There are two families of Morse-Bott tori of slope $(n, 1)$ accumulating to the suspension of each homoclinic orbit of Y, and no other orbits of the same slope. When $s < 0$ (resp. $s > 0$) the Morse-Bott tori are positive (resp. negative).*

Proof. By Remark 4.1, $R_{\lambda_0|_N}$ is tangent to the level sets of a, viewed as a function on N. The closures of the homoclinic trajectories of $h_{+,0}$ and $h_{-,0}$ times S^1_ϕ are singular tori $\mathcal{T}_+$ and $\mathcal{T}_-$ that are tangent to $R_{\lambda_0|_N}$. By (C4), the region between $\{-1\} \times T^2$ and $\mathcal{T}_+$ is foliated by tori, each of which is foliated by $R_{\lambda_0|_N}$ so that the slope rotates clockwise with positive derivative as s increases. Symmetrically, the region between $\mathcal{T}_-$ and $\{1\} \times T^2$ is foliated by tori and the slope induced by $R_{\lambda_0|_N}$ rotates counterclockwise as s increases. Moreover, whenever such a slope is rational, the foliation given by $R_{\lambda_0|_N}$ has an S^1-family of (closed) orbits, and the S^1-family is Morse-Bott by (C4). The case of slope $(n, 1)$ is a special case. $\square$

The lemma, informally speaking, says that the Reeb vector field in the buffer zone makes (up to a perturbation) a *windshield wiper* movement from vertical to horizontal and then to vertical again.

Remark 4.4. *There can be very long orbits that wind around the two horizontal elliptic orbits $e_{-,0}$ and $e_{+,0}$ and have zero intersection number with Σ. They can be excluded from the ECH chain complex of Y by an easy direct limit argument applied to a sequence of contact forms that do not have horizontal orbits of action $\leq L$ as $L \to \infty$, besides multiples of $e_{-,0}, h_{-,0}, e_{+,0}, h_{+,0}$.*

4.1.2. *A Morse-Bott contact form on Y.* Let S be a compact oriented surface of genus $g \geq 3$ with connected boundary. We pick a closed disk $D_1 \subset S$ and a larger closed disk D_2 such that $D_1 \subset \mathrm{int}(D_2)$. Let (r, θ) be polar coordinates on D_2 so that $D_i = \{r \leq i\}$ for $i = 1, 2$. We define a contact form

$$\lambda_0 = g(r)dt + h(r)d\theta \tag{4.5}$$

on $D_2 \times S^1_t$, where $g, h : [0, 2] \to \mathbb{R}$ satisfy:

- $gh' - g'h > 0$ (contact condition; the Reeb vector field R_{λ_0} is parallel to $h'\partial_t - g'\partial_\theta$);
- (g, h) makes less than a π-rotation in a counterclockwise manner from $(g(0), h(0)) = (-1, 0)$ to $(g(2), h(2))$ with $g(r) = 1$ near $r = 2$ and $h(2) < 0$;
- $(g'(0), h'(0)) = (0, -1)$ and $(g'(2), h'(2)) = (0, 1)$ (R_{λ_0} is vertical at $\{r = 0, 2\}$);
- $(g'(1), h'(1)) = (1, 0)$ (R_λ is horizontal at $\{r = 1\}$);
- $h'g'' - g'h'' > 0$ (Morse-Bott condition; hence R_{λ_0} rotates counterclockwise with nonzero derivative in the basis $(\partial_\theta, \partial_t)$).

Then we choose a one-form β on $S \setminus \mathrm{int}(D_2)$ and a Morse function $f : S \setminus \mathrm{int}(D_2) \to \mathbb{R}$ close to 1 such that

- $d\beta$ is an area form;
- the Liouville vector field of β points into S along ∂S and into D_2 along ∂D_2;
- f has a Morse-Bott minimum along ∂S;
- f has $2g$ index one critical points in the interior of $S \setminus \mathrm{int}(D_2)$;
- f has a Morse-Bott maximum along ∂D_2; and
- the contact form $f dt + \beta$ agrees with the contact form given by Equation (4.5) and the contact form λ_0 on Y'.

Then we define λ_0 on Y by $f dt + \beta$ on $(S \setminus \mathrm{int}(D_2)) \times S^1$, by Equation (4.5) on $D_2 \times S^1$ and by λ_0 on Y'.

Convention 4.6. *Given a torus parallel to $\partial D_2 \times S^1$ with induced (θ, t)-coordinates, we define the slope of a curve tangent to $q\partial_\theta + p\partial_t$ (or isotopic to such a curve) to be $(-q, p)$.*

4.1.3. *Morse-Bott perturbations and excavating the ball.* Fix an unbounded, monotonically increasing sequence of positive real numbers L_i, $i \geq 1$, such that L_i is not the period of an orbit of R_{λ_0} and fix a sequence of small functions $f_i : Y \to \mathbb{R}$ that perturb all the Morse-Bott tori of period less than L_i and slope $(n, 1)$ or $(\pm 1, 0)$ (computed with respect to Conventions 4.2 and 4.6) contained in $N \cup (D_2 \setminus \mathrm{int}(D_1)) \times S^1$ into an elliptic-hyperbolic pair of nondegenerate orbits as in

[CGH0, Section 4] (see also [Bou]), and leaving the nondegenerate orbits of period less than L_i unchanged. Then the Reeb vector field $R_{f_i\lambda_0}$ of $f_i\lambda_0$ has two nondegenerate orbits, one elliptic and one hyperbolic, for every Morse-Bott torus of R_{λ_0} of period less than L_i.

We additionally assume that each L_i is larger than the period of the simple Reeb orbits foliating $\partial D_1 \times S^1$, and that the perturbed orbits e_0 and h_0 (where e_0 is elliptic and h_0 is hyperbolic as usual) are supported on $\partial D_1 \times \{0\}$ and $\partial D_1 \times \{\frac{1}{2}\}$, respectively. The open disk $\mathrm{int}(D_1) \times \{0\}$ is negatively transverse to the flow of $R_{f_i\lambda_0}$ for every i, and therefore we can take a sequence[2] of closed balls B_i with concave corners such that $B_{i+1} \subset \mathrm{int}(B_i)$ as follows: we choose a small solid torus neighborhood $N_i(e_0)$ of e_0 whose boundary is tangent to the Reeb flow of $f_i\lambda_0$, and define the ball B_i to be the union of $N_i(e_0)$ together with a very small thickening of the disk $D_1 \times \{0\}$. See Figure 3.

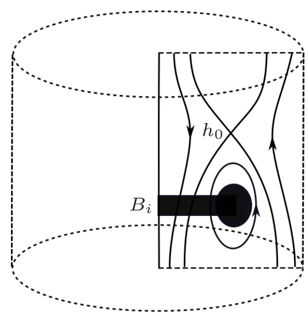

FIGURE 3. The concave ball B_i in $D_2 \times S^1$, obtained by rotating the shaded region about the vertical central axis.

