Non-commutative Divergence and the Turaev Cobracket

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Abstract

The divergence map, an important ingredient in the algebraic description of the Turaev cobracket on a connected oriented compact surface with boundary, is reformulated in the context of non-commutative geometry and is generalised to give a similar algebraic description of the Turaev cobracket on a closed surface. We also look into a relation between the Satoh trace and the divergence map on a free Lie algebra, using a non-commutative analogue of a flat connection.

0. Introduction

The Turaev cobracket is a loop operation introduced in [Tur91], which endows a structure of an involutive Lie bialgebra [Cha04] on the space of non-trivial free loops on an oriented surface together with the Goldman bracket (the involutivity is due to Chas [Cha04]). These topologically defined operations have several interesting algebraic descriptions ([Mas18] and [KK16], for example), a combinatorial description ([Cha04]) and also relate to the necklace Lie bialgebra in quiver theory.

Since connected oriented and closed surfaces are distinguished by their fundamental group, one would expect to recover a significant amount of topological information from the group, including those loop operations. In that case, an algebraic description of the Goldman bracket was given by Dmitry Vaintrob [Vai07b], and it is the one induced from the commutator bracket of derivations combined with the Poincaré–Van den Bergh duality.

In the case of connected oriented compact surfaces with non-empty boundary, another algebraic description of the Turaev cobracket is given in [AKKN18] using a non-commutative analogue of the divergence map. The Turaev cobracket, together with the divergence map, is deeply related to the Enomoto–Satoh trace originally introduced in [ES10]. For the detail and the relation to mapping class groups of surfaces, see Theorem 1.14 and Section 9 of [AKKN18].

The goal of this paper is to generalise their divergence map and algebraic description to the case of connected oriented and closed surfaces. Let $\mathbb{K}$ be a unital ring, Σ a connected oriented and closed surface and $\pi = \pi_1(\Sigma)$ the fundamental group of a surface. The symbol $|\cdot|$ denotes the cyclic quotient, so that the space $|\mathbb{K}\pi|$ is the free $\mathbb{K}$-module spanned by the homotopy classes of free loops on Σ. The main result is the following:

Theorem. *We have a $\mathbb{K}$-linear map ("the divergence map")*

$$\mathrm{Div}\colon \mathrm{HH}^1(\mathbb{K}\pi) \to |\mathbb{K}\pi/\mathbb{K}1|^{\otimes 2}$$

from the first Hochschild cohomology of the group ring $\mathbb{K}\pi$, constructed from a flat homological connection (see Definition 24), such that the composition

$$|\mathbb{K}\pi| \xrightarrow{v} \mathrm{HH}^1(\mathbb{K}\pi) \xrightarrow{\mathrm{Div}} |\mathbb{K}\pi/\mathbb{K}1|^{\otimes 2}.$$

with the Vaintrob's map v (denoted by ρ in [Vai07a]) is the Turaev cobracket.

For more precise statement, see Theorem 27 in the body.

Along the way, we introduce a reformulation of the divergence map in terms of non-commutative geometry in the sense of Kontsevich and Ginzburg (and many others), which is the most important ingredient in this paper; see Definition 9. This leads us to a conceptual understanding of the Satoh trace on the automorphism group of a free group. It is combinatorially defined in their original paper [Sat12], but it can be realised as the divergence map associated with a non-commutative version of a flat connection; for the detail, see Section 5.

For a pre-existing interpretation of the Satoh trace, we refer to the paper [MS20] by Massuyeau and Sakasai, where they introduce several variants of the trace map and relate it to the Magnus representation of the group of certain automorphisms on a free Lie algebra.

Organisation of the paper. In Section 1, we recall the Turaev cobracket and the divergence map defined in [AKKN18]. Sections 2-4 are the introduction to the language of non-commutative geometry and a reformulation of the divergence map. In Section 5, which is logically independent with later sections, we look into divergence maps in geometry over Lie operad and its relation with the Satoh trace. Finally, the case of closed surfaces is dealt with in Sections 6 and 7.

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Conventions. $\mathbb{K}$ is a unital commutative ring. All $\mathbb{K}$-algebras contain $\mathbb{K}1$ in their centre. Unadorned tensor products are always over $\mathbb{K}$.

1. The Turaev Cobracket and its Algebraic Description

Let Σ be a connected oriented surface possibly with boundary, and $\pi = \pi_1(\Sigma)$ its fundamental group. Put $|\mathbb{K}\pi| = \mathbb{K}\pi/[\mathbb{K}\pi, \mathbb{K}\pi]$, the cyclic quotient of the algebra $\mathbb{K}\pi$, which is a free $\mathbb{K}$-module spanned by the homotopy classes of free loops on Σ. The Turaev cobracket is a map $\delta\colon |\mathbb{K}\pi| \to |\mathbb{K}\pi/\mathbb{K}1|^{\otimes 2}$ defined by, for a generically immersed free loop $\alpha\colon [0,1]/\{0,1\} \to \Sigma$,

$$\delta(\alpha) = \sum_{\substack{t_1 \neq t_2 \in [0,1] \\ \alpha(t_1)=\alpha(t_2)}} \operatorname{sign}(\alpha; t_1, t_2)\, \alpha|_{[t_1,t_2]} \otimes \alpha|_{[t_2,t_1]}\,,$$

where $\operatorname{sign}(\alpha; t_1, t_2)$ is the local intersection number with respect to the orientation of Σ. This map is well-defined up to birth-deaths of monogons, hence takes its value in $|\mathbb{K}\pi/\mathbb{K}1|^{\otimes 2}$. If Σ admits a framing fr (i.e., a smooth non-vanishing vector field), we can upgrade it to the map

$$\delta^{\mathsf{fr}}\colon |\mathbb{K}\pi| \to |\mathbb{K}\pi|^{\otimes 2}$$

by taking a rotation-free representative of α.

Now assume that Σ is compact. Then, an algebraic description of δ^{fr} involving the non-commutative divergence is given in [AKKN18] as follows. Firstly, σ is defined as the based version of the Goldman bracket

$$\sigma\colon |\mathbb{K}\pi| \to \operatorname{Der}_{\mathbb{K}}(\mathbb{K}\pi)$$

from the space of free loops to the space of $\mathbb{K}$-linear derivations on $\mathbb{K}\pi$. It is given by

$$\sigma(\alpha)(x) = \sum_{p \in \alpha\cap x} \operatorname{sign}(\alpha, x; p)\, \alpha *_p x$$

for generic representatives of a free loop α and $x \in \pi$. Here $\alpha *_p x$ is a based loop obtained by traversing x until it reaches p, then going along α, and finally following the rest of x. (In [AKKN18], σ is described as the map induced from the double bracket κ, but we omit the detail here.) Next, take a free generating system $\mathcal{C} = (x_i)_{1\le i\le n}$ of π as in Figure 10 of [AKKN18], which induces an isomorphism $\pi \cong F_n$. The associated divergence is defined by

$$\operatorname{Div}^{\mathcal{C}}(f) = \sum_{1\le i\le n} |\partial_i(f(x_i)) - 1 \otimes x_i^{-1} f(x_i)| \in |\mathbb{K}\pi| \otimes |\mathbb{K}\pi|\,,$$

where ∂_i is the double derivation (see the end of the next section) defined by $\partial_i(x_j) = \delta_{ij} \cdot 1 \otimes 1$.

Theorem 1. (Theorem 5.16, [AKKN18]) *Let fr be a framing such that all generators x_i are rotation-free. Then, the composite $\operatorname{Div}^{\mathcal{C}} \circ \sigma$ is equal to the framed Turaev cobracket δ^{fr}.*

2. Preliminaries on Non-commutative Geometry

In this section, we recall some definitions in non-commutative geometry. Let A be a unital associative $\mathbb{K}$-algebra with the multiplication map $\mu\colon A \otimes A \to A$, and $A^{\mathrm{e}} = A \otimes A^{\mathrm{op}}$ its enveloping algebra. We identify A-bimodules with *left* A^{e}-modules.

Definition 2.

- The left A^{e}-module structure on A is given by

$$(x \otimes y) \cdot a = xay \qquad \text{for } a \in A \text{ and } x \otimes y \in A^{\mathrm{e}}\,.$$

- The left A^{e}-module structure on $A \otimes A$, the *outer* structure, is given by

$$(x \otimes y) \cdot (a \otimes b) = xa \otimes by \qquad \text{for } a \otimes b \in A \otimes A \text{ and } x \otimes y \in A^{\mathrm{e}},$$

  while the right A^{e}-module structure, the *inner* structure, is given by $(a \otimes b) \cdot (x \otimes y) = ax \otimes yb$. The natural identification $A^{\mathrm{e}} \cong A \otimes A$ is an isomorphism of A^{e}-bimodules.

- $\Omega^1 A = \mathrm{Ker}(\mu\colon A \otimes A \to A)$ is the space of *non-commutative* 1*-forms*. This is a left A^{e}-submodule of $A \otimes A$, since μ is a left A^{e}-module homomorphism.

- $\Omega^\bullet A = \bigoplus_{m \geq 0} (\Omega^1 A)^{\otimes_A m}$ is the tensor algebra over A generated by 1-forms, which is a graded A-algebra.

Let $\bar{A} = A/\mathbb{K}1$. Then $\Omega^1 A$ is canonically isomorphic to $A \otimes \bar{A}$ as a left A-module by the map given by

$$A \otimes \bar{A} \to \Omega^1 A\colon a_0 \otimes [a_1] \mapsto a_0 da_1 := a_0 \otimes a_1 - a_0 a_1 \otimes 1\,.$$

Its inverse is given by the projection $A \otimes A \twoheadrightarrow A \otimes \bar{A}$. Similarly, $\Omega^1 A$ is also isomorphic to $\bar{A} \otimes A$ as a right A-module. With this notation, every element of $\Omega^\bullet A$ can be written as a linear combination of elements of the form $a_0 da_1 \ldots da_n$, abbreviating the tensor symbol.

Definition 3. An algebra A is said to be *formally smooth* if it is finitely generated as a $\mathbb{K}$-algebra with A^{e}-projective $\Omega^1 A$.

This condition is convenient yet very restrictive: it forces $\Omega^1 A$ to be a dualisable (i.e., finitely generated and projective) module.

Example 4. (1) A finitely generated free associative algebra $A = \mathbb{K}\langle z_1, \ldots, z_n \rangle$ is formally smooth. In fact, we have a standard A^{e}-free resolution of A:

$$0 \longrightarrow A \otimes \mathbb{K}\{z_1, \ldots, z_n\} \otimes A \xrightarrow{\ \delta\ } A \otimes A \xrightarrow{\ \mu\ } A \longrightarrow 0\,,$$

$$\delta(1 \otimes z_i \otimes 1) = 1 \otimes z_i - z_i \otimes 1 = dz_i\,,$$

which shows that $\Omega^1 A$ is an A^{e}-free module with the basis $(dz_i)_{1 \leq i \leq n}$.

