DYNAMICS OF CAYLEY FORMS

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ABSTRACT. The most natural first-order PDE's to be imposed on a Cayley 4-form in eight dimensions is the condition that it is closed. As is well-known, this implies integrability of the Spin(7) structure defined by the Cayley form, as well as Ricci-flatness of the associated metric. We address the question as to what the most natural second-order in derivatives set of conditions is. We start at the linearised level, and construct the most general diffeomorphism invariant second order in derivatives Lagrangian that is quadratic in the perturbations of the Cayley form. We find that there is a one-parameter family of such Lagrangians, and that Euler-Lagrange equations following from any generic one are elliptic modulo gauge. We then describe a non-linear completion of the linear story. To this end, we parametrise the intrinsic torsion of a Spin(7) structure by a 3-form, and show that this 3-form is completely determined by the exterior derivative of the Cayley form. We then construct an action functional, which depends on the Cayley 4-form and and auxiliary 3-form as independent variables. There is a unique functional whose Euler-Lagrange equation for the auxiliary 3-form states that it is equal to the torsion 3-form. There is, however, a more general one-parameter family of functionals that can be constructed, and we show how the linearisation of these functionals reproduces the linear story. For any member of our family of theories, the Euler-Lagrange equations are written only using the operator of exterior differentiation of forms, and do not require the knowledge of the metric-compatible Levi-Civita connection. Geometrically, there is a preferred member in the family of Lagrangians, and we propose that its Euler-Lagrange equations are the most natural second-order equations to be satisfied by Cayley forms. Our construction also leads to a natural geometric flow in the space of Cayley forms, defined as the gradient flow of our action functional.

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1. Introduction

A Spin(7) structure on an 8-dimensional manifold is defined to be a 4-form of a special algebraic type. Such a 4-form is known as a Cayley form, and its $GL(8,\mathbb{R})$ stabiliser is Spin(7). An 8-manifold admits a Spin(7) structure if it is spin. However, since most of the considerations in this paper are local, we do not need to concern ourselves with assumptions about M.

As is well-known since [1], a Spin(7) structure is integrable if and only if the associated Cayley 4-form Φ is closed $d\Phi = 0$. This in turn implies that the metric determined by Φ is Ricci-flat. It is clear that $d\Phi = 0$ gives the most geometrically motivated set of first-order PDE's on the Cayley form. In this paper we address the question of what the most natural second-order PDE's are. We describe a certain construction, inspired by the Plebanski formalism [2], see also [3] Chapter 5, for four-dimensional General Relativity. The result of the construction is a unique action functional for Φ, whose Euler-Lagrange equations are a set of second-order PDE's on it. As it will become clear from the construction and the equations it results in, these equations poses some desirable properties. They are elliptic modulo (diffeomorphism) gauge. They are also constructed solely from the operator of the exterior differentiation of forms, so one never needs to know the metric determined by Φ and its associated covariant derivative to write them down.

The main outcome of our construction is the action

$$S[\Phi, C] = \int_M \Phi \wedge (dC - 6C \wedge_\Phi C) + \frac{\lambda}{6} v_\Phi + \text{constraint terms.} \tag{1.1}$$

Here $\Phi \in \Lambda^4(M)$ is a Cayley form, and we have included in the action a set of constraint terms whose purpose is to guarantee that Φ is of the correct algebraic type. An easy comparison between the dimension of the space of 4-forms $\dim(\Lambda^4) = 70$ and the dimension of the orbit $\dim(\mathrm{GL}(8,\mathbb{R})/\mathrm{Spin}(7)) = 43$ shows that there are 27 independent constraints to be satisfied. We will never need to specify these constraints explicitly, as only the variation of these terms with respect to Φ matters for the Euler-Lagrange equations, and this can be determined by a different argument, see below. The object $C \in \Lambda^3(M)$ is what we refer to as the auxiliary 3-form. The Euler-Lagrange equations for C are algebraic, and determine C in terms of the exterior derivative of Φ, see below. After this solution is substituted back into the action, one gets a second order in derivatives action for Φ only. The term λv_Φ is a "cosmological constant" term, with $\lambda \in \mathbb{R}$ being a parameter and v_Φ being the volume form for Φ, which can be taken to be $v_\Phi = (1/14)\Phi \wedge \Phi$. Finally, $C \wedge_\phi C$ is a 4-form constructed from two copies of C, as well as the (inverse) metric determined by Φ. In the index notations that we will be using in this article, it is given by

$$(C \wedge_\Phi C)_{abcd} := C_{abp}C_{cdq}g^{pq}. \tag{1.2}$$

Even though (1.1) is the most natural action for Cayley forms for reasons to become clear below, there are two independent and quadratic in $C \in \Lambda^3$ scalars that can be constructed, as follows from the fact that there are precisely two irreducible representations in the decomposition of Λ^3 into irreducibles of Spin(7), see (2.15) and (2.16). A particular combination of these two independent invariants of C appears in (1.1). We can, however, consider a more general family of Lagrangians given by

$$S_\kappa[\Phi, C] = \int_M \Phi \wedge (dC - 6C \wedge_\Phi C) + \frac{\kappa}{6}(C)^2 v_\Phi + \frac{\lambda}{6} v_\Phi + \text{constraint terms}. \tag{1.3}$$

Our analysis of the linearised theory below will show that there is a one-parameter family of diffeomorphism invariant Lagrangians that are second order in derivatives and quadratic in the perturbations of the Cayley form. We will verify that the linearisation of (1.3) reproduces the one-parameter family of linearised Lagrangians, thus showing that (1.3) gives the non-linear completion of the most general diffeomorphism invariant linear Lagrangian. However, for reasons to be explained now, the $\kappa = 0$ action functional (1.1) is the geometrically preferred one. The argument that fixes this action proceeds through a series of propositions.

The fact that the dimension of the space where the intrinsic torsion of a Spin(7) structure lies equals to the dimension of the space of 3-form is known. However, the paper [4], which was an important precursor to our construction, uses a different parametrisation. The analog of Lemma 2.10 of [4] in our parametrisation is the following statement:

Proposition 1.1. The intrinsic torsion of a Spin(7) structure, measured by $\nabla_i \Phi_{abcd}$, where ∇ is the metric-compatible covariant derivative, lies in $\Lambda^1 \otimes \Lambda^4_7$. For the notations explaining Λ^4_7 and the decomposition of the space of forms into irreducible components see below. The intrinsic torsion can be parametrised by an object $T \in \Lambda^3$ so that

$$\nabla_i \Phi_{abcd} = 4T_{i[a}{}^p \Phi_{|p|bcd]}. \tag{1.4}$$

Here the index p of T_{aip} is raised with the metric determined by Φ.

It turns out that the torsion 3-form is completely determined by the exterior derivative $d\Phi$. This is the content of the following proposition:

Proposition 1.2. The Hodge dual of the projection of (1.4) on the space of 5-forms can be written as

$$\star d\Phi = \frac{2}{5} J_3(T), \tag{1.5}$$

where J_3 is a certain operator $J_3 : \Lambda^3 \to \Lambda^3$ defined by Φ, see (2.13). The operator J_3 is invertible, and so T is completely determined by $d\Phi$.

We now have the proposition linking the action (1.1) and the relation (1.5) between the torsion 3-form and the exterior derivative of the Cayley form:

Proposition 1.3. The Euler-Lagrange equation arising from (1.1) by extemising it with respect to C is $C = T$.

One can rephrase this by saying that (1.1) is precisely the first-order action dependant of both Φ, C that leads to $C = T$ as the C field equation. Importantly, there is no ambiguity in the construction of the action once we demand that $C = T$ is to follow. The Euler-Lagrange equation for C that follows from (1.3) gives instead (6.8). So, the 3-form C that is the critical point of the action (1.3) is not the intrinsic torsion T of the Spin(7) structure, but is only related to it. At present, it appears that the action (1.1) is the unique geometrically preferred action in the more general family (1.3).

The next proposition describes the Euler-Lagrange equations resulting by varying (1.1) with respect to Φ:

Proposition 1.4. The Euler-Lagrange equations resulting from extremisation of (1.1) with respect to Φ can be written as:

$$\partial_{[a}T_{bcd]} - \frac{3}{2}T_{[ab}{}^{p}T_{cd]p} - \frac{1}{8}(TT)_{[a|e|}\Phi^{e}{}_{bcd]} + \frac{\lambda}{84}\Phi_{abcd} = \Psi_{[ab}{}^{pq}\Phi_{|pq|cd]}, \tag{1.6}$$

where Ψ^{abcd} is an arbitrary symmetric tracefree matrix in $\Lambda^2_7 \otimes_S \Lambda^2_7$ and

$$(TT)_{ab} := \Phi^{ijkl}T_{ija}T_{klb} - \frac{1}{7}g_{ab}\Phi^{ijkl}T_{ij}{}^{p}T_{klp} \tag{1.7}$$

is a symmetric matrix quadratic in the torsion. We can also write the Euler-Lagrange equations in form notations as

$$d\Phi - 6T \wedge_\Phi T - \frac{1}{16}K(TT) + \frac{\lambda}{42}\Phi = \Psi(\Phi). \tag{1.8}$$

Here K is the map from the space of symmetric tensors to Λ^4 described in (2.36), and $\Psi(\Phi)$ is a general 4-form in Λ^4_{27}.

Proposition 1.5. An alternative way of writing the field equations is to project both sides on the Λ^4_{1+7+35} component in Λ^4. This gives the following set of equations

$$\begin{aligned}\frac{1}{4}\Phi_b{}^{pqr}\nabla_a T_{pqr} - \frac{3}{4}\Phi_b{}^{pqr}\nabla_r T_{apq} - \frac{3}{2}\Phi_b{}^{pqr}T_{ap}{}^{s}T_{qrs} - \frac{3}{4}\Phi^{pqrs}T_{apq}T_{brs} \\ +\frac{1}{2}g_{ab}\left(\lambda + \frac{3}{8}\Phi^{pqrs}T_{pq}{}^{p}T_{rsp}\right) = 0.\end{aligned} \tag{1.9}$$

The equations here are written in terms of ∇, but they have the same form with ∇ replaced by the partial derivative operator ∂. We note that the left-hand side is not automatically ab symmetric, and the anti-symmetric part of these equations are non-trivial. The anti-symmetric part can be shown to lie in Λ^2_7, and so the total number of independent second-order differential equations is $36 + 7 = 43$, the dimension of the space of Cayley forms.

We can also characterise what the field equations imply for the Riemann curvature of the metric defined by Φ. This is done in the main text. We will see that the metrics defined by Φ that are the critical points of our action functional are not in general Einstein.

There is another Lagrangian in the family (1.3), namely one corresponding to $\kappa = -2$, which is special. As a computation shows, the linearised Lagrangian in this case is just that for a metric perturbation. The other field that parametrises the perturbation of Φ, namely one living in Λ^2_7, does not receive any kinetic terms at this value of κ. One can rephrase this by saying that the linearisation of the $\kappa = -2$ Lagrangian is the same as the linearisation of the Einstein-Hilbert Lagrangian for the metric. One could then be led to believe that the non-linear theory for $\kappa = -2$ is just that describing Einstein metrics. This is, however, not the case, as is confirmed by calculations. There are very interesting differences only visible at the non-linear level, still to be better understood.

The set of second-order PDE's (1.6) is the main result of our construction. We propose these equations as the most natural set of second-order PDE's for a Cayley form to satisfy. There is also a natural geometric flow in the space of Cayley forms that our construction defines.

The gradient of the action functional (1.1) with respect to Φ, with $C = T$, is given by the Hodge dual of the 4-form on the left-hand-side of (1.8), projected to the space Λ^4_{1+7+35}. Our linearised analysis shows that this geometric flow is elliptic modulo gauge, and thus by standard arguments has good short time existence properties.

More work is needed to get better intuition about the properties of both the geometrically preferred $\kappa = 0$ as well as $\kappa = -2$ equations. We hope that this work will follow. It is worth remarking already now, however, that construction of a similar type to that described in this paper is possible also for other G-structures, in various dimensions. It would be particularly interesting to perform a similar analysis and construct actions for 3-forms in 7-dimensions, building on the work [5].

Many of the tensor computations in the paper are performed using symbolic manipulation with xAct Mathematica package [6]. The stated representation theoretic facts are obtained using the Mathematical package LieART [7].

