A GAUSS–BONNET FORMULA FOR THE RENORMALIZED AREA OF MINIMAL SUBMANIFOLDS OF POINCARÉ–EINSTEIN MANIFOLDS

JEFFREY S. CASE, C ROBIN GRAHAM, TZU-MO KUO, AARON J. TYRRELL, AND ANDREW WALDRON

Abstract. Assuming the extrinsic Q-curvature admits a decomposition into the Pfaffian, a scalar conformal submanifold invariant, and a tangential divergence, we prove that the renormalized area of an even-dimensional minimal submanifold of a Poincaré–Einstein manifold can be expressed as a linear combination of its Euler characteristic and the integral of a scalar conformal submanifold invariant. We derive such a decomposition of the extrinsic Q-curvature in dimensions two and four, thereby recovering and generalizing results of Alexakis–Mazzeo and Tyrrell, respectively. We also conjecture such a decomposition for general natural submanifold scalars whose integral over compact submanifolds is conformally invariant, and verify our conjecture in dimensions two and four. Our results also apply to the area of a compact even-dimensional minimal submanifold of an Einstein manifold.

Contents

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1. Introduction

Poincaré–Einstein manifolds are a generalization of hyperbolic space which have been intensely studied in both geometry and physics. These are asymptotically hyperbolic manifolds (X^n, g_+) with $\mathrm{Ric}(g_+) = -(n-1)g_+$. Although their volume is infinite, they have a renormalized volume $\mathcal{V}$, which is an invariant when n is even. Their minimal submanifolds $Y^k \subset X^n$ with prescribed boundary at infinity are also fundamental objects. Their area is infinite, but they have a renormalized area $\mathcal{A}$, which is an invariant when k is even.

Chang, Qing, and Yang [CQY08] derived a formula of Gauss–Bonnet type for the renormalized volume: if (X^n, g_+) is Poincaré–Einstein with $n \geq 4$ even, then

$$\mathcal{V} = c_n \chi(X) + \int_X \mathcal{Z}\, \mathrm{dV}, \qquad c_n = \frac{(-2\pi)^{n/2}}{(n-1)!!}. \tag{1.1}$$

Here $\chi(X)$ denotes the Euler characteristic of X and $\mathcal{Z}$ is a natural scalar that is pointwise conformally invariant of weight $-n$. The result for $n = 4$ was first proved by Anderson [And01]. In this case $\mathcal{Z} = -|W|^2/24$, where W denotes the Weyl tensor. Conformal invariance implies convergence of the integral in Equation (1.1): the expression $\mathcal{Z}\, \mathrm{dV}$ does not change upon conformally rescaling the metric, so it equals the same expression when evaluated on a compactification of g_+.

A main ingredient in Chang, Qing, and Yang's derivation of Equation (1.1) is a result of Alexakis [Ale12], motivated by a conjecture of Deser and Schwimmer [DS93], that establishes in even dimension n a decomposition of any natural scalar I whose integral over compact manifolds is conformally invariant. Namely, any such scalar can be written

$$I = c\, \mathrm{Pf} + \mathcal{Z}_I + \mathrm{div}\, V, \tag{1.2}$$

where $c \in \mathbb{R}$, Pf denotes the Pfaffian of the curvature tensor, $\mathcal{Z}_I$ is a natural scalar that is pointwise conformally invariant of weight $-n$, and V is a natural vector field. In their proof of Equation (1.1), Chang, Qing, and Yang applied Alexakis' result with I equal to Branson's [Bra95] critical Q-curvature. The proof shows that the conformal invariant in (1.1) is given by $\mathcal{Z} = \frac{(-1)^{n/2}}{(n-1)!}\mathcal{Z}_Q$.

Our main result is an analogue of Equation (1.1) for even-dimensional minimal submanifolds of Poincaré–Einstein manifolds, assuming a special case of a submanifold version of Alexakis' result. We first formulate this submanifold version as a conjecture.

Conjecture 1.1. Let $k, n \in \mathbb{N}$ with k even and $n > k$. Suppose that I is a natural scalar on k-dimensional submanifolds Y of n-dimensional Riemannian manifolds (X, g) whose integral over compact Y is invariant under conformal rescaling of g. Then there is a natural submanifold scalar $\mathcal{W}_I$ that is pointwise conformally invariant of weight $-k$, a natural submanifold vector field V, and a constant $c \in \mathbb{R}$ so that

$$I = c\, \overline{\mathrm{Pf}} + \mathcal{W}_I + \overline{\mathrm{div}}\, V. \tag{1.3}$$

Here $\overline{\mathrm{div}}$ and $\overline{\mathrm{Pf}}$ are the divergence operator and the Pfaffian of the Riemannian curvature tensor, respectively, for the metric induced on Y by g.

By convention, manifolds and submanifolds are without boundary unless otherwise specified. See Section 2 for a discussion of natural submanifold tensors.

An extrinsic submanifold version of Branson's critical Q-curvature was defined by Case, Graham, and Kuo [CGK23]. For $k, n \in \mathbb{N}$ with k even and $n > k$, this Q-curvature is a natural submanifold scalar whose integral over compact Y is conformally invariant. It will be reviewed in Section 2. Henceforth in this paper, by Q we will always mean this critical extrinsic submanifold Q-curvature.

Our main theorem is the following.

Theorem 1.2. *Let $k, n \in \mathbb{N}$ with k even and $n > k$. Suppose that* (1.3) *holds for $I = Q$ on k-dimensional submanifolds of n-manifolds, with $\mathcal{W}_Q$ and V as in the statement of Conjecture 1.1. If (X^n, g_+) is a Poincaré–Einstein manifold, Y^k is a smooth compact manifold with boundary, and $i \colon Y \to (X, g_+)$ is a polyhomogeneous minimal immersion, then*

$$\mathcal{A} = c_k \chi(Y) + \frac{(-1)^{k/2}}{(k-1)!} \int_Y \mathcal{W}_Q \, \mathrm{dA}, \tag{1.4}$$

where $\mathcal{A}$ denotes the renormalized area of Y and c_k is as in Equation (1.1).

Importantly, the conformal invariance of $\mathcal{W}_Q$ implies that $\int_Y \mathcal{W}_Q \, \mathrm{dA}$ converges.

Explicit formulas for Q for $k = 2, 4$ are derived in [CGK23]. The formula [CGK23, Equation (5.14)] for $k = 2$ already exhibits a decomposition of the form (1.3):

$$Q = \overline{\mathrm{Pf}} + \mathcal{W}_Q, \qquad \mathcal{W}_Q = \frac{1}{2}|\mathring{L}|^2 - W^T.$$

Here $|\mathring{L}|^2$ is the squared norm of the trace-free part of the second fundamental form and $W^T := W(e_1, e_2, e_1, e_2)$ is the tangential component of the background Weyl tensor, where e_1, e_2 is an orthonormal basis for TY. In this case, the resulting Gauss–Bonnet formula (1.4) was derived by Alexakis and Mazzeo [AM10] using another method.