(2) The group algebra $A = \mathbb{K}F_n$ of a free group is also formally smooth. The following is the proof the author learned from Florian Naef. First of all, we fix a free generating system $(x_i)_{1 \leq i \leq n}$ of F_n. Then we have an exact sequence of left A-modules:

$$0 \longrightarrow \bigoplus_{1 \leq i \leq n} A \otimes \mathbb{K} \cdot (x_i - 1) \xrightarrow{\ \iota\ } A \xrightarrow{\ \varepsilon\ } \mathbb{K} \longrightarrow 0\,,$$

where ι is the summation map and ε is the augmentation map. The inverse of ι is given by the direct sum of the Fox derivatives

$$\frac{\partial}{\partial x_i} : \mathrm{Ker}\,\varepsilon \to A \otimes \mathbb{K} \cdot (x_i - 1)\,,$$

which is uniquely specified by the formulae

$$\frac{\partial}{\partial x_i}(ab) = \frac{\partial}{\partial x_i}(a) + a\frac{\partial}{\partial x_i}(b) \quad \text{and} \quad \frac{\partial}{\partial x_i}(x_j) = \delta_{ij}\,,$$

and satisfies the equation

$$a - \varepsilon(a) = \sum_{1\le i\le n} \frac{\partial}{\partial x_i}(a)\cdot(x_i - 1)\,.$$

Now define a functor as follows:

$$\Phi_A : (\text{left } A\text{-modules}) \to (A\text{-bimodules}) : M \mapsto \Phi_A(M),\ \ \varphi \mapsto \varphi\otimes \mathrm{id}_A\ . \tag{1}$$

Here we set $\Phi_A(M) = M\otimes A$ as a $\mathbb{K}$-module with the A^{e}-action given by, for $a,x,y\in A$ and $m\in M$,

$$x\cdot(a\otimes m)\cdot y = x^{(1)}a\otimes x^{(2)}my,$$

where $\Delta(x) = x^{(1)}\otimes x^{(2)}$ is the coproduct of A. This functor is exact since A is $\mathbb{K}$-flat. The functor can be analogously defined for an arbitrary Hopf algebra A. By applying this functor to the resolution above, we obtain an acyclic complex

$$0 \longrightarrow \Phi_A\left(\bigoplus_{1\le i\le n} A\cdot(x_i-1)\right) \xrightarrow{\mathrm{incl}\otimes\mathrm{id}} \Phi_A(A) \xrightarrow{\varepsilon\otimes\mathrm{id}} \Phi_A(\mathbb{K}) \longrightarrow 0\,,$$

In addition, we have canonical isomorphisms of A-bimodules: for $x,y\in F_n$ and $k\in\mathbb{K}$,

$$\begin{aligned} \Phi_A(A) &\cong A^{\otimes 2} : x\otimes y \mapsto xy^{-1}\otimes y \text{ and}\\ \Phi_A(\mathbb{K}) &\cong A \quad\ : x\otimes k \mapsto xk\,. \end{aligned}$$

Transporting differentials using these isomorphisms, we have a left A^{e}-free resolution of A:

$$0 \longrightarrow \bigoplus_{1\le i\le n} A\otimes\mathbb{K}\cdot(x_i-1)\otimes A \xrightarrow{\delta'} A^{\otimes 2} \xrightarrow{\mu} A \longrightarrow 0\,,$$

$$\delta'(1\otimes(x_i-1)\otimes 1) = x_i\otimes x_i^{-1} - 1\otimes 1 = -dx_i\,x_i^{-1}\,.$$

Therefore $\Omega^1 A$ is an A^{e}-free module with the basis $(dx_i\,x_i^{-1})_{1\le i\le n}$.

We need some more definitions from non-commutative differential geometry.

Definition 5.

- For a $\mathbb{K}$-subalgebra R of A and an A-bimodule M, an R-linear derivation on A into M is an R-linear map $f\colon A\to M$ such that

$$f(ab) = f(a)\cdot b + a\cdot f(b) \quad \text{for } a,b\in A.$$

  The space of all such derivations is denoted by $\mathrm{Der}_R(A,M)$. Set $\mathrm{Der}_R(A) = \mathrm{Der}_R(A,A)$.

- For $f\in\mathrm{Der}_{\mathbb{K}}(A)$, the *contraction map* $i_f\colon \Omega^\bullet A\to\Omega^\bullet A$ is the A-linear, degree (-1) derivation defined by

$$i_f(da) = f(a) \quad \text{for } a\in A.$$

  This gives a canonical isomorphism of $\mathbb{K}$-modules

$$\mathrm{Der}_{\mathbb{K}}(A) \to \mathrm{Hom}_{A^{\mathrm{e}}}(\Omega^1 A, A)\colon f\mapsto i_f\,.$$

  Conversely, the pair $(\Omega^1 A, d)$ is characterised by the following universal property: for any A-bimodule M and a $\mathbb{K}$-linear derivation $f\colon A\to M$, there is a unique A^{e}-module map $i_f\colon\Omega^1 A\to M$ such that $i_f\circ d = f$ holds.

- $\mathrm{DDer}_{\mathbb{K}}(A) = \mathrm{Hom}_{A^e}(\Omega^1 A, A^e)$ is the space of $\mathbb{K}$-linear *double derivations* on A. Equivalently, a double derivation is a $\mathbb{K}$-linear map $\theta\colon A \to A \otimes A$ satisfying

$$\theta(ab) = \theta(a) \cdot b + a \cdot \theta(b) \quad \text{ for } a, b \in A.$$

- The *exterior derivative* $d\colon \Omega^\bullet A \to \Omega^\bullet A$ is the $\mathbb{K}$-linear, degree 1 derivation defined by

$$d(a_0 da_1 \cdots da_n) = da_0 da_1 \cdots da_n \quad \text{ for } a_i \in A.$$

- For $f \in \mathrm{Der}_{\mathbb{K}}(A)$, the *Lie derivative* $L_f = [d, i_f]\colon \Omega^\bullet A \to \Omega^\bullet A$ is the $\mathbb{K}$-linear, degree 0 derivation. For instance, $L_f(a_0 da_1) = f(a_0) da_1 + a_0 df(a_1)$ holds. The map

$$L\colon \mathrm{Der}_{\mathbb{K}}(A) \to \mathrm{Der}_{\mathbb{K}}(\Omega^\bullet A)^{(0)}\colon f \mapsto L_f$$

  is a Lie algebra homomorphism. Here the superscript (0) denotes the degree 0 part.

- The Hochschild homology and cohomology of A are defined by

$$\mathrm{HH}_\bullet(A) = \mathrm{Tor}_\bullet^{A^e}(A, A) \quad \text{and} \quad \mathrm{HH}^\bullet(A) = \mathrm{Ext}^\bullet_{A^e}(A, A).$$

3. Connections and Divergences

In this section, we recall the definition of a non-commutative connection, which we use to further define a non-commutative divergence map. Let B be another unital associative $\mathbb{K}$-algebra.

Definition 6. Let M be a left B-module.

- A $\mathbb{K}$-linear map $\nabla\colon M \to \Omega^1 B \otimes_B M$ is said to be a *connection* on M if it satisfies the Leibniz rule:

$$\nabla(b \cdot m) = db \otimes m + b \cdot \nabla(m) \quad \text{for } b \in B \text{ and } m \in M\,.$$

  This extends to a degree 1 derivation on $\Omega^\bullet B \otimes_B M$ by

$$\nabla(\omega \otimes m) = d\omega \otimes m + (-1)^p \omega \cdot \nabla(m) \quad \text{for } \omega \in \Omega^p A \text{ and } m \in M\,.$$

- The *curvature* of a connection ∇ is defined by $R = \nabla^2\colon M \to \Omega^2 B \otimes_B M$, which is a B-module map.

- A connection ∇ is *flat* if the curvature R vanishes identically.

If M is $\mathbb{K}$-free, M admits a ($\mathbb{K}$-linear) connection if and only if M is B-projective (see, for example, Corollary 8.2 of [CQ95]). The existence of flat connections is more subtle; one sufficient condition is that M is B-free. In fact, connections on free modules are uniquely specified by their values on a free basis. One necessary condition, on the other hand, is the vanishing of Chern classes; for the definition, see [Gin05].

Definition 7. Let ∇ be a connection on a B-module M. Its *push-out* by a $\mathbb{K}$-algebra homomorphism $\psi : B \to C$ is a connection $\psi_* \nabla$ on $C \otimes_B M$ defined by the composition

$$\begin{aligned} \psi_*\nabla\colon C \otimes_B M &\to \Omega^1 C \otimes_B M \cong \Omega^1 C \otimes_C (C \otimes_B M)\,, \\ 1 \otimes m &\mapsto \psi(\nabla(m))\,. \end{aligned}$$

This is well-defined, since, for $b \in B$ and $m \in M$,

$$\begin{aligned} \psi_*\nabla(1 \otimes bm) &= \psi(db \otimes m + b\nabla(m)) \\ &= d\psi(b) \otimes m + \psi(b)\psi(\nabla(m)) \\ &= \psi_*\nabla(\psi(b) \otimes m)\,. \end{aligned}$$

Next, we recall the trace of a module endomorphism over a non-commutative ring. First of all, the dual space $M^* = \mathrm{Hom}_B(M, B)$ is naturally a right B-module by

$$(\theta \cdot b)(m) = \theta(m)b \quad \text{for } \theta\colon M \to B\,,\ b \in B \text{ and } m \in M.$$

Let $|B| = \mathrm{HH}_0(B) = B/[B, B]$ be the *trace space* of B. Denoting $Z(B)$ the centre of B, there are well-defined maps of $Z(B)$-modules

$$\begin{aligned} \iota\colon M^* \otimes_B M \to \mathrm{End}_B(M)\colon \theta \otimes m \mapsto\ & (m' \mapsto \theta(m')m) \text{ and} \\ \mathrm{ev}\colon M^* \otimes_B M \to |B|\colon \theta \otimes m \mapsto\ & |\theta(m)|\,. \end{aligned}$$

If M is a dualisable B-module, ι gives an isomorphism. The composite

$$\mathrm{Tr} = \mathrm{ev} \circ \iota^{-1}\colon\ \mathrm{End}_B(M) \xrightarrow{\iota^{-1}} M^* \otimes_B M \xrightarrow{\mathrm{ev}} |B|\,,$$

is known as the *Hattori–Stallings trace* [Hat65, Sta65].

Proposition 8. *Let M and N be dualisable B-modules. Then, for B-module maps $f : M \to N$ and $g : N \to M$, we have $\mathrm{Tr}(f \circ g) = \mathrm{Tr}(g \circ f)$.*

Proof. For $f = \theta \otimes n \in M^* \otimes_B N$ and $g = \varphi \otimes m \in N^* \otimes_B M$, we have

$$\begin{aligned} f \circ g &= \varphi \otimes \theta(m)n\,, \\ g \circ f &= \theta \otimes \varphi(n)m, \text{ and} \\ \mathrm{Tr}(f \circ g) &= |\theta(m)\varphi(n)| = |\varphi(n)\theta(m)| = \mathrm{Tr}(g \circ f). \end{aligned}$$

This completes the proof. □

To define the divergence, suppose that a Lie algebra $\mathfrak{d}$ is acting on dualisable M by derivation. Namely, $\varphi\colon \mathfrak{d} \to \mathrm{Der}_{\mathbb{K}}(B)$ and $\rho\colon \mathfrak{d} \to \mathrm{End}_{\mathbb{K}}(M)$ are given Lie algebra homomorphisms satisfying

$$\rho(f)(bm) = \varphi(f)(b) \cdot m + b \cdot \rho(f)(m) \quad \text{for } f \in \mathfrak{d}, b \in B \text{ and } m \in M.$$

In this setting, $|B|$ is a naturally a $\mathfrak{d}$-module by the composition

$$\mathfrak{d} \xrightarrow{\varphi} \mathrm{Der}_{\mathbb{K}}(B) \to \mathrm{End}_{\mathbb{K}}(|B|)$$

of Lie algebra homomorphisms.

Definition 9. Let ∇ be a connection on M. The *non-commutative divergence* associated with ∇ and (φ, ρ) is defined by

$$\mathrm{Div}^{(\nabla,\varphi,\rho)}\colon \mathfrak{d} \to |B|\colon f \mapsto \mathrm{Tr}(\rho(f) - (i_{\varphi(f)} \otimes \mathrm{id}_M) \circ \nabla)\,.$$

Now assume that A is formally smooth until the end of this section, so that $\Omega^1 A$ is dualisable and

$$\mathrm{DDer}_{\mathbb{K}}(A) \otimes_{A^{\mathrm{e}}} \Omega^1 A \cong \mathrm{End}_{A^{\mathrm{e}}}(\Omega^1 A)$$

holds. The trace map now takes a form $\mathrm{Tr}\colon\ \mathrm{End}_{A^{\mathrm{e}}}(\Omega^1 A) \to |A^{\mathrm{e}}|$.