2. Decomposition of the spaces of forms

2.1. **Basic algebra.** Similar to [8] and [4], we use the index notation, which is very useful for encoding various relations satisfied by the Cayley form. The basic algebraic relation satisfied by the 4-form Φ_{abcd} is

$$\begin{aligned}\Phi_{ijkp}\Phi_{abcp} = g_{ia}g_{jb}g_{kc} + g_{ib}g_{jc}g_{ka} + g_{ic}g_{ja}g_{kb} - g_{ia}g_{jc}g_{kb} - g_{ic}g_{jb}g_{ka} - g_{ib}g_{ja}g_{kc} \\ -g_{ia}\Phi_{jkbc} - g_{ja}\Phi_{kibc} - g_{ka}\Phi_{ijbc} \\ -g_{ib}\Phi_{jkca} - g_{jb}\Phi_{kica} - g_{kb}\Phi_{ijca} \\ -g_{ic}\Phi_{jkab} - g_{jc}\Phi_{kiab} - g_{kc}\Phi_{ijab}.\end{aligned} \tag{2.1}$$

One more contraction of this gives

$$\Phi_{ijpq}\Phi_{abpq} = 6g_{ia}g_{jb} - 6g_{ib}g_{ja} - 4\Phi_{ijab}. \tag{2.2}$$

Yet one more contraction gives

$$\Phi_{ipqr}\Phi_{apqr} = 42g_{ia}. \tag{2.3}$$

The 4-form Φ is self-dual

$$\frac{1}{4!}\epsilon_{ijkl}{}^{abcd}\Phi_{abcd} = \Phi_{ijkl}. \tag{2.4}$$

Useful consequences of self-duality are

$$\epsilon^{aijklpqr}\Phi_{bpqr} = 30\delta_b^{[a}\Phi^{ijkl]}, \tag{2.5}$$

and

$$\epsilon^{ijklmnpq}\Phi_{abpq} = 60\delta_a^{[i}\delta_b^{j}\Phi^{klmn]}, \tag{2.6}$$

and

$$\epsilon^{ijklmnpr}\Phi_{abcr} = 210\delta_a^{[i}\delta_b^{j}\delta_c^{k}\Phi^{lmnp]}, \tag{2.7}$$

2.2. **Identity.** The following non-trivial identity

$$-2\Phi_{[ijk}{}^{[a}\Phi_{l]}{}^{bcd]} - 3\Phi_{[ij}{}^{[ab}\Phi_{kl]}{}^{cd]} + 42\Phi_{[ij}{}^{[ab}\delta_k^c\delta_{l]}^{d]} + \Phi_{ijkl}\Phi^{abcd} = 0 \tag{2.8}$$

can be checked by multiplying with δ_a^i and using the algebra (2.1) to check that the result is zero.

2.3. **Decomposition of Λ^2.** We introduce the following operator on 2-forms

$$J_2(\beta)_{ij} = \frac{1}{2}\Phi_{ij}{}^{ab}\beta_{ab}. \tag{2.9}$$

Using (2.2) we see that

$$(J_2)^2 = 3\mathbb{I} - 2J_2. \tag{2.10}$$

This means that the eigenvalues of J_2 are $-3, 1$. The eigenspace of eigenvalue -3 is Λ^2_7, and eigenspace of eigenvalue 1 is Λ^2_{21}. The two projectors are

$$\pi_7 = \frac{1}{4}(\mathbb{I} - J_2), \qquad \pi_{21} = \frac{1}{4}(3\mathbb{I} + J_2). \tag{2.11}$$

For later purposes we note that

$$J_2^{-1} = \frac{1}{3}(2\mathbb{I} + J_2). \tag{2.12}$$

2.4. **Decomposition of Λ^3.** We introduce the following operator on 3-forms

$$J_3(\gamma)_{ijk} = \frac{1}{2}(\Phi_{ij}{}^{pq}\gamma_{kpq} + \Phi_{jk}{}^{pq}\gamma_{ipq} + \Phi_{ki}{}^{pq}\gamma_{jpq}) = \frac{3}{2}\Phi_{[ij}{}^{pq}\gamma_{k]pq} \tag{2.13}$$

A calculation using (2.1) gives

$$(J_3)^2 = 6\mathbb{I} - 5J_3. \tag{2.14}$$

This means that the eigenvalues of J_3 are $-6, 1$. The eigenspace of eigenvalue -6 is Λ^3_8, and eigenspace of eigenvalue 1 is Λ^3_{48}. The elements of the space Λ^3_8 are of the form

$$\Lambda^3_8 = \{X^p\Phi_{pijk}, X \in TM\}, \tag{2.15}$$

and

$$\Lambda^3_{48} = \{\gamma \in \Lambda^3 : \gamma \wedge \Phi = 0\}. \tag{2.16}$$

We note that

$$\pi_{48} = \frac{6}{7}\left(\mathbb{I} + \frac{1}{6}J_3\right), \qquad \pi_8 = \frac{1}{7}\left(\mathbb{I} - J_3\right). \tag{2.17}$$

We also note that

$$J_3^{-1} = \frac{1}{6}(J_3 + 5\mathbb{I}). \tag{2.18}$$

2.5. **Decomposition of Λ^4.** We introduce the following operator on 4-forms

$$J_4(\sigma)_{ijkl} = 3\Phi_{[ij}{}^{pq}\sigma_{kl]pq} = \tag{2.19}$$
$$\frac{1}{2}(\Phi_{ij}{}^{pq}\sigma_{pqkl} + \Phi_{ki}{}^{pq}\sigma_{pqjl} + \Phi_{il}{}^{pq}\sigma_{pqjk} + \Phi_{kl}{}^{pq}\sigma_{pqij} + \Phi_{jl}{}^{pq}\sigma_{pqki} + \Phi_{jk}{}^{pq}\sigma_{pqil}).$$

We have the following relation, also to be found in [4]

$$(J_4)^2(\sigma)_{ijkl} = \frac{1}{2}(\Phi_{ij}{}^{ab}\Phi_{kl}{}^{cd} + \Phi_{ki}{}^{ab}\Phi_{jl}{}^{cd} + \Phi_{il}{}^{ab}\Phi_{jk}{}^{cd})\sigma_{abcd} + 6\sigma_{ijkl} - 8J_4(\sigma)_{ijkl} = \tag{2.20}$$
$$\frac{3}{2}\Phi_{[ij}{}^{ab}\Phi_{kl]}{}^{cd}\sigma_{abcd} - 24\Phi_{[ij}{}^{ab}\sigma_{kl]ab} + 6\sigma_{ijkl}.$$

We also have the following result for the cube of this operator

$$(J_4)^3(\sigma)_{ijkl} = -6\Phi^a_{[ijk}\Phi^{bcd}_{l]}\sigma_{abcd} - 15\Phi_{[ij}{}^{ab}\Phi_{kl]}{}^{cd}\sigma_{abcd} + 258\Phi_{[ij}{}^{ab}\sigma_{kl]ab} - 24\sigma_{ijkl}. \tag{2.21}$$

Using the identity (2.8) we can rewrite this as

$$(J_4)^3(\sigma)_{ijkl} = -6\Phi_{[ij}{}^{ab}\Phi_{kl]}{}^{cd}\sigma_{abcd} + 132\Phi_{[ij}{}^{ab}\sigma_{kl]ab} - 3\Phi_{ijkl}\Phi^{abcd}\sigma_{abcd} - 24\sigma_{ijkl}. \tag{2.22}$$

Finally, for the fourth power of this operator we have

$$(J_4)^4(\sigma)_{ijkl} = 87\Phi^a_{[ijk}\Phi^{bcd}_{l]}\sigma_{abcd} + \frac{345}{2}\Phi_{[ij}{}^{ab}\Phi_{kl]}{}^{cd}\sigma_{abcd} - 2643\Phi_{[ij}{}^{ab}\sigma_{kl]ab} + \frac{9}{2}\Phi_{ijkl}\Phi^{abcd}\sigma_{abcd} + 168\sigma_{ijkl}. \tag{2.23}$$

Using (2.8) we can rewrite this as

$$(J_4)^4(\sigma)_{ijkl} = 42\Phi_{[ij}{}^{ab}\Phi_{kl]}{}^{cd}\sigma_{abcd} - 816\Phi_{[ij}{}^{ab}\sigma_{kl]ab} + 48\Phi_{ijkl}\Phi^{abcd}\sigma_{abcd} + 168\sigma_{ijkl}. \tag{2.24}$$

This shows that

$$(J_4)^4 + 16(J_4)^3 + 36(J_4)^2 - 144J_4 = 0, \tag{2.25}$$

or in other words

$$(J_4 + 12\mathbb{I})(J_4 + 6\mathbb{I})(J_4 - 2\mathbb{I})J_4 = 0. \tag{2.26}$$

This shows that the operator J_4 has eigenvalues $-12, -6, 2, 0$, see also [4]. The eigenspaces are the irreducible parts of the space of 4-forms

$$\Lambda^4_1 = \{\sigma \in \Lambda^4 : J_4(\sigma) = -12\sigma\}, \qquad \Lambda^4_{27} = \{\sigma \in \Lambda^4 : J_4(\sigma) = 2\sigma\},$$
$$\Lambda^4_7 = \{\sigma \in \Lambda^4 : J_4(\sigma) = -6\sigma\}, \qquad \Lambda^4_{35} = \{\sigma \in \Lambda^4 : J_4(\sigma) = 0\}. \tag{2.27}$$

This characterisation will follow after we characterise each of the irreducible parts below.

2.6. **Projector on Λ^4_{27}.** In what follows it will be useful to have the projector on Λ^4_{27} explicitly. It is clear that it is a multiple of $J_4(J_4 + 12\mathbb{I})(J_4 + 6\mathbb{I})$. Taking into account the eigenvalues of J_4 on different subspaces, it is not difficult to check that the required multiple is $1/224$. Thus, we have

$$\pi_{27} = \frac{1}{224}J_4(J_4 + 12\mathbb{I})(J_4 + 6\mathbb{I}) = \frac{1}{224}((J_4)^3 + 18(J_4)^2 + 72J_4). \tag{2.28}$$

Using (2.22) and (2.20) we get

$$\pi_{27}(\sigma)_{ijkl} = \frac{3}{32}\left(\Phi_{[ij}{}^{ab}\Phi_{kl]}{}^{cd}\sigma_{abcd} - 4\Phi_{[ij}{}^{ab}\sigma_{kl]ab} - \frac{1}{7}\Phi_{ijkl}\Phi^{abcd}\sigma_{abcd} + 4\sigma_{ijkl}\right). \tag{2.29}$$

We have explicitly checked that this projector kills 4-forms of the form

$$H_{[i}{}^p\Phi_{jkl]p}, \qquad H \in \Lambda^1 \otimes \Lambda^1, \tag{2.30}$$

which lie in Λ^4_{1+35+7}. This characterisation of Λ^4_{1+35+7} is subject of the next two subsections.

For later purposes, we note that $\mathbb{I} - \pi_{27}$ projects out the Λ^4_{27} component of any 4-form, and is given by

$$\mathbb{I} - \pi_{27} = \frac{1}{32}\left(20\sigma_{ijkl} - 3\Phi_{[ij}{}^{ab}\Phi_{kl]}{}^{cd}\sigma_{abcd} + 12\Phi_{[ij}{}^{ab}\sigma_{kl]ab} + \frac{3}{7}\Phi_{ijkl}\Phi^{abcd}\sigma_{abcd}\right). \tag{2.31}$$

We also note that π_{27} can be understood in a simple way. Indeed, taking a 4-form $\sigma_{ijkl} \in \Lambda^4$, we can interpret this as an object in $\Lambda^2 \otimes_S \Lambda^2$, and apply the projector π_7 on the indices ij and on the indices kl. After this the result can be projected back to Λ^4 by antisymmetrising the indices. The result of this operation is

$$\left((\pi_7\sigma\pi_7)\Big|_{\Lambda^4}\right)_{ijkl} = \frac{1}{64}\left(\Phi_{[ij}{}^{ab}\Phi_{kl]}{}^{cd}\sigma_{abcd} - 4\Phi_{[ij}{}^{ab}\sigma_{kl]ab} + 4\sigma_{ijkl}\right). \tag{2.32}$$

This contains almost all the terms in $\pi_{27}(\sigma)$. The only term present in (2.29) and absent in $(\pi_7\sigma\pi_7)\Big|_{\Lambda^4}$ is the third term in (2.29), whose purpose is to make the result tracefree. So, we can write

$$\pi_{27}(\sigma) = 6(\pi_7\sigma\pi_7)\Big|_{\Lambda^4} - \text{trace}, \tag{2.33}$$

where the last term just removes the trace of the first. This gives a simple and useful interpretation of the projector π_{27}.

2.7. **An operator from Λ^2 to Λ^4_7 and its inverse.** Let us introduce the following operator

$$\Lambda^2 \ni \beta_{ij} \to K(\beta)_{ijkl} = 4\beta_{[i|p|}\Phi^p{}_{jkl]} \in \Lambda^4. \tag{2.34}$$

It can now be checked that

$$K \circ \pi_{21} = 0, \qquad K \circ \pi_7 = K. \tag{2.35}$$

This shows that the image of K is Λ^4_7. This makes sense, because the image $K(\beta) \in \Lambda^4$ is precisely the orbit of the basic 4-form Φ under the action of the Lie algebra $\mathfrak{spin}(8)$. The statement that $K \circ \pi_{21} = 0$ is just the statement that Φ is Spin(7) invariant.