When $k = 4$, the formula for Q in [CGK23, Equation (5.14)] is not decomposed in the form (1.3). The main issue is recognizing $\mathcal{W}_Q$, since the obvious scalar conformal submanifold invariants constructed from the trace-free second fundamental form and the Weyl tensor are not sufficient to produce such a decomposition for Q. In Subsection 5.1 below we identify four non-obvious scalar conformal submanifold invariants of weight -4 for submanifolds of general dimension and codimension. One of them, which we denote $\mathcal{I}$, takes the following form when $k = 4$:

$$\mathcal{I} = -3\overline{\Delta}\mathsf{G} - 4\langle \mathsf{F} - \mathsf{G} i^* g, \overline{\mathsf{P}} + \mathsf{F} \rangle - 2\overline{\nabla}^{\alpha}\big(\mathcal{C}_{\alpha} - \mathsf{D}^{\beta\alpha'}\mathring{L}_{\alpha\beta\alpha'}\big) - \frac{2}{n-4}\mathcal{B}_{\alpha}{}^{\alpha} + 2|\mathsf{D}|^2.$$

Here F is the conformally invariant Fialkow tensor given in Equation (4.9), G is its trace, $\overline{\mathsf{P}}$ is the Schouten tensor for the induced metric, $\mathring{L}$ the trace-free second fundamental form, and

$$\begin{aligned}
\mathsf{D}_{\alpha\alpha'} &:= \mathsf{P}_{\alpha\alpha'} - \overline{\nabla}_{\alpha} H_{\alpha'}, \\
\mathcal{C}_{\alpha} &:= C_{\beta\alpha}{}^{\beta} - H^{\alpha'} W_{\beta\alpha}{}^{\beta}{}_{\alpha'}, \\
\mathcal{B}_{\alpha}{}^{\alpha} &:= B_{\alpha}{}^{\alpha} - 2(n-4) H^{\alpha'} C_{\alpha\alpha'}{}^{\alpha} + (n-4) H^{\alpha'} H^{\beta'} W_{\alpha'\alpha\beta'}{}^{\alpha},
\end{aligned}$$

where P is the background Schouten tensor, H is the mean curvature vector, and B, C, and W are the background Bach, Cotton, and Weyl tensors, respectively. (Our notational conventions are described in Section 4.) In Subsection 5.2 we show that the formula [CGK23, Equation (5.14)] for Q when $k = 4$ can be rewritten in the form (1.3) with

$$\mathcal{W}_Q = -\frac{1}{4}|\overline{W}|^2 + \mathcal{I} + 2|\mathsf{F}|^2 - 2\mathsf{G}^2,$$

where $|\overline{W}|^2$ is the squared length of the Weyl tensor of the induced metric. Theorem 1.2 then immediately implies the following:

Corollary 1.3. *Let $i\colon Y^4 \to (X^n, g_+)$ be a polyhomogeneous minimal immersion into a Poincaré–Einstein manifold of dimension $n \geq 5$. Then*

$$\mathcal{A} = \frac{4\pi^2}{3}\chi(Y) - \frac{1}{6}\int_Y \left(\frac{1}{4}|\overline{W}|^2 - \mathcal{I} - 2|\mathsf{F}|^2 + 2\mathsf{G}^2\right) \mathrm{dA}\,. \tag{1.5}$$

Tyrrell [Tyr23] derived a formula equivalent to Equation (1.5) in the $n = 5$ hypersurface case. Tyrrell's integrand is a natural submanifold scalar that agrees with our integrand for minimal hypersurfaces of Poincaré–Einstein manifolds. But his integrand is not conformally invariant and an analysis of the asymptotics of its summands was required to establish convergence of the integral. See Remark 5.10 below for further discussion.

Decompositions of the form (1.3) are not unique when $k = 4$. In Proposition 5.2 we identify two scalar conformal submanifold invariants $\mathcal{K}_1$ and $\mathcal{K}_2$ of weight -4 that are divergences when $k = 4$. Theorem 1.2 implies that the Gauss–Bonnet formula (1.4) holds for any $\mathcal{W}_Q$ that arises in such a decomposition for Q.

The main reason that it is important to have a conformally invariant integrand in Equations (1.1) and (1.4) is to guarantee convergence of the integral. The formulas then give a concrete expression for the outcome of the renormalization process applied to volume or area. There are other formulas of Gauss–Bonnet type in this setting, for instance in [Alb09] for Poincaré–Einstein manifolds and in [TT20] for minimal submanifolds. But these formulas involve renormalized quantities other than just the volume or area. A drawback of Equations (1.1) and (1.4) is that they require identifying a conformal invariant in the decomposition of Q-curvature to become explicit.

Our proof of Theorem 1.2 follows the same outline as the proof in [CQY08] of Equation (1.1). But our geometric situation is more complicated and we introduce two modifications which simplify the argument. Important properties of the extrinsic Q-curvature which enter are its linear transformation law in terms of a critical extrinsic GJMS operator and the fact that both of these objects have an explicit factorization for minimal submanifolds of Einstein manifolds. These properties are established in [CGK23]. The other main ingredient in the proof is the existence and properties of what we call the scattering potential. This is a solution of a linear scalar equation on the submanifold whose asymptotic expansion contains a term which produces the renormalized area when integrated over the boundary. The scattering potential was introduced in [FG02] in the setting of renormalized volume. The scattering potential determines a specific compactification of the metric induced on Y by g_+, called the scattering compactification, whose Q-curvature vanishes identically. (In the setting of Poincaré–Einstein manifolds, this is sometimes called the Fefferman–Graham compactification.) In our formulation of the argument, the decomposition of Q is applied to a geodesic compactification and the renormalized area arises via the integrated asymptotics of the scattering potential upon conformally transforming to the scattering compactification.

The following theorem is a result analogous to Theorem 1.2 for compact minimal submanifolds of Einstein manifolds, in which the renormalized area is replaced by the area. In this case there are no convergence issues and the result follows immediately upon integrating the decomposition (1.3) for the extrinsic Q-curvature determined by the background Einstein metric itself.

Theorem 1.4. *Let $k, n \in \mathbb{N}$ with k even and $n > k$. Suppose that (1.3) holds for $I = Q$ on k-dimensional submanifolds of n-manifolds, with $\mathcal{W}_Q$ and V as in the statement of Conjecture 1.1. If (X^n, g) satisfies $\mathrm{Ric}(g) = \lambda(n-1)g$ and if $i\colon Y^k \to (X, g)$ is a minimal immersion of a compact manifold, then*

$$\lambda^{k/2} A = \frac{(2\pi)^{k/2}}{(k-1)!!}\chi(Y) + \frac{1}{(k-1)!}\int_Y \mathcal{W}_Q \,\mathrm{dA}, \tag{1.6}$$

where A denotes the area of Y.

In particular, if $\dim Y = 2$, then

$$\lambda A = 2\pi\chi(Y) + \int_Y \left(\frac{1}{2}|\mathring{L}|^2 - W^T\right)\mathrm{dA}, \tag{1.7}$$

and if $\dim Y = 4$, then

$$\lambda^2 A = \frac{4\pi^2}{3}\chi(Y) - \frac{1}{6}\int_Y \left(\frac{1}{4}|\overline{W}|^2 - \mathcal{I} - 2|\mathsf{F}|^2 + 2\mathsf{G}^2\right)\mathrm{dA}\,. \tag{1.8}$$

Equation (1.7) follows immediately by integrating the Gauss equation (4.8c). It has been used, for example, in the classification of minimal surfaces in S^3 of index at most five [Urb90, p. 991] and in the study of a class of immersed surfaces in self-dual Einstein manifolds [Fri84, Section 2].

Our second result is the verification of Conjecture 1.1 for submanifolds of dimension two or four.

Theorem 1.5. *Conjecture 1.1 is true when $k = 2$ and $k = 4$.*

The main ingredient in our proof of Theorem 1.5 is the identification of a well-chosen spanning set for the space of natural submanifold scalars of weight $-k$ modulo scalar conformal submanifold invariants and tangential divergences. When $k = 4$, we need to use two non-obvious scalar conformal submanifold invariants $\mathcal{I}$ and $\mathcal{J}$ identified in Subsection 5.1 to eliminate potential elements from the spanning set. The cardinality of our spanning set is 3 when $k = 2$ and 33 when $k = 4$. We then calculate directly the conformal variation of the integral of a linear combination of the elements of this set to argue that if this integral is conformally invariant, then the linear combination must be proportional to the Pfaffian modulo a conformal invariant.

Various cases of Conjecture 1.1 have been considered in the literature. Mondino and Nguyen [MN18] discussed natural submanifold scalars whose integrals are conformally invariant. They gave a characterization of a family of such scalars involving the curvature and second fundamental form, but not their derivatives, for surfaces of codimension at most two and hypersurfaces in general dimension. Juhl [Juh23] verified Conjecture 1.1 when $k = 4$ for the singular Yamabe extrinsic Q-curvature on hypersurfaces derived by Gover and Waldron [GW21] and conjectured the decomposition (1.3) of this invariant for even $k > 4$.

The invariants that enter the Gauss–Bonnet formula (1.4) are even; i.e. they are unchanged under changes of orientation. Throughout this paper we only consider even invariants.