We let $Y_{B_i} := Y \setminus \mathrm{int}(B_i)$ and Γ_{B_i} a closed, connected 1-manifold in $\partial B_i \cap \partial N_i(e_0)$ parallel to e_0. Then $(Y_{B_i}, \Gamma_{B_i}, f_i\lambda_0)$ is a sutured contact manifold. We also set $Y''_{B_i} = Y'' \setminus \mathrm{int}(B_i)$. We make the following observation, which is immediate from the construction:

Claim 4.7. *All the Reeb orbits of $f_i\lambda_0$ in Y_{B_i} intersect Σ nonnegatively, and the only orbit of $f_i\lambda_0$ in $Y''_{B_i} := Y'' \setminus \mathrm{int}(B_i)$ that does not intersect Σ is h_0.*

Let $ECC^{<L_i}(Y_{B_i}, \Gamma_{B_i}, f_i\lambda_0|\Sigma)$ be the ECH chain complex generated by orbit sets of total action less that L_i which intersect Σ once algebraically, and let $ECH^{<L_i}(Y_{B_i}, \Gamma_{B_i}, f_i\lambda_0|\Sigma)$ be its homology. By Morse-Bott theory there are canonical inclusions

$$ECC^{<L_i}(Y_{B_i}, \Gamma_{B_i}, f_i\lambda_0|\Sigma) \hookrightarrow ECC^{<L_j}(Y_{B_j}, \Gamma_{B_i}, f_j\lambda_0|\Sigma)$$

for $j > i$; see [CGH0, Yao1, Yao2]. We observe that the Morse-Bott correspondence of [Yao1] applies to Morse-Bott cascades of planar holomorphic curves, and this hypothesis is satisfied here by [HS1].

[2] We cannot choose the ball once and for all i because the support of the perturbations f_i near $\partial D_1 \times S^1$ must shrink as i increases.

We define

$$\mathfrak{C} = \varinjlim ECC^{<L_i}(Y_{B_i}, \Gamma_{B_i}, f_i\lambda_0|\Sigma)$$

and denote by $H\mathfrak{C}$ its homology. Since homology commutes with direct limits we have

$$\varinjlim ECH^{<L_i}(Y_{B_i}, \Gamma_{B_i}, f_i\lambda_0|\Sigma) \simeq H\mathfrak{C}.$$

Lemma 4.8. *$H\mathfrak{C}$ is isomorphic to $\widehat{ECH}(Y, \xi|\Sigma)$.*

Proof. Fix a reference sutured manifold (Y_{B_0}, Γ_{B_0}), where B_0 is a closed ball with concave corners and is a slight enlargement of B_1. Fix diffeomorphisms $\phi_i\colon (Y_{B_0}, \Gamma_{B_0}) \to (Y_{B_i}, \Gamma_{B_i})$ such that $\phi_i = \mathrm{id}$ outside a fixed small neighborhood of B_0 and consider the contact forms $\lambda_i = \phi_i^*(f_i\lambda_0)$. Then

$$ECH^{<L_i}(Y_{B_i}, \Gamma_{B_i}, f_i\lambda_0|\Sigma) \simeq ECH^{<L_i}(Y_{B_0}, \Gamma_{B_0}, \lambda_i|\Sigma)$$

tautologically, and by Lemma 10.2.1 and Corollary 3.2.3 of [CGH0], we have

$$ECH(Y_{B_0}, \Gamma_{B_0}, \xi|\Sigma) \simeq \varinjlim ECH^{<L_i}(Y_{B_0}, \Gamma_{B_0}, \lambda_i|\Sigma).$$

Hence $ECH(Y_{B_0}, \Gamma_{B_0}, \xi|\Sigma) \simeq H\mathfrak{C}$. Finally

$$ECH(Y_{B_0}, \Gamma_{B_0}, \xi|\Sigma) \simeq \widehat{ECH}(Y, \xi|\Sigma)$$

by [CGH0, Theorem 10.3.1]. □

4.1.4. *List of orbits contributing to $\mathfrak{C}$.* We describe a *partially defined* trivialization τ of ξ with respect to which we compute the Conley-Zehnder indices of the orbits. (Here "partially defined" means the trivializations do not extend globally to a trivialization of ξ.)

1. On $(S \setminus \mathrm{int}(D_2)) \times S^1$, τ comes from the fibration: More specifically, let τ' be a trivialization of $T(S \setminus \mathrm{int}(D_2))$. Then let τ be the pullback of τ' to ξ on $(S \setminus \mathrm{int}(D_2)) \times S^1$.
2. On $(\mathrm{int}(D_2) \setminus \mathrm{int}(D_{1/2})) \times S^1$, τ has first component $-\partial_r$.
3. On the buffer region $N = [-1, 1]_s \times T^2_{\phi,t} \subset Y'$ of $\partial Y' = \{s = +1\}$, τ has first component ∂_s.

Now we describe the orbits in Y'' and in the buffer zone N which contribute to the generators of the chain complex $\mathfrak{C}$. We can regard them equivalently as nondegenerate orbits of the Reeb flow of the perturbed contact form $f_i\lambda_0$ for i sufficiently large, or as possibly degenerate orbits of R_{λ_0} via the Morse-Bott correspondence. In computing the Conley-Zehnder index the first point of view will be taken, while for every other aspect, we will switch from one to the other without mention. Whenever we say "orbit" without further specification, this convention has always to be understood. For the next several pages we encourage the reader to refer to Figure 4 for a more graphical description. Summarizing the above construction of λ_0, we have:

Lemma 4.9. *The following is the list of orbits in Y'' which can appear in orbit sets that generate $\mathfrak{C}$, where the Conley-Zehnder indices μ_{CZ} are computed with respect to τ:*

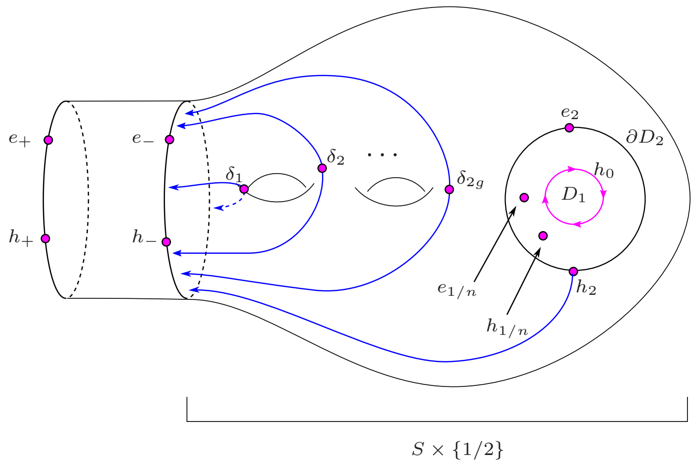

FIGURE 4. The orbits that intersect $S \times \{0\}$, given by pink dots. All the orbits except for h_0 intersect $S \times \{0\}$ once; the orbits besides $h_0, e_{1/n}, h_{1/n}$ are "vertical", i.e., parallel to the S^1-fibers; the orbit h_0 in pink lies on $S \times \{1/2\}$ and bounds $D_1 \times \{0\}$. The downward gradient trajectories of f are given in blue.

- *Over* $\mathrm{int}(S \setminus D_2)$*,* $2g$ *vertical hyperbolic orbits* $\delta_1, \dots, \delta_{2g}$ *of* $\mu_{CZ} = 0$*, where* g *is the genus of* S*.*
- *Over* ∂D_2*, a vertical (i.e., of slope* $(0,1)$*)* $\mu_{CZ} = 1$ *elliptic orbit* e_2 *and a* $\mu_{CZ} = 0$ *hyperbolic orbit* h_2*.*
- *For every* $n \in \mathbb{Z}_{>0}$*, a* $\mu_{CZ} = 1$ *elliptic orbit* $e_{1/n}$ *and a* $\mu_{CZ} = 0$ *hyperbolic orbit* $h_{1/n}$ *in* $(\mathrm{int}(D_2) \setminus D_1) \times S^1$*, both of slope* $(n,1)$ *with respect to* (θ, t)*-coordinates.*
- *Over* ∂D_1*, a* $\mu_{CZ} = 0$ *hyperbolic orbit* h_0 *of slope* $(1,0)$*.*

See Figure 4.