The operator K to Λ^4 can be generalised and applied to a general tensor from $\Lambda^1 \otimes \Lambda^1$. In particular, it can be applied to a symmetric tensor $h_{ij} \in S^2\Lambda^1$

$$S^2\Lambda^1 \ni h_{ij} \to K(h)_{ijkl} = 4h_{[i|p|}\Phi^p{}_{jkl]} \in \Lambda^4. \tag{2.36}$$

The operator K is injective on its image, and the image of $S^2\Lambda^1$ in Λ^4 is $\Lambda^4_1 \oplus \Lambda^4_{35}$.

To find the inverse of K on Λ^4_7 let us consider

$$\Lambda^4 \ni \sigma_{ijkl} \to K'(\sigma)_{ij} = \frac{1}{2}\Phi_i{}^{pqr}\sigma_{jpqr} - \frac{1}{2}\Phi_j{}^{pqr}\sigma_{ipqr} \in \Lambda^2. \tag{2.37}$$

We then have

$$\pi_{21} \circ K' = 0, \qquad K' \circ K = 96\pi_7. \tag{2.38}$$

This means that K' is (a multiple of) the inverse of K on Λ^4_7.

2.8. **Characterisation of Λ^4_{1+35+7}.** We can apply the map K to a general element $H_{ij} \in \Lambda^1 \otimes \Lambda^1$

$$K(H)_{ijkl} := 4H^p_{[i}\Phi_{jkl]p}. \tag{2.39}$$

We already know that $\pi_{27}(K(H)) = 0$, and so the result of this map lies in Λ^4_{1+35+7}. We also know that the map K applied to the symmetric part of H lies in Λ^4_{1+35}, and to the anti-symmetric part in Λ^4_7. A computation gives the following result

$$J_4(K(H))_{ijkl} = -3(H_{[i}{}^p - H^p{}_{[i})\Phi_{ijk]p} - 6\Phi_{ijkl}H_p{}^p. \tag{2.40}$$

This shows that when H is symmetric tracefree $H_{ij} = H_{(ij)}, H_p{}^p = 0$ we have $J_4(K(H)) = 0$. This shows that Λ^4_{35} is the eigenspace of J_4 of eigenvalue 0. When H_{ij} is anti-symmetric, we have $J_4(K(H)) = -6K(H)$, and so Λ^4_7 is eigenspace of eigenvalue -6. When $H_{ij} = g_{ij}$, we have $J_4(K(H)) = -12K(H)$, and thus Λ^4_1 is eigenspace of eigenvalue -12. This gives the characterisation described above in (2.27).

2.9. **Characterisation of Λ^4_{27}.** It will be useful to have an explicit parametrisation of a general element of Λ^4_{27}, similar to have we already have a parametrisation of a general element of the other irreducible subspaces Λ^4_{1+35+7}. For this purpose, let us take a symmetric tracefree matrix in $\Lambda^2_7 \otimes_S \Lambda^2_7$

$$\Psi^{abcd} \in \Lambda^2_7 \otimes_S \Lambda^2_7. \tag{2.41}$$

This matrix must satisfy a number of requirements:

$$\Psi^{abcd} = \Psi^{[ab][cd]} = \Psi^{[cd][ab]}, \qquad \Psi_{ab}{}^{ab} = 0, \qquad \pi_7\Psi = \Psi = \Psi\pi_7. \tag{2.42}$$

We can then construct a 4-form that we denote as $\Psi\Phi$ as

$$(\Psi\Phi)_{abcd} := \Psi_{[ab}{}^{pq}\Phi_{cd]pq}. \tag{2.43}$$

Let us determine the projection of this to Λ^4_{1+35+7}. A calculation gives

$$(\Psi\Phi)_{ipqr}\Phi^{apqr} = \delta^a_i\left(\Psi_{qr}{}^{qp} - \frac{1}{2}\Psi_{pqrs}\Phi^{pqrs}\right) + 4\Psi_{ip}{}^{ap} + \Phi_{ipqr}\Psi^{apqr} - \Psi_{ipqr}\Phi^{apqr}. \tag{2.44}$$

The first term here is zero because Ψ is tracefree $\Psi_{qr}{}^{qp} = 0$, and also $\pi_7\Psi = \Psi$ means $\pi_{21}\Psi = 0$, which implies

$$3\Psi_{ijkl} + \frac{1}{2}\Phi_{ij}{}^{pq}\Psi_{pqkl} = 0. \tag{2.45}$$

So, if $\Psi_{qr}{}^{qp} = 0$ then also $\Psi_{pqrs}\Phi^{pqrs} = 0$. On the other hand, if we contract jl in this expression we get

$$\Psi_k{}^{pqr}\Phi_{ipqr} = -6\Psi_{ipk}{}^p. \tag{2.46}$$

The right-hand side is ik symmetric, and thus the left-hand side must also be ik symmetric. This shows that the last two terms in (2.44) cancel. It remains to characterise $\Psi_{ip}{}^{ap}$. To do this, we compute $0 = \pi_{21}\Psi\pi_{21}$

$$0 = 3\Psi_{ijkl} + \frac{3}{2}\Phi_{ij}{}^{pq}\Psi_{pqkl} + \frac{3}{2}\Psi_{ij}{}^{pq}\Phi_{pqkl} + \frac{1}{4}\Phi_{ij}{}^{pq}\Phi_{kl}{}^{rs}\Psi_{pqrs}. \tag{2.47}$$

Taking the jl contraction of this we get

$$0 = 8\Psi_{ipk}{}^p + 2\Psi_k{}^{pqr}\Phi_{ipqr} + 2\Psi_i{}^{pqr}\Phi_{kpqr} + \frac{1}{2}g_{ik}\left(\Psi_{pq}{}^{pq} - \frac{1}{2}\Psi_{pqrs}\Phi^{pqrs}\right). \tag{2.48}$$

Using $\Psi_{qr}{}^{qp} = 0, \Psi_{pqrs}\Phi^{pqrs} = 0$ as well as (2.46) here we see that $\Psi_{ipk}{}^p = 0$. All in all, all the terms in (2.44) are zero and the object (2.43) is in Λ^4_{27}. This gives the desired parametrisation of a general element of Λ^4_{27}.

3. Linearised theory

In this section we address the question as to what is the most general diffeomorphism invariant action that can be constructed for the fields living in the representations Λ^4_{1+7+35} of the group Spin(7). Our analysis in this section is at the linearised level, where we can use representation theory to write down all possible action terms with arbitrary coefficients, and then use diffeomorphism invariance to relate the coefficients. We will see that the action is not unique, unlike in the case of just metrics. There are two independent possible linearised actions that can be constructed. Non-linear completion of the theories described here is the subject of the following sections.

3.1. **The usual metric only case.** This story is standard, and works in exactly the same way in any dimension. We review it for completeness, and for establishing the main idea of the calculation to follow in the Spin(7) case. For concreteness, we do calculations in dimension eight, but the story repeats itself with no changes in any dimension.

In the usual gravity case we have linearised fields transforming with respect to the Lorentz group SO(8). The metric perturbation contains two irreducible representations $\mathbf{1}, \mathbf{35}_v$. The subscript v stands for "vector", to distinguish them from also possible spinor representations. This is a standard notation at least in some literature. As is also standard, we refer to the irreducible representations by their dimensions written in bold face. Let us denote the fields in representations $\mathbf{1}, \mathbf{35}_v$ by $h, \tilde{h}_{ab}$ respectively. We are interested in an action that contains two derivatives. It will be useful to think in terms of Fourier transform, and denote the derivative by its Fourier transform p^a at intermediate stages of the computation. The two most obvious action terms one can construct are of the type $p^2h^2, p^2(\tilde{h}_{ab})^2$. To analyse the other possible terms we need to decompose the product of two derivatives into irreducibles, taking into account that they commute. We have

$$\mathbf{8}_v \otimes_S \mathbf{8}_v = \mathbf{1} + \mathbf{35}_v. \tag{3.1}$$

The trivial representation here corresponds to p^2, and the other representation is p_ap_b with the trace removed. The p^2 terms were already taken into account, so we only need to consider the possible couplings between p_ap_b and the two other factors of either h or $\tilde{h}$. There is no term with p_ap_b and two factors of h. There is clearly a mixed term $hp^ap^b\tilde{h}_{ab}$. To determine possible terms with two factors of $\tilde{h}$ we need

$$\mathbf{35}_v \otimes_S \mathbf{35}_v = \mathbf{1} + \mathbf{35}_v + \mathbf{294}_v + \mathbf{300}. \tag{3.2}$$

There is only a single occurrence of $\mathbf{35}_v$ here, which means that there is only a single term that does not reduce to $p^2(\tilde{h}_{ab})^2$, and this is $(p^a\tilde{h}_{ab})^2$.

All in all, there are just 4 possible terms in the action that one can write. We now go back to the notation that uses the derivative operators, and write the linear combination of the above four terms with arbitrary coefficients

$$\mathcal{L} = \frac{1}{2}\tilde{h}^{bc}\partial^a\partial_a\tilde{h}_{bc} + \frac{\alpha}{2}h\partial^a\partial_a h + \beta h\partial^a\partial^b\tilde{h}_{ab} + \gamma(\partial^a\tilde{h}_{ab})^2. \tag{3.3}$$

We have chosen the coefficient in front of the first term to be $1/2$, which we can always do by changing an overall coefficient in front of the Lagrangian $\tilde{h}_{ab}$. At this stage it will be more convenient to introduce the fields

$$h_{ab} := \tilde{h}_{ab} + \frac{1}{D}\eta_{ab}h, \tag{3.4}$$

so that $\tilde{h}_{ab}$ is the tracefree part of h_{ab} and $h = \eta^{ab}h_{ab}$, with D being the dimension we work in. It is clear that the Lagrangian retains the same general form, except that the coefficients change. We will give the new coefficients the same name, hoping it will not lead to any confusion. The Lagrangian in terms of h_{ab} is

$$\mathcal{L} = \frac{1}{2}(\partial_a h_{bc})^2 + \frac{\alpha}{2}(\partial_a h)^2 - \beta h\partial^a\partial^b h_{ab} - \gamma(\partial^a h_{ab})^2. \tag{3.5}$$

We now demand diffeomorphism invariance of the action, with the field transformation properties being

$$h_{ab} = \partial_{(a}\xi_{b)}, \qquad \delta h = \partial^c\xi_c. \tag{3.6}$$

We now the variation, and set coefficients in front of independent terms to zero, allowing integration by parts. This results in the following set of coefficients

$$\gamma = \beta = 1, \quad \alpha = -1. \tag{3.7}$$

Thus, the unique (modulo field rescaling) Lagrangian that is diffeomorphism invariant reads

$$\mathcal{L}_{GR} = \frac{1}{2}(\partial_a h_{bc})^2 - \frac{1}{2}(\partial_a h)^2 - h\partial^a\partial^b h_{ab} - (\partial^a h_{ab})^2, \tag{3.8}$$

which is the standard result.

3.2. **Gauge-fixing.** Let us also derive the standard gauge-fixed form of the Lagrangian. Completing the square in the $(\partial^a h_{ab})^2$ part, we can rewrite the Lagrangian as

$$\mathcal{L}_{GR} = \frac{1}{2}(\partial_a h_{bc})^2 - \frac{1}{4}(\partial_a h)^2 - (\partial^a(h_{ab} - \frac{1}{2}\eta_{ab}h))^2. \tag{3.9}$$

If we gauge-fix the diffeomorphisms by setting

$$\partial^a(h_{ab} - \frac{1}{2}\eta_{ab}h) = 0, \tag{3.10}$$

we get a simple linear combination of the terms containing $\partial^a\partial_a$ only. This means that Euler-Lagrange equations following from this Lagrangian are elliptic modulo gauge, a desirable property.

3.3. **The case of** Spin(7) **structures.** Let us now consider 3 fields in irreducible representations of Spin(7) given by $\mathbf{1}, \mathbf{7}, \mathbf{35}$. These are precisely the representations appearing in a tangent vector to the orbit of Cayley forms. We will refer to these fields as $h, \xi, \tilde{h}$ respectively. The decomposition into irreducibles is now dictated by the Spin(7) representation theory. There are again terms involving p^2, which are $p^2h^2, p^2\xi^2, p^2\tilde{h}^2$. To determine other possible terms we need to consider the (symmetric) product of two derivatives. We have

$$\mathbf{8} \otimes_S \mathbf{8} = \mathbf{1} + \mathbf{35}, \tag{3.11}$$

which is unchanged from the Spin(8) case. The trivial representation here corresponds to p^2, and so we only need to consider the **35** representation. This must couple to the product of two fields from the list $h, \xi, \tilde{h}$. The non-trivial such decompositions are

$$\begin{aligned} \mathbf{7} \otimes_S \mathbf{7} &= \mathbf{1} + \mathbf{27}, \\ \mathbf{7} \otimes \mathbf{35} &= \mathbf{21} + \mathbf{35} + \mathbf{189}, \\ \mathbf{35} \otimes_S \mathbf{35} &= \mathbf{1} + \mathbf{27} + \mathbf{35} + \mathbf{105} + \mathbf{168} + \mathbf{294}. \end{aligned} \tag{3.12}$$

This means that, in addition to the usual terms $hp^a p^b \tilde{h}_{ab}, (p^a \tilde{h}_{ab})^2$, there is a new term of the type $p^a p^b \xi \tilde{h}$. It is easy to write down this term by noting that the representation **7** appears in the anti-symmetric part of the tensor product

$$\mathbf{35} \otimes_A \mathbf{35} = \mathbf{7} + \mathbf{21} + \mathbf{35} + \mathbf{189} + \mathbf{378}. \tag{3.13}$$

We already know that the best way to describe a field in representation **7** is by using a field in Λ^2_7. Thus, let us introduce an object

$$\xi_{ab} \in \Lambda^2_7. \tag{3.14}$$

We can then construct the term coupling $\xi_{ab}, \tilde{h}_{ab}$ as $p_a p^c h_{cb} \xi^{ab}$.