This paper is organized as follows:

In Section 2 we define natural submanifold tensors and differential operators and review properties of the extrinsic Q-curvature and the scattering compactification which we will use in the proof of Theorem 1.2.

In Section 3 we prove Theorems 1.2 and 1.4.

In Section 4 we review background material concerning Riemannian and conformal submanifold geometry and fix our notational conventions. In Subsection 4.3 we also introduce three tensors $\mathcal{P}$, $\mathcal{C}$, and $\mathcal{B}$ that are modifications of projections of the background Schouten, Cotton, and Bach tensors, respectively. These tensors play an important role in our subsequent analysis; one reason for their importance is that their conformal transformation laws only involve tangential derivatives of the conformal factor.

Section 5 contains three subsections. In Subsection 5.1 we introduce four non-obvious scalar conformal submanifold invariants $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{I}$, $\mathcal{J}$ of weight -4 in general dimension and codimension. In Subsection 5.2 we use $\mathcal{I}$ to derive a decomposition in general dimension of the fourth-order Q-curvature defined in [CGK23]; this specializes to the form (1.3) when $k = 4$. Corollary 1.3 and Equation (1.8) are immediate consequences. In Subsection 5.3 we state without proof how some invariants previously found in special cases by Blitz, Gover, and Waldron [BGW21], Juhl [Juh23], Astaneh and Solodukhin [AS21], and Chalabi, Herzog, O'Bannon, Robinson, and Sisti [CHO$^+$22] can be written in terms of our invariants constructed in Subsection 5.1, thus generalizing their constructions to general dimension and codimension.

In Section 6 we prove Theorem 1.5. As indicated above, the proof uses our invariants $\mathcal{I}$ and $\mathcal{J}$ from Subsection 5.1.

In Appendix A we give the details of the computations omitted in Subsection 5.3.

2. Extrinsic Q-curvature and scattering compactification

2.1. **Natural submanifold tensors and extrinsic Q-curvature.** We will be dealing with immersions $i\colon Y^k \to (X^n, g)$ into Riemannian manifolds. The pullback bundle i^*TX admits the g-orthogonal splitting $i^*TX = TY \oplus NY$. We use the induced metrics on TY and NY to identify these bundles with their respective duals, T^*Y and N^*Y. The Levi-Civita connection ∇ of g determines connections $\overline{\nabla}$ on TY and NY by projection. On TY this connection is the Levi-Civita connection of i^*g. We use bars to denote intrinsic quantities on Y. For example, Rm denotes the curvature tensor of g and $\overline{\mathrm{Rm}}$ the curvature tensor of i^*g; also, Δ denotes the Laplacian of g and $\overline{\Delta}$ the Laplacian of i^*g. Our sign convention is $\Delta = \sum \partial_i^2$ on Euclidean space. The second fundamental form $L : S^2TY \to NY$ is defined by $L(U,V) = (\nabla_U V)^\perp$. If $i\colon Y^k \to (X^n, [g])$ is an immersion into a conformal manifold, then there is an induced conformal class $i^*[g]$ on Y. In $i^*[g]$, we allow rescalings by arbitrary functions in $C^\infty(Y)$, not just pullbacks of functions in $C^\infty(X)$.

Let k, $n \in \mathbb{N}$ with $n > k$. A natural tensor on k-dimensional submanifolds of n-dimensional Riemannian manifolds, or a **natural submanifold tensor**, is an assignment to each immersion $i\colon Y^k \to (X^n, g)$ of a section of $(T^*Y)^{\otimes r} \otimes (N^*Y)^{\otimes s}$ for some integers $r, s \geq 0$ which can be expressed as an $\mathbb{R}$-linear combination of partial contractions of tensors

$$\pi_1(\nabla^{M_1} \mathrm{Rm}) \otimes \cdots \otimes \pi_p(\nabla^{M_p} \mathrm{Rm}) \otimes \overline{\nabla}^{N_1} L \otimes \cdots \otimes \overline{\nabla}^{N_q} L \otimes \pi(g^{\otimes P}). \tag{2.1}$$

Here M_j, N_j, and P denote powers, and π and π_j denote restriction to Y followed by projection to either TY or NY in each index. The tensor displayed at (2.1) is viewed as covariant in all indices (that is, with all indices lowered), and the contractions are taken with respect to the metrics induced by g on TY and NY for partial pairings of tangential and normal indices. A natural submanifold tensor T has **weight** $w \in \mathbb{R}$ if $T^{c^2 g} = c^w T^g$ for all $c > 0$. A natural submanifold tensor T is **conformally invariant** of weight w, or a **conformal submanifold invariant**, if $T^{e^{2\Upsilon} g} = e^{w i^* \Upsilon} T^g$ for all immersions $i\colon Y^k \to (X^n, g)$ and all $\Upsilon \in C^\infty(X)$.

A (linear) **natural submanifold differential operator** on k-dimensional submanifolds of n-dimensional Riemannian manifolds is an assignment to each immersion $i\colon Y^k \to (X^n, g)$ of a differential operator

$$D\colon C^\infty(Y) \to C^\infty(Y; (T^*Y)^{\otimes r} \otimes (N^*Y)^{\otimes s})$$

for some integers $r, s \geq 0$ which can be expressed as an $\mathbb{R}$-linear combination of partial contractions of terms of the form

$$\pi_1(\nabla^{M_1} \mathrm{Rm}) \otimes \cdots \otimes \pi_p(\nabla^{M_p} \mathrm{Rm}) \otimes \overline{\nabla}^{N_1} L \otimes \cdots \otimes \overline{\nabla}^{N_q} L \otimes \pi(g^{\otimes P}) \otimes \overline{\nabla}^{Q}.$$

(In this paper, we only need to consider natural operators acting on scalars.) For both natural submanifold tensors and natural submanifold differential operators, some of the free indices can be raised to view the tensor as contravariant in these indices.

Case, Graham, and Kuo [CGK23] constructed **extrinsic GJMS operators** and **extrinsic Q-curvatures** associated to a submanifold of a conformal manifold which satisfy covariance relations under conformal change. The objects of critical weight are the ones that are relevant to the proof of Theorem 1.2. As discussed in [CGK23], the construction applies to immersed submanifolds and the conclusion can be formulated as follows:

Proposition 2.1 ([CGK23, Theorem 1.1])**.** *Let $(X^n, [g])$, $n \geq 3$, be a conformal manifold and let $i\colon Y^k \to X^n$ be an immersion with $2 \leq k < n$ and k even. For each $h \in i^*[g]$, there is a formally self-adjoint differential operator $P_k\colon C^\infty(Y) \to C^\infty(Y)$ and a scalar function $Q \in C^\infty(Y)$ such that $P_k(1) = 0$,*

$$P_k = (-\overline{\Delta})^{k/2} + \mathrm{l.\,o.\,t.}, \tag{2.2}$$

and if $\widehat{h} = e^{2\Upsilon} h$ with $\Upsilon \in C^\infty(Y)$, then

$$\begin{aligned} e^{k\Upsilon} P_k^{\widehat{h}} &= P_k^h, \\ e^{k\Upsilon} Q^{\widehat{h}} &= Q^h + P_k^h(\Upsilon). \end{aligned} \tag{2.3}$$

In Equation (2.2), l. o. t. denotes a differential operator on Y of order at most $k-2$.

Note that P_k and Q depend on the conformal class $[g]$ on X and the choice of $h \in i^*[g]$. We can also view P_k and Q as determined simply by a choice of metric g on X, since g determines $[g]$ and the induced metric $h = i^*g$. When viewed this way, P_k is a natural submanifold differential operator and Q is a natural submanifold scalar. (A different definition of natural submanifold differential operators and scalars is used in [CGK23]. The definitions will be shown to be equivalent in [GK].)