The following is the list of orbits in the buffer zone $N = [-1,1] \times T^2$ *that intersect* Σ *at most once, where the Conley-Zehnder indices* μ_{CZ} *are computed with respect to* τ*:*

- *On* $\{1\} \times T^2 = \partial S \times S^1$*, a* $\mu_{CZ} = -1$ *vertical elliptic orbit* e_- *and a* $\mu_{CZ} = 0$ *vertical hyperbolic orbit* h_-*.*
- *On* $\{-1\} \times T^2$*, a* $\mu_{CZ} = 1$ *vertical elliptic orbit* e_+ *and a* $\mu_{CZ} = 0$ *vertical hyperbolic orbit* h_+*.*
- *For every* $n \in \mathbb{Z}_{>0}$*, a* $\mu_{CZ} = -1$ *elliptic orbit* $e_{-,1/n}$ *and a* $\mu_{CZ} = 0$ *hyperbolic orbit* $h_{-,1/n}$ *in* $(0,1) \times T^2$*, both of slope* $(n,1)$ *with respect to* (ϕ, t)*-coordinates.*
- *For every* $n \in \mathbb{Z}_{>0}$*, a* $\mu_{CZ} = 1$ *elliptic orbit* $e_{+,1/n}$ *and a* $\mu_{CZ} = 0$ *hyperbolic orbit* $h_{+,1/n}$ *in* $(-1,0) \times T^2$*, both of slope* $(n,1)$ *with respect to* (ϕ, t)*-coordinates.*
- *Four horizontal (i.e., of slope* $(\pm 1, 0)$*) orbits* $e_{-,0}$*,* $h_{-,0}$*,* $h_{+,0}$*, and* $e_{+,0}$ *of* $\mu_{CZ} = -1, 0, 0, 1$*, respectively; see Figure 2.*

Remark 4.10. *Recall that a Morse-Bott perturbation also creates uncontrollable very long orbits, but they do not contribute to the direct limit, and therefore do not appear in the generators of $\mathfrak{C}$.*

4.2. **Proof of Theorem 3.2.** In this section we will regard the holomorphic curves contributing to the differential of $\mathfrak{C}$ either as J_i-holomorphic curves in $\mathbb{R} \times Y_{B_i}$ for an almost complex structure J_i which is tailored to $(Y_{B_i}, \Gamma_0, f_i\lambda_0)$ (see [CGHH, Section 3.1]) for i sufficiently large, or as Morse-Bott cascades consisting of J-holomorphic maps in $\mathbb{R} \times Y$ for an almost complex structure J adapted to λ_0, augmented by gradient flow trajectories in the Morse-Bott tori. Since the two types of moduli spaces are in canonical bijection by Morse-Bott theory provided that the almost complex structures J_i are chosen to be suitable perturbations of J, we will switch from one point of view to the other without explicit mention, very much as we do for Reeb orbits. For this reason the almost complex structure will usually be omitted from the notation.

4.2.1. *The slope.* Topological constraints on holomorphic curves derived from the positivity of intersections will play a central role in the proof of Theorem 3.2. Here we develop some basic tools. We start by recalling the standard orientation convention for the transverse intersection of two surfaces in a 3-manifold.

Convention 4.11. *If C and T are transversely intersecting oriented surfaces in an oriented 3-manifold Z (in that order), then at $x \in C \cap T$, let (n, v) be an oriented basis for T_xC, where n is an oriented normal to T and $v \in T_x(C \cap T)$. Then v orients $T_x(C \cap T)$.*

Definition 4.12 (Slope $s\ell(u,T)$)**.** *Let $u : \dot{F} \to \mathbb{R} \times Z$ be a nontrivial finite energy holomorphic curve in the symplectization of a contact manifold Z, C the projection to Z of the image of u, and $T \subset Z$ an oriented 2-torus which:*

(1) *is foliated by (closed) Reeb orbits;*
(2) *does not contain any orbits that the ends of u limit to; and*
(3) *has a neighborhood which is foliated by tori which in turn are foliated by Reeb orbits.*

Then the slope $s\ell(u,T)$ of u along T *is defined as follows: The projection C has a finite number of singularities and, away from those, is an immersion that is transverse to the Reeb vector field. If T does not contain a singular point of the projection, then C is transverse to T and its intersection $C \cap T$ is an immersed oriented 1-manifold in T. Its homology class in $H_1(T;\mathbb{Z})$ is the slope $s\ell(u,T)$ of u along T. If T contains a singular point of the projection, then $s\ell(u,T)$ is defined as $s\ell(u,T')$ for a sufficiently close parallel torus T'.*

Claim 4.13. *The positivity of intersections immediately implies the following:*

(1) *if C is the projection to Z of the image of u, then $C \cap T$ is positively transverse to the Reeb vector field outside of its singular points, i.e., (v, R) is an oriented basis, where v orients $T(C \cap T)$; and*
(2) *if T is foliated by (closed) Reeb orbits in the homology class σ, then*
$$s\ell(u,T) \cdot \sigma > 0.$$

See [CGH0, Section 5.2] for a more detailed account. One should observe that, while the image of a holomorphic curve is canonically oriented, the torus T is not, and the sign of $s\ell(u,T)$ depends on the orientation of T. However, the sign of an intersection point also depends on the orientation of T, and therefore the claim holds for both orientations.

Remark 4.14. *We usually write the slope with respect to a chosen basis of $H_1(T;\mathbb{Z})$. We recall that our convention is the following:*

- *if T is parallel to $\partial Y'$, then we take $(1,0) = -\partial S$ and $(0,1)$ to be the fiber $[0,1]/\sim$ (Convention 4.2);*
- *if T is parallel to $\partial D_2 \times S^1$ we take $(1,0) = -\partial D_2$ and $(0,1)$ to be the fiber $S^1 = [0,1]/\sim$ (Convention 4.6).*

4.2.2. *Decomposing the differential.* A generator $\boldsymbol{\gamma}$ of $\mathfrak{C}$ splits as $\boldsymbol{\gamma} = \boldsymbol{\gamma}_0 \cup \boldsymbol{\gamma}_1$, where $\boldsymbol{\gamma}_0$ is an orbit set in $\operatorname{int}(Y')$ and $\boldsymbol{\gamma}_1$ is an orbit set in Y''. Note that by construction e_- and h_- belong to Y'', while every other orbit in the buffer zone N belongs to $\operatorname{int}(Y')$. Since $[\boldsymbol{\gamma}]\cdot[\Sigma] = 1$ and no orbit intersects Σ negatively,[3] we are left with two possibilities:

(0) $[\boldsymbol{\gamma}_0]\cdot[\Sigma] = 0$ and $[\boldsymbol{\gamma}_1]\cdot[\Sigma] = 1$, or
(1) $[\boldsymbol{\gamma}_0]\cdot[\Sigma] = 1$ and $[\boldsymbol{\gamma}_1]\cdot[\Sigma] = 0$.