It is now clear that there are just two additional terms that can be constructed from ξ_{ab}, which can be written as $\xi^{ab} \partial^c \partial_c \xi_{ab}$ and $\partial_b \tilde{h}^{ba} \partial^c \xi_{ca}$. Note that we can also write the second term as $\partial_b h^{ba} \partial^c \xi_{ca}$, because the trace part of h_{ab} does not couple to ξ_{ab}. As before, we now write a general linear combination of all the possible terms, with arbitrary coefficients:

$$\begin{aligned} \mathcal{L} = \frac{\rho}{2}(\partial_a h_{bc})^2 + \frac{\alpha}{2}(\partial_a h)^2 - \beta h \partial^a \partial^b h_{ab} - \gamma(\partial^a h_{ab})^2 \\ + \frac{\lambda}{2}(\partial_a \xi_{bc})^2 - \mu \partial_b h^{ba} \partial^c \xi_{ca}. \end{aligned} \tag{3.15}$$

For reasons that will become clear later it will be convenient to put an arbitrary coefficient ρ also in front of the first term.

3.4. **Some identities.** Let us now explain why the term $(\partial^a \xi_{ab})^2$ is not added to the Lagrangian (3.15). The representation theory tells us that there is no representation **35** in the decomposition $\mathbf{7} \otimes_S \mathbf{7}$, and so this term must be a multiple of $\xi^{bc} \partial^a \partial_a \xi_{bc}$. Let us confirm that. Using the fact that $\xi \in \Lambda^2_7$ we have

$$\xi_{ab} = -\frac{1}{6} \Phi_{ab}{}^{pq} \xi_{pq}, \tag{3.16}$$

and so

$$\xi_b{}^a \xi_{ca} = \frac{1}{36}\left(-28 \xi_b{}^a \xi_{ca} + 8 g_{bc} (\xi_{ab})^2\right). \tag{3.17}$$

From this we get

$$\xi_b{}^a \xi_{ca} = \frac{1}{8} g_{bc} (\xi_{ab})^2. \tag{3.18}$$

This explains why the term $(\partial^a \xi_{ab})^2$ is already contained in the $\xi^{bc} \partial^a \partial_a \xi_{bc}$ term in the Lagrangian (3.15) and does not need to be added as a separate term.

3.5. **Elliptic modulo gauge.** We note that the general action is elliptic modulo gauge for any choice of the parameters. Indeed, for $\gamma \neq 0$ we can rewrite it as

$$\begin{aligned} \mathcal{L} = \frac{\rho}{2}(\partial_a h_{bc})^2 + \left(\frac{\alpha}{2} + \frac{\beta^2}{4\gamma}\right)(\partial_a h)^2 + \left(\frac{\lambda}{2} + \frac{\mu^2}{32\gamma}\right)(\partial_a \xi_{bc})^2 \\ -\gamma(\partial^a (h_{ab} - \frac{\beta}{2\gamma} \eta_{ab} h + \frac{\mu}{2\gamma} \xi_{ab}))^2. \end{aligned} \tag{3.19}$$

This means that there is a gauge in which the action is given by terms only involving the Laplacian.

3.6. **The transformation properties under diffeomoprhisms.** To determine the diffeomoprhism transformation rules for all the fields we recall that h_{ab} and ξ_{ab} appear from a certain projection of the perturbation of the 4-form. If we call this perturbation $\phi \in \Lambda^4$, the fact that this 4-form is a tangent vector to the orbit of Cayley 4-forms means that $\phi \in \Lambda^4_{1+7+35}$. Let us define the fields h_{ab}, ξ_{ab} as

$$\tilde{h}_{ab} = \frac{1}{24}(\phi_{(a}{}^{pqr}\Phi_{b)pqr} - \frac{1}{8}\eta_{ab}\phi^{pqrs}\Phi_{pqrs}), \quad \xi_{ab} = \frac{1}{24}\phi_{[a}{}^{pqr}\Phi_{b]pqr}, \quad h = \frac{1}{168}\phi^{pqrs}\Phi_{pqrs}. \tag{3.20}$$

A calculation shows that the inverse of this map is the following parametrisation of ϕ_{abcd}

$$\phi_{abcd} = -4(h_{[a}{}^p + \frac{1}{4}\xi_{[a}{}^p)\Phi_{bcd]p}, \tag{3.21}$$

and this formula explains the choice of prefactors in (3.20). We note that

$$\frac{1}{96}\phi^{abcd}\phi_{abcd} = h^{ab}h_{ab} + \frac{3}{4}h^2 + \frac{1}{4}\xi^{ab}\xi_{ab}, \tag{3.22}$$

where it is used that $\xi_{ab} \in \Lambda^2_7$.

Under diffeomoprhisms

$$\delta\phi = i_\xi d\Phi + d i_\xi \Phi. \tag{3.23}$$

We assume that the background 4-form is closed (in fact constant), so that there is only the second term. Then

$$\delta\phi_{abcd} = -4\partial_{[a}\xi^p\Phi_{bcd]p}, \tag{3.24}$$

and so $(1/4)\delta\xi_{ab} = \pi_7(\partial_{[a}\xi_{b]})$ giving

$$\delta h_{ab} = \partial_{(a}\xi_{b)}, \qquad \delta\xi_{ab} = \partial_{[a}\xi_{b]} - \frac{1}{2}\Phi_{ab}{}^{pq}\partial_p\xi_q. \tag{3.25}$$

3.7. **Determining the diffeomorphism invariant Lagrangian.** The variation of the Lagrangian (3.15), modulo surface terms, is given by

$$\begin{aligned}\delta\mathcal{L} = (\rho - \gamma - \frac{\mu}{2})\partial^a h_{ab}\partial^2\xi^b + (-\beta + \gamma - \frac{\mu}{2})\partial^a\partial^b h_{ab}(\partial\xi) \\ -(\alpha + \beta)\partial^2 h(\partial\xi) + (4\lambda - \frac{\mu}{2})\partial^a\xi_{ab}\partial^2\xi^b.\end{aligned} \tag{3.26}$$

Here $\partial^2 = \partial^a\partial_a$ and $(\partial\xi) = \partial^a\xi_a$. We have used the fact that ξ_{ab} is in Λ^2_7, and so $(1/2)\Phi_{ab}{}^{pq}\xi_{pq} = -3\xi_{ab}$. Setting to zero the coefficients in front of the independent parts we get a system of equations. The solution depends on two of the parameters, for which we can take ρ, μ. Then

$$\alpha = -\rho + \mu, \quad \beta = \rho - \mu, \quad \gamma = \rho - \frac{\mu}{2}, \quad \lambda = \frac{\mu}{8}. \tag{3.27}$$

It is clear that the resulting diffeomorphism invariant Lagrangian is the sum of two separately invariant terms

$$\mathcal{L} = \rho\mathcal{L}_{GR} + \mu\mathcal{L}', \tag{3.28}$$

where

$$\mathcal{L}' = \frac{1}{2}(\partial_a h)^2 + h\partial^a\partial^b h_{ab} + \frac{1}{2}(\partial^a h_{ab})^2 + \frac{1}{16}(\partial_c\xi_{ab})^2 - \partial_b h^{ba}\partial^c\xi_{ca}. \tag{3.29}$$

We thus observe that the linearised action in the case of Spin(7) structures is not unique. There are two linearly independent such actions, and the general action is given by their linear combination. One of the parameters can always be absorbed into the perturbation of the 4-form field, but the other parameter remains.

4. Intrinsic torsion

We now proceed to our construction of the non-linear theories completing the linear story described above. The purpose of this section is to recall the definition of the intrinsic torsion of a Spin(7) structure and establish some necessary for the following facts.

4.1. **Characterisation of the intrinsic torsion.** We start with the following proposition, whose proof can also be found in [4].

Proposition 4.1. The intrinsic torsion of a Spin(7) structure, measured by $\nabla_a\Phi_{ijkl}$, where ∇_a is the metric-compatible covariant derivative, takes values in $\Lambda^1\otimes\Lambda^4_7$. Using the isomorphism $\Lambda^2_7\sim\Lambda^4_7$ provided by the operator K, see (2.34), the intrinsic torsion can be parametrised by an object in $\Lambda^1\otimes\Lambda^2_7$. Explicitly,

$$\nabla_a\Phi_{ijkl} = T_{a;ip}\Phi_{pjkl} - T_{a;jp}\Phi_{pkli} + T_{a;kp}\Phi_{plij} - T_{a;lp}\Phi_{pijk}, \qquad T_{a;ij}\in\Lambda^1\otimes\Lambda^2_7. \tag{4.1}$$

Proof. The proof of this proposition consists in showing that the projections of $\nabla_a\Phi_{ijkl}\in\Lambda^1\otimes\Lambda^4$ to all other irreducible components of Λ^4 apart from Λ^4_7 vanish. It is given in [4], and similar computations in the case of G_2 structures are spelled out in [5]. We spell out an alternative, completely explicit proof, which is made possible by our knowledge of the projections to Λ^4_{35+1} and the expression (2.29) for the projector to Λ^4_{27}. The projection to Λ^4_{35+1} is obtained by computing

$$2\Phi_{(i}{}^{pqr}\nabla_{|a|}\Phi_{j)pqr} = \nabla_a\Phi_i{}^{pqr}\Phi_{jpqr} = 42\nabla_a g_{ij} = 0. \tag{4.2}$$

For the projection on Λ^4_{27} the computation is a bit more involved. First, we need some identities. We have, on one hand

$$\begin{aligned} &\nabla_p(\Phi_{[ij}{}^{ab}\Phi_{kl]}{}^{cd}\Phi_{abcd}) = \\ &\Phi_{[kl}{}^{cd}\Phi_{|abcd}\nabla_{p|}(\Phi_{ij]}{}^{ab}) + \Phi_{[ij}{}^{ab}\Phi_{|abcd}\nabla_{p|}(\Phi_{kl]}{}^{cd}) + \Phi_{[ij}{}^{ab}\Phi_{kl]}{}^{cd}\nabla_p\Phi_{abcd} = \\ &12\delta^a_{[k}\delta^b_l\nabla_{|p|}\Phi_{ij]ab} - 4\Phi_{[kl}{}^{ab}\nabla_{|p|}\Phi_{ij]ab} + 12\delta^a_{[i}\delta^b_j\nabla_{|p|}\Phi_{kl]ab} - 4\Phi_{[ij}{}^{ab}\nabla_{|p|}\Phi_{kl]ab} \\ &+\Phi_{[ij}{}^{ab}\Phi_{kl]}{}^{cd}\nabla_p\Phi_{abcd} = 24\nabla_p\Phi_{ijkl} - 8\Phi_{[ij}{}^{ab}\nabla_{|p|}\Phi_{kl]ab} + \Phi_{[ij}{}^{ab}\Phi_{kl]}{}^{cd}\nabla_p\Phi_{abcd}. \end{aligned}$$

On the other hand

$$\nabla_p(\Phi_{[ij}{}^{ab}\Phi_{kl]}{}^{cd}\Phi_{abcd}) = 28\nabla_p\Phi_{ijkl}. \tag{4.3}$$

Thus, we have

$$\Phi_{[ij}{}^{ab}\Phi_{kl]}{}^{cd}\nabla_p\Phi_{abcd} = 4\nabla_p\Phi_{ijkl} + 8\Phi_{[ij}{}^{ab}\nabla_{|p|}\Phi_{kl]ab}. \tag{4.4}$$

We also have

$$\nabla_p(\Phi_{[ij}{}^{ab}\Phi_{kl]ab}) = 2\Phi_{[ij}{}^{ab}(\nabla_{|p|}\Phi_{kl]ab}). \tag{4.5}$$

On the other hand,

$$\nabla_p(\Phi_{[ij}{}^{ab}\Phi_{kl]ab}) = -4\nabla_p\Phi_{ijkl}, \tag{4.6}$$

and so

$$\Phi_{[ij}{}^{ab}(\nabla_{|p|}\Phi_{kl]ab}) = -2\nabla_p\Phi_{ijkl}, \qquad \Phi_{[ij}{}^{ab}\Phi_{kl]}{}^{cd}\nabla_p\Phi_{abcd} = -12\nabla_p\Phi_{ijkl}. \tag{4.7}$$

Using (2.29), these identities, as well as $\Phi^{abcd}\nabla_p\Phi_{abcd}=0$, it is easy to see that

$$\pi_{27}(\nabla_p\Phi_{ijkl}) = 0. \tag{4.8}$$

Finally, to establish (4.1) we just need to recall that a general element of Λ^4_7 can be parametrised as $K(\beta), \beta\in\Lambda^2_7$, where $K:\Lambda^2_7\to\Lambda^4_7$ is the map introduced in (2.34). We thus have

$$\nabla_a\Phi_{ijkl} = -4T_{a;[i|p|}\Phi_{jkl]}{}^p, \tag{4.9}$$

where $T_{a;ij}\in\Lambda^1\otimes\Lambda^2_7$. This is precisely the formula (4.1). $\square$

4.2. **Parametrisation by the torsion 3-form.** As is known, see e.g. [9] Example 3.4., the spaces $\Lambda^1 \otimes \Lambda^2_7$ and Λ^3 are isomorphic. We can make this isomorphism explicit, in one direction, by parametrising the intrinsic torsion $T_{a;ij}$ as follows

$$T_{a;ij} = \pi_7(T_{aij}) = \frac{1}{4} T_{aij} - \frac{1}{8} \Phi_{ij}{}^{kl} T_{akl}, \qquad T_{aij} \in \Lambda^3. \tag{4.10}$$

An explicit relation in the other direction is

$$T_{aij} = \frac{4}{3} T_{a;ij} + 4T_{[a;ij]} + T_{[a;kl]} \Phi_{ij}{}^{kl} + \frac{2}{9} T_{k;lm} \Phi^{klm}{}_{[i} g_{j]a}. \tag{4.11}$$

Using this parametrisation, we can rewrite (4.1) in terms of the torsion 3-form.