Proposition 2.1 implies that if Y is compact, then the total Q-curvature is conformally invariant:

$$\int_Y Q^{\widehat{h}}\,\mathrm{dA}_{\widehat{h}} = \int_Y Q^h\,\mathrm{dA}_h\,.$$

An essential property of the operators and Q-curvatures constructed in [CGK23] is their factorization for minimal submanifolds of Einstein manifolds. We rely on the specialization of this fact in the critical-order case:

Proposition 2.2 ([CGK23, Theorem 1.2])**.** *Let $i\colon Y^k \to (X^n, g)$ be a minimal immersion, where k is even and g is Einstein with $\mathrm{Ric}(g) = \lambda(n-1)g$. Then*

$$\begin{aligned} P_k &= \prod_{j=1}^{k/2} \left(-\overline{\Delta} + \lambda(\tfrac{k}{2}+j-1)(\tfrac{k}{2}-j)\right), \\ Q &= \lambda^{k/2}(k-1)!. \end{aligned}$$

2.2. **Asymptotics and scattering compactification.** If X is a manifold with nonempty boundary, by a **boundary identification** we mean a diffeomorphism from a collar neighborhood of ∂X to $\partial X \times [0,\epsilon)_r$ for some $\epsilon > 0$, for which $\partial X \ni p \to (p, 0)$. A function or tensor defined on X is **polyhomogeneous** if in a boundary identification it has an asymptotic expansion in powers of r and nonnegative integral powers of $\log r$ whose coefficients are smooth on ∂X. This is an informal formulation; see, for example, [Gri01] for a detailed presentation. This polyhomogeneity condition is independent of the choice of boundary identification.

Let (X^n, g_+), $n \geq 3$, be a Poincaré–Einstein manifold with conformal infinity $(\partial X, \mathfrak{c})$. That is, X is a compact connected manifold with nonempty boundary, g_+ is an asymptotically hyperbolic metric satisfying $\mathrm{Ric}(g_+) = -(n-1)g_+$, and $\mathfrak{c} = \left[r^2 g_+|_{T\partial X}\right]$, where r is a defining function for ∂X. A representative $g_{(0)} \in \mathfrak{c}$ uniquely determines [GL91, Lemma 5.2] a defining function r near ∂X such that $r^2 g_+|_{T\partial X} = g_{(0)}$ and $|dr|^2_{r^2 g_+} = 1$. We call r the **geodesic defining function** and $r^2 g_+$ the **geodesic compactification** determined by $g_{(0)}$. The metric $g_{(0)}$ also determines, for some $\epsilon > 0$, a boundary identification with respect to which

$$r^2 g_+ = dr^2 + g_r.$$

Here [CDLS05, Theorem A; BH14, Théorème 1] g_r is a one-parameter family of metrics on ∂X which is smooth if n is even or $n = 3$, and is polyhomogeneous if $n > 3$ is odd. The expansion of g_r has the form [FG12, Theorem 4.8]

$$\begin{aligned} &g_r = g_{(0)} + g_{(2)} r^2 + \cdots, && n = 3, \\ &g_r = g_{(0)} + \cdots + g_{(n-2)} r^{n-2} + g_{(n-1)} r^{n-1} + \cdots, && n \text{ even}, \\ &g_r = g_{(0)} + \cdots + g_{(n-3)} r^{n-3} + \kappa r^{n-1}\log r + g_{(n-1)} r^{n-1} + \cdots, && \text{otherwise}, \end{aligned} \tag{2.4}$$

where the coefficients $g_{(j)}$ and κ are smooth symmetric 2-tensors on ∂X. Terms r^j do not occur for odd $j < n-1$.

Let (X^n, g_+) be Poincaré–Einstein and let Y^k be a compact manifold with nonempty boundary. Let $i\colon Y \to X$ be an immersion that is C^∞ on the interior $\mathring{Y}$, is a C^1 embedding in a neighborhood of ∂Y, satisfies $i(\partial Y) \subset \partial X$ with $i(Y)$ transverse to ∂X, and is such that $i|_{\partial Y}$ is a C^∞ embedding into ∂X. If $(x^\alpha, u^{\alpha'})$,

$1 \le \alpha \le k-1$, $1 \le \alpha' \le n-k$, are local coordinates on ∂X with $i(\partial Y) = \{u = 0\}$, then in the boundary identification induced by $g_{(0)} \in \mathfrak{c}$, we may write $i(Y)$ in the form $i(Y) = \{u = u(x,r)\}$. We say that i is a **polyhomogeneous immersion** if the graphing map $u(x,r)$ is polyhomogeneous. This condition is independent of the choice of coordinates (x,u) on ∂X. Since $r^2 g_+$ is itself polyhomogeneous, the transition maps relating boundary identifications determined by different choices of $g_{(0)}$ themselves have polyhomogeneous expansions, so the condition that i is a polyhomogeneous immersion is also independent of the choice of $g_{(0)}$.

Our analysis requires that the minimal immersions under consideration are polyhomogeneous (at least to some order). We will simply assume this to be the case. There are results establishing the polyhomogeneity of minimal submanifolds under certain initial regularity hypotheses [AM10, Proposition 2.2; HJ23, Theorem 1.1; Mar21, Theorem 3.1].

We need a global invariant description of the asymptotics of minimal submanifolds. We use the normal exponential map of $\Sigma := i(\partial Y)$ for this purpose as in [GR20]. Let $i : Y^k \to (X^n, g_+)$ be a polyhomogeneous immersion. Choose $g_{(0)} \in \mathfrak{c}$. Use the boundary identification determined by $g_{(0)}$ to identify a neighborhood of ∂X in X with $\partial X \times [0,\epsilon)_r$. For $r \ge 0$ small, let $\Sigma_r \subset \partial X$ denote the slice of $i(Y)$ at height r; i.e. $i(Y) \cap (\partial X \times \{r\}) = \Sigma_r \times \{r\}$. Then Σ_r is a smooth submanifold of ∂X of dimension $k-1$ and $\Sigma_0 = \Sigma$. The normal exponential map of Σ with respect to $g_{(0)}$, denoted $\exp_\Sigma$, defines a diffeomorphism of a neighborhood of the zero section in $N\Sigma$ to a neighborhood of Σ in ∂X. So there is a unique section $U_r \in \Gamma(N\Sigma)$ near $r = 0$ such that $\exp_\Sigma\{U_r(p) : p \in \Sigma\} = \Sigma_r$. This defines a polyhomogeneous 1-parameter family U_r of sections of $N\Sigma$ for which we have

$$i(Y) = \bigl\{ \bigl(\exp_\Sigma U_r(p), r \bigr) : p \in \Sigma, r \ge 0 \bigr\}. \tag{2.5}$$

The inverse normal exponential map can be used to define a boundary identification for $i(Y)$. Define a diffeomorphism ψ from a neighborhood of Σ in $i(Y)$ to $\Sigma \times [0,\epsilon)$ for some $\epsilon > 0$ by

$$\psi(q,r) := (\pi((\exp_\Sigma)^{-1} q), r), \qquad (q,r) \in i(Y) \subset \partial X \times [0,\epsilon),$$

where $\pi \colon N\Sigma \to \Sigma$ is the canonical projection. Then ψ is a boundary identification for $i(Y)$.

Choose local coordinates $\{x^\alpha : 1 \le \alpha \le k-1\}$ for Σ and a local frame $\{e_{\alpha'}(x) : 1 \le \alpha' \le n-k\}$ for $N\Sigma$. The map $\exp_\Sigma \bigl(u^{\alpha'} e_{\alpha'}(x) \bigr) \mapsto (x,u)$ defines a geodesic normal coordinate system for ∂X near Σ with respect to which $\Sigma = \{u = 0\}$. Extend the coordinates (x,u) to $\partial X \times [0,\epsilon_0) \subset X$ to be constant in r. In these coordinates, the diffeomorphism ψ is given by $\psi(x,u,r) = (x,r)$. Express the section U_r in (2.5) as $U_r(x) = u^{\alpha'}(x,r) e_{\alpha'}(x)$; then $i(Y)$ is locally the graph $\{u = u(x,r)\}$. A function defined on $i(Y)$ near ∂X can be uniquely extended to a neighborhood of $i(Y)$ in X near ∂X by making it independent of the u variables. Since varying u gives the fibers of $\pi : N\Sigma \to \Sigma$, this extension is independent of the choice of coordinates and is globally defined near ∂X.