We denote by $\mathfrak{C}_0$ the subspace generated by the orbit sets of type (0) and by $\mathfrak{C}_1$ the subspace generated by the orbit sets of type (1). We write the differential $\partial\colon \mathfrak{C} \to \mathfrak{C}$ with respect to the decomposition $\mathfrak{C} = \mathfrak{C}_0 \oplus \mathfrak{C}_1$ as a matrix

$$\partial = \begin{pmatrix} \partial_{0,0} & \partial_{1,0} \\ \partial_{0,1} & \partial_{1,1} \end{pmatrix}.$$

For $j = 0,1$ we introduce sets $\mathcal{P}'_j \subset H_1(\operatorname{int}(Y');\mathbb{Z})$ and $\mathcal{P}''_j \subset H_1(Y'';\mathbb{Z}) \simeq H_1(Y''_{B_i};\mathbb{Z})$ consisting of homology classes A such that $A\cdot[\Sigma] = j$. We denote

$$ECC(-,\mathcal{P}^\star_j) = \bigoplus_{A\in\mathcal{P}^\star_j} ECC(-,A),$$

where $\mathcal{P}^\star_j$ stands for either $\mathcal{P}'_j$ or $\mathcal{P}''_j$. It is clear that

$$\mathfrak{C}_0 = \varinjlim \left(ECC^{<L_i}(\operatorname{int}(Y'), f_i\lambda_0, \mathcal{P}'_0) \otimes ECC^{<L_i}(Y''_{B_i}, \Gamma_{B_i}, f_i\lambda_0, \mathcal{P}''_1)\right),$$
$$\mathfrak{C}_1 = \varinjlim \left(ECC^{<L_i}(\operatorname{int}(Y'), f_i\lambda_0, \mathcal{P}'_1) \otimes ECC^{<L_i}(Y''_{B_i}, \Gamma_{B_i}, f_i\lambda_0, \mathcal{P}''_0)\right).$$

For the moment the identifications above are only as vector spaces; later we will prove that they are identifications as chain complexes. Inspired by these identifications, we will write $\boldsymbol{\gamma}_0 \otimes \boldsymbol{\gamma}_1$ for $\boldsymbol{\gamma}_0 \cup \boldsymbol{\gamma}_1$.

First we prove a preliminary lemma.

Lemma 4.15. *The only holomorphic curve contributing to the differential of $\mathfrak{C}$ whose projection to Y is not contained in $Y \setminus (\operatorname{int}(D_1)\times S^1)$, after removing all covers of trivial cylinders, is a holomorphic plane completely contained in $D^1 \times S^1$ and asymptotic to h_0.*

[3] Those which did have been intercepted by the balls B_i.

Proof. Since $\mathfrak{C}$ has no generators inside $\mathrm{int}(D_1) \times S^1$, the projection of such a curve intersected with $\partial D_1 \times S^1$ must be homologous to a multiple of the meridian. Then by Claim 4.13 it must have ends at h_0. The only possibility for such a curve to have index $I = 1$ is to be a holomorphic plane asymptotic to h_0 together, possibly, with covers of trivial cylinders. Such a holomorphic plane was constructed in [We2, Section 3.1] in the Morse-Bott setting, and the transition from the Morse-Bott setting to the nondegenerate one is the "easy case" of the Morse-Bott correspondence, since there is no need to glue Morse trajectories. □

Lemma 4.16. $\partial_{0,1} = 0$.

Proof. We will show that there is no ECH index 1 holomorphic curve from an orbit set $\boldsymbol{\gamma}_0 \otimes \boldsymbol{\gamma}_1$ of $\mathfrak{C}_0$ to an orbit set $\boldsymbol{\gamma}_0' \otimes \boldsymbol{\gamma}_1'$ of $\mathfrak{C}_1$. The element $\boldsymbol{\gamma}_1$ is one of the following list:

$$\begin{array}{ccccccc} e_-, & h_-, & \delta_i, & e_2, & h_2, & e_{1/n}, & h_{1/n}, \\ e_-h_0, & h_-h_0, & \delta_i h_0, & e_2 h_0, & h_2 h_0, & e_{1/n}h_0, & h_{1/n}h_0, \end{array} \tag{4.17}$$

and the element $\boldsymbol{\gamma}_1'$ is either h_0 or $\emptyset$. Moreover every holomorphic curve contributing to $\partial_{0,1}$ projects to $Y \setminus (\mathrm{int}(D_1) \times S^1)$ by Lemma 4.15. We introduce the notation $\boldsymbol{\gamma}_1^\flat$ to denote $\boldsymbol{\gamma}_1$ with e_- and h_- removed.

Let u be a J-holomorphic curve from $\boldsymbol{\gamma}_0 \otimes \boldsymbol{\gamma}_1$ to $\boldsymbol{\gamma}_0' \otimes \boldsymbol{\gamma}_1'$. We analyze how the curve u approaches the buffer region in Y' from the Y''-side using the following homological argument: Take a torus T parallel to and oriented in the same way as $\partial Y'$ and slightly inside Y''; we may assume that T and its nearby tori are linearly foliated by Reeb orbits by adjusting the construction of the contact form. Let $Z \subset Y''$ be a slight retraction of $Y'' \setminus (\mathrm{int}(D_1) \times S^1)$ obtained by excising the thickened torus between $\partial Y''$ and T, and let u_Z be the projection to Z of the restriction of u to $\mathbb{R} \times Z$. Then $u_Z \cap T$ is homologous to $\boldsymbol{\gamma}_1^\flat - \boldsymbol{\gamma}_1'$ in $H_1(Z)$ via the surface u_Z. Let b be the homology class of $-\partial S$ and f the homology class of the S^1-fiber in $H_1(Z) \simeq H_1(S \setminus \mathrm{int}(D_1)) \oplus H_1(S^1)$. Since $[\boldsymbol{\gamma}_1^\flat] = nb + f$, $n \geq 0$, or b, or 0, and $[\gamma_1'] = 0$ or b, there are five possibilities:

(1) $s\ell(u, T) = (0, 1)$, in which case u cannot cross $\partial Y''$ since it is blocked by the vertical flow along $\partial Y''$; see the Blocking Lemma 5.2.3 in [CGH0]. Then u has an end at e_- or h_- and therefore does not contribute to $\partial_{0,1}$.

(2) $s\ell(u, T) = (n, 1)$, $n \geq 1$, in which case u is either stopped inside the buffer zone by a negative orbit of slope $(n, 1)$, or has a negative end at an orbit of slope $(n-k, 1)$, $0 < k \leq n$. In the latter case, $s\ell(u, \{s_0\} \times T^2) = (k, 0)$, where $s_0 > 0$ is smaller than the s-value of the torus foliated by orbits of slope $(n-k, 1)$. Then u is blocked in the buffer zone by k orbits of slope $(1, 0)$. By this we mean the hyperbolic orbit must have multiplicity at most 1 but the elliptic orbit can have multiplicity k or $k-1$ and the same number of ends limiting to it.

(3) $s\ell(u, T) = (1, 0)$, in which case u has positive ends at h_0 and either at e_- or h_-. Then $s\ell(u, \{s_1\} \times T^2) = (1, 1)$ for s_1 slightly smaller than 1, and u is blocked in the buffer zone by a negative orbit of slope $(1, 1)$.

(4) $s\ell(u,T)=(-1,1)$ or $(-1,0)$, in which case u has a negative end at h_0 and no orbits $e_{1/n},h_{1/n}$ at the positive end. We consider $s\ell(u,\{r=2-\epsilon\})$, where $\{r=2-\epsilon\}$ is a torus in $D_2\times S^1$ which is close to the boundary. The slope $s\ell(u,\{r=2-\epsilon\})$ must be $(-1,0)$ due to the negative end h_0 and the absence of orbits $e_{1/n},h_{1/n}$ at the positive end, but $(-1,0)$ is not positively transverse to the Reeb vector field, contradicting Claim 4.13.