Proposition 4.2. In the parametrisation of the intrinsic torsion by a torsion 3-form, we have

$$\nabla_a \Phi_{ijkl} = T_{aip} \Phi_{pjkl} - T_{ajp} \Phi_{pkli} + T_{akp} \Phi_{plij} - T_{alp} \Phi_{pijk}, \qquad T_{aij} \in \Lambda^3. \tag{4.12}$$

Note that this is the same formula for the covariant derivative of the basic 4-form, but now with the torsion 3-form instead of the object $T_{a;ij} \in \Lambda^1 \otimes \Lambda^2_7$.

4.3. **Connection with skew-symmetric torsion.** As is known, see e.g. [9], any Spin(7) structure on an 8-dimensional manifold admits a unique connection with totally skew-symmetric torsion. Such a connection is given by

$$\tilde{\nabla}_a X_i = \nabla_a X_i - T_{aip} X_p. \tag{4.13}$$

It is then clear that the relation (4.12) can be interpreted as the statement that the 4-form is parallel with respect to $\tilde{\nabla}$

$$\tilde{\nabla}_a \Phi_{ijkl} = 0. \tag{4.14}$$

4.4. **Torsion 3-form from the exterior derivative of the Cayley form.**

Proposition 4.3. The torsion 3-form is completely determined by the exterior derivative $d\Phi$. Explicitly, we have

$$T = \frac{5}{2} J_3^{-1}(\star(d\Phi)), \tag{4.15}$$

where J_3 is the operator in 3-forms introduced in (2.13), and $\star(d\Phi)$ is the Hodge dual of $d\Phi$.

Proof. On one hand, we have

$$\star(d\Phi)_{mnr} = \frac{1}{5!} \epsilon_{mnr}{}^{aijkl} \partial_a \Phi_{ijkl}. \tag{4.16}$$

On the other hand, substituting here the right-hand-side of (4.12) we have

$$\frac{1}{5!} \epsilon_{mnr}{}^{aijkl} \partial_a \Phi_{ijkl} = \frac{1}{30} \epsilon_{mnr}{}^{aijkl} T_{aip} \Phi_{pjkl}. \tag{4.17}$$

Now, using (2.5) we get

$$\epsilon_{mnr}{}^{aijkl} T_{aip} \Phi_{pjkl} = 6(\Phi_{mn}{}^{pq} T_{rpq} + \Phi_{nr}{}^{pq} T_{mpq} + \Phi_{rm}{}^{pq} T_{npq}) = 12 J_3(T)_{mnr}. \tag{4.18}$$

This means we have

$$\star(d\Phi)_{mnr} = \frac{2}{5} J_3(T)_{mnr}. \tag{4.19}$$

Now, the operator J_3 is invertible, with inverse given by (2.18). This proves the proposition. □

5. Riemann curvature identities

Having described the intrinsic torsion and its relation with the covariant and exterior derivatives of the Cayley form, we can obtain very useful characterisations of (some parts of) the Riemann curvature. This material is rather standard, except that we use the parametrisation of the torsion by a 3-form.

5.1. **Irreducible parts of the Riemann tensor.** The Riemann tensor is an object with values in $\Lambda^2 \otimes_S \Lambda^2$, with Λ^4 removed. Given that $\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{21}$, it is easy to compute the decomposition of $\Lambda^2 \otimes_S \Lambda^2$ into irreducibles using the known in the literature formulas for the tensor products of irreducible representations of Spin(7). We denote representations using the corresponding dimension written in bold face. We need the following tensor product decompositions:

$$\begin{aligned} \mathbf{7} \otimes_S \mathbf{7} &= \mathbf{1} \oplus \mathbf{27}, \\ \mathbf{7} \otimes \mathbf{21} &= \mathbf{105} \oplus \mathbf{35} \oplus \mathbf{7}, \\ \mathbf{21} \otimes_S \mathbf{21} &= \mathbf{1} \oplus \mathbf{27} \oplus \mathbf{35} \oplus \mathbf{168}. \end{aligned} \tag{5.1}$$

Taking into account that

$$\Lambda^4 = \mathbf{1} \oplus \mathbf{7} \oplus \mathbf{27} \oplus \mathbf{35}, \tag{5.2}$$

we see that Riemann curvature gets decomposed into the following irreducible components

$$\text{Riemann} = \mathbf{1} \oplus \mathbf{27} \oplus \mathbf{35} \oplus \mathbf{105} \oplus \mathbf{168}. \tag{5.3}$$

Of these the Ricci part is

$$\text{Ricci} = \mathbf{1} \oplus \mathbf{35}, \tag{5.4}$$

and the Weyl part is

$$\text{Weyl} = \mathbf{27} \oplus \mathbf{105} \oplus \mathbf{168}. \tag{5.5}$$

Our next task is to characterise which parts of the Riemann curvature can be extracted from the intrinsic torsion.

5.2. **Part of Riemann curvature from the torsion.** We now take the commutator of two covariant derivatives applied to the basic 4-form to get

$$4R_{ab[i}{}^p \Phi_{|p|jkl]} = 2\nabla_{[a}\nabla_{b]}\Phi_{ijkl} = 8\nabla_{[a}(T_{b][i|p|}\Phi^p{}_{jkl]}). \tag{5.6}$$

Applying the product rule and using (4.12) one more time we get

$$\begin{aligned} 4R_{ab[i}{}^p \Phi_{|p|jkl]} &= 4\nabla_a(T_{b[i|p|})\Phi^p{}_{jkl]} - 4\nabla_b(T_{a[i|p|})\Phi^p{}_{jkl]} \\ &\quad + 4T_{a[i}{}^p T_{|bp|}{}^q \Phi_{jkl]q} - 4T_{b[i}{}^p T_{|ap|}{}^q \Phi_{jkl]q}. \end{aligned} \tag{5.7}$$

5.3. **Identity for the divergence of the torsion 3-form.** Before we proceed any further, a useful consequence of this identity is obtained by multiplying it with $\epsilon^{mnabijkl}$, and using (2.5). On the left-hand side we get identically zero, by properties of the Riemann curvature. The right-hand side is non-trivial and so we get

$$2\nabla_a T_{bi[m}\Phi_{n]}{}^{abi} + \nabla^a T_{abi}\Phi^{bi}{}_{mn} + 2T_{ai}{}^p T_{bp[m}\Phi_{n]}{}^{abi} - 2T_a{}^{pq}T_{bpq}\Phi_{mn}{}^{ab} = 0. \tag{5.8}$$

We would now like to extract from here the divergence $\nabla^a T_{amn}$ of the torsion 3-form in terms of other quantities. Applying to this expression $(1/4)(\mathbb{I} + J_2)$, we get

$$\nabla^a T_{amn} = \frac{1}{2}\Phi_{[m}{}^{abc}\nabla_{n]}T_{abc} - \frac{1}{2}\Phi_{[m}{}^{abc}\nabla_{|a|}T_{n]bc} - \Phi_{[m}{}^{abc}T_{n]a}{}^p T_{bcp}. \tag{5.9}$$

We can rewrite this in a different form, by applying the projection to Λ^2_7. We get

$$\nabla^a T_{amn} - \frac{1}{2}\Phi_{mn}{}^{pq}\nabla^a T_{apq} = \frac{1}{2}\Phi_{[m}{}^{abc}\nabla_{n]}T_{abc} - \frac{3}{2}\Phi_{[m}{}^{abc}\nabla_{|a|}T_{n]bc} - 2\Phi_{[m}{}^{abc}T_{n]a}{}^p T_{bcp}. \tag{5.10}$$

It is useful to rewrite this as the divergence of the original torsion. Using (4.12) we have

$$4\nabla^a T_{a;mn} = \nabla^a T_{amn} - \frac{1}{2}\Phi_{mn}{}^{pq}\nabla^a T_{apq} - \Phi_{[m}{}^{abc}T_{n]a}{}^p T_{bcp}, \tag{5.11}$$

and thus

$$4\nabla^a T_{a;mn} = \frac{1}{2}\Phi_{[m}{}^{abc}\nabla_{n]}T_{abc} - \frac{3}{2}\Phi_{[m}{}^{abc}\nabla_{|a|}T_{n]bc} - 3\Phi_{[m}{}^{abc}T_{n]a}{}^p T_{bcp}. \tag{5.12}$$

5.4. **Component of the Riemann curvature.** Thinking about R_{abcd} as an object in $\Lambda^2 \otimes_S \Lambda^2$ (with a copy of Λ^4 removed), and recalling the operator K introduced in (2.34), we see that the object on the left-hand side of (5.7) is valued in $\Lambda^2 \otimes \Lambda^4_7$. We can then apply the inverse operator K' to obtain

$$R_{abij} - \frac{1}{2}\Phi_{ij}{}^{pq} R_{abpq} = \nabla_a T_{bij} - \nabla_b T_{aij} - \frac{1}{2}\Phi_{ij}{}^{cd}\nabla_a T_{bcd} + \frac{1}{2}\Phi_{ij}{}^{cd}\nabla_b T_{acd} \tag{5.13}$$
$$+ T_{ai}{}^{p} T_{bjp} - T_{bi}{}^{p} T_{ajp} - \Phi_{ij}{}^{pq} T_{ap}{}^{k} T_{bqk}.$$

Both sides of this equality can be checked to be in Λ^2_7 with respect to indices ij, by applying the projector to Λ^2_{21} and seeing that the result is identically zero. This computation makes it obvious that all apart from the **168** part of the Weyl curvature are determined by the intrinsic torsion. Indeed, all parts but this one come from $\Lambda^2 \otimes \Lambda^2_7$, and this is precisely what the part of the Riemann curvature tensor that the intrinsic torsion determines.

5.5. **Ricci curvature scalar.** Before we use the facts above to obtain a formula for the Ricci curvature, let us note that there are two different ways to extract the Ricci scalar from here. One is to contract the indices with $g^{ai}g^{bj}$. The other is to contract it with $-(1/6)\Phi^{abij}$. Both of these give

$$R = -\Phi^{abcd}\nabla_a T_{bcd} + T^{abc}T_{abc} + \Phi^{abcd} T_{ab}{}^{p} T_{cdp}. \tag{5.14}$$

Note that this only depends on the exterior derivative dT of the torsion 3-form.

5.6. **Extracting the Ricci curvature.** We can extract the Ricci tensor from (5.13) by multiplying with $\Phi_c{}^{bij}$, and applying the Bianchi identity $R_{a[bij]} = 0$ to get

$$-\frac{1}{2}\Phi_{ij}{}^{pq} R_{abpq} \Phi_c{}^{bij} = -6R_{ac}. \tag{5.15}$$

Doing the same operations with the right-hand side and we get

$$R_{ab} = -\nabla^c T_{abc} - \frac{1}{2}\Phi_b{}^{ijk}\nabla_a T_{ijk} + \frac{1}{2}\Phi_b{}^{ijk}\nabla_i T_{ajk} + T_a{}^{pq} T_{bpq} + \Phi_b{}^{ijk} T_{ai}{}^{p} T_{jkp}. \tag{5.16}$$

This is not explicitly symmetric in ab, and must therefore become symmetric when T_{ijk} is given by its expression (4.15). And indeed, the anti-symmetric part of the right-hand side vanishes in view of (5.9). Thus, the Ricci curvature is given by

$$R_{ab} = -\frac{1}{2}\Phi_{(a}{}^{ijk}\nabla_{b)} T_{ijk} + \frac{1}{2}\Phi_{(a}{}^{ijk}\nabla_{|i|} T_{b)jk} + T_a{}^{pq} T_{bpq} + \Phi_{(a}{}^{ijk} T_{b)i}{}^{p} T_{jkp}. \tag{5.17}$$

We have now proven the result known since [1]: when $d\Phi = 0$ the metric is Ricci-flat. Indeed, by (4.15) $d\Phi = 0$ implies $T = 0$, which in turn gives $R_{ab} = 0$ by (5.17). Note that, unlike the Ricci scalar (5.14), the Ricci tensor depends on the full covariant derivative of the torsion 3-form.