In addition, set $B = A^{\mathrm{e}}$, $M = \Omega^1 A$, $\mathfrak{d} = \mathrm{Der}_{\mathbb{K}}(A)$,

$$\begin{aligned} \varphi\colon\ \mathrm{Der}_{\mathbb{K}}(A) \to \mathrm{Der}_{\mathbb{K}}(A^{\mathrm{e}}) \quad &\colon\ f \mapsto \tilde{f} := f \otimes \mathrm{id}_{A^{\mathrm{op}}} + \mathrm{id}_A \otimes f, \text{ and} \\ \rho\colon\ \mathrm{Der}_{\mathbb{K}}(A) \to \mathrm{End}_{\mathbb{K}}(\Omega^1 A)&\colon\ f \mapsto L_f\,. \end{aligned}$$

The condition above reads

$$L_f(x \cdot da \cdot y) = (f(x) \otimes y + x \otimes f(y)) \cdot da + (x \otimes y) \cdot L_f(da),$$

which amounts to saying that L_f is a derivation. Thus we obtain the divergence

$$\mathrm{Div}^{\nabla}\colon\ \mathrm{Der}_{\mathbb{K}}(A) \to |A^{\mathrm{e}}|\colon f \mapsto \mathrm{Tr}(L_f - (i_{\tilde{f}} \otimes \mathrm{id}_{\Omega^1 A}) \circ \nabla)$$

associated with a connection $\nabla\colon \Omega^1 A \to \Omega^1 A^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} \Omega^1 A$. In this case, we have an interpretation of it in terms of the horizontal lift:

Definition-Lemma 10. *For a connection $\nabla \colon \Omega^1 A \to \Omega^1 A^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} \Omega^1 A$, there is a unique $\mathbb{K}$-linear map*

$$(-)^{\mathrm{H}} \colon \operatorname{Der}_{\mathbb{K}}(A) \to \operatorname{Der}_{\mathbb{K}}(\Omega^{\bullet} A)^{(0)} \colon f \mapsto f^{\mathrm{H}}$$

satisfying the following properties: for $f \in \operatorname{Der}_{\mathbb{K}}(A)$,

(1) $f^{\mathrm{H}}(a) = f(a)$ in $A = \Omega^0 A$ for $a \in A$; and

(2) $f^{\mathrm{H}}(\alpha) = (i_{\tilde{f}} \otimes \mathrm{id}_{\Omega^1 A}) \circ \nabla(\alpha)$ in $A^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} \Omega^1 A \cong \Omega^1 A$ for $\alpha \in \Omega^1 A$.

The map $(-)^{\mathrm{H}}$ is called the horizontal lift by ∇.

Proof. The uniqueness follows from the fact that $\Omega^{\bullet} A$ is generated as a $\mathbb{K}$-algebra by A and $\Omega^1 A$. Then, as $\Omega^{\bullet} A$ is the tensor algebra of $\Omega^1 A$ over A, it suffices to show that (1) and (2) are compatible to see that f^{H} is well-defined. We compute, for $x, y \in A$ and $\alpha \in \Omega^1 A$,

$$\begin{aligned} f^{\mathrm{H}}(x\alpha y) &= (i_{\tilde{f}} \otimes \mathrm{id}_{\Omega^1 A}) \circ \nabla((x \otimes y) \cdot \alpha) \\ &= (i_{\tilde{f}} \otimes \mathrm{id}_{\Omega^1 A})(d(x \otimes y) \otimes \alpha + (x \otimes y) \cdot \nabla(\alpha)) \\ &= \tilde{f}(x \otimes y) \cdot \alpha + (x \otimes y) \cdot f^{\mathrm{H}}(\alpha) \\ &= f(x)\alpha y + x f^{\mathrm{H}}(\alpha) y + x\alpha f(y), \end{aligned}$$

which completes the proof. □

For the original definition of the horizontal lift in Riemannian geometry, see [YP67].

There is a simple criterion for $\operatorname{Div}^{\nabla}$ to be a Lie algebra 1-cocycle with coefficients in $|A^{\mathrm{e}}|$.

Proposition 11. *If ∇ is a flat connection, then $\operatorname{Div}^{\nabla}$ is a Lie algebra 1-cocycle.*

We need some preparation for the proof. First, define the action of $\operatorname{Der}_{\mathbb{K}}(A)$ on $\operatorname{DDer}_{\mathbb{K}}(A)$ by

$$f \cdot \theta = [f, \theta] := (f \otimes \mathrm{id} + \mathrm{id} \otimes f) \circ \theta - \theta \circ f \quad \text{for } f \in \operatorname{Der}_{\mathbb{K}}(A) \text{ and } \theta \in \operatorname{DDer}_{\mathbb{K}}(A)$$

with the commutator taken in the space $\operatorname{Der}_{\mathbb{K}}(T(A))$, where $T(A)$ denotes the tensor algebra generated by A over $\mathbb{K}$. This makes a canonical isomorphism of $\mathbb{K}$-modules $\operatorname{DDer}_{\mathbb{K}}(A) \otimes_{A^{\mathrm{e}}} \Omega^1 A \cong \operatorname{End}_{A^{\mathrm{e}}}(\Omega^1 A)$ into that of $\operatorname{Der}_{\mathbb{K}}(A)$-modules.

Lemma 12. *The trace map Tr is a $\operatorname{Der}_{\mathbb{K}}(A)$-module homomorphism.*

Proof. For $\theta \otimes da \in \operatorname{DDer}_{\mathbb{K}}(A) \otimes_{A^{\mathrm{e}}} \Omega^1 A \cong \operatorname{End}_{A^{\mathrm{e}}}(\Omega^1 A)$ and $f \in \operatorname{Der}_{\mathbb{K}}(A)$, we have

$$\begin{gathered} f \cdot (\theta \otimes da) = [f, \theta] \otimes da + \theta \otimes L_f(da), \text{ so that} \\ \operatorname{Tr}(f \cdot (\theta \otimes da)) = |[f, \theta](a) + \theta(f(a))| = |f(\theta(a))| = f \cdot \operatorname{Tr}(\theta \otimes da). \end{gathered}$$

Hence follows the $\operatorname{Der}_{\mathbb{K}}(A)$-equivariance of the trace. □

Lemma 13. *Let R be the curvature of ∇ and $f, g \in \operatorname{Der}_{\mathbb{K}}(A)$. Then we have $[f, g]^{\mathrm{H}} = [f^{\mathrm{H}}, g^{\mathrm{H}}] + i_{\tilde{f}} i_{\tilde{g}} R$ in $\operatorname{End}_{\mathbb{K}}(\Omega^1 A)$.*

Proof. For $\alpha \in \Omega^1 A$, put $\nabla \alpha = \sum \omega \otimes \beta$ for some $\omega \in \Omega^1 A^{\mathrm{e}}$ and $\beta \in \Omega^1 A$. Dropping the summation symbol and denoting $i_{\tilde{f}} \otimes \mathrm{id}_{\Omega^1 A}$ simply by $i_{\tilde{f}}$, we have

$$\begin{aligned} [f^{\mathrm{H}}, g^{\mathrm{H}}]\alpha &= f^{\mathrm{H}}(i_{\tilde{g}} \nabla \alpha) - g^{\mathrm{H}}(i_{\tilde{f}} \nabla \alpha) \\ &= f^{\mathrm{H}}(i_{\tilde{g}}(\omega \otimes \beta)) - g^{\mathrm{H}}(i_{\tilde{f}}(\omega \otimes \beta)) \\ &= f^{\mathrm{H}}(i_{\tilde{g}}\omega \cdot \beta) - g^{\mathrm{H}}(i_{\tilde{f}}\omega \cdot \beta) \\ &= \tilde{f}(i_{\tilde{g}}\omega) \cdot \beta + i_{\tilde{g}}\omega \cdot f^{\mathrm{H}}(\beta) - \tilde{g}(i_{\tilde{f}}\omega) \cdot \beta - i_{\tilde{f}}\omega \cdot g^{\mathrm{H}}(\beta) \\ &= \tilde{f}(i_{\tilde{g}}\omega) \cdot \beta + i_{\tilde{g}}\omega \cdot i_{\tilde{f}} \nabla \beta - \tilde{g}(i_{\tilde{f}}\omega) \cdot \beta - i_{\tilde{f}}\omega \cdot i_{\tilde{g}} \nabla \beta. \end{aligned}$$

On the other hand, we have

$$R(\alpha) = \nabla(\omega \otimes \beta) = d\omega \otimes \beta - \omega\nabla\beta, \text{ and}$$
$$i_{\tilde{f}} i_{\tilde{g}} R(\alpha) = i_{\tilde{f}} i_{\tilde{g}} d\omega \cdot \beta - i_{\tilde{g}}\omega \cdot i_{\tilde{f}}\nabla\beta + i_{\tilde{f}}\omega \cdot i_{\tilde{g}}\nabla\beta.$$

Therefore, we have

$$\begin{aligned}\left([f^{\mathrm{H}}, g^{\mathrm{H}}] + i_{\tilde{f}} i_{\tilde{g}} R\right)(\alpha) &= (\tilde{f}(i_{\tilde{g}}\omega) - \tilde{g}(i_{\tilde{f}}\omega) + i_{\tilde{f}} i_{\tilde{g}} d\omega) \cdot \beta \\ &= [i_{\tilde{f}}, L_{\tilde{g}}]\omega \cdot \beta = i_{[\tilde{f},\tilde{g}]}\omega \cdot \beta \\ &= i_{[\tilde{f},\tilde{g}]}\nabla\alpha = [f,g]^{\mathrm{H}}(\alpha).\end{aligned}$$

This completes the proof. ☐

Proof of Proposition 11. Consider the following split sequence of Lie algebras:

$$0 \longrightarrow \mathrm{Der}_A(\Omega^\bullet A)^{(0)} \xrightarrow{\mathrm{incl}} \mathrm{Der}_{\mathbb{K}}(\Omega^\bullet A)^{(0)} \underset{L}{\overset{\mathrm{res}_A}{\rightleftarrows}} \mathrm{Der}_{\mathbb{K}}(A) \longrightarrow 0\,,$$

where res_A is the restriction to A. Since any A-linear derivation on $\Omega^\bullet A$ is uniquely determined by the restriction on $\Omega^1 A$, the space $\mathrm{Der}_A(\Omega^\bullet A)^{(0)}$ is isomorphic to $\mathrm{End}_{A^e}(\Omega^1 A)$ as a Lie algebra.

Now put $c = \mathrm{incl}^{-1} \circ ((-)^{\mathrm{H}} - L)$. By Lemma 13 and the flatness of ∇, $(-)^{\mathrm{H}}$ is a Lie algebra homomorphism. Then c satisfies the non-abelian 1-cocycle condition

$$c([f,g]) = f \cdot c(g) - g \cdot c(f) + [c(f), c(g)]$$

since c is the difference between two Lie algebra homomorphisms. Applying the trace on both sides and using Proposition 8 and Lemma 12, $\mathrm{Div}^{\nabla} = -\mathrm{Tr} \circ c$ is indeed a Lie algebra 1-cocycle. ☐

Remark 14. Proposition 11 remains true for the general case $\mathrm{Div}^{(\nabla,\varphi,\rho)}$ in Definition 9; the proof is almost identical, just less conceptual.

Similar formulations of a connection and the divergence map in the usual differential geometry can be seen in the context of Lie algebroid theory. For example, a connection is defined using an analogous exact sequence to above in Definition 3.8 of [LM10], and the corresponding divergence formula can be found in Proposition 3.11 of [Xu99].

An operad-theoretic formulation of a divergence map, on the other hand, has been studied in [Pow21], where they defined the standard divergence map on a free $\mathcal{O}$-algebra for a reduced operad $\mathcal{O}$. The standard divergences are also discussed here in the next two sections, only in the case of operads controlling associative algebras and Lie algebras. They correspond to the ones associated to the canonical flat connections naturally defined from free generating systems in our setting, as we will see below.