If $i \colon Y^k \to (X^n, g_+)$ is a polyhomogeneous minimal immersion, the form of its expansion can be formally calculated [GW99, GR20] from the minimal submanifold equation $H = 0$. Henceforth we assume that k is even. In this case,

$$U_r = U_{(2)} r^2 + U_{(4)} r^4 + \cdots + U_{(k)} r^k + U_{(k+1)} r^{k+1} + O(r^{k+2} \log r), \tag{2.6}$$

where the $U_{(j)}$ are global sections of $N\Sigma$. Equivalently, in local coordinates (x, u, r) as above, the expansion of $u(x, r)$ takes the form

$$u = u_{(2)}r^2 + u_{(4)}r^4 + \cdots + u_{(k)}r^k + u_{(k+1)}r^{k+1} + O(r^{k+2}\log r), \tag{2.7}$$

where the $u_{(j)}$ are functions of x. The log terms in (2.6) and (2.7) come only from the log terms in g_r: If n is even or $n = 3$, then g_r is smooth, and hence the expansion of $u(x, r)$ has no log terms. If $n > 3$ is odd, then the $r^{n-1}\log r$ term in the expansion of g_r can generate a $r^{n+1}\log r$ term in the expansions of U_r and $u(x, r)$. Thus the $O(r^{k+2}\log r)$ terms are actually $O(r^{k+2})$ unless $n > 3$ is odd and $n = k + 1$.

Set $h_+ := i^*g_+$. The metric $\overline{h} := r^2h_+$ can be written in the coordinates (x, r):

$$\overline{h} = \overline{h}_{00}dr^2 + 2\overline{h}_{\alpha 0}drdx^\alpha + \overline{h}_{\alpha\beta}dx^\alpha dx^\beta, \tag{2.8}$$

where

$$\begin{aligned} \overline{h}_{\alpha\beta} &= g_{\alpha\beta} + 2g_{\alpha'(\alpha}u^{\alpha'}{}_{,\beta)} + g_{\alpha'\beta'}u^{\alpha'}{}_{,\alpha}u^{\beta'}{}_{,\beta}, \\ \overline{h}_{\alpha 0} &= g_{\alpha\alpha'}u^{\alpha'}{}_{,r} + g_{\alpha'\beta'}u^{\alpha'}{}_{,\alpha}u^{\beta'}{}_{,r}, \\ \overline{h}_{00} &= 1 + g_{\alpha'\beta'}u^{\alpha'}{}_{,r}u^{\beta'}{}_{,r}. \end{aligned} \tag{2.9}$$

We have used a '0' index for the r-direction and commas denote partial derivatives with respect to the coordinates (x^α, r) on Y. The components of $\overline{h}$ and the derivatives of u are evaluated at (x, r). The above formulas for the components of $\overline{h}$ were obtained from the pullback of r^2g_+ upon writing

$$g_r = g_{\alpha\beta}(x, u, r)dx^\alpha dx^\beta + 2g_{\alpha\alpha'}(x, u, r)dx^\alpha du^{\alpha'} + g_{\alpha'\beta'}(x, u, r)du^{\alpha'}du^{\beta'},$$

and all g_{ij} in (2.9) are understood to be evaluated at $(x, u(x, r), r)$.

The expansions of $\overline{h}$ are obtained by substituting (2.4) and (2.7) into (2.9). Observe first that $\overline{h}_{\alpha 0} = 0$ and $\overline{h}_{00} = 1$ at $r = 0$. Thus $|dr|^2_{\overline{h}} = 1$ on ∂Y, so h_+ is asymptotically hyperbolic. One also finds that the first odd term in the expansions of $\overline{h}_{\alpha\beta}$ and $\overline{h}_{00}$ occurs at order $k + 1$, and the first even term in the expansion of $\overline{h}_{\alpha 0}$ occurs at order $k + 2$. (Once again there can be log terms if $n > 3$ is odd: in this case the expansion of $\overline{h}_{\alpha\beta}$ can have a $r^{n-1}\log r$ term, the expansion of $\overline{h}_{\alpha 0}$ can have a $r^n\log r$ term, and the expansion of $\overline{h}_{00}$ can have a $r^{n+1}\log r$ term.)

Set $h_{(0)} := i^*g_{(0)}$. Let ρ be the geodesic defining function for h_+ determined by $h_{(0)}$ and let (y, ρ), $y \in \partial Y$, be the associated boundary identification for Y. For this discussion we identify Y with $i(Y)$ near ∂Y. The map $(x, r) \to (y, \rho)$ relating the the two boundary identifications is uniquely determined by the requirement that

$$\rho^2 h_+ = d\rho^2 + h_\rho \tag{2.10}$$

relative to the (y, ρ) boundary identification. The defining function ρ is determined by the eikonal equation $|d\rho|^2_{\rho^2h_+} = 1$ and then y is extended to Y by following the gradient flow of $\mathrm{grad}_{\rho^2h_+}(\rho)$ (see [GL91, Lemma 5.2 and the subsequent paragraph]). Analysis of the eikonal and gradient flow equations as in [Gui05, Lemma 2.1] shows that $y = y(x, r)$, $\rho = \rho(x, r)$, where the expansions of y and ρ have the form

$$\begin{aligned} y(x, r) &= x + y_{(2)}r^2 + \cdots + y_{(k-2)}r^{k-2} + O(r^k\log r), \\ \rho(x, r) &= r\big(1 + \rho_{(3)}r^2 + \cdots + \rho_{(k-1)}r^{k-2} + O(r^k\log r)\big). \end{aligned} \tag{2.11}$$

Here $y \in \partial Y$ is described in terms of its x-coordinate, and the coefficients $y_{(2j)}$ and $\rho_{(2j+1)}$ are functions of x.

The expansions of the inverse map have the same form:

$$\begin{aligned} x(y,\rho) &= y + x_{(2)}\rho^2 + \cdots + x_{(k-2)}\rho^{k-2} + O(\rho^k \log \rho), \\ r(y,\rho) &= \rho\big(1 + r_{(3)}\rho^2 + \cdots + r_{(k-1)}\rho^{k-2} + O(\rho^k \log \rho)\big), \end{aligned} \tag{2.12}$$

where the coefficients $x_{(2j)}$, $r_{(2j+1)}$ are functions of y. Now pull back $h_+ = r^{-2}\overline{h}$ by the transformation (2.12) and use the parity of the components of $\overline{h}$ discussed above to deduce that h_ρ in Equation (2.10) has the expansion

$$h_\rho = h_{(0)} + h_{(2)}\rho^2 + \cdots + h_{(k-2)}\rho^{k-2} + O(\rho^k \log \rho). \tag{2.13}$$

There are no log terms in any of the expansions (2.11), (2.12), (2.13) if n is even or $n = 3$. If $n > 3$ is odd, then the first log term in these expansions occurs at order $n-1$. Thus the remainder terms are actually $O(r^k)$ in (2.11) or $O(\rho^k)$ in (2.12), (2.13) unless $n > 3$ is odd and $n = k+1$.

The renormalized area $\mathcal{A}$ of Y was defined in [GW99] as the constant term in the asymptotic expansion of $\mathrm{Area}_Y\{r > \epsilon\}$ as $\epsilon \to 0$, and it was shown that $\mathcal{A}$ is independent of the choice of geodesic defining function r for g_+ on X. The same argument shows that $\mathcal{A}$ also equals the constant term in the asymptotic expansion of $\mathrm{Area}_Y\{\rho > \epsilon\}$, where as above ρ is a geodesic defining function for h_+ on Y. The argument only uses the parity of the terms in the expansions (2.11), (2.12) of r and ρ in terms of one another and the parity of the terms in (2.13). Since ρ is an intrinsic geodesic defining function on Y, the constant term in the asymptotic expansion of $\mathrm{Area}_Y\{\rho > \epsilon\}$ can also be interpreted as a renormalized volume of the asymptotically hyperbolic manifold (Y, h_+).