6. The action

This section is central to the whole paper. We consider a one-parameter family of action functionals of the Cayley form Φ and an auxiliary 3-form C. The Euler-Lagrange equations for C are algebraic. There is a member in the family of actions for which the Euler-Lagrange equation for C equates it with the intrinsic torsion T of the Spin(7) structure. But we will treat an arbitrary member of the family of actions, and derive the arising from it second order PDE's for Φ.

6.1. **A one-parameter family of action functionals.** The action we want to construct is a functional of $\Phi \in \Lambda^4$ and $C \in \Lambda^3$. It will contain a term imposing the constraints that guarantee that Φ is the of algebraic type of the Cayley form. We will never need to specify what these constraints are, as we will only need their consequences. Given Φ, C there is a natural top form that can be constructed, which is $\Phi \wedge dC$. We take the integral of this to be our "kinetic" i.e. derivative containing term. We then want to add to the Lagrangian terms quadratic in C, such that the variation of the action with respect to C gives a set of linear equations for C.

The representation theoretic fact $\Lambda^3 = \Lambda^3_8 \oplus \Lambda^3_{48}$ implies that there are two linearly independent quadratic invariants that can be constructed from a 3-form C. A computation gives

$$\begin{aligned} \pi_8(C)_{abc}C^{abc} &= \frac{1}{7}(C_{abc})^2 - \frac{3}{14}\Phi^{abcd}C_{ab}{}^pC_{cdp}, \\ \pi_{48}(C)_{abc}C^{abc} &= \frac{6}{7}(C_{abc})^2 + \frac{3}{14}\Phi^{abcd}C_{ab}{}^pC_{cdp}. \end{aligned} \tag{6.1}$$

This shows that the two linearly independent quadratic invariants constructed from C can be taken to be $(C_{abc})^2$ and $\Phi^{abcd}C_{ab}{}^pC_{cdp}$. The coefficient in front of one of these can always be chosen as desired by rescaling the C field. This leads us to consider the following one-parameter family of action functionals

$$S[\Phi, C] = \int \Phi \wedge (dC - 6C \wedge_g C) + \frac{\kappa}{6}(C)^2 v_\Phi + \frac{\lambda}{6}v_\Phi + \text{constraint terms}. \tag{6.2}$$

The choice of coefficients here will be convenient for what follows. The constant λ is a "cosmological constant" term that can be set to zero if desired. The object $C \wedge_g C$ is the 4-form

$$C \wedge_\Phi C := \frac{1}{4!}g^{pq}C_{ijp}C_{klq}\, dx^i \wedge dx^j \wedge dx^k \wedge dx^l, \tag{6.3}$$

and $v_\Phi = (1/14)\Phi \wedge \Phi$ is the volume form. Written in index notations the action becomes

$$S[\Phi, C] = \frac{1}{3!}\int \left(\frac{1}{4!}\tilde{\epsilon}^{ijklabcd}\Phi_{ijkl}(\partial_a C_{bcd} - \frac{3}{2}g^{pq}C_{abp}C_{cdq}) + \kappa(C_{abc})^2 v_g + \lambda v_g\right) d^8x. \tag{6.4}$$

The constraint terms are omitted for brevity. The object $\tilde{\epsilon}^{ijklabcd}$ is the density weight one totally anti-symmetric tensor. This exists on any manifold, and does not need a metric for its definition. Using the self-duality (2.4) of Φ we can see that the two C-invariants added to the Lagrangian are indeed $(C_{abc})^2$ and $\Phi^{abcd}C_{ab}{}^pC_{cdp}$.

6.2. **The variation with respect to the torsion 3-form.** The variation of the action with respect to C is given by

$$\delta_C S = \frac{1}{3!}\int v_g \left(-5\frac{1}{5!}\epsilon^{ijklabcd}\partial_a\Phi_{ijkl} - 3\Phi^{aecd}C_{ae}{}^b + 2\kappa C^{bcd}\right)\delta C_{bcd} d^8x, \tag{6.5}$$

where v_g is the volume form for g, and we used the self-duality of the basic 4-form in the second term. The resulting Euler-Lagrange equation is therefore

$$5\frac{1}{5!}\epsilon_{bcd}{}^{aijkl}\partial_a\Phi_{ijkl} - 2J_3(C)_{bcd} + 2\kappa C_{bcd} = 0. \tag{6.6}$$

When $\kappa = 0$, comparing to (4.19), we see that $C = T$. The coefficient in front of the second term in the action was selected so that this happens. In general we have

$$J_3(T) = J_3(C) - \kappa C. \tag{6.7}$$

For a general κ this relation can be inverted

$$C = \frac{6T + \kappa J_3(T)}{6 - (5+\kappa)\kappa}, \tag{6.8}$$

which shows that $\kappa = 1, -6$ are the values when the relation cannot be inverted. These are of course also the eigenvalues of J_3. We are particularly interested in the case when $\kappa = 0$, where $C = T$, and $\kappa = -2$ where

$$C = \frac{1}{2}T - \frac{1}{6}J_3(T). \tag{6.9}$$

6.3. **Variation of the metric with respect to the 4-form.** To vary the action with respect to the 4-form, we need a formula for the variation of g^{ij} with respect to the 4-form Φ_{ijkl}. The best way to obtain this is to consider a variation of the metric, thought of as an $\mathrm{GL}(8,\mathbb{R})$ transformation. As we have already discussed in (2.36), such a transformation effected by a symmetric 8×8 matrix h_{ij} induces a change in the basic 4-form given by

$$K(h)_{ijkl} = 4h_{[i|p|}\Phi^p{}_{jkl]}. \tag{6.10}$$

It will be more convenient, however, to consider the variation of Φ^{ijkl}. We have

$$\delta\Phi_{ijkl} = 4\alpha\delta g_{[i|p|}\Phi^p{}_{jkl]}. \tag{6.11}$$

The coefficient of proportionality α should be fixable by taking the variation of any of the algebraic relations satisfied by Φ. For example we have

$$\Phi_{abcd}\Phi_{ijkl}g^{ia}g^{jb}g^{kc}g^{ld} = 336. \tag{6.12}$$

Varying this gives

$$2\delta\Phi_{ijkl}\Phi^{ijkl} + 4\cdot 42\delta g^{ia}g_{ia} = 0, \tag{6.13}$$

where we used (2.3). Using (6.11) we have

$$4\alpha\delta g^{[i|p|}\Phi_p{}^{jkl]}\Phi_{ijkl} + 2\cdot 42\delta g_{ia}g^{ia} = 0. \tag{6.14}$$

Using (2.3) again this becomes

$$2\alpha\delta g_{ij}g^{ij} + \delta g^{ij}g_{ij} = 0, \tag{6.15}$$

which shows that $\alpha = 1/2$. Thus, we have

$$\delta\Phi_{ijkl} = 2\delta g_{[i|p|}\Phi^p{}_{jkl]}. \tag{6.16}$$

As a check of consistency of these expressions, we also compute

$$\begin{aligned}\delta(\Phi^{ijkl}) = \delta(g^{ia}g^{jb}g^{kc}g^{ld}\Phi_{abcd}) = g^{ia}g^{jb}g^{kc}g^{ld}\delta\Phi_{abcd} + 4\delta^{[i|a}g^{jb}g^{kc}g^{l]d}\Phi_{abcd} = \\ 2\delta g^{[i|p|}\Phi_p{}^{jkl]} - 4\delta g^{[i|p|}\Phi_p{}^{jkl]} = -2\delta g^{[i|p|}\Phi_p{}^{jkl]} = -g^{ia}g^{jb}g^{kc}g^{ld}\delta\Phi_{abcd}.\end{aligned} \tag{6.17}$$

This is analogous to the relation that we have for the metric

$$\delta g^{ij} = -g^{ia}g^{jb}\delta g_{ab}. \tag{6.18}$$

We now extract δg_{ij} in terms of $\delta\Phi_{ijkl}$. To do so we multiply the above expression by $\Phi^a{}_{jkl}$. We get

$$\delta\Phi_{(i|jkl|}\Phi_{a)}{}^{jkl} = 12\delta g_{ia} + 9\delta g_{pq}g^{pq}g_{ia}. \tag{6.19}$$

One more contraction gives

$$\delta g_{pq}g^{pq} = \frac{1}{84}\delta\Phi_{ijkl}\Phi^{ijkl}, \tag{6.20}$$

and so

$$\delta g_{ij} = \frac{1}{12}(\delta\Phi_{(i|pqr|}\Phi_{j)}{}^{pqr} - \frac{3}{28}g_{ij}\delta\Phi_{pqrs}\Phi^{pqrs}). \tag{6.21}$$

Because the variation of the 4-form with all upper indices is given by minus the variation of the form with the lower indices, and the same is true for the metric variation, we can also write

$$\delta g^{ij} = \frac{1}{12}(\delta\Phi^{(i|pqr|}\Phi^{j)}{}_{pqr} - \frac{3}{28}g^{ij}\delta\Phi^{pqrs}\Phi_{pqrs}), \tag{6.22}$$

which is the form of the relation that will be used later.

6.4. **Variation of the action with respect to the 4-form.** We now derive the other half of the Euler-Lagrange equations. We first rewrite the action in terms of Φ^{abcd}

$$S[\Phi, C] = \frac{1}{3!}\int \left(v_g \Phi^{abcd}(\partial_a C_{bcd} - \frac{3}{2} g^{pq} C_{abp} C_{cdq}) + \kappa (C_{abc})^2 v_g + \lambda v_g \right) d^8x. \tag{6.23}$$

and then vary with respect to Φ^{abcd}. We have

$$\begin{aligned} \delta_\Phi S[\Phi, T] = \frac{1}{3!}\int v_g \Big(&\delta\Phi^{abcd}(\partial_a C_{bcd} - \frac{3}{2} g^{pq} C_{abp} C_{cdq}) - \Phi^{abcd}\frac{3}{2}\delta g^{pq} C_{abp} C_{cdq} \\ &-\frac{1}{2}\delta g^{pq} g_{pq} \Phi^{abcd}(\partial_a C_{bcd} - \frac{3}{2} g^{pq} C_{abp} C_{cdq}) - \frac{1}{2}\delta g^{pq} g_{pq}(\kappa (C_{abc})^2 + \lambda)\Big) d^8x. \end{aligned} \tag{6.24}$$

We now substitute (6.22). The last term in the first line becomes

$$\left(-\frac{1}{8}\Phi^{ijkl} C_{ija} C_{kle} \Phi^e{}_{bcd} + \frac{3}{8\cdot 28}\Phi^{ijkl} C_{ij}{}^p C_{klp} \Phi_{abcd} \right) \delta\Phi^{abcd}. \tag{6.25}$$

Thus, the variation of the action with respect to Φ^{abcd} is

$$\begin{aligned} E_{abcd} = \partial_{[a} C_{bcd]} - \frac{3}{2} C_{[ab}{}^p C_{cd]p} - \frac{1}{8}\Phi^{ijkl} C_{ij[a} C_{|kle|} \Phi^e{}_{bcd]} + \frac{3}{8\cdot 28}\Phi^{ijkl} C_{ij}{}^p C_{klp}\Phi_{abcd} \\ -\frac{1}{2\cdot 84}\Phi_{abcd}\left(\Phi^{ijkl}(\partial_i C_{jkl} - \frac{3}{2} C_{ij}{}^p C_{klp}) + \kappa (C_{ijk})^2 + \lambda \right). \end{aligned} \tag{6.26}$$

This does not need to be zero, as it the action also contains terms imposing the constraints guaranteeing that Φ_{abcd} is of the correct algebraic type. The constraint terms produce a variation that is an arbitrary tensor in Λ^4_{27}. So, we can only deduce that the Λ^4_{35+1} and Λ^4_7 projection of the above vanishes. Before we extract these projections, it is worth evaluating the trace of the field equations. We have

$$\Phi^{abcd} E_{abcd} = -2\lambda - 2\kappa (C_{abc})^2 - \Phi^{abcd}(\partial_a C_{bcd} - \frac{3}{4} C_{ab}{}^p C_{cdp}). \tag{6.27}$$

This is the projection of the field equations onto Λ^4_1, which must vanish. We therefore get the following consequence of the field equations

$$\Phi^{abcd}(\partial_a C_{bcd} - \frac{3}{4} C_{ab}{}^p C_{cdp}) + 2\lambda + 2\kappa (C_{abc})^2 = 0. \tag{6.28}$$

We can use this to simplify E_{abcd}. We have

$$\Phi^{ijkl}(\partial_i C_{jkl} - \frac{3}{2} C_{ij}{}^p C_{klp}) + \kappa (C_{ijk})^2 + \lambda = -\lambda - \kappa (C_{ijk})^2 - \frac{3}{4}\Phi^{ijkl} C_{ij}{}^p C_{klp}, \tag{6.29}$$

and so we can rewrite

$$\begin{aligned} E'_{abcd} = \partial_{[a} C_{bcd]} - \frac{3}{2} C_{[ab}{}^p C_{cd]p} - \frac{1}{8}\Phi^{ijkl} C_{ij[a} C_{|kle|}\Phi^e{}_{bcd]} \\ +\frac{1}{56}\Phi_{abcd}\Phi^{ijkl} C_{ij}{}^p C_{klp} + \frac{1}{2\cdot 84}\Phi_{abcd}(\lambda + \kappa (C_{ijk})^2). \end{aligned} \tag{6.30}$$

The Λ^4_{35+1+7} projections of this vanish when the Λ^4_{35+1+7} projections of E_{abcd} vanish and vice versa, so this gives an equivalent encoding of field equations.