4. Examples of Flat Connections

In this short section, we look into two examples of flat connections.

Firstly, we consider the case $A = \mathbb{K}\langle z_1, \dots, z_n\rangle$. Define a connection on $\Omega^1 A$ by $\nabla_z(dz_i) = 0$ for all i. Then this connection is obviously flat. Now that $(dz_i)_{1\le i\le n}$ is a A^e-free basis of $\Omega^1 A$, we can take its dual basis $(\partial_i)_{1\le i\le n}$, which comprises of double derivations. More precisely, they are given by the formula

$$\partial_j(z_i) = \delta_{ij} \cdot 1 \otimes 1 \quad \text{ for } 1 \le i, j \le n.$$

For $f \in \mathrm{Der}_{\mathbb{K}}(A)$, we compute

$$(L_f - f^{\mathrm{H}})(dz_i) = df(z_i) - i_{\tilde{f}}\nabla_z(dz_i) = \sum_j \partial_j(f(z_i)) \cdot dz_j, \text{ and}$$
$$\mathrm{Div}^{\nabla_z}(f) = \mathrm{Tr}(\mathrm{incl}^{-1} \circ (L_f - f^{\mathrm{H}})) = \sum_i |\partial_i(f(z_i))|\,,$$

which recovers the standard (double) divergence map.

Next, we investigate the case $A = \mathbb{K}F_n$. Recall that the divergence $\mathrm{Div}^{\mathcal{C}}$ associated with a free generating system $\mathcal{C} = (x_i)_{1\le i\le n}$ is given by

$$\mathrm{Div}^{\mathcal{C}}(f) = \sum_i |\partial_i(f(x_i)) - 1 \otimes x_i^{-1} f(x_i)| \in |A| \otimes |A|.$$

Here ∂_i is now defined by $\partial_i(x_j) = \delta_{ij} \cdot 1 \otimes 1$. Since $(dx_i\, x_i^{-1})_{1\le i\le n}$ is a A^{e}-free basis of $\Omega^1 A$, we define a canonical connection $\nabla_{\mathcal{C}}$ associated with $\mathcal{C}$ by the formula $\nabla_{\mathcal{C}}(dx_i\, x_i^{-1}) = 0$ for all i, which is also flat. Denoting $\bar{x}_i \in A^{\mathrm{op}}$ the corresponding element to $x_i \in A$ so that $\bar{x}_i \bar{x}_j = \overline{x_j x_i}$ holds in A^{op}, this is equivalent to

$$0 = \bar{x}_i^{-1} \nabla_{\mathcal{C}}(dx_i) + d\bar{x}_i^{-1} \otimes dx_i, \text{ or } \nabla_{\mathcal{C}}(dx_i) = d\bar{x}_i\, \bar{x}_i^{-1} \otimes dx_i.$$

Now we compute the associated divergence:

$$\begin{aligned}(L_f - f^{\mathrm{H}})(dx_i) &= df(x_i) - i_{\bar{f}} \nabla_{\mathcal{C}}(dx_i) \\ &= \sum_j \partial_j(f(x_i)) \cdot dx_j - (f(\bar{x}_i)\bar{x}_i^{-1}) \cdot dx_i, \text{ and} \\ \mathrm{Div}^{\nabla_{\mathcal{C}}}(f) &= \sum_i |\partial_i(f(x_i)) - f(\bar{x}_i)\bar{x}_i^{-1}|,\end{aligned}$$

which coincides with $\mathrm{Div}^{\mathcal{C}}(f)$ using the identification $|A^{\mathrm{e}}| \cong |A| \otimes |A|$.

5. Geometry over the Lie Operad and the Satoh Trace

The two connections in the previous section, actually, is induced from connections involving geometry over the Lie operad, as we shall see below. We will not use any operad explicitly, however. At the last of this section, we briefly discuss the relation between the divergence map and the Satoh trace. Also, this section can be skipped as there are no logical consequences in later chapters.

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$, and $U\mathfrak{g}$ its enveloping algebra with the standard coproduct Δ, the antipode S, and the counit ε given by

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad S(x) = -x, \quad \text{and} \quad \varepsilon(x) = 0 \quad \text{for } x \in \mathfrak{g}.$$

Put $\tilde{\Delta} = (\mathrm{id} \otimes S) \circ \Delta \colon U\mathfrak{g} \to (U\mathfrak{g})^{\mathrm{e}}$, so that it is an algebra homomorphism. Denote by $\mathbb{P}(U\mathfrak{g})$ the primitive part of $U\mathfrak{g}$, which is isomorphic to $\mathfrak{g}$ as a Lie algebra.

Definition 15.

- $\mathrm{Der}_{\mathrm{Lie}}(\mathfrak{g}, M)$ is the space of all derivations (i.e., 1-cocycles) on $\mathfrak{g}$ into M. Set $\mathrm{Der}_{\mathrm{Lie}}(\mathfrak{g}) = \mathrm{Der}_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{g})$.
- For $f \in \mathrm{Der}_{\mathrm{Lie}}(\mathfrak{g})$, denote by Uf the natural extension of f on $U\mathfrak{g}$.
- The space of *Lie 1-forms* $\Omega^1_{\mathrm{Lie}}\mathfrak{g}$, together with the *exterior derivative* $d \colon \mathfrak{g} \to \Omega^1_{\mathrm{Lie}}\mathfrak{g}$, is a $U\mathfrak{g}$-module defined by the following universal property: for any $\mathfrak{g}$-module M and a derivation $f \colon \mathfrak{g} \to M$, there is a unique $U\mathfrak{g}$-module homomorphism $i_f \colon \Omega^1_{\mathrm{Lie}}\mathfrak{g} \to M$ such that $i_f \circ d = f$ holds.
- For $f \in \mathrm{Der}_{\mathrm{Lie}}(\mathfrak{g})$, the *Lie derivative* $L_f \colon \Omega^1_{\mathrm{Lie}}\mathfrak{g} \to \Omega^1_{\mathrm{Lie}}\mathfrak{g}$ is defined by the formula

$$L_f(a \otimes dx) = (Uf)(a) \otimes dx + a \otimes df(x) \quad \text{for } a \in U\mathfrak{g} \text{ and } x \in \mathfrak{g},.$$

Concretely, we can realise $\Omega^1_{\mathrm{Lie}}\mathfrak{g}$ as the quotient

$$(U\mathfrak{g} \otimes d\mathfrak{g})/\langle 1 \otimes d[x,y] = x \otimes dy - y \otimes dx\rangle$$

of the free (left) $U\mathfrak{g}$-module spanned by the symbols dx, by the submodule generated by the Leibniz relation.

Definition 16. A Lie algebra $\mathfrak{g}$ is said to be *formally smooth* if $\Omega^1_{\mathrm{Lie}}\mathfrak{g}$ is dualisable as a $U\mathfrak{g}$-module.

Formally smooth Lie algebras are of cohomological dimension one (see Proposition 21.1.19 of [Gin05]), and one example is a finitely generated free Lie algebra $L(z_1, \dots, z_n)$. The existence of cohomologically one-dimensional non-free Lie algebras is an open problem over a field of characteristic zero; for the reference, see Introduction of [Zus19] and references there. If $\mathfrak{g}$ is formally smooth, $U\mathfrak{g}$ is automatically formally smooth as an associative algebra as the functor (1) admits a right adjoint.

Now suppose that $\mathfrak{g}$ is formally smooth and we are given a connection $\nabla' \colon \Omega^1_{\mathrm{Lie}}\mathfrak{g} \to \Omega^1 U\mathfrak{g} \otimes_{U\mathfrak{g}} \Omega^1_{\mathrm{Lie}}\mathfrak{g}$. We consider the divergence map, defined in Definition 9, associated with the following data: $B = U\mathfrak{g}$, $M = \Omega^1_{\mathrm{Lie}}\mathfrak{g}$, $\mathfrak{d} = \mathrm{Der}_{\mathrm{Lie}}(\mathfrak{g})$,

$$\begin{aligned} \varphi &\colon \mathrm{Der}_{\mathrm{Lie}}(\mathfrak{g}) \to \mathrm{Der}_{\mathbb{K}}(U\mathfrak{g}) \quad \colon f \mapsto Uf, \text{ and} \\ \rho &\colon \mathrm{Der}_{\mathrm{Lie}}(\mathfrak{g}) \to \mathrm{End}_{\mathbb{K}}(\Omega^1_{\mathrm{Lie}}\mathfrak{g}) \colon f \mapsto L_f. \end{aligned}$$

From these, we obtain the associated divergence map

$$\mathrm{div}^{\nabla'} \colon \mathrm{Der}_{\mathrm{Lie}}(\mathfrak{g}) \to |U\mathfrak{g}|.$$

This *single* divergence map is related to the double ones in the associative setting via the algebra homomorphism $\tilde{\Delta}$. To see this, we need some preparation. Put $A = U\mathfrak{g}$.

Definition-Lemma 17. *The module map $U \colon \Omega^1_{\mathrm{Lie}}\mathfrak{g} \to \Omega^1 A$ over an algebra homomorphism $\tilde{\Delta}$ given by*

$$U(1 \otimes dx) = dx \quad \textit{for } x \in \mathfrak{g}$$

is well-defined. Similarly, the module map $\mathbb{P} \colon \Omega^1 A \to \Omega^1_{\mathrm{Lie}}\mathfrak{g}$ over an algebra homomorphism $\mathrm{id} \otimes \varepsilon \colon A^{\mathrm{e}} \to A$ given by

$$\mathbb{P}(dx) = 1 \otimes dx \quad \textit{for } x \in \mathbb{P}(A)$$

is well-defined. The composition $\mathbb{P} \circ U$ is the identity map.

Proof. For the first one, we only have to check the Leibniz relation. For $x, y \in \mathfrak{g}$, we have

$$\begin{aligned} U(1 \otimes d[x, y]) &= d(xy - yx) \\ &= dx\, y + xdy - dy\, x - ydx \\ &= \tilde{\Delta}(x) \cdot dy - \tilde{\Delta}(y) \cdot dx \\ &= U(x \otimes dy - y \otimes dx), \end{aligned}$$

which proves the first claim. For the second one, it is sufficient to check that the equation $[x, y] = xy - yx$ in $U\mathfrak{g}$ is respected. We compute,

$$\begin{aligned} \mathbb{P}(d[x, y]) &= 1 \otimes d[x, y] \\ &= x \otimes dy - y \otimes dx \\ &= x \cdot \mathbb{P}(dy) - y \cdot \mathbb{P}(dx) \\ &= (\mathrm{id} \otimes \varepsilon) \circ \tilde{\Delta}(x) \cdot \mathbb{P}(dy) - (\mathrm{id} \otimes \varepsilon) \circ \tilde{\Delta}(y) \cdot \mathbb{P}(dx) \\ &= \mathbb{P}(\tilde{\Delta}(x) \cdot dy - \tilde{\Delta}(y) \cdot dx) \\ &= \mathbb{P}(d(xy - yx)). \end{aligned}$$

This completes the proof. $\square$

Definition-Lemma 18. *Let $\nabla' \colon \Omega^1_{\mathrm{Lie}}\mathfrak{g} \to \Omega^1 A \otimes_A \Omega^1_{\mathrm{Lie}}\mathfrak{g}$ be a connection. Then, there is a unique connection $U\nabla' \colon \Omega^1 A \to \Omega^1 A^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} \Omega^1 A$ which lifts ∇', namely, makes the diagram on the left side commute. Similarly, let*

$\nabla\colon \Omega^1 A \to \Omega^1 A^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} \Omega^1 A$ *be a connection. Then, there is a unique connection* $\mathbb{P}(\nabla)\colon \Omega^1_{\mathrm{Lie}}\mathfrak{g} \to \Omega^1 A \otimes_A \Omega^1_{\mathrm{Lie}}\mathfrak{g}$ *which makes the diagram on the right side commute.*

$$
\begin{array}{ccc}
\Omega^1_{\mathrm{Lie}}\mathfrak{g} & \xrightarrow{\nabla'} & \Omega^1 A \otimes_A \Omega^1_{\mathrm{Lie}}\mathfrak{g} \\
\big\downarrow{\scriptstyle U} & & \big\downarrow{\scriptstyle \tilde{\Delta}\otimes U} \\
\Omega^1 A & \xrightarrow{U\nabla'} & \Omega^1 A^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} \Omega^1 A
\end{array}
\qquad\qquad
\begin{array}{ccc}
\Omega^1_{\mathrm{Lie}}\mathfrak{g} & \xrightarrow{\mathbb{P}(\nabla)} & \Omega^1 A \otimes_A \Omega^1_{\mathrm{Lie}}\mathfrak{g} \\
{\scriptstyle \mathbb{P}}\big\uparrow & & {\scriptstyle (\mathrm{id}\otimes\varepsilon)\otimes\mathbb{P}}\big\uparrow \\
\Omega^1 A & \xrightarrow{\nabla} & \Omega^1 A^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} \Omega^1 A
\end{array}
$$

The functors U and $\mathbb{P}$ preserve flat connections.