(5) $s\ell(u,T)=0$, in which case $\boldsymbol{\gamma}_1=h_-$, e_-, h_-h_0, or e_-h_0. The trapping lemma [CGH0, Lemma 5.3.2] implies that either u consists of a holomorphic cylinder from h_- to e_-, and therefore does not contribute to $\partial_{0,1}$; or u_Z intersects T, and this is incompatible with $s\ell(u,T)=0$ by Claim 4.13.

Hence we are left with Cases (2) and (3).

We explain how to compute $I_{ECH}(u)$ in Case (3). The projection C of the embedded surface $u(\dot F)$ to $Y\setminus(\mathrm{int}(D_1)\times S^1)$ from $\boldsymbol{\gamma}^+=h_0e_-$ or h_0h_- to an orbit $\boldsymbol{\gamma}^-=e_{-,1/1}$ or $h_{-,1/1}$ in N of slope $(1,1)$ is constructed by surgering a horizontal section over an enlargement of $S\setminus\mathrm{int}(D_2)$ together with an annulus. (Surgering with an annulus changes χ by -1.) Since C is embedded and all the orbits involved are simple,

$$I_{ECH}(u)=\mathrm{ind}(u)=-\chi(\dot F)+2\langle c_1(\xi,\tau),C\rangle+\mu_{CZ}(\boldsymbol{\gamma}^+)-\mu_{CZ}(\boldsymbol{\gamma}^-). \tag{4.18}$$

Here $c_1(\xi,\tau)$ is the first Chern class of ξ relative to the trivialization τ. Then $\chi(\dot F)=-2g-1$, $\langle c_1(\xi,\tau),C\rangle=-2g$, $\mu_{CZ}(h_0)=0$, $\mu_{CZ}(h_-)=0$, $\mu_{CZ}(e_-)=-1$, $\mu_{CZ}(e_{-,1/1})=-1$, $\mu_{CZ}(h_{-,1/1})=0$, and

$$I_{ECH}(u)\leq(2g+1)-4g-1-0\leq-2g<0,$$

since $g\geq 3$. This is a contradiction.

Next we consider Case (2). In this case C goes from the orbit set $\boldsymbol{\gamma}^+=h_0e_{1/(n-1)}$, $h_0h_{1/(n-1)}$, $e_{1/n}$, $h_{1/n}$, or a vertical orbit ($\neq e_-$ or h_-) times h_0 to the orbit set $\boldsymbol{\gamma}^-=e_{-,1/n}$, $h_{-,1/n}$, $e_{-,1/(n-k)}e_{-,0}^{k-1}h_{-,0}$, $h_{-,1/(n-k)}e_{-,0}^{k-1}h_{-,0}$, $e_{-,1/(n-k)}e_{-,0}^{k}$, or $h_{-,1/(n-k)}e_{-,0}^{k}$, where $e_{-,0}$ and $h_{-,0}$ are the slope $(1,0)$ orbits. When C is from $\boldsymbol{\gamma}^+=e_{1/n}$ or $h_{1/n}$ to $\boldsymbol{\gamma}^-=e_{-,1/n}$ or $h_{-,1/n}$, C is an n-fold cover of an enlargement of $S\setminus\mathrm{int}(D_2)$ and has $\chi=2ng$. If $\boldsymbol{\gamma}^+$ is changed to h_0 times an orbit, then C is modified by surgering with an annulus. (This changes χ by -1 as before.) If $\boldsymbol{\gamma}^-$ is changed to $e_{-,1/(n-k)}e_{-,0}^{k-1}h_{-,0}$, $h_{-,1/(n-k)}e_{-,0}^{k-1}h_{-,0}$, $e_{-,1/(n-k)}e_{-,0}^{k}$, or $h_{-,1/(n-k)}e_{-,0}^{k}$, then C is modified by adding $k-1$ branch points and surgering with an annulus. (This changes χ by $-k$.) Even though $e_{-,0}$ may have multiplicity ≥ 0, Formula (4.18) still holds because all ends at $e_{-,0}$ are simple by the partition condition and the multiples of $e_{-,0}$ still have Conley-Zehnder index -1. The number of ends l satisfies $2\leq l\leq 3+k<3+n$, $\chi(\dot F)=2-2ng-l$, $\langle c_1(\xi,\tau),C\rangle=-2ng$, $\mu_{CZ}(\boldsymbol{\gamma}^+)\leq 1$, and $\mu_{CZ}(\boldsymbol{\gamma}^-)\geq -k-1$. Thus we have

$$I_{ECH}(u)\leq(2ng+l-2)-4ng+1-(-k-1)\leq-2ng+2n+3<0,$$

since $g\geq 3$. This is also a contradiction. $\square$

A consequence of $\partial_{0,1}=0$ is that $\partial_{0,0}^2=\partial_{1,1}^2=0$ and $\partial_{1,0}$ is a chain map.

4.2.3. *Computation of the homologies of* $\mathfrak{C}_0$ *and* $\mathfrak{C}_1$.

Lemma 4.19. $H_*(\mathfrak{C}_1, \partial_{1,1}) = 0$

Proof. The elements of $\mathfrak{C}_1$ are linear combinations of elements of the form $\boldsymbol{\gamma} \otimes h_0$ or $\boldsymbol{\gamma} \otimes \emptyset$. By the proof of Case (4) of Lemma 4.16, no holomorphic curve in Y''_{B_i} has h_0 as a negative end, and by the argument of Case (3) of Lemma 4.16, the only holomorphic curve with a positive end at h_0 and no other positive end in Y'' is the holomorphic plane over $(D_1 \times S^1) \setminus B_0$. Thus we can decompose $\partial_{1,1}$ as

$$\partial_{1,1}(\boldsymbol{\gamma} \otimes \emptyset) = \partial'\boldsymbol{\gamma} \otimes \emptyset,$$
$$\partial_{1,1}(\boldsymbol{\gamma} \otimes h_0) = \partial'\boldsymbol{\gamma} \otimes h_0 + \boldsymbol{\gamma} \otimes \emptyset,$$

where ∂' is the differential in $ECC(\mathrm{int}(Y'), \lambda_0)$. The map $K\colon \mathfrak{C}_1 \to \mathfrak{C}_1$ defined by

$$K(\boldsymbol{\gamma} \otimes \emptyset) = \boldsymbol{\gamma} \otimes h_0, \quad K(\boldsymbol{\gamma} \otimes h_0) = 0$$

satisfies $\partial_{1,1} \circ K + K \circ \partial_{1,1} = \mathrm{id}$, and therefore $H_*(\mathfrak{C}_1, \partial_{1,1}) = 0$ □

The following lemma enumerates the holomorphic curves that are involved in the calculation of $H_*(\mathfrak{C}_0, \partial_{0,0})$.