6.5. **Extracting Λ^4_{35+1+7} projections.** To understand the implications of the field equations we extract the Λ^4_{35+1} and Λ^4_7 projections. This gives

$$\begin{aligned} \Phi_b{}^{pqr} E'_{apqr} = \frac{1}{4}\Phi_b{}^{pqr}\nabla_a C_{pqr} - \frac{3}{4}\Phi_b{}^{pqr}\nabla_r C_{apq} - \frac{3}{2}\Phi_b{}^{pqr} C_{ap}{}^s C_{qrs} - \frac{3}{4}\Phi^{pqrs} C_{apq} C_{brs} \\ +\frac{1}{4} g_{ab}\left(\lambda + \kappa (C_{pqr})^2 + \frac{3}{4}\Phi^{pqrs} C_{pq}{}^p C_{rsp} \right). \end{aligned} \tag{6.31}$$

Its ab symmetrisation and anti-symmetrisation compute the Λ^4_{35+1} and Λ^4_7 parts respectively. We wrote the derivatives here as the covariant derivatives, for the computations to follow.

6.6. **Rewriting the $\kappa = 0$ field equations - antisymmetric part.** For $\kappa = 0$ we have $C = T$. Let us understand the arising field equations. We start with the anti-symmetric part. Taking (twice) the anti-symmetric part of the field equations (6.31) we get

$$\frac{1}{2}\Phi_{[a}{}^{pqr}\nabla_{b]}T_{pqr} - \frac{3}{2}\Phi_{[a}{}^{pqr}\nabla_{|r|}T_{b]pq} - 3\Phi_{[a}{}^{pqr}T_{b]p}{}^{s}T_{qrs} = 0. \tag{6.32}$$

With the help of the curvature identity (5.12) we can rewrite this as

$$\nabla^r T_{r;ab} = 0, \tag{6.33}$$

which is just vanishing of the divergence of the original torsion. This also makes it manifest that this equation is Λ^2_7 valued. Note also that this equation does not hold automatically. It is a non-trivial field equation to be imposed, and it becomes a second order PDE on the original 4-form. It can be interpreted as the evolution equation for the Λ^4_7 part of the Cayley form perturbation, as is confirmed by the linearised analysis below.

6.7. **Rewriting the $\kappa = 0$ field equations - symmetric part.** For the analysis of the symmetric part, we take (twice) the symmetric part of (6.31), also writing it with the opposite sign

$$-\frac{1}{2}\Phi_{(a}{}^{pqr}\nabla_{b)}T_{pqr} + \frac{3}{2}\Phi_{(a}{}^{pqr}\nabla_{|r|}T_{b)pq} + 3\Phi_{(a}{}^{pqr}T_{b)p}{}^{s}T_{qrs} + \frac{3}{2}\Phi^{pqrs}T_{apq}T_{brs} \tag{6.34}$$
$$+\frac{3}{8}g_{ab}\Phi^{ijkl}T_{ij}{}^{p}T_{klp} + \lambda g_{ab} = 0.$$

Contract the resulting equation with g^{ab} we get (6.28). Comparing this with (5.14) we see that this is *not* the condition that the Ricci scalar is constant. Rather, using (5.14), we can rewrite this equation as

$$R = T_{abc}T^{abc} + \frac{1}{4}\Phi^{abcd}T_{ab}{}^{p}T_{cdp} + 4\lambda. \tag{6.35}$$

A computation shows that this can be rewritten as

$$R = T^{abc}(T + \frac{1}{6}J_3(T))_{abc} = \frac{7}{6}(T^{48}_{abc})^2 + 4\lambda. \tag{6.36}$$

Here $T_{48} = \pi_{48}(T)$ is the Λ^3_{48} part of the torsion 3-form. We thus see that the curvature scalar is sourced just by this part of the torsion.

For the complete symmetric part of the equation, comparing this with (5.17), we can see that the second order part here does not reduce to that in R_{ab}. The comparison with (5.17) suggests that we can rewrite (6.34) as

$$3R_{ab} + \Phi_{(a}{}^{pqr}\nabla_{b)}T_{pqr} - 3T_a{}^{pq}T_{bpq} + \frac{3}{2}\Phi^{pqrs}T_{apq}T_{brs} + \frac{3}{8}g_{ab}\Phi^{ijkl}T_{ij}{}^{p}T_{klp} + \lambda g_{ab} = 0. \tag{6.37}$$

We thus see that the field equations *do not* state that the metric is Einstein. Instead, there are extra contributions coming from the torsion 3-form, and its derivatives. Note that the covariant derivative appears in this equation in such a way that, while both R_{ab} and $\Phi_{(a}{}^{pqr}\nabla_{b)}T_{pqr}$ do depend on it, the specific combination of these terms that appears does not depend on ∇. This will become more pronounced once we rewrite the field equations as a condition that a certain 4-form vanishes.

6.8. **Different ways of writing the field equations.** We note that we can introduce a symmetric tensor

$$T_{ab} := \Phi^{ijkl}T_{ija}T_{klb} - \frac{1}{7}g_{ab}\Phi^{ijkl}T_{ij}{}^{p}T_{klp}. \tag{6.38}$$

The 4-form encoding the field equations can then be written very compactly as

$$E'_{abcd} = \partial_{[a}T_{bcd]} - \frac{3}{2}T_{[ab}{}^{p}T_{cd]p} - \frac{1}{8}T_{[a|e|}\Phi^{e}{}_{bcd]} + \frac{\lambda}{84}\Phi_{abcd}. \tag{6.39}$$

The field equations are then the statement that this equals to an arbitrary tensor in Λ^4_{27}, which we know can be parametrised as (2.43). So, we get one of the possible ways of writing the field equations

$$\partial_{[a}T_{bcd]} - \frac{3}{2}T_{[ab}{}^pT_{cd]p} - \frac{1}{8}T_{[a|e|}\Phi^e{}_{bcd]} + \frac{\lambda}{84}\Phi_{abcd} = \Psi_{[ab}{}^{pq}\Phi_{|pq|cd]}, \tag{6.40}$$

where Ψ^{abcd} is an arbitrary symmetric tracefree matrix in $\Lambda^2_7 \otimes_S \Lambda^2_7$.

6.9. **Yet another rewriting the field equations.** Yet another way of writing the field equations, potentially useful, is obtained by computing $\Phi_{abc}{}^s\Phi_s{}^{pqr}E'_{dpqr}$, and anti-symmetrising on $abcd$. This gives a 4-form that is projected onto the Λ^4_{35+1} and Λ^4_7 parts, eliminating the Λ^4_{27} part of E'_{abcd} that does not need to be zero. For a general 4-form we have

$$\frac{1}{6}\Phi_{abc}{}^s\Phi_s{}^{pqr}\sigma_{dpqr} = (\mathbb{I} - \frac{1}{2}J_4)(\sigma)_{abcd}, \tag{6.41}$$

explicitly showing that the Λ^4_{27} component is projected away. We now apply this projector to the 4-form E'_{abcd} to get the following 4-form field equations

$$\begin{aligned} &\nabla_{[a}T_{bcd]} - \frac{3}{4}\Phi_{[ab}{}^{pq}\nabla_cT_{d]pq} - \frac{3}{4}\Phi_{[ab}{}^{pq}\nabla_{|p|}T_{cd]q} \\ &-\frac{3}{2}T_{[ab}{}^pT_{cd]p} + \frac{3}{4}\Phi_{[ab}{}^{pq}T_{cd]}{}^rT_{pqr} - \frac{1}{8}\Phi_{[abc}{}^p\Phi^{ijkl}T_{d]ij}T_{klp} - \frac{3}{2}\Phi_{[ab}{}^{pq}T_{c|p|}{}^rT_{d]qr} \\ &+\frac{1}{32}\Phi_{abcd}\Phi^{ijkl}T_{ij}{}^pT_{klp} + \frac{\lambda}{12}\Phi_{abcd} = 0. \end{aligned} \tag{6.42}$$

Since the first line here can be rewritten as

$$(\mathbb{I} - \frac{1}{2}J_4)(\nabla_{[a}T_{bcd]}), \tag{6.43}$$

we see that the operator that appears in the field equations is built from the usual partial derivative, rather than the covariant one.

6.10. **Analysis of the $\kappa = -2$ field equations.** In the general κ case, we can rewrite the field equations (6.31) in terms of the intrinsic torsion 3-form T, using the relation between C and T. However, the arising general κ results are too cumbersome. Using as the motivation the computation of the linearised action in the last section, we now specialise to the particularly interesting case $\kappa = -2$, when the linearised action coincides with that of GR. Our intention is to see whether the full non-linear equations of the theory in this case also reduce to the Einstein condition.

We substitute C in the form (6.9) to (6.31) and whenever the derivatives get applied to the basic 4-form, evaluate them using (4.12). The resulting field equations are as follows

$$\begin{aligned} &\frac{1}{4}\Phi_b{}^{pqr}\nabla_aT_{pqr} - \frac{1}{2}\Phi_b{}^{pqr}\nabla_rT_{apq} + \frac{1}{4}\Phi_a{}^{pqr}\nabla_rT_{bpq} + \frac{1}{2}(\nabla^pT_{abp} - \frac{1}{2}\Phi_{ab}{}^{cd}\nabla^pT_{cdp}) \\ &-\frac{23}{24}\Phi_b{}^{pqr}T_{ap}{}^sT_{qrs} - \frac{5}{24}\Phi_a{}^{pqr}T_{bp}{}^sT_{qrs} - \frac{1}{4}\Phi^{pqrs}T_{apq}T_{brs} + \frac{1}{6}T_a{}^{pq}T_{bpq} \\ &-\frac{1}{24}\Phi^{pqrs}T_{abp}T_{qrs} + \frac{1}{48}\Phi_{ab}{}^{pq}\Phi^{ijkl}T_{pqi}T_{jkl} - \frac{1}{24}\Phi_a{}^{pqr}\Phi_b^{ijk}T_{pqi}T_{rjk} \\ &+\frac{1}{4}g_{ab}\left(\lambda - \frac{17}{12}(T_{pqr})^2 + \frac{17}{24}\Phi^{pqrs}T_{pq}{}^pT_{rsp} + \frac{1}{2}\Phi^{pqrs}\nabla_pT_{qrs}\right) = 0. \end{aligned} \tag{6.44}$$

We now use (5.10) to simplify the first line. We also separate the symmetric and anti-symmetric parts. We get

$$\begin{aligned} &\frac{1}{4}\Phi_{(a}{}^{pqr}\nabla_{b)}T_{pqr} - \frac{1}{4}\Phi_{(a}{}^{pqr}\nabla_rT_{b)pq} \\ &-\frac{7}{6}\Phi_{(a}{}^{pqr}T_{b)p}{}^sT_{qrs} - \frac{1}{4}\Phi^{pqrs}T_{apq}T_{brs} + \frac{1}{6}T_a{}^{pq}T_{bpq} - \frac{1}{24}\Phi_a{}^{pqr}\Phi_b^{ijk}T_{pqi}T_{rjk} \\ &+\frac{1}{4}g_{ab}\left(\lambda - \frac{17}{12}(T_{pqr})^2 + \frac{17}{24}\Phi^{pqrs}T_{pq}{}^pT_{rsp} + \frac{1}{2}\Phi^{pqrs}\nabla_pT_{qrs}\right) = 0 \end{aligned} \tag{6.45}$$

for the symmetric part and

$$-\frac{1}{4}\Phi_{[a}{}^{pqr}T_{b]p}{}^{s}T_{qrs}-\frac{1}{24}\Phi^{pqrs}(T_{abp}-\frac{1}{2}\Phi_{ab}{}^{cd}T_{cdp})T_{qrs}=0 \tag{6.46}$$

for the anti-symmetric part.