Proof. For the first one, it automatically follows that

$$U\nabla'(dx) = (\tilde{\Delta} \otimes U) \circ \nabla'(1 \otimes dx) \quad \text{for } x \in \mathfrak{g}\,.$$

The uniqueness follows from the Leibniz property of $U\nabla'$ together with the fact that $U\mathfrak{g}$ is generated, as an associative algebra, by $\mathfrak{g}$. For the well-definedness, just as the definition-lemma above, it is sufficient to check that the equation $[x, y] = xy - yx$ in $U\mathfrak{g}$ is respected. We have, for $x, y \in \mathfrak{g}$,

$$
\begin{aligned}
U\nabla'(d[x,y]) &= (\tilde{\Delta} \otimes U) \circ \nabla'(1 \otimes d[x,y]) \\
&= (\tilde{\Delta} \otimes U) \circ \nabla'(x \otimes dy - y \otimes dx) \\
&= (\tilde{\Delta} \otimes U)\big(dx \otimes (1 \otimes dy) + x \cdot \nabla'(1 \otimes dy) - dy \otimes (1 \otimes dx) - y \cdot \nabla'(1 \otimes dx)\big) \\
&= d\tilde{\Delta}(x) \otimes dy + \tilde{\Delta}(x) \cdot U\nabla'(dy) - d\tilde{\Delta}(y) \otimes dx - y \cdot U\nabla'(dx) \\
&= U\nabla'(\tilde{\Delta}(x) \cdot dy - \tilde{\Delta}(y) \cdot dx) \\
&= U\nabla'(d(xy - yx)).
\end{aligned}
$$

This shows the first claim. For the second one, it also follows that

$$\mathbb{P}(\nabla)(1 \otimes dx) = ((\mathrm{id} \otimes \varepsilon) \otimes \mathbb{P}) \circ \nabla(dx) \quad \text{for } x \in \mathfrak{g}\,,$$

and the uniqueness follows from the surjectivity of $\mathbb{P}$. For the well-definedness, we check the invariance under the Leibniz rule. We have

$$
\begin{aligned}
\mathbb{P}(\nabla)(1 \otimes d[x,y]) &= ((\mathrm{id} \otimes \varepsilon) \otimes \mathbb{P}) \circ \nabla(\tilde{\Delta}(x) \cdot dy - \tilde{\Delta}(y) \cdot dx) \\
&= ((\mathrm{id} \otimes \varepsilon) \otimes \mathbb{P})\big(d\tilde{\Delta}(x) \otimes dy + \tilde{\Delta}(x) \cdot \nabla(dy) - d\tilde{\Delta}(y) \otimes dx - y \cdot \nabla(dx)\big) \\
&= dx \otimes (1 \otimes dy) + x \cdot \mathbb{P}(\nabla)(1 \otimes dy) - dy \otimes (1 \otimes dx) - y \cdot \mathbb{P}(\nabla)(1 \otimes dx) \\
&= \mathbb{P}(\nabla)(x \otimes dy - y \otimes dx)\,.
\end{aligned}
$$

Lastly, if ∇' is flat, the composition

$$(\tilde{\Delta} \otimes U) \circ \nabla' \circ \nabla' = (U\nabla') \circ (U\nabla') \circ U$$

is zero. Since the image of U generates $\Omega^1 A$ as an A^{e}-module, $(U\nabla')^2 = 0$ holds. Conversely, if ∇ is flat, the composition

$$((\mathrm{id} \otimes \varepsilon) \otimes \mathbb{P}) \circ \nabla \circ \nabla = \mathbb{P}(\nabla) \circ \mathbb{P}(\nabla) \circ \mathbb{P}$$

is zero, and $\mathbb{P}(\nabla)^2 = 0$ follows, again, from the surjectivity of $\mathbb{P}$. This completes the proof. $\square$

Remark 19. The correspondence in above definition-lemmas is inspired by the theorem of Milnor–Moore, which states that $A \cong U\mathbb{P}(A)$ holds for a complete topological (or graded) Hopf algebra A admitting a suitable filtration.

In the following, we give a relation between div and Div in the special case $\mathfrak{g} = L(z_1, \ldots, z_n)$, as this is virtually the only example of a formally smooth Lie algebra.

Proposition 20. *Let $\mathfrak{g} = L(z_1, \ldots, z_n)$, and $\nabla'\colon \Omega^1_{\mathrm{Lie}}\mathfrak{g} \to \Omega^1 A \otimes_A \Omega^1_{\mathrm{Lie}}\mathfrak{g}$ be a connection. Then*

$$\mathrm{Div}^{U\nabla'}(Uf) = \tilde{\Delta}(\mathrm{div}^{\nabla'}(f))$$

holds in $|A^{\mathrm{e}}|$, for $f \in \mathrm{Der}_{\mathrm{Lie}}(\mathfrak{g})$.

Proof. First of all, we have, for $f \in \mathrm{Der}_{\mathrm{Lie}}(\mathfrak{g})$,

$$U \circ L_f = L_{Uf} \circ U \colon \Omega^1_{\mathrm{Lie}}\mathfrak{g} \to \Omega^1 A, \text{ and}$$
$$\tilde{\Delta} \circ i_{Uf} = i_{\widetilde{Uf}} \circ \tilde{\Delta} \colon \Omega^1 A \to A^{\mathrm{e}}\,.$$

From now on, we employ the Einstein summation convention. We put $\nabla'(1 \otimes dz_i) = \omega_i^j \otimes (1 \otimes dz_j)$ for some $\omega_i^j \in \Omega^1 A$ and $L_f(1 \otimes dz_i) = f^i \otimes dz_i$ for some $f^i \in A$. Then, in $\Omega^1 A$,

$$\tilde{\Delta}(f^i) \cdot dz_i = U(f^i \otimes dz_i) = U(L_f(1 \otimes dz_i)) = L_{Uf}(U(1 \otimes dz_i)) = L_{Uf}(dz_i)$$

holds. In addition, we have

$$(U\nabla')(dz_i) = \tilde{\Delta}(\omega_i^j) \otimes dz_j$$

by definition. Therefore, we have

$$\begin{aligned}
\mathrm{Div}^{U\nabla'}(Uf) &= \sum_i |\tilde{\Delta}(f^i) - i_{\widetilde{Uf}}(\tilde{\Delta}(\omega_i^i))| \\
&= \tilde{\Delta}\left(\sum_i |f^i - i_{Uf}(\omega_i^i)|\right) \\
&= \tilde{\Delta}(\mathrm{div}^{\nabla'}(f))
\end{aligned}$$

This completes the proof. □

Let us get back to the flat connections in the previous section. First of all, the flat connection $\nabla_z(dz_i) = 0$ on $\Omega^1\mathbb{K}\langle z_1, \dots, z_n\rangle$ is induced from $\nabla'_z(1 \otimes dz_i) = 0$ on $\Omega^1_{\mathrm{Lie}}L(z_1, \dots, z_n)$, while the other one, $\nabla_{\mathcal{C}}(dx_i\, x_i^{-1}) = 0$ on $\Omega^1\mathbb{K}F_n$, is *not*, as $\mathbb{K}F_n$ cannot be written in the form $U\mathfrak{g}$ for any $\mathfrak{g}$. However, after the completion by the maximal ideal $\mathrm{Ker}(\varepsilon \colon \mathbb{K}F_n \to \mathbb{K})$, the completion $\widehat{\mathbb{K}F_n}$ is isomorphic, as a complete Hopf algebra, to the completed free associative algebra $\mathbb{K}\langle\langle z_1, \dots, z_n\rangle\rangle$ by the following map:

$$\widehat{\mathbb{K}F_n} \cong \mathbb{K}\langle\langle z_1, \dots, z_n\rangle\rangle \colon x_i \mapsto e^{z_i}\,.$$

Under this identification, we have

$$\begin{aligned}
dx_i\, x_i^{-1} &= 1 \otimes 1 - x_i \otimes x_i^{-1} = \tilde{\Delta}(1 - x_i) = \tilde{\Delta}(1 - e^{z_i}) \\
&= \tilde{\Delta}\left(\frac{e^{z_i} - 1}{z_i}\right)\tilde{\Delta}(-z_i) = \tilde{\Delta}\left(\frac{e^{z_i} - 1}{z_i}\right) \cdot dz_i = U\left(\frac{e^{z_i} - 1}{z_i} \otimes dz_i\right)
\end{aligned}$$

which shows that (the continuous extension of) $\nabla_{\mathcal{C}}$ is induced from the connection $\nabla'_{\mathcal{C}}$ on $\Omega^1_{\mathrm{Lie}}L(\!(z_1, \dots, z_n)\!)$ defined by

$$\nabla'_{\mathcal{C}}\left(\frac{e^{z_i} - 1}{z_i} \otimes dz_i\right) = 0.$$

Note that all constructions above work just as well in the topological setting, provided everything is continuous.

Finally, recall the Satoh trace map introduced in [Sat12]. Let $H = \mathbb{K}\{z_1, \dots, z_n\}$ be a free $\mathbb{K}$-module, $H^* = \mathrm{Hom}_{\mathbb{K}}(H, \mathbb{K})$ its dual space, and $\mathfrak{g} = L(z_1, \dots, z_n)$. Then $A = U\mathfrak{g}$ is isomorphic to $\mathbb{K}\langle z_1, \dots, z_n\rangle$ as a Hopf algebra. The Satoh trace map $\mathrm{Tr}_{\mathrm{Satoh}}$ is defined as the composition

$$\mathrm{Tr}_{\mathrm{Satoh}} \colon H^* \otimes L(H) \xrightarrow{\mathrm{incl}} H^* \otimes A \xrightarrow{\mathrm{cont}} A \xrightarrow{\mathrm{proj}} |A|,$$

where the contraction map is defined by

$$\mathrm{cont}(z_j^* \otimes z_{i_1} \cdots z_{i_r}) = \delta_{i_r, j} \cdot z_{i_2} \cdots z_{i_r}\,.$$

Satoh showed in [Sat12] that the kernel of $\mathrm{Tr}_{\mathrm{Satoh}}$ stably coincides with the image of the Johnson homomorphism on the automorphism group of a free group. On the other hand, the map $\mathrm{Tr}_{\mathrm{Satoh}}$ is exactly the map $\mathrm{div}^{\nabla'_z}$ (up to some isomorphisms), as is also observed in Section 9 of [AKKN18].

For an introduction into geometry over an operad, see the last section of [Gin05], for example.

6. Perfect Modules and Homological Connections

In this section, we define a divergence associated with a connection on a not-necessarily-projective module. Since the existence of a connection (more or less) implies projectivity, we have to modify the definition to work with them.

Conventions. A projective resolution of a module M is of the form $P = (\cdots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} M \to 0)$. We set $P_{-1} = M$.