Lemma 4.20. *The list of all connected $I = 1$ holomorphic curves in $\mathbb{R} \times Y''$ with ends in $\mathcal{P}''_0 \cup \mathcal{P}''_1$ consists of:*

(A) *Two cylinders each from δ_i to e_-, a cylinder from h_2 to e_-, a cylinder from e_2 to h_-, and two cylinders each from e_2 to h_2 and h_- to e_- that correspond to gradient trajectories of a Morse perturbation of f on $S \setminus \mathrm{int}(D_2)$.*

(B) *Two cylinders each from $e_{1/n}$ to $h_{1/n}$ and pairs-of-pants in $\mathbb{R} \times (D_2 \setminus \mathrm{int}(D_1)) \times S^1$ from $e_2 h_0$ to $e_{1/1}$; $h_2 h_0$ to $h_{1/1}$; $e_{1/n} h_0$ to $e_{1/(n+1)}$; and $h_{1/n} h_0$ to $h_{1/(n+1)}$. The pairs-of-pants all belong to moduli spaces of cardinality 1 mod 2 (after quotienting by target $\mathbb{R}$-translations).*

(C) *A holomorphic plane over $(D_1 \times S^1) \setminus B_0$ with a positive end at h_0.*

Proof. Let u be a connected holomorphic curve in $\mathbb{R} \times Y''$ with positive ends at $\mathcal{P}''_0 \cup \mathcal{P}''_1$. We first note that u either projects to $(S \setminus \mathrm{int}(D_1)) \times S^1$ or to $D_1 \times S^1$ by Lemma 4.15, and in the latter case it is a holomorphic plane with a positive end at h_0. Next we show that if u projects to $Y'' \setminus (D_1 \times S^1)$, then it either projects to $Y'' \setminus (\mathrm{int}(D_2) \times S^1)$ or to $(D_2 \setminus D_1) \times S^1$: Suppose the projection of u intersects both regions. Since in $Y'' \setminus (\mathrm{int}(D_2) \times S^1)$ we consider only orbits in the homology class of the S^1-fiber, $s\ell(u, \partial D_2 \times S^1)$ can only be one of $(0,-1), (0,0), (0,1)$. However none of the three values is possible by Claim 4.13 because the Reeb vector field on $\partial D_2 \times S^1$ has slope $(0,1)$.

(A), (B), and (C) correspond to u with $I(u) = 1$ in $Y'' \setminus (\mathrm{int}(D_2) \times S^1)$, $(D_2 \setminus D_1) \times S^1$, and $D_1 \times S^1$, respectively.

(A) There exists an adapted almost complex structure J on $\mathbb{R} \times (S \setminus \mathrm{int}(D_2)) \times S^1$ such that there is a bijection between gradient trajectories $\delta : \mathbb{R} \to S \setminus \mathrm{int}(D_2)$ of f modulo domain $\mathbb{R}$-translation and finite energy J-holomorphic cylinders Z_δ in $\mathbb{R} \times (S \setminus \mathrm{int}(D_2)) \times S^1$ that project to $\mathrm{Im}(\delta)$, modulo target $\mathbb{R}$-translation.

(B) A perturbation of the Morse-Bott torus $\{r = r_n\}$ containing $e_{1/n}$ and $h_{1/n}$ gives the two cylinders from $e_{1/n}$ to $h_{1/n}$. The remaining curves follow from adapting Hutchings-Sullivan [HS1, Theorem 3.5], but a few remarks are in place:

(a) In [HS1, Theorem 3.5], there could be curves with more than one negative puncture or more than two positive punctures (the latter are obtained by the "double rounding" operations). Those curves are not considered here because the homology class of their ends is not one we are considering here; see the complete list of orbits given by Lemma 4.9.
(b) The computation in [HS1] is made for perturbations of negative Morse-Bott tori and the actual computation we use is the dual one from [HS2].
(c) The work [HS1] is done in the context of periodic Floer homology and requires a "d-regularity" assumption on the almost complex structure and Reeb vector field. However in [HS2] the argument is extended to ECH where d-regularity is not needed.

(C) is immediate from the first paragraph of the proof. □

We define
$$\mathfrak{C}''_0 = \varinjlim ECC^{<L_i}(Y''_{B_i}, \Gamma_{B_i}, f_i\lambda_0, \mathcal{P}''_1).$$
As a vector space it is generated by the orbit sets of the list (4.17) and its differential $\partial''_{0,0}$, which is determined by Lemma 4.20, is:

(1) $\partial''_{0,0}(\gamma h_0) = \gamma$, where $\gamma = e_-, h_-, \delta_i$,
(2) $\partial''_{0,0}(e_2) = h_-$,
(3) $\partial''_{0,0}(h_2) = e_-$,
(4) $\partial''_{0,0}(e_2h_0) = e_{1/1} + e_2 + h_-h_0$,
(5) $\partial''_{0,0}(h_2h_0) = h_{1/1} + h_2 + e_-h_0$,
(6) $\partial''_{0,0}(e_{1/n}h_0) = e_{1/(n+1)} + e_{1/n}$,
(7) $\partial''_{0,0}(h_{1/n}h_0) = h_{1/(n+1)} + h_{1/n}$,

and vanishes on all other generators.

Let $\partial'_{0,0}$ be the differential on $ECC(\mathrm{int}(Y'), \lambda_0, \mathcal{P}'_0)$. We recall that
$$\mathfrak{C}_0 = ECC(\mathrm{int}(Y'), \lambda_0, \mathcal{P}'_0) \otimes \mathfrak{C}''_0$$
as a vector space. In the next lemma we prove that the differential splits.

Lemma 4.21. *For every generator $\boldsymbol{\gamma}_0 \otimes \boldsymbol{\gamma}_1$ of $\mathfrak{C}_0$ we have*
$$\partial_{0,0}(\boldsymbol{\gamma}_0 \otimes \boldsymbol{\gamma}_1) = \partial'_{0,0}(\boldsymbol{\gamma}_0) \otimes \boldsymbol{\gamma}_1 + \boldsymbol{\gamma}_0 \otimes \partial''_{0,0}(\boldsymbol{\gamma}_1).$$

Proof. Let u be a connected holomorphic curve from $\boldsymbol{\gamma}_0 \otimes \boldsymbol{\gamma}_1$ to $\boldsymbol{\gamma}'_0 \otimes \boldsymbol{\gamma}'_1$ that contributes to $\partial_{0,0}$. We show that the projection of u cannot intersect a torus $T' \subset Y'$ that is parallel to $\partial Y'$, foliated by Reeb orbits, and separates $\boldsymbol{\gamma}_0, \boldsymbol{\gamma}'_0$ from $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}'_1$. Suppose this is not the case. Since $[\boldsymbol{\gamma}_0] \cdot [\Sigma] = [\boldsymbol{\gamma}'_0] \cdot [\Sigma] = 0$, we have $s\ell(u, T') = (k, 0)$, where $k > 0$ by the positivity of intersections with the vertical Reeb orbits in $\partial Y'$. Since h_0 cannot be at a negative end by an argument similar to that of Case (4) of Lemma 4.16, the remaining possibilities for $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}'_1$ are:

(a) $\boldsymbol{\gamma}_1$ consists of $e_{1/n}$ or $h_{1/n}$ where $n > 0$ and $\boldsymbol{\gamma}'_1$ consists of a vertical orbit, $e_{1/n'}$, or $h_{1/n'}$ where $n > n'$; and

(b) γ_1 consists of h_0 and a vertical orbit, $e_{1/n}$, or $h_{1/n}$ where $n > 0$ and γ'_1 consists of a vertical orbit, $e_{1/n'}$, or $h_{1/n'}$ where $n+1 > n'$.

Both (a) and (b) can be ruled out as in Case (2) of Lemma 4.16 by considering the ECH index of u. $\square$

Finally we compute $H_*(\mathfrak{C}_0, \partial_{0,0})$:

Lemma 4.22. $H_*(\mathfrak{C}_0, \partial_{0,0}) = ECH(\mathrm{int}(Y'), \lambda_0, \mathcal{P}'_0) \otimes \langle [e_{1/1}], [h_{1/1}] \rangle$.

Proof. By Lemma 4.21 and the Künneth formula we have

$$H_*(\mathfrak{C}_0, \partial_{0,0}) \simeq ECH(\mathrm{int}(Y'), \lambda_0, \mathcal{P}'_0) \otimes H_*(\mathfrak{C}''_0, \partial''_{0,0}),$$

and therefore the proof of the lemma reduces to the computation of $H_*(\mathfrak{C}''_0, \partial''_{0,0})$.