In [FG02, Theorems 4.1 and 4.3], the scattering theory of [GZ03] was applied to show that the renormalized volume of an asymptotically hyperbolic approximately Einstein manifold can be calculated as an integral over the boundary of a function that appears in the asymptotic expansion of a solution of a particular linear scalar equation. The same argument applies to any asymptotically hyperbolic metric that is even to the appropriate order. In particular, it applies to calculate the renormalized area of a minimal submanifold:

Proposition 2.3. *Let $i \colon Y^k \to (X^n, g_+)$ be a polyhomogeneous minimal immersion of an even-dimensional manifold into a Poincaré–Einstein manifold and let $h_{(0)}$ be a representative of the conformal infinity of $(Y, h_+ := i^* g_+)$. Then there is a unique $v \in C^\infty(\mathring{Y})$ such that*

$$\begin{cases} -\Delta_{h_+} v = k-1, & \text{in } \mathring{Y}, \\ v = \log \rho + o(1), & \text{near } \partial Y, \end{cases} \tag{2.14}$$

where ρ is the geodesic defining function determined by $h_{(0)}$. Moreover, near ∂Y,

$$v = \log \rho + F, \tag{2.15}$$

where F is polyhomogeneous with expansion

$$F = F_{(2)}\rho^2 + \cdots + F_{(k-2)}\rho^{k-2} + B\rho^{k-1} + O(\rho^k \log \rho). \tag{2.16}$$

The renormalized area of Y is given by:

$$\mathcal{A} = \int_{\partial Y} B|_{\partial Y} \, d\sigma, \tag{2.17}$$

where $d\sigma$ is the volume form of $h_{(0)}$.

Some remarks are in order. The existence of v uses scattering theory for the metric h_+. If $n > 3$ is odd, then $\rho^2 h_+$ is polyhomogeneous but not necessarily smooth. The extension of the scattering theory to the case of polyhomogeneous metrics is addressed in [CMY22]. Also, the same comments as above apply to the remainder term in (2.16). Namely, F is smooth if n is even or $n = 3$, and if $n > 3$ is odd, then the first log term occurs at order $n-1$, so that the remainder term is $O(\rho^k)$ unless $n = k+1$. Taking this into consideration, when $k = 2$, Equation (2.16) should be interpreted as $F = B\rho + O(\rho^2)$.

We call v the **scattering potential** determined by $h_{(0)}$. Equations (2.15) and (2.16) imply that e^v is a (polyhomogeneous) defining function for ∂Y. Therefore $\widehat{h} := e^{2v}h_+$ is a compactification of h_+, which we call the **scattering compactification**. The scattering compactification of a Poincaré–Einstein metric was introduced in [CQY08], where it was observed that it has vanishing Branson's Q-curvature. The attempt to find an analogy in the setting of minimal submanifolds of Poincaré–Einstein manifolds led to the introduction of the extrinsic Q-curvature in [CGK23], and the corresponding vanishing statement is likewise essential for the proof of Theorem 1.2:

Proposition 2.4. *Let $i\colon Y^k \to (X^n, g_+)$ be a polyhomogeneous minimal immersion of an even-dimensional manifold into a Poincaré–Einstein manifold and let $h_{(0)}$ be a representative of the conformal infinity of $(Y, h_+ := i^*g_+)$. Let v be the associated scattering potential and set $\widehat{h} := e^{2v}h_+$. Then $Q^{\widehat{h}} = 0$.*

Proof. Applying Equation (2.3), Proposition 2.2 and Equation (2.14) to h_+ and $\widehat{h} = e^{2v}h_+$ yields

$$\begin{aligned} e^{kv}Q^{\widehat{h}} &= Q^{h_+} + P_k^{h_+}v \\ &= (-1)^{\frac{k}{2}}(k-1)! + \left(\prod_{j=1}^{(k-2)/2}\left(-\Delta_{h_+} - \left(\tfrac{k-2}{2}+j\right)\left(\tfrac{k}{2}-j\right)\right)\right)(-\Delta_{h_+}v) \\ &= 0. \end{aligned}$$

□

3. Proofs of Theorems 1.2 and 1.4

We turn first to the proof of Theorem 1.2. Our first objective is to compute the coefficient c of the Pfaffian in the decomposition (1.3) for $I = Q$. Our normalization of the Pfaffian is such that the Chern–Gauss–Bonnet formula reads

$$(2\pi)^{k/2}\chi(Y) = \int_Y \mathrm{Pf}\,\mathrm{dV} \tag{3.1}$$

for compact Riemannian manifolds (Y^k, h) of even dimension k.

Lemma 3.1. *Let $k, n \in \mathbb{N}$ with k even and $n > k$. Suppose that there is a constant $c_{n,k} \in \mathbb{R}$, a scalar conformal submanifold invariant $\mathcal{W}_Q$, and a natural submanifold vector field V such that*

$$Q = c_{n,k}\,\overline{\mathrm{Pf}} + \mathcal{W}_Q + \overline{\mathrm{div}}\,V \tag{3.2}$$

for all Riemannian manifolds (X^n, g) and embedded submanifolds $Y^k \subset X^n$. Then

$$c_{n,k} = \frac{(k-1)!}{(k-1)!!}.$$

Proof. Consider an equatorial sphere $i\colon S^k \to (S^n, g)$ in the round n-sphere. By stereographic projection, the equatorial sphere is locally equivalent to the embedding $\mathbb{R}^k \hookrightarrow (\mathbb{R}^n, |dx|^2)$ of $\mathbb{R}^k$ as an affine subspace in flat Euclidean n-space. The latter is a totally geodesic submanifold of a flat manifold, and hence, by naturality, $\mathcal{W}_Q^{|dx|^2} = 0$. By conformal invariance, $\mathcal{W}_Q^g = 0$. Proposition 2.2 implies that $Q^{i^*g} = (k-1)!$. We conclude by integration that

$$2(2\pi)^{k/2} c_{n,k} = c_{n,k} \int_{S^k} \overline{\mathrm{Pf}}\, \mathrm{dA} = (k-1)!\, \mathrm{Vol}(S^k) = (k-1)! \times \frac{2(2\pi)^{k/2}}{(k-1)!!}.$$

Therefore $c_{n,k} = \frac{(k-1)!}{(k-1)!!}$. □

We introduce two simplifications to the argument of Chang, Qing, and Yang to help deal with the more complicated submanifold setting. The first is that we make a second conformal change so as to apply the conjectured decomposition (1.3) to the geodesic compactification instead of the scattering compactification. This gives a more direct route to the term producing the renormalized area in the Chern–Gauss–Bonnet formula. The second is that we provide a simpler version of the parity argument for the vanishing of the other terms.

Proof of Theorem 1.2. Set $h_+ = i^*g_+$. Pick a metric $g_{(0)} \in \mathfrak{c}$ and set $h_{(0)} = i^*g_{(0)}$. Let ρ be the geodesic defining function for h_+ determined by $h_{(0)}$. Regard ρ as defined on $i(Y)$ near $i(\partial Y)$. Extend ρ to a neighborhood of $i(Y)$ in X near ∂X by requiring it to be independent of the u variables in coordinates (x, u, r) as described in Section 2. Now choose some positive smooth extension of ρ to a neighborhood of $i(Y)$ in all of X and set $g := \rho^2 g_+$ and $h := i^*g$. Then $h = d\rho^2 + h_\rho$ near ∂Y in the boundary identification determined by $h_{(0)}$.

Let v be the scattering potential determined by $h_{(0)}$ as in Proposition 2.3 and let $\widehat{h} = e^{2v}h_+$ be the associated scattering compactification. Extend F so that Equation (2.15) holds on all of Y; then $e^v = \rho e^F$, and so $\widehat{h} = e^{2F}h$ on all of Y. Extend F to a neighborhood of $i(Y)$ in X near ∂X by requiring it to be independent of the u variables.