For a B-module N and $f \in \mathrm{Der}_{\mathbb{K}}(B)$, a $\mathbb{K}$-linear map $u : N \to N$ is called an *f-derivation* if

$$u(bn) = f(b)n + bu(n)$$

holds for $b \in B$ and $n \in N$. Now suppose $\mathrm{Der}_{\mathbb{K}}(B)$ is acting on M by derivation, which is the same thing as the action of each $f \in \mathrm{Der}_{\mathbb{K}}(B)$ is itself an f-derivation on M.

Proposition 21. *Let (P, ∂) be a projective resolution of M. Then, the action $f : M \to M$ of $f \in \mathrm{Der}_{\mathbb{K}}(B)$ admits a lift $\lambda[f] : P \to P$ satisfying $[\partial, \lambda[f]] = 0$, which is an f-derivation and is unique up to homotopy.*

Proof. Fix $f \in \mathrm{Der}_{\mathbb{K}}(B)$. First, we take an arbitrary family of f-dervations $(\gamma_n : P_n \to P_n)_n$. In fact, if P_n is free, we may take γ_n as the direct sum of f's. For a general projective module P_n, we realise it as a summand of a free module, and then restrict the map to each P_n to obtain γ_n.

Now using $(\gamma_n)_n$, we construct maps by following steps: first,

$$\lambda[f]_{-1} = f : M \to M,$$

and, for $n \geq 0$,

$$\begin{aligned}
&\varphi[f]_n = \lambda[f]_{n-1} \circ \partial_n - \partial_n \circ \gamma_n : P_n \to P_{n-1}, \\
&\tilde{\varphi}[f]_n : P_n \to P_n \text{ so that } \partial_n \circ \tilde{\varphi}[f]_n = \varphi[f]_n, \text{ and} \\
&\lambda[f]_n = \tilde{\varphi}[f]_n + \gamma_n : P_n \to P_n.
\end{aligned}$$

We check that this inductive procedure is well-defined. Let $n \geq 0$ and suppose that $\lambda[f]_{n-1}$ is an f-derivation. Then, $\varphi[f]_n$ is a B-module map: for $b \in B$ and $p \in P_n$,

$$\begin{aligned}
\varphi[f]_n(bp) &= \lambda[f]_{n-1}(\partial_n(bp)) - \partial_n(\gamma_n(bp)) \\
&= \lambda[f]_{n-1}(b\partial_n(p)) - \partial_n(f(b)p + b\gamma_n(p)) \\
&= f(b)\partial_n(p) + b\lambda[f]_{n-1}(\partial_n(p)) - f(b)\partial_n(p) + b\partial_n(\gamma_n(p)) \\
&= b\varphi[f]_n(p).
\end{aligned}$$

Next, we check $\partial_{n-1} \circ \varphi[f]_n = 0$. If $n = 0$, this is true since ∂_{-1} is defined to be zero. If $n \geq 1$, we have

$$\begin{aligned}
\partial_{n-1} \circ \varphi[f]_n &= \partial_{n-1} \circ (\lambda[f]_{n-1} \circ \partial_n - \partial_n \circ \gamma_n) \\
&= \partial_{n-1} \circ (\tilde{\varphi}[f]_{n-1} + \gamma_{n-1}) \circ \partial_n \\
&= (\varphi[f]_{n-1} + \partial_{n-1} \circ \gamma_{n-1}) \circ \partial_n \\
&= (\lambda[f]_{n-2} \circ \partial_{n-1} - \partial_{n-1} \circ \gamma_{n-1} + \partial_{n-1} \circ \gamma_{n-1}) \circ \partial_n \\
&= 0,
\end{aligned}$$

which shows that we can take a lift of $\varphi[f]_n$. Lastly, it is clear that $\lambda[f]_n$ is an f-derivation. This shows the well-definedness. Next, we have

$$\begin{aligned}
\partial_n \circ \lambda[f]_n &= \partial_n \circ (\tilde{\varphi}[f]_n + \gamma_n) \\
&= \varphi[f]_n + \partial_n \circ \gamma_n \\
&= \lambda[f]_{n-1} \circ \partial_n - \partial_n \circ \gamma_n + \partial_n \circ \gamma_n \\
&= \lambda[f]_{n-1} \circ \partial_n,
\end{aligned}$$

which is briefly denoted by $[\partial, \lambda[f]] = 0$.

Now suppose we have two such lifts $\lambda[f]$ and $\lambda[f]'$. Then the difference $\lambda[f] - \lambda[f]'$ is a B-module map, and at the same time, is a lift of the zero map $0 : M \to M$. Hence this is null-homotopic and this completes the proof. $\square$

Definition 22. Let (C, ∂) be a chain complex of B-modules.

- We put $\mathrm{IHom}^0_B(C, C) = \{(\psi_n : C_n \to C_n)_n : B\text{-module maps}\}$, the space of degree zero maps. IHom stands for internal hom-set.
- (C, ∂) is called a *perfect complex* if it is of finite length and each C_n is B-dualisable. In this case, the trace map is defined by

$$\mathrm{Tr} : \mathrm{IHom}^0_B(C, C) \to |B| : (\psi_n)_n \mapsto \sum_n (-1)^n \mathrm{Tr}(\psi_n).$$

Lemma 23. *Let (C, ∂) be a perfect complex of B-modules and $(h_n : C_n \to C_{n+1})_n$ be a family of B-module maps. Then we have $\mathrm{Tr}([\partial, h]) = 0$.*

Proof. By definition, we have

$$\begin{aligned}\mathrm{Tr}([\partial, h]) &= \sum_n (-1)^n \mathrm{Tr}(\partial_{n+1} \circ h_n + h_{n-1} \circ \partial_n) \\ &= \sum_n (-1)^n \mathrm{Tr}(\partial_{n+1} \circ h_n) + \sum_n (-1)^{n+1} \mathrm{Tr}(h_n \circ \partial_{n+1}).\end{aligned}$$

Since $\mathrm{Tr}(h_n \circ \partial_{n+1}) = \mathrm{Tr}(\partial_{n+1} \circ h_n)$ by proposition 8, we obtain $\mathrm{Tr}([\partial, h]) = 0$. $\square$

Now we can define a connection on a B-module and the divergence associated with it.

Definition 24. Let M be a B-module.

- A *homological connection* on M is a pair (P, ∇) where P is a projective resolution of M and $\nabla = \{\nabla_n : P_n \to \Omega^1 B \otimes_B P_n\}_{n \geq 0}$ is a family of usual connections.
- The *curvature* of (P, ∇) is defined to be $R = \{(\nabla_n)^2 : P_n \to \Omega^2 B \otimes_B P_n\}_{n \geq 0}$, which is a family of B-module maps. A homological connection (P, ∇) is *flat* if the curvature R is the zero map.

Note that the collection ∇ of connections above is *not* required to be cochain maps.

Now suppose that a Lie algebra $\mathfrak{g}$ and a derivation action $(\varphi\colon \mathfrak{g} \to \mathrm{Der}_{\mathbb{K}}(B), \rho\colon \mathfrak{g} \to \mathrm{End}_{\mathbb{K}}(M))$ on M are given. Note that $\rho(f)$ is a $\varphi(f)$-derivation.

Definition 25. Let (P, ∇) be a homological connection on M with P perfect. The associated *divergence* $\mathrm{Div}^{(P,\nabla,\varphi,\rho)}$ is defined by

$$\mathrm{Div}^{(P,\nabla,\varphi,\rho)}\colon \mathfrak{g} \to |B| : f \mapsto \sum_{n \geq 0} (-1)^n \mathrm{Tr}\left(\lambda[\rho(f)]_n - (i_{\varphi(f)} \otimes \mathrm{id}) \circ \nabla_n\right).$$

This is well-defined by Proposition 21 and the lemma above.

An algebra A is said to be *homologically smooth* if A is finitely generated as a $\mathbb{K}$-algebra with perfect $\Omega^1 A$. In this case, we obtain a divergence map $\mathrm{Div}^\nabla\colon \mathrm{Der}_{\mathbb{K}}(A) \to |A^{\mathrm{e}}|$, associated with ∇, just as in Section 3.

7. The Closed Surface Case

Let $\Sigma_{g,1}$ be an oriented surface of genus g with one boundary component, and pick a base point on the boundary. Denote by ζ the boundary class in $\pi_1(\Sigma_{g,1})$ and fix an isomorphism

$$\pi_1(\Sigma_{g,1}) \cong F_{2g} = \langle a_i, b_i \, (1 \le i \le g) \rangle$$

so that $\zeta = (a_1, b_1) \cdots (a_g, b_g)$ holds. Here $(x, y) = xyx^{-1}y^{-1}$ is the group commutator. Then the closed surface Σ_g is obtained by capping the boundary, with the base point induced from $\Sigma_{g,1}$. Put $\pi = \pi_1(\Sigma_g)$.

Definition 26. The map $v\colon |\mathbb{K}\pi| \to \mathrm{HH}^1(\mathbb{K}\pi)$ is defined by

$$v(\alpha)(x) = \sum_{p \in \alpha \cap x} \mathrm{sign}(\alpha, x; p)\, \alpha *_p x$$

for generic representatives of a free loop α and $x \in \pi$. Its well-definedness is due to D. Vaintrob (Lemma 1 of [Vai07a]).

In this section, we will show the following theorem, which gives an algebraic description of the Turaev cobracket on a closed surface. Put $R = \mathbb{K}\langle\zeta\rangle$, the subalgebra of $\mathbb{K}F_n$ generated by ζ.

Theorem 27. *Let $\mathcal{C} = (a_i, b_i)_{1 \le i \le g}$ be a free generating system of $\pi_1(\Sigma_{g,1})$ with $\zeta = (a_1, b_1) \cdots (a_g, b_g)$. Then we have a commutative diagram*

$$\begin{tikzcd}
|\mathbb{K}F_n| \arrow[r, "\sigma"] \arrow[dd, two heads] & \mathrm{Der}_R(\mathbb{K}F_n) \arrow[r, "\mathrm{Div}^{\mathcal{C}}"] \arrow[d] & |\mathbb{K}F_n|^{\otimes 2} \arrow[d, two heads] \\
 & \mathrm{Der}_{\mathbb{K}}(\mathbb{K}\pi) \arrow[d] & |\mathbb{K}\pi|^{\otimes 2} \arrow[d] \\
|\mathbb{K}\pi| \arrow[r, "v"] & \mathrm{HH}^1(\mathbb{K}\pi) \arrow[r, "\mathrm{Div}^{\nabla'}"] & |\mathbb{K}\pi/\mathbb{K}1|^{\otimes 2}.
\end{tikzcd}$$

for some homological connection ∇' on $\Omega^1\mathbb{K}\pi$. Therefore, the composition $\mathrm{Div}^{\nabla'} \circ v$ is equal to the Turaev cobracket δ.

To prove the theorem above, we first construct a homological connection ∇'. To do so, we have to choose a free resolution of $\mathbb{K}\pi$.