6.11. **The trace.** It will be useful for the later to compute the trace of the field equations. We get

$$2\lambda-\frac{11}{4}(T_{abc})^2-\frac{1}{8}\Phi^{abcd}T_{ab}{}^{p}T_{cdp}+\frac{3}{2}\Phi^{abcd}\nabla_a T_{bcd}=0. \tag{6.47}$$

Using (5.14) we can rewrite this as

$$\frac{3}{2}R=2\lambda-\frac{5}{4}(T_{abc})^2+\frac{11}{8}\Phi^{abcd}T_{ab}{}^{p}T_{cdp}. \tag{6.48}$$

6.12. **An identity.** Contracting (2.8) with $T_{bcd}T^{jkl}$ we get the following identity

$$\frac{1}{4}\Phi_a{}^{pqr}\Phi_b^{ijk}T_{pqi}T_{rjk}=-2\Phi_{(a}{}^{pqr}T_{b)p}{}^{s}T_{qrs}+\frac{1}{2}g_{ab}\Phi^{ijkl}T_{ij}{}^{p}T_{klp}-\frac{1}{2}\Phi^{pqrs}T_{apq}T_{brs} \tag{6.49}$$
$$+\frac{1}{12}\Phi_a{}^{pqr}\Phi_b^{ijk}T_{pqr}T_{ijk}.$$

Using this in the symmetric part of the field equations we can transform it to

$$\frac{1}{4}\Phi_{(a}{}^{pqr}\nabla_{b)}T_{pqr}-\frac{1}{4}\Phi_{(a}{}^{pqr}\nabla_r T_{b)pq} \tag{6.50}$$
$$-\frac{5}{6}\Phi_{(a}{}^{pqr}T_{b)p}{}^{s}T_{qrs}-\frac{1}{6}\Phi^{pqrs}T_{apq}T_{brs}+\frac{1}{6}T_a{}^{pq}T_{bpq}-\frac{1}{72}\Phi_a{}^{pqr}\Phi_b^{ijk}T_{pqr}T_{ijk}$$
$$+\frac{1}{4}g_{ab}\left(\lambda-\frac{17}{12}(T_{pqr})^2+\frac{3}{8}\Phi^{pqrs}T_{pq}{}^{p}T_{rsp}+\frac{1}{2}\Phi^{pqrs}\nabla_p T_{qrs}\right)=0.$$

We can now rewrite this in terms of the Ricci tensor using (5.17). We get

$$R_{ab}=-\frac{2}{3}\Phi_{(a}{}^{pqr}T_{b)p}{}^{s}T_{qrs}-\frac{1}{3}\Phi^{pqrs}T_{apq}T_{brs}+\frac{4}{3}T_a{}^{pq}T_{bpq}-\frac{1}{36}\Phi_a{}^{pqr}\Phi_b^{ijk}T_{pqr}T_{ijk} \tag{6.51}$$
$$+\frac{1}{2}g_{ab}\left(\lambda-\frac{17}{12}(T_{pqr})^2+\frac{3}{8}\Phi^{pqrs}T_{pq}{}^{p}T_{rsp}+\frac{1}{2}\Phi^{pqrs}\nabla_p T_{qrs}\right).$$

We can also use (5.14) to rewrite this as

$$R_{ab}+\frac{1}{4}g_{ab}R=-\frac{2}{3}\Phi_{(a}{}^{pqr}T_{b)p}{}^{s}T_{qrs}-\frac{1}{3}\Phi^{pqrs}T_{apq}T_{brs}+\frac{4}{3}T_a{}^{pq}T_{bpq}-\frac{1}{36}\Phi_a{}^{pqr}\Phi_b^{ijk}T_{pqr}T_{ijk} \tag{6.52}$$
$$+\frac{1}{2}g_{ab}\left(\lambda-\frac{11}{12}(T_{pqr})^2+\frac{7}{8}\Phi^{pqrs}T_{pq}{}^{p}T_{rsp}\right).$$

This makes it clear that the $\kappa=-2$ non-linear equations *do not coincide* with Einstein equations. Rather, these are Einstein equations with "stress-energy" tensor sourced by the intrinsic torsion. Better understanding of these equations requires further work.

6.13. **Rewriting the $\kappa=-2$ field equations - antisymmetric part.**

$$\Phi_{[a}{}^{pqr}T_{b]p}{}^{s}T_{qrs}+\frac{1}{6}\Phi^{pqrs}(T_{abp}-\frac{1}{2}\Phi_{ab}{}^{cd}T_{cdp})T_{qrs}=0. \tag{6.53}$$

The expression in brackets contains a multiple of the projector π_7, so it is in Λ^2_7. The first term can also be checked to be in Λ^2_7 by computing the π_{21} projection and verifying that it is identically zero. Moreover, it can be checked that the above expression is invariant under the change

$$T_{abc}\to T_{abc}+\Phi_{abcd}V^d, \tag{6.54}$$

which means that it only depends on the Λ^3_{48} part of T_{abc}. This means we can write this equation as

$$\Phi_{[a}{}^{pqr}\tilde{T}_{b]p}{}^{s}\tilde{T}_{qrs}=0. \tag{6.55}$$

where $\tilde{T} = \pi_{48}(T)$. There is precisely one copy of the **7** representation in the tensor product $\mathbf{48} \otimes \mathbf{48}$, and the field equation (6.55) states that $\pi_7(\tilde{T} \otimes \tilde{T}) = 0$.

7. Linearisation

We now compute the linearisation of the general action (6.2) and verify that it gives the most general diffeomoprhism invariant linearised theory (3.28).

7.1. **Linearisation of the non-linear action.** We start with the full action without the cosmological constant part, and without the constraint terms, which we assume to be satisfied

$$S[\Phi, C] = \frac{1}{3!} \int \left(\frac{1}{4!} \Phi_{ijkl} (\partial_a C_{bcd} - \frac{3}{2} g^{pq} C_{abp} C_{cdq}) \tilde{\epsilon}^{ijklabcd} + \kappa (C_{abc})^2 v_g \right) d^8 x. \tag{7.1}$$

We will then linearise around the background given by $C = 0$ and a constant Φ. Denoting the variation of Φ by ϕ and of C by c we get for the first variation

$$S^{(2)}[\phi, c] = \frac{1}{3!} \int \left(\frac{1}{4!} \epsilon^{ijklabcd} \phi_{ijkl} \partial_a c_{bcd} - \frac{3}{2} \Phi^{abcd} c_{ab}{}^p c_{cdp} + \kappa (c_{abc})^2 \right), \tag{7.2}$$

where the volume element $d^8 x$ is omitted for compactness. A calculation shows

$$J_3(c)_{abc} c^{abc} = \frac{3}{2} \Phi^{abcd} c_{ab}{}^p c_{cdp}, \tag{7.3}$$

which it makes it easy to derive the Euler-Lagrange equation for c_{abc}, which is given by

$$\frac{1}{4!} \epsilon_{bcd}{}^{aijkl} \partial_a \phi_{ijkl} = 2 J_3(c)_{bcd} - 2\kappa c_{bcd}. \tag{7.4}$$

We can now integrate by parts in (7.2) in the first term to rewrite it in terms of the torsion. We see that it is given by $(2J_3(c)_{bcd} - 2\kappa c_{bcd}) c^{bcd}$. This means that the linearised action written in terms of ϕ only is given by

$$S^{(2)}[\phi] = \frac{1}{6} \int J_3(c)_{abc} c^{abc} - \kappa (c_{abc})^2. \tag{7.5}$$

We note that the linearised action is manifestly diffeomoprhism invariant. Indeed, the linearised 4-form transforms as (3.24). This is clearer in the form notations

$$\delta \phi = d i_\xi \Phi. \tag{7.6}$$

Thus, the variation is an exact form. The formula (7.4) shows that the linearised torsion is obtained from the Hodge dual of the exterior derivative of the linearised 4-form, which is clearly diffeomorphism invariant. So, any linearised action written in terms of the linearised torsion is diffeomoprhism invariant.

7.2. **Rewriting of the linearised action.** We now rewrite the linearised action explicitly in terms of ϕ. To compute t_{abc} explicitly we need to compute the action of J_3 on the left-hand side in (7.4). We have

$$J_3(t_{abc}) = \frac{1}{48} \epsilon_{abc}{}^{pijkl} \partial_p \phi_{ijkl}, \tag{7.7}$$

and

$$J_3(\frac{1}{48} \epsilon_{abc}{}^{pijkl} \partial_p \phi_{ijkl}) = \frac{1}{2} \Phi_{[a}{}^{pqr} \partial_b \phi_{c]pqr} + \frac{3}{4} \Phi_{[a}{}^{pqr} \partial_{|p|} \phi_{bc]qr}. \tag{7.8}$$

This means that

$$t_{abc} = \frac{5}{288} \epsilon_{abc}{}^{pijkl} \partial_p \phi_{ijkl} + \frac{1}{12} \Phi_{[a}{}^{pqr} \partial_b \phi_{c]pqr} + \frac{1}{8} \Phi_{[a}{}^{pqr} \partial_{|p|} \phi_{bc]qr}. \tag{7.9}$$

For completeness, we state the result of computation of the Λ^3_8 part of the torsion 3-form. We have

$$t_{abc} \Phi^{mabc} = \frac{1}{48} \Phi^{abcd} \partial^m \phi_{abcd} + \frac{1}{12} \Phi^{abcd} \partial_a \phi_{bcd}{}^m. \tag{7.10}$$

A long calculation using (2.7) gives

$$\int J_3(t_{abc})t^{abc} = \int \frac{5}{96}\phi^{abcd}\partial^p\partial_p\phi_{abcd} + \frac{1}{32}\Phi^{abcd}\phi_{ab}{}^{pq}\partial^e\partial_e\phi_{cdpq} \tag{7.11}$$
$$+\frac{5}{24}(\partial^a\phi_{abcd})^2 + \frac{1}{16}\Phi^{abcd}\partial^i\phi_{abip}\partial^j\phi_{cdj}{}^p - \frac{1}{8}\Phi^{abci}(\partial_i\partial^r\phi_{crpq})\phi_{ab}{}^{pq}.$$

We cannot exhibit this in a form that is elliptic modulo gauge without using additional properties of the perturbation ϕ_{abcd}. To this end, we now pass to the parametrisation of ϕ by fields h, ξ.

7.3. **Evaluation of the linearised action.** We now use the parametrisation (3.21). In this parametrisation, using (2.5) gives

$$J_3(t)_{abc} := \frac{1}{48}\epsilon_{abc}{}^{pijkl}\partial_p\phi_{ijkl} = \tag{7.12}$$
$$-\frac{1}{2}\Phi_{bcd}{}^a\partial^i(h_{ia} - \frac{1}{4}\xi_{ia}) + \frac{1}{2}\Phi_{bcd}{}^i\partial_i h - \frac{3}{2}\Phi_{[bc}{}^{ai}\partial_i(h_{d]a} - \frac{1}{4}\xi_{d]a}).$$

We have introduced the notation $J_3(t)$ for this quantity. We also have

$$J_3(t) = J_3(c) - \kappa c, \tag{7.13}$$

and so we need to compute $J_3(t)_{abc}c^{abc}$. We have

$$c = \frac{6t + \kappa J_3(t)}{6 - 5\kappa - \kappa^2}. \tag{7.14}$$

This means that we can write the linearised action as

$$S^{(2)}[\phi] = \int \mathcal{L}^{(2)} = \frac{1}{6(6-5\kappa-\kappa^2)}\int \kappa J_3(t)J_3(t) + 6J_3(t)t = \tag{7.15}$$
$$\frac{1}{6(6-5\kappa-\kappa^2)}\int \kappa J_3(t)J_3(t) + J_3(t)(J_3 + 5\mathbb{I})J_3(t).$$

A computation gives

$$\left(1 - \frac{5\kappa}{6} - \frac{\kappa^2}{6}\right)\mathcal{L}^{(2)} = \frac{1}{2}(1+\frac{\kappa}{6})(\partial_a h_{bc})^2 - \frac{1}{6}(1-\frac{\kappa}{2})(\partial_a h)^2 - \frac{1}{3}(1-\frac{\kappa}{2})h\partial^a\partial^b h_{ab} \tag{7.16}$$
$$-\frac{2}{3}(\partial^a h_{ab})^2 + \frac{1}{24}(1+\frac{\kappa}{2})(\partial_a\xi_{bc})^2 - \frac{2}{3}(1+\frac{\kappa}{2})\partial_b h^{ba}\partial^c\xi_{ca}.$$

This is the diffeomorphism invariant Lagrangian of the type (3.15) with (3.27) and

$$\rho = 1 + \frac{\kappa}{6}, \quad \mu = \frac{2}{3}\left(1 + \frac{\kappa}{2}\right). \tag{7.17}$$

This shows that the linearisation of our general action gives the linearisation of the Einstein-Hilbert Lagrangian for $\kappa = -2$.

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