Proposition 2.4 followed by Equation (2.3) gives

$$0 = e^{kF} Q^{\widehat{h}} = Q^h + P_k^h(F).$$

Now write the assumed decomposition (1.3) for Q^h and use Lemma 3.1 to obtain

$$\frac{(k-1)!}{(k-1)!!}\overline{\mathrm{Pf}}^h + \mathcal{W}_Q^g + \overline{\mathrm{div}}^h\, V^g + P_k^h(F) = 0. \tag{3.3}$$

On a manifold with boundary, there is an additional boundary integral in the Chern–Gauss–Bonnet formula (3.1), but it vanishes if the boundary is totally geodesic. The expansion (2.13) of h_ρ implies that ∂Y is totally geodesic for h. So integrating Equation (3.3) gives

$$\frac{(k-1)!}{(k-1)!!}(2\pi)^{k/2}\chi(Y) + \int_Y \mathcal{W}_Q\, \mathrm{dA} + \int_{\partial Y} \langle \mathsf{n}, V^g\rangle\, d\sigma + \int_Y P_k^h(F)\, \mathrm{dA} = 0, \tag{3.4}$$

where $\mathsf{n} = -\partial_\rho$ is the h-outward pointing unit normal along ∂Y. The proof will be completed by showing that $\langle \mathsf{n}, V^g\rangle = 0$ and $\int_Y P_k^h(F)\, \mathrm{dA} = -(-1)^{k/2}(k-1)!\, \mathcal{A}$.

By definition, $V = V^g$ is a linear combination of partial contractions with one free raised tangential index of tensors of the form (2.1). The condition that $\overline{\operatorname{div}}\, V$ has weight $-k$ is equivalent to

$$\sum_{i=1}^{p}(M_i + 2) + \sum_{j=1}^{q}(N_j + 1) = k - 1.$$

It follows that each $M_i \le k-3$ and each $N_j \le k-2$. Since $\nabla^M \operatorname{Rm}$ and $\overline{\nabla}^N L$ depend on at most $M+2$ and $N+1$ derivatives of g, respectively, we see that V depends on at most $k-1$ derivatives of g. Also, since $\overline{\nabla}^N L$ depends on at most $N+2$ derivatives of (defining functions for) Y, we see that V depends on at most k derivatives of Y.

Consider the expansion at $r=0$ of $g = (\rho/r)^2(dr^2 + g_r)$ in the (x,u,r) coordinates. Using the expansion (2.4) for g_r, the expansion (2.11) for ρ/r, and the fact that ρ was extended to be independent of u, it follows that the expansion of g is even through order $k-2$ and has no r^{k-1} term. An evenness argument implies $\langle \mathsf{n}, V^g\rangle = 0$: Define g^0 by truncating the expansion of g at order $k-1$. Likewise, define U_r^0 by truncating the expansion (2.6) of U_r at order k (keeping the term of order k). Define Y^0 by (2.5) with U_r replaced by U_r^0 and define V^0 to be the natural vector field determined by g^0 and Y^0. Then $V = V^0$ at $r=0$. Since they are polynomial and even in r, both g^0 and U^0 extend to $r \in (-\epsilon,\epsilon)$ and are invariant under the reflection $\mathcal{R}(p,r) := (p,-r)$ on $\partial X \times (-\epsilon,\epsilon)_r$. Extend Y^0 to $r<0$ by (2.5) with Y replaced by Y^0 and U_r replaced by U_r^0. Naturality implies that $\mathcal{R}^*V^0 = V^0$. Hence $\mathcal{R}^*V = V$ at $r=0$. But $\mathcal{R}^*\mathsf{n} = -\mathsf{n}$, and hence $\mathcal{R}^*\langle \mathsf{n}, V\rangle = -\langle \mathsf{n}, V\rangle$ at $r=0$. We conclude that $\langle \mathsf{n}, V\rangle = 0$.

Now F has the expansion (2.16) in the boundary identification (y,ρ) induced by $h_{(0)}$. This is related to the (x,r) boundary identification on Y by (2.11). Since $y(x,r)$ and ρ/r have even expansions to order $k-2$ with no r^{k-1} term and $\rho/r = 1$ on ∂Y, it follows that the expansion of F in the (x,r) boundary identification has the same form:

$$F = \widetilde{F}_{(2)}r^2 + \cdots + \widetilde{F}_{(k-2)}r^{k-2} + Br^{k-1} + O(r^k \log r),$$

where the $\widetilde{F}_{(2j)}$ are functions of x and B is the same as in (2.16). This expansion also holds in a neighborhood of $i(Y)$ in X near ∂X since F was extended to be independent of the u variables.

Recall that P_k is a natural formally self-adjoint operator with leading term $(-\overline{\Delta})^{k/2}$ that annihilates constants. Therefore there is a natural TY-valued differential operator $T\colon C^\infty(Y) \to C^\infty(Y;TY)$ of order at most $k-3$ such that

$$P_k = (-\overline{\Delta})^{k/2} + \overline{\operatorname{div}} \circ T.$$

Therefore

$$\int_Y P_k^h(F)\,\mathrm{dA} = \int_{\partial Y} \Big(-\langle \mathsf{n}, \operatorname{grad}_h(-\Delta_h)^{(k-2)/2}(F^0 + Br^{k-1})\rangle + \langle \mathsf{n}, T(F^0)\rangle \Big)\, d\sigma,$$

where $F^0 := \widetilde{F}_{(2)}r^2 + \cdots + \widetilde{F}_{(k-2)}r^{k-2}$. On the one hand, the evenness argument above shows that

$$\langle \mathsf{n}, \operatorname{grad}_h(\Delta_h)^{(k-2)/2}(F^0)\rangle = 0, \qquad \langle \mathsf{n}, T(F^0)\rangle = 0.$$

On the other hand, there is a differential operator T' of transverse order strictly less than $k-1$ so that

$$\begin{aligned}\langle \mathsf{n}, \operatorname{grad}_h(-\Delta_h)^{(k-2)/2}(Br^{k-1})\rangle &= \big((-1)^{k/2}\partial_r^{k-1} + T'\big)(Br^{k-1})\big|_{r=0} \\ &= (-1)^{k/2}(k-1)!\,B.\end{aligned}$$

Integrating using Equation (2.17) shows that $\int_Y P_k^h(F)\,\mathrm{dA} = -(-1)^{k/2}(k-1)!\,\mathcal{A}$, as claimed. □

The proof of Theorem 1.4 is much simpler:

Proof of Theorem 1.4. Proposition 2.2 shows that $Q^{i^*g} = \lambda^{k/2}(k-1)!$. Recalling Lemma 3.1, the assumed decomposition (1.3) reads

$$\lambda^{k/2} = \frac{1}{(k-1)!!}\overline{\mathrm{Pf}} + \frac{1}{(k-1)!}\mathcal{W}_Q + \frac{1}{(k-1)!}\,\overline{\operatorname{div}}\,V.$$

The result follows upon integrating over Y. □

4. Riemannian and conformal submanifold geometry

4.1. **Conventions from Riemannian geometry.** Let (X^n, g) be a Riemannian manifold. We always assume that $n \geq 3$. The **Riemann curvature tensor** is determined by

$$\nabla_a\nabla_b\tau_c - \nabla_b\nabla_a\tau_c = R_{abc}{}^d\tau_d \tag{4.1}$$

for all one-forms τ_a, where ∇_a is the Levi-Civita connection, we raise and lower indices using g, and we employ abstract index notation [PR84]. The **Ricci tensor** and **scalar curvature** of g are the contractions $R_{ab} := R_{acb}{}^c$ and $R := R_a{}^a$, respectively, of R_{abcd}. The **Weyl tensor** is

$$W_{abcd} := R_{abcd} - \mathsf{P}_{ac}g_{bd} - \mathsf{P}_{bd}g_{ac} + \mathsf{P}_{ad}g_{bc} + \mathsf{P}_{bc}g_{ad}, \tag{4.2}$$

where

$$\mathsf{P}_{ab} := \frac{1}{n-2}\left(R_{ab} - \mathsf{J}g_{ab}\right)$$

is the **Schouten tensor** and $\mathsf{J} := \frac{1}{2(n-1)}R$. Note that $\mathsf{J} = \mathsf{P}_a{}^a$. Recall that the Weyl tensor is trace-free; i.e. $W_{acb}{}^c = 0$. As discussed further below, the interest in W_{abcd} stems from its conformal invariance. The **Cotton tensor** and the **Bach tensor** are

$$\begin{aligned} C_{abc} &:= \nabla_a\mathsf{P}_{bc} - \nabla_b\mathsf{P}_{ac}, \\ B_{ab} &:= \nabla^c C_{cab} + W_{acbd}\mathsf{P}^{cd}, \end{aligned}$$

respectively. Note that our convention for the Cotton tensor differs from that used in [CGK23]. Clearly $C_{abc} = -C_{bac}$. We use square brackets to denote skew symmetrization of the enclosed indices; e.g.

$$C_{[ab]c} := \frac{1}{2}(C_{abc} - C_{bac}) = C_{abc}.$$

The Bianchi identities imply that

$$\nabla_{[a}W_{bc]}{}^{de} = -2C_{[ab}{}^{[d}g_{c]}{}^{e]}, \tag{4.3}$$

$$\nabla^e W_{abec} = (n-3)C_{abc}, \tag{4.4}$$

and also that

$$C_{ba}{}^{b} = 0, \qquad C_{[abc]} = 0,$$
$$B_a{}^a = 0, \qquad B_{[ab]} = 0.$$

We refer to Equation (4.3) as the Weyl–Bianchi identity.