Lemma 28. *Let $V = \mathbb{K}\{a_i, b_i\}_{1 \le i \le g}$ be a free $\mathbb{K}$-module of rank $2g$. We have a $(\mathbb{K}\pi)^{\mathrm{e}}$-free resolution of $\mathbb{K}\pi$:*

$$0 \longrightarrow \mathbb{K}\pi \otimes \mathbb{K}\pi \xrightarrow{d_1} \mathbb{K}\pi \otimes V \otimes \mathbb{K}\pi \xrightarrow{d_0} \mathbb{K}\pi \otimes \mathbb{K}\pi \xrightarrow{\mu} \mathbb{K}\pi \longrightarrow 0\,, \tag{2}$$

$$\begin{aligned}
d_1(1 \otimes 1) = \sum_{1 \le i \le g} (a_1, b_1) \cdots (a_{i-1}, b_{i-1}) \Big(& 1 \otimes a_i \otimes b_i a_i^{-1} b_i^{-1} + a_i \otimes b_i \otimes a_i^{-1} b_i^{-1} \\
& - a_i b_i a_i^{-1} \otimes a_i \otimes a_i^{-1} b_i^{-1} - a_i b_i a_i^{-1} b_i^{-1} \otimes b_i \otimes b_i^{-1} \Big) (a_{i+1}, b_{i+1}) \cdots (a_g, b_g),
\end{aligned}$$

$$d_0(1 \otimes v \otimes 1) = 1 \otimes v - v \otimes 1.$$

Proof. We follow a method by R. Lyndon [Lyn50]. By Section 11 of [Lyn50], we have a left $\mathbb{K}\pi$-free resolution of $\mathbb{K}$ using the Fox derivative:

$$0 \longrightarrow \mathbb{K}\pi \xrightarrow{\partial_1} \mathbb{K}\pi \otimes V \xrightarrow{\partial_0} \mathbb{K}\pi \xrightarrow{\varepsilon} \mathbb{K} \longrightarrow 0\,,$$

$$\begin{aligned}
\partial_1(1) &= \sum_{1 \le i \le g} \left(\frac{\partial \zeta}{\partial a_i} \otimes a_i + \frac{\partial \zeta}{\partial b_i} \otimes b_i \right) \\
&= \sum_{1 \le i \le g} (a_1, b_1) \cdots (a_{i-1}, b_{i-1}) \left(1 \otimes a_i + a_i \otimes b_i - a_i b_i a_i^{-1} \otimes a_i - a_i b_i a_i^{-1} b_i^{-1} \otimes b_i \right),
\end{aligned}$$

$$\partial_0(1 \otimes c) = c - 1 \quad \text{for } c \in \{a_i, b_i\}_{1 \le i \le g}\,.$$

Applying the functor (1) and isomorphisms defined in Example in Section 2, we obtain a left $(\mathbb{K}\pi)^{\mathrm{e}}$-free resolution

$$0 \longrightarrow \mathbb{K}\pi \otimes \mathbb{K}\pi \xrightarrow{\partial_1'} \mathbb{K}\pi \otimes V \otimes \mathbb{K}\pi \xrightarrow{\partial_0'} \mathbb{K}\pi \otimes \mathbb{K}\pi \xrightarrow{\mu} \mathbb{K}\pi \longrightarrow 0\,,$$

$$\begin{aligned}\partial_1'(1\otimes 1) &= \sum_{1\le i\le g} (a_1,b_1)\cdots(a_{i-1},b_{i-1})\left(1\otimes a_i\otimes a_ib_ia_i^{-1}b_i^{-1} + a_i\otimes b_i\otimes b_ia_i^{-1}b_i^{-1}\right.\\ &\qquad \left.-a_ib_ia_i^{-1}\otimes a_i\otimes b_i^{-1} - a_ib_ia_i^{-1}b_i^{-1}\otimes b_i\otimes 1\right)(a_{i+1},b_{i+1})\cdots(a_g,b_g),\\ \partial_0'(1\otimes c\otimes 1) &= c\otimes c^{-1} - 1\otimes 1 \quad \text{for } c\in\{a_i,b_i\}_{1\le i\le g}\,.\end{aligned}$$

Finally, swapping the second-to-left term by the automorphism

$$\begin{aligned}\tau\colon \mathbb{K}\pi\otimes V\otimes\mathbb{K}\pi &\to \mathbb{K}\pi\otimes V\otimes\mathbb{K}\pi\\ 1\otimes c\otimes 1 &\mapsto 1\otimes c\otimes c^{-1} \quad \text{for } c\in\{a_i,b_i\}_{1\le i\le g}\end{aligned}$$

gives the proclaimed resolution: $d_1 = \tau\circ\partial_1'$ and $d_0 = -\partial_0'\circ\tau^{-1}$. □

Now we define a homological connection on $\Omega^1\mathbb{K}\pi$. Truncating (2) yields a resolution

$$0 \longrightarrow \mathbb{K}\pi\otimes\mathbb{K}\pi \xrightarrow{d_1} \mathbb{K}\pi\otimes V\otimes\mathbb{K}\pi \xrightarrow{d_0} \Omega^1\mathbb{K}\pi \longrightarrow 0\,.$$

On each degree, we set

$$\nabla_0'(1\otimes a_i\otimes a_i^{-1}) = 0,\ \nabla_0'(1\otimes b_i\otimes b_i^{-1}) = 0,\ \text{and } \nabla_1'(1\otimes 1) = 0\,.$$

∇_0' is the push-out of $\nabla_{\mathcal{C}}$ by the natural map $p\colon \mathbb{K}F_n \twoheadrightarrow \mathbb{K}\pi$. To compute the associated divergence, we need to lift the Lie derivatives on $\Omega^1\mathbb{K}\pi$.

Lemma 29. *Let $f\in \mathrm{Der}_R(\mathbb{K}F_n)$, and $\bar f\in\mathrm{Der}_{\mathbb{K}}(\mathbb{K}\pi)$ the induced derivation. Then $\bar f$-derivations defined by*

$$\begin{aligned}\lambda[\bar f]_0\colon \mathbb{K}\pi\otimes V\otimes\mathbb{K}\pi\to\mathbb{K}\pi\otimes V\otimes\mathbb{K}\pi\colon 1\otimes v\otimes 1 &\mapsto \sum_{c\in\{a_i,b_i\}_{1\le i\le g}} p(\partial_c' f(v))\otimes c\otimes p(\partial_c'' f(v)) \ \text{and}\\ \lambda[\bar f]_1\colon \mathbb{K}\pi\otimes\mathbb{K}\pi\to\mathbb{K}\pi\otimes\mathbb{K}\pi\colon \qquad 1\otimes 1&\mapsto 0\end{aligned}$$

is a lift of $L_{\bar f}\colon \Omega^1\mathbb{K}\pi\to\Omega^1\mathbb{K}\pi$.

Proof. First of all, $L_{\bar f}\circ d_0 = d_0\circ\lambda[\bar f]_0$ is clear from the construction. Next, we have

$$\begin{aligned}(\lambda[\bar f]_0\circ d_1)(1\otimes 1) &= \lambda[\bar f]_0\left(\sum_{1\le i\le g}(a_1,b_1)\cdots(a_{i-1},b_{i-1})\Big(1\otimes a_i\otimes b_ia_i^{-1}b_i^{-1} + a_i\otimes b_i\otimes a_i^{-1}b_i^{-1}\right.\\ &\qquad \left. - a_ib_ia_i^{-1}\otimes a_i\otimes a_i^{-1}b_i^{-1} - a_ib_ia_i^{-1}b_i^{-1}\otimes b_i\otimes b_i^{-1}\Big)(a_{i+1},b_{i+1})\cdots(a_g,b_g)\right)\\ &= \sum_{1\le i\le g}\bar f((a_1,b_1)\cdots(a_{i-1},b_{i-1}))\Big(1\otimes a_i\otimes b_ia_i^{-1}b_i^{-1} + a_i\otimes b_i\otimes a_i^{-1}b_i^{-1}\\ &\qquad - a_ib_ia_i^{-1}\otimes a_i\otimes a_i^{-1}b_i^{-1} - a_ib_ia_i^{-1}b_i^{-1}\otimes b_i\otimes b_i^{-1}\Big)(a_{i+1},b_{i+1})\cdots(a_g,b_g)\\ &\quad + (a_1,b_1)\cdots(a_{i-1},b_{i-1})\lambda[\bar f]_0\Big(1\otimes a_i\otimes b_ia_i^{-1}b_i^{-1} + a_i\otimes b_i\otimes a_i^{-1}b_i^{-1}\\ &\qquad - a_ib_ia_i^{-1}\otimes a_i\otimes a_i^{-1}b_i^{-1} - a_ib_ia_i^{-1}b_i^{-1}\otimes b_i\otimes b_i^{-1}\Big)(a_{i+1},b_{i+1})\cdots(a_g,b_g)\\ &\quad + (a_1,b_1)\cdots(a_{i-1},b_{i-1})\Big(1\otimes a_i\otimes b_ia_i^{-1}b_i^{-1} + a_i\otimes b_i\otimes a_i^{-1}b_i^{-1}\\ &\qquad - a_ib_ia_i^{-1}\otimes a_i\otimes a_i^{-1}b_i^{-1} - a_ib_ia_i^{-1}b_i^{-1}\otimes b_i\otimes b_i^{-1}\Big)\bar f((a_{i+1},b_{i+1})\cdots(a_g,b_g))\,.\end{aligned}$$

Using the isomorphism $\Omega^1\mathbb{K}F_n \cong \mathbb{K}F_n \otimes V \otimes \mathbb{K}F_n : dv \mapsto 1 \otimes v \otimes 1$, the above equals

$$\begin{aligned}
&\sum_{1\le i\le g} \bar{f}((a_1,b_1)\cdots(a_{i-1},b_{i-1}))d(a_i,b_i)(a_{i+1},b_{i+1})\cdots(a_g,b_g)\\
&\qquad + (a_1,b_1)\cdots(a_{i-1},b_{i-1})df(a_i,b_i)(a_{i+1},b_{i+1})\cdots(a_g,b_g)\\
&\qquad + (a_1,b_1)\cdots(a_{i-1},b_{i-1})d(a_i,b_i)\bar{f}((a_{i+1},b_{i+1})\cdots(a_g,b_g))\\
&= (p\otimes \mathrm{id}_V \otimes p)(df(\zeta)),
\end{aligned}$$

which is zero as $f(\zeta)=0$. This concludes the proof. ☐

Lemma 30. *Let $x \in \mathbb{K}F_n$, $\bar{x} \in \mathbb{K}\pi$ its image and $\mathrm{ad}_{\bar{x}} = [\bar{x}\,,\cdot]$ the adjoint action by $\bar{x}$ on $\mathbb{K}\pi$. Then $\mathrm{ad}_{\bar{x}}$-derivations defined by*

$$\lambda[\mathrm{ad}_{\bar{x}}]_0 : \mathbb{K}\pi\otimes V\otimes \mathbb{K}\pi \to \mathbb{K}\pi\otimes V\otimes\mathbb{K}\pi : 1\otimes v\otimes 1 \mapsto \sum_{c\in\{a_i,b_i\}_{1\le i\le g}} p(\partial'_c[x,v])\otimes c\otimes p(\partial''_c[x,v]) \text{ and}$$

$$\lambda[\mathrm{ad}_{\bar{x}}]_1 : \mathbb{K}\pi\otimes\mathbb{K}\pi \to \mathbb{K}\pi\otimes\mathbb{K}\pi : \qquad 1\otimes 1\mapsto \bar{x}\otimes 1 - 1\otimes\bar{x}$$

is a lift of $L_{\mathrm{ad}_{\bar{x}}} : \Omega^1\mathbb{K}\pi \to \Omega^1\mathbb{K}\pi$.