First we observe that the orbit sets $e_{1/n}, h_{1/n}, e_{1/n}h_0, h_{1/n}h_0$, $n \in \mathbb{Z}_{>0}$, form a subcomplex $(\mathfrak{C}'''_0, \partial'''_{0,0})$ with homology $H_*(\mathfrak{C}'''_0, \partial'''_{0,0}) = \langle [e_{1/1}, h_{1/1}] \rangle$. Note that $[e_{1/1}] = [e_{1/n}]$ and $[h_{1/1}] = [h_{1/n}]$ for every $n \in \mathbb{Z}_{>0}$.

The quotient complex $\mathfrak{C}''_0 / \mathfrak{C}'''_0$ can be identified with the mapping cone of

$$h_0 C_*(S \setminus \mathrm{int}(D_2), \partial S) \xrightarrow{h_0\gamma \mapsto \gamma} C_*(S \setminus \mathrm{int}(D_2), \partial S),$$

where $C_*(S \setminus \mathrm{int}(D_2), \partial S)$ is the Morse complex of a Morse perturbation of the Morse-Bott function f. This mapping cone is clearly acyclic because the map $h_0\gamma \mapsto \gamma$ is an isomorphism. The lemma then follows. $\square$

4.2.4. *Completion of the proof of Theorem 3.2.* We now complete the computation of $\widehat{ECH}(Y, \xi|\Sigma)$. By Lemma 4.16, the chain complex $\mathfrak{C}$ can be written as the cone of $\mathfrak{C}_1 \xrightarrow{\partial_{1,0}} \mathfrak{C}_0$. Using the corresponding exact sequence on homology and Lemmas 4.19, 4.22 and 4.8 we obtain that:

$$\widehat{ECH}(Y, \xi|\Sigma) \simeq ECH(\mathrm{int}(Y'), \lambda_0, \mathcal{P}'_0) \otimes \langle [e_{1/1}], [h_{1/1}] \rangle.$$

Let $\widetilde{Y}'$ be the manifold obtained by excising a thin collar $\mathcal{C}$ of $\partial Y'$ so that $\partial \widetilde{Y}'$ is foliated by orbits of R_{λ_0} of irrational slope. We assume that all the orbits of R_{λ_0} in $\mathcal{C}$ intersect Σ many times, so that

$$ECH(\mathrm{int}(Y'), \lambda_0, \mathcal{P}'_0) \simeq ECH(\widetilde{Y}', \lambda_0, \mathcal{P}'_0).$$

We also consider a contact form $\widetilde{\lambda}$ on $\widetilde{Y}'$ obtained from λ by a modification on a slight enlargement $\mathcal{C}'$ of $\mathcal{C}$ such that:

- $R_{\widetilde{\lambda}} = R_{\lambda_0}$ near $\partial \widetilde{Y}'$;
- $\widetilde{\lambda} = \lambda$ on $\mathrm{int}(Y') \setminus \mathrm{int}(\mathcal{C}')$ which contains all the orbit sets in $\mathcal{P}'_0$; and
- all Reeb orbits in $\mathrm{int}(\mathcal{C}')$ intersect Σ many times.

Then $ECH(\mathrm{int}(Y'), \lambda, \mathcal{P}'_0) \simeq ECH(\widetilde{Y}', \widetilde{\lambda}, \mathcal{P}'_0)$. Moreover,

$$ECH(\widetilde{Y}', \widetilde{\lambda}, \mathcal{P}'_0) \simeq ECH(\widetilde{Y}', \lambda_0, \mathcal{P}'_0),$$

by [CGH0, Proposition 7.2.1] and

$$ECH(\mathrm{int}(Y'), \lambda, \mathcal{P}'_0) \simeq ECH(M, \Gamma, \xi)$$

by [CGHH, Theorem 1.9]. Putting the isomorphisms together yields

$$ECH(\mathrm{int}(Y'), \lambda_0, \mathcal{P}'_0) \simeq ECH(M, \Gamma, \xi).$$

In [CGHH], Theorem 1.9 is proven modulo (i) the invariance of sutured ECH with respect to the contact form and the almost complex structure and (ii) the existence of cobordism maps in sutured ECH that are similar to the ones given by Hutchings-Taubes [HT3] in the closed case. The invariance (i) and the existence of cobordism maps with good properties (see [CGHH, Section 10.4] for the precise requirements) (ii) are both given in [CGH0, Theorem 10.2.2].

Therefore we obtain

$$\widehat{ECH}(Y, \xi|\Sigma) \simeq ECH(M, \Gamma, \xi) \otimes \langle [e_{1/1}], [h_{1/1}] \rangle,$$

completing the proof of Theorem 3.2.

5. Decomposition along Spin^c-structures

In this section we describe how the isomorphism between sutured Floer homology and sutured ECH behaves with respect to the decomposition along relative Spin^c-structures. Let (M, Γ, ξ) be a sutured contact manifold and let $\psi \colon R_+ \to R_-$ be a diffeomorphism which, near the boundary, coincides with the identification induced by the coordinates in the neighborhood $U(\Gamma)$. Let $i_\pm \colon R_\pm \to M$ be the natural inclusions and let $K_\psi \subset H_1(M)$ be given by

$$K_\psi = \mathrm{Im}(i_{-*} \circ \psi_* - i_{+*}).$$

Let $M_\psi := M/(x \sim \psi(x))$ be the 3-manifold with torus boundary obtained by gluing R_+ to R_- using ψ, and which contains a distinguished surface R corresponding to R_+ and R_-. Using the Mayer-Vietoris sequence one computes that

$$H_1(M_\psi; \mathbb{Z}) \simeq (H_1(M; \mathbb{Z})/K_\psi) \oplus \mathbb{Z},$$

where the $\mathbb{Z}$-factor is generated by a cycle γ that intersects R once.

Theorem 5.1. *Let (M, Γ, ξ) be a sutured contact 3-manifold and $\psi : R_+ \to R_-$ a diffeomorphism as above. Then, for every $A \in H_1(M; \mathbb{Z})$,*

$$\bigoplus_{c \in A + K_\psi} ECH(M, \Gamma, \xi, c) \simeq \bigoplus_{c \in A + K_\psi} SFH(-M, -\Gamma, \mathfrak{s}_\xi + PD(c)).$$

Proof. First assume that Γ is connected. Let (Y_ψ, ξ_ψ) be the contact closure of (M, Γ, ξ) as defined in Section 3.2. Here it is convenient to record the gluing diffeomorphism ψ in the notation and to distinguish ξ from its extension. For every $A \in H_1(M; \mathbb{Z})$ we denote by $[A]$ its image in $H_1(M; \mathbb{Z})/K_\psi$ and define $\overline{A} = [A] + \gamma$. Here we identify $H_1(M_\psi; \mathbb{Z})$ with its image in $H_1(Y_\psi; \mathbb{Z})$ because the map induced by the inclusion is injective. An inspection of the proof of Theorem 3.2 gives the following refinement of the isomorphism (3.3):

$$\bigoplus_{c \in A + K_\psi} (ECH(M, \Gamma, \xi, c) \oplus ECH(M, \Gamma, \xi, c)[1]) \simeq ECH(Y_\psi, \xi_\psi, \overline{A}).$$

Similarly, by Proposition 19, Corollary 20, Theorem 21, and Theorem 24 of Lekili [Le] (note that in [Le] M is denoted Y, while Y_ψ is denoted Y_n), we have

$$\bigoplus_{c\in A+K_\psi}(SFH(-M,-\Gamma,\mathfrak{s}_\xi+PD(c))\oplus SFH(-M,-\Gamma,\mathfrak{s}_\xi+PD(c))[1])$$
$$\simeq\ \widehat{HF}(-Y_\psi,\mathfrak{s}_{\xi_\psi}+PD(\overline{A})).$$

Although [Le, Theorem 24] does not explicitly mention the decomposition of $\widehat{HF}(M,\Gamma)$ along relative Spin^c-structures, the proof first uses [Le, Theorem 21] that keeps track of them, followed by the identification of $SFH(M,\Gamma)$ and the ad hoc homology $QFH'(Y_\psi)$ where they are not carefully tracked. The only thing to point out is that this second step is actually done by an isomorphism between chain complexes that automatically respects Spin^c-structures.