Proof. As in the previous lemma, $L_{\mathrm{ad}_{\bar{x}}}\circ d_0 = d_0\circ\lambda[\mathrm{ad}_{\bar{x}}]_0$ is clear. Next, we have

$$\begin{aligned}
(\lambda[\mathrm{ad}_{\bar{x}}]_0\circ d_1)(1\otimes 1) = &\sum_{1\le i\le g} [\bar{x},(a_1,b_1)\cdots(a_{i-1},b_{i-1})]\Big(1\otimes a_i\otimes b_ia_i^{-1}b_i^{-1} + a_i\otimes b_i\otimes a_i^{-1}b_i^{-1}\\
&\qquad - a_ib_ia_i^{-1}\otimes a_i\otimes a_i^{-1}b_i^{-1} - a_ib_ia_i^{-1}b_i^{-1}\otimes b_i\otimes b_i^{-1}\Big)(a_{i+1},b_{i+1})\cdots(a_g,b_g)\\
&+ (a_1,b_1)\cdots(a_{i-1},b_{i-1})\lambda[\mathrm{ad}_{\bar{x}}]_0\Big(1\otimes a_i\otimes b_ia_i^{-1}b_i^{-1} + a_i\otimes b_i\otimes a_i^{-1}b_i^{-1}\\
&\qquad - a_ib_ia_i^{-1}\otimes a_i\otimes a_i^{-1}b_i^{-1} - a_ib_ia_i^{-1}b_i^{-1}\otimes b_i\otimes b_i^{-1}\Big)(a_{i+1},b_{i+1})\cdots(a_g,b_g)\\
&+ (a_1,b_1)\cdots(a_{i-1},b_{i-1})\Big(1\otimes a_i\otimes b_ia_i^{-1}b_i^{-1} + a_i\otimes b_i\otimes a_i^{-1}b_i^{-1}\\
&\qquad - a_ib_ia_i^{-1}\otimes a_i\otimes a_i^{-1}b_i^{-1} - a_ib_ia_i^{-1}b_i^{-1}\otimes b_i\otimes b_i^{-1}\Big)[\bar{x},(a_{i+1},b_{i+1})\cdots(a_g,b_g)],
\end{aligned}$$

which is equal to, using the isomorphism as before,

$$\begin{aligned}
&\sum_{1\le i\le g} [\bar{x},(a_1,b_1)\cdots(a_{i-1},b_{i-1})]d(a_i,b_i)(a_{i+1},b_{i+1})\cdots(a_g,b_g)\\
&\qquad + (a_1,b_1)\cdots(a_{i-1},b_{i-1})d[\bar{x},(a_i,b_i)](a_{i+1},b_{i+1})\cdots(a_g,b_g)\\
&\qquad + (a_1,b_1)\cdots(a_{i-1},b_{i-1})d(a_i,b_i)[\bar{x},(a_{i+1},b_{i+1})\cdots(a_g,b_g)]\\
&= \sum_{1\le i\le g} \bar{x}(a_1,b_1)\cdots(a_{i-1},b_{i-1})d(a_i,b_i)(a_{i+1},b_{i+1})\cdots(a_g,b_g)\\
&\qquad - (a_1,b_1)\cdots(a_{i-1},b_{i-1})\bar{x}d(a_i,b_i)(a_{i+1},b_{i+1})\cdots(a_g,b_g)\\
&\qquad + (a_1,b_1)\cdots(a_{i-1},b_{i-1})d\bar{x}(a_i,b_i)(a_{i+1},b_{i+1})\cdots(a_g,b_g)\\
&\qquad + (a_1,b_1)\cdots(a_{i-1},b_{i-1})\bar{x}d(a_i,b_i)(a_{i+1},b_{i+1})\cdots(a_g,b_g)\\
&\qquad - (a_1,b_1)\cdots(a_{i-1},b_{i-1})(a_i,b_i)d\bar{x}(a_{i+1},b_{i+1})\cdots(a_g,b_g)\\
&\qquad - (a_1,b_1)\cdots(a_{i-1},b_{i-1})d(a_i,b_i)\bar{x}(a_{i+1},b_{i+1})\cdots(a_g,b_g)\\
&\qquad + (a_1,b_1)\cdots(a_{i-1},b_{i-1})d(a_i,b_i)\bar{x}(a_{i+1},b_{i+1})\cdots(a_g,b_g)\\
&\qquad - (a_1,b_1)\cdots(a_{i-1},b_{i-1})d(a_i,b_i)(a_{i+1},b_{i+1})\cdots(a_g,b_g)\bar{x}\\
&= \sum_{1\le i\le g} \bar{x}(a_1,b_1)\cdots(a_{i-1},b_{i-1})d(a_i,b_i)(a_{i+1},b_{i+1})\cdots(a_g,b_g)\\
&\qquad - (a_1,b_1)\cdots(a_{i-1},b_{i-1})d(a_i,b_i)(a_{i+1},b_{i+1})\cdots(a_g,b_g)\bar{x}\\
&= \bar{x}d_1(1\otimes 1) - d_1(1\otimes 1)\bar{x}\\
&= (d_1\circ\lambda[\mathrm{ad}_{\bar{x}}]_1)(1\otimes 1).
\end{aligned}$$

This concludes the proof. ☐

Lemma 31. *For $y \in \mathbb{K}F_n$, we have* $\sum_{1\le i\le n} |\partial_i[y, x_i]| = (n+1)|y \otimes 1 - 1 \otimes y|$ *in* $|A| \otimes |A|$.

Proof. Write $y = x_{i_1}^{\varepsilon_1} \cdots x_{i_r}^{\varepsilon_r}$ for some $\varepsilon_k \in \{1, -1\}$. Then,

$$\sum_{1\le i\le n} |\partial_i[y, x_i]| = \sum_{1\le i\le n} |\partial_i(yx_i - x_i y)| = \sum_{1\le i\le n} |\partial_i(y)x_i - x_i\partial_i(y) + y \otimes 1 - 1 \otimes y|.$$

The first two terms on the RHS read, putting $e_i = (\varepsilon_i - 1)/2$,

$$\begin{aligned}
\sum_{1\le i\le n} |\partial_i(y)x_i - x_i\partial_i(y)| &= \sum_{1\le i\le n} |\partial_i(x_{i_1}^{\varepsilon_1} \cdots x_{i_r}^{\varepsilon_r})x_i - x_i\partial_i(x_{i_1}^{\varepsilon_1} \cdots x_{i_r}^{\varepsilon_r})| \\
&= \sum_{1\le k\le r} |x_{i_1}^{\varepsilon_1} \cdots \partial_{i_k}(x_{i_k}^{\varepsilon_k}) \cdots x_{i_r}^{\varepsilon_r} x_{i_k} - x_{i_k}\partial_{i_k}(x_{i_k}^{\varepsilon_k}) \cdots x_{i_r}^{\varepsilon_r}| \\
&= \sum_{1\le k\le r} |\varepsilon_{i_k}(x_{i_1}^{\varepsilon_1} \cdots x_{i_{k-1}}^{\varepsilon_{k-1}} x_{i_k}^{e_k} \otimes x_{i_k}^{e_k} x_{i_{k+1}}^{\varepsilon_{k+1}} \cdots x_{i_r}^{\varepsilon_r} x_{i_k} - x_{i_k} x_{i_1}^{\varepsilon_1} \cdots x_{i_{k-1}}^{\varepsilon_{k-1}} x_{i_k}^{e_k} \otimes x_{i_k}^{e_k} \cdots x_{i_r}^{\varepsilon_r})| \\
&= \sum_{1\le k\le r} |x_{i_1}^{\varepsilon_1} \cdots x_{i_{k-1}}^{\varepsilon_{k-1}} \otimes x_{i_k}^{\varepsilon_k} \cdots x_{i_r}^{\varepsilon_r} - x_{i_1}^{\varepsilon_1} \cdots x_{i_k}^{\varepsilon_k} \otimes x_{i_{k+1}}^{\varepsilon_{k+1}} \cdots x_{i_r}^{\varepsilon_r}| \\
&= |y \otimes 1 - 1 \otimes y|.
\end{aligned}$$

This completes the proof. ∎

Proof of Theorem 27. The left square is commutative by the construction of σ and v. Next, by Lemmas 30 and 31, the divergence of an inner derivation takes its value in $(\mathbb{K}\pi \otimes 1 + 1 \otimes \mathbb{K}\pi)$, so the map $\mathrm{Div}^{\nabla'}$ descends to $\mathrm{HH}^1(\mathbb{K}\pi) \to |\mathbb{K}\pi/\mathbb{K}1|^{\otimes 2}$. By Lemma 29, we have, for $f \in \mathrm{Der}_R(\mathbb{K}F_n)$,

$$\mathrm{Div}^{\nabla'}(\bar{f}) = \sum_{c\in\{a_i,b_i\}_{1\le i\le g}} |\partial_c f(c)| = p(\mathrm{Div}^{\nabla^C}(f))$$

modulo $(\mathbb{K}\pi \otimes 1 + 1 \otimes \mathbb{K}\pi)$. Thus, the right square is also commutative; this completes the proof. ∎

References

[AKKN18] Anton Alekseev, Nariya Kawazumi, Yusuke Kuno, and Florian Naef. The Goldman–Turaev Lie bialgebra and the Kashiwara–Vergne problem in higher genera. 04 2018. arXiv:1804.09566v3.

[Cha04] Moira Chas. Combinatorial Lie bialgebras of curves on surfaces. *Topology*, 43(3):543–568, 2004. arXiv:math/0105178.

[CQ95] Joachim Cuntz and Daniel Quillen. Algebra extensions and nonsingularity. *Journal of the American Mathematical Society*, 8(2):251–289, 1995.

[ES10] Naoya Enomoto and Takao Satoh. New series in the Johnson cokernels of the mapping class groups of surfaces. 12 2010. arXiv:1012.2175.

[Gin05] Victor Ginzburg. Lectures on noncommutative geometry. 2005. arXiv:math/0506603.

[Hat65] Akira Hattori. Rank element of a projective module. *Nagoya Mathematical Journal*, 25:113–120, 1965.

[KK16] Nariya Kawazumi and Yusuke Kuno. The Goldman–Turaev Lie bialgebra and the Johnson homomorphisms. In Athanase Papadopoulos, editor, *Handbook of Teichmüller theory*, volume V, pages 97–165. EMS Publishing House, Zurich, 2016. arXiv:1304.1885.

[LM10] Serge Lazzarini and Thierry Masson. Connections on Lie algebroids and on derivation-based noncommutative geometry. *CPT-P020-2010*, 03 2010. arXiv:1003.6106, doi:10.1016/j.geomphys.2011.11.002.

[Lyn50] Roger C. Lyndon. Cohomology theory of groups with a single defining relation. *Annals of Mathematics*, 52(3):650–665, 1950.

[Mas18] Gwenael Massuyeau. Formal descriptions of Turaev's loop operations. *Quantum Topol.*, 9:39–117, 2018. arXiv:1511.03974.

[MS20] Gwenael Massuyeau and Takuya Sakasai. Morita's trace maps on the group of homology cobordisms. *J. Topol. Anal.*, 12:3:775–818, 2020. arXiv:1606.08244.

[Pow21] Geoffrey Powell. On derivations of free algebras over operads and the generalized divergence. 05 2021. arXiv:2105.09123.

[Sat12] Takao Satoh. On the lower central series of the IA-automorphism group of a free group. *Journal of Pure and Applied Algebra*, 216(3):709–717, 2012. doi:10.1016/j.jpaa.2011.08.006.

[Sta65] John Stallings. Homology and central series of groups. *Journal of Algebra*, 2(2):170–181, 1965. doi:10.1016/0021-8693(65)90017-7.

[Tur91] Vladimir G. Turaev. Skein quantization of Poisson algebras of loops on surfaces. *Annales scientifiques de l'École Normale Supérieure*, Ser. 4, 24(6):635–704, 1991. doi:10.24033/asens.1639.

[Vai07a] Dmitry Vaintrob. A relationship between Hochschild cohomology and the Goldman bracket for compact oriented two-dimensional manifolds. 2007. URL: https://web.mit.edu/rsi/www/pdfs/papers/2006/2006-mitkav.pdf.

[Vai07b] Dmitry Vaintrob. The string topology BV algebra, Hochschild cohomology and the Goldman bracket on surfaces. 2007. arXiv:math/0702859.

[Xu99] Ping Xu. Gerstenhaber algebras and BV-algebras in Poisson geometry. *Communications in Mathematical Physics*, 200(3):545–560, Feb 1999. doi:10.1007/s002200050540.

[YP67] K. Yano and E. M. Patterson. Horizontal lifts from a manifold to its cotangent bundle. *J. Math. Soc. Japan*, 19(2):185–198, 1967.

[Zus19] Pasha Zusmanovich. On Lie p-algebras of cohomological dimension one. *Indagationes Mathematicae*, 30(2):288–299, 2019. doi:10.1016/j.indag.2018.11.005.