Now by the isomorphism for closed manifolds [CGH1],

$$\widehat{HF}(-Y_\psi,\mathfrak{s}_{\xi_\psi}+PD(\overline{A}))\simeq\widehat{ECH}(Y_\psi,\xi_\psi,\overline{A}),$$

and this concludes the proof of the theorem if Γ is connected. The general case is obtained by observing that the isomorphisms in the proof of Lemma 3.1 behave well with respect to homology classes and relative Spin^c-structures. $\Box$

We now turn our attention to knot Floer homology.

Corollary 5.2. *Let K be a null-homologous knot in a closed manifold M and ξ a contact structure that is compatible with the sutured manifold $(M(K),\Gamma_K)$. Then*

$$ECH(M(K),\Gamma_K,\xi,A)\simeq\widehat{HFK}(-M,-K,\underline{\mathfrak{s}}_\xi+\mathrm{PD}(i_*(A))),$$

where $A\in H_1(M)$, $\mathfrak{s}_\xi$ is the canonical relative Spin^c-structure of ξ, $\underline{\mathfrak{s}}_\xi$ is its extension to a Spin^c-structure on $M_0(K)$, and $i_*\colon H_1(M(K))\to H_1(M_0(K))$ is the isomorphism induced by the inclusion $i\colon M(K)\to M_0(K)$.

If K is fibered and $\mathfrak{h}$ is an area-preserving representative of the monodromy with zero flux, then

$$PFH^\sharp(\mathfrak{h},A)\simeq\widehat{HFK}(-M,-K,\underline{\mathfrak{s}}_\xi+\mathrm{PD}(i_*(A))).$$

Proof. Knot Floer homology can be identified with the sutured Heegaard Floer homology of the knot complement with two meridian sutures. Then the corollary follows from Theorem 5.1 by observing that, when R_+ and R_- are annuli, $K_\psi=\{0\}$ for every choice of gluing diffeomorphism ψ. The statement about periodic Floer homology follows from that of ECH and Lemma 2.7. $\Box$

Proof of Corollaries 1.7 and 1.9. Corollaries 1.7 and 1.9 are just weaker formulations of Corollary 5.2. $\Box$

Proof of Corollary 1.12. The proof is exactly the same as the proof of Corollary 5.2, which is based only on formal properties that holds also for monopole Floer homology; see [KM2, Section 5]. $\Box$

6. A DYNAMICAL CHARACTERIZATION OF PRODUCT SUTURED MANIFOLDS

In this section we prove dynamical results which were announced in [CH], as corollaries of Theorem 1.3. The following answers a question of Pardon.

Theorem 6.1. *If $(M, \Gamma, \xi = \ker \lambda)$ is a taut balanced sutured contact manifold whose Reeb vector field R_λ has no orbit, then (M, ξ) is a product tight sutured contact manifold $(S \times [0,1], \xi)$, where ξ is $[0,1]$-invariant. Moreover, if S is planar and R_λ has no orbit, then every orbit of R_λ flows from $S \times \{0\}$ to $S \times \{1\}$; in particular R_λ has no trapped orbits.*

Proof. If R_λ has no orbit, then $ECH(M, \Gamma, \lambda) \simeq \mathbb{F}\langle[\emptyset]\rangle$ and hence

$$HF(-M, -\Gamma) \simeq \mathbb{F}.$$

Moreover, by Hofer [Hof] (applied without modification to our sutured situation thanks to the control on holomorphic curves given by [CGHH, Proposition 5.20]), M is irreducible and ξ is tight. By [Ju2, Theorem 9.7] and the irreducibility of M, (M, Γ) is a product sutured manifold $(S \times [0,1], \partial S \times [0,1])$. (We remark that it is also possible to prove this result directly using the theory of end-periodic diffeomorphisms of end-periodic surfaces.)

Next we show that ξ is $[0,1]$-invariant. We decompose $S \times [0,1]$ along a collection of compression disks of the form $a_1 \times [0,1], \dots, a_k \times [0,1]$, where $\{a_1, \dots, a_k\}$ is a basis of arcs for S. Each circle $\partial(a_i \times [0,1])$ intersects the dividing set $\partial S \times \{\frac{1}{2}\}$ in exactly two points, i.e., $(S \times [0,1], \partial S \times [0,1])$ is product disk decomposable. Hence, by the usual convex surface theory, there is a unique tight contact structure on $(S \times [0,1], \partial S \times [0,1])$, and it is $[0,1]$-invariant.

It remains to prove that the Reeb vector field R_λ itself flows from $S \times \{0\}$ to $S \times \{1\}$ when S is planar. We use the well-known technique of foliating $\mathbb{R} \times S \times [0,1]$ by holomorphic curves, due to Eliashberg-Hofer [EH] when S is a disk, and to Wendl [We] when S is a more general planar surface. For that, we embed our product as a part of an open book decomposition. We have a page S_0 transverse to the Reeb vector field to start the foliation by holomorphic curves asymptotic to the binding and, even if the contact form is not adapted, there is no possibility of breaking since all orbits intersect the pages positively. Since the Reeb flow is transverse to the foliation, there must be a first return map on S_0 and the conclusion follows. □

Question 6.2. *Can one prove that if there is no orbit in $S \times [0,1]$, then there is also no trapped orbit even when S is not planar?*

In the higher-dimensional case, such a normalization theorem does not hold, as shown by Geiges, Röttgen and Zehmisch in [GRZ] where they exhibit a situation with trapped orbits without periodic ones in a product sutured contact manifold.

Finally, we relate the Reeb dynamics and the *depth* of the sutured manifold, i.e., the minimum number of steps in a sutured hierarchy needed to get to a product sutured manifold. This is also the minimal depth of a supported foliation.

Theorem 6.3. *If $(M, \Gamma, \xi = \ker \lambda)$ is a taut balanced irreducible sutured contact manifold of depth greater than $2k$ with $H_2(M) = 0$ and if R_λ is nondegenerate and has no elliptic orbit, then it has at least $k+1$ hyperbolic orbits.*

Proof. Under the hypothesis of the theorem, Juhász [Ju3, Theorem 4] shows that

$$\mathrm{rk}\, HF(-M, -\Gamma) \geq 2^{k+1}.$$

By our isomorphism, the ECH chain complex must have rank $\geq 2^{k+1}$. When there are no elliptic orbits, this implies the existence of at least $k+1$ hyperbolic orbits for R_λ. $\square$

Notice that every Reeb vector field can be perturbed to possess only hyperbolic orbits up to a certain action threshold L [CGH1, Theorem 2.5.2], typically a number going to infinity with L whenever there is an elliptic orbit to start with.

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