4.2. **Riemannian submanifolds.** Let $i\colon Y^k \to (X^n, g)$ be an immersion into a Riemannian manifold. We always assume that $1 \leq k < n$ and $n \geq 3$. As above, we use abstract indices, with a lowercase Latin letter $(a, b, c, \dots)$ labeling a section of i^*TX or its dual. Recall the g-orthogonal splitting $i^*TX = TY \oplus NY$. We use a lowercase Greek letter $(\alpha, \beta, \gamma, \dots)$ to label a section of TY or its dual, and we use a primed lowercase Greek letter $(\alpha', \beta', \gamma', \dots)$ to label a section of NY or its dual. We also use Greek or primed Greek indices implicitly to denote composition with the projections to TY or NY, respectively, or their duals. With these conventions, g_{ab} denotes the metric on X, while $g_{\alpha\beta}$ and $g_{\alpha'\beta'}$ denote the induced metrics on TY and NY, respectively. Note that $g_{\alpha\alpha'} = 0$.

Recall from Section 2 that the Levi-Civita connection of g_{ab} determines connections $\overline{\nabla}_\alpha$ on TY and NY by projection. Moreover, the connection $\overline{\nabla}_\alpha$ on TY is the Levi-Civita connection of $g_{\alpha\beta}$.

Our convention from Section 2 for the **second fundamental form** $L_{\alpha\beta\alpha'}$ of Y is that if τ_a is a section of i^*T^*X with projections τ_α and $\tau_{\alpha'}$, then

$$\nabla_\alpha \tau_{\alpha'} = \overline{\nabla}_\alpha \tau_{\alpha'} + L_{\alpha\beta\alpha'}\tau^\beta. \tag{4.5}$$

Recall that $L_{\alpha\beta\alpha'} = L_{\beta\alpha\alpha'}$. Since the splitting $i^*TX = TY \oplus NY$ is g-orthogonal, we deduce that

$$\nabla_\alpha \tau_\beta = \overline{\nabla}_\alpha \tau_\beta - L_{\alpha\beta\alpha'}\tau^{\alpha'}. \tag{4.6}$$

These identities extend to higher rank tensors by the Leibniz rule; e.g.

$$\nabla_\beta \tau_{\alpha\alpha'} = \overline{\nabla}_\beta \tau_{\alpha\alpha'} - L_{\alpha\beta}{}^{\beta'}\tau_{\beta'\alpha'} + L_\beta{}^\gamma{}_{\alpha'}\tau_{\alpha\gamma}.$$

The **mean curvature** of Y is the vector field

$$H^{\alpha'} := \frac{1}{k} L_\alpha{}^{\alpha\alpha'}.$$

We denote by

$$\mathring{L}_{\alpha\beta\alpha'} := L_{\alpha\beta\alpha'} - H_{\alpha'} g_{\alpha\beta}$$

the trace-free part of the second fundamental form. As discussed further below, the interest in $\mathring{L}_{\alpha\beta\alpha'}$ stems from its conformal invariance.

When a function $u \in C^\infty(X)$ is given, we denote $u_{ab\cdots c} := \nabla_c \cdots \nabla_b \nabla_a u$. Thus, by the conventions described above,

$$u_{\alpha\beta} = \nabla_\beta \nabla_\alpha u = \overline{\nabla}_\beta \overline{\nabla}_\alpha u - L_{\alpha\beta\alpha'}\nabla^{\alpha'} u = \overline{\nabla}_\beta \overline{\nabla}_\alpha u - L_{\alpha\beta\alpha'} u^{\alpha'}.$$

The Gauss–Codazzi–Ricci equations (see, e.g., [DT19, Section 1.3]) and their various contracted forms are obtained by combining Equations (4.1), (4.5), and (4.6):

$$\begin{aligned}
R_{\alpha\beta\gamma\delta} &= \overline{R}_{\alpha\beta\gamma\delta} - L_{\alpha\gamma\alpha'}L_{\beta\delta}{}^{\alpha'} + L_{\alpha\delta\alpha'}L_{\beta\gamma}{}^{\alpha'}, \\
R_{\alpha\beta} &= \overline{R}_{\alpha\beta} + R_{\alpha\alpha'\beta}{}^{\alpha'} + L_{\alpha\gamma\alpha'}L_{\beta}{}^{\gamma\alpha'} - kH^{\alpha'}L_{\alpha\beta\alpha'}, \\
R &= \overline{R} + 2R_{\alpha'}{}^{\alpha'} + L_{\alpha\beta\alpha'}L^{\alpha\beta\alpha'} - k^2H_{\alpha'}H^{\alpha'} - R_{\alpha'\beta'}{}^{\alpha'\beta'}, \\
R_{\alpha\beta\alpha'\gamma} &= 2\overline{\nabla}_{[\alpha}L_{\beta]\gamma\alpha'}, \\
R_{\alpha\alpha'} &= R_{\alpha\beta'\alpha'}{}^{\beta'} - \overline{\nabla}^{\beta}L_{\beta\alpha\alpha'} + k\overline{\nabla}_{\alpha}H_{\alpha'}, \\
R_{\alpha\beta\alpha'\beta'} &= \overline{R}_{\alpha\beta\alpha'\beta'} - L^{\gamma}{}_{\alpha\alpha'}L_{\gamma\beta\beta'} + L^{\gamma}{}_{\alpha\beta'}L_{\gamma\beta\alpha'},
\end{aligned}$$

where $\overline{R}_{\alpha\beta\alpha'\beta'}$ is the curvature of the connection $\overline{\nabla}_{\alpha}$ on NY. It is useful to rewrite these in terms of the Schouten tensor, the Weyl tensor, and the second fundamental form. To that end, set

$$\mathsf{D}_{\alpha\alpha'} := \mathsf{P}_{\alpha\alpha'} - \overline{\nabla}_{\alpha}H_{\alpha'}. \tag{4.7}$$

If $k \geq 3$, then the Gauss–Codazzi–Ricci equations are equivalent to (cf. [Fia44])

$$W_{\alpha\beta\gamma\delta} = \overline{W}_{\alpha\beta\gamma\delta} - \mathring{L}_{\alpha\gamma\alpha'}\mathring{L}_{\beta\delta}{}^{\alpha'} + \mathring{L}_{\alpha\delta\alpha'}\mathring{L}_{\beta\gamma}{}^{\alpha'} - 2\mathsf{F}_{\alpha[\gamma}g_{\delta]\beta} + 2\mathsf{F}_{\beta[\gamma}g_{\delta]\alpha}, \tag{4.8a}$$

$$\mathsf{P}_{\alpha\beta} = \overline{\mathsf{P}}_{\alpha\beta} - H^{\alpha'}\mathring{L}_{\alpha\beta\alpha'} - \frac{1}{2}H^{\alpha'}H_{\alpha'}g_{\alpha\beta} + \mathsf{F}_{\alpha\beta}, \tag{4.8b}$$

$$\mathsf{J} = \overline{\mathsf{J}} + \mathsf{P}_{\alpha'}{}^{\alpha'} - \frac{k}{2}H^{\alpha'}H_{\alpha'} + \mathsf{G}, \tag{4.8c}$$

$$W_{\alpha\beta\alpha'\gamma} = 2\overline{\nabla}_{[\alpha}\mathring{L}_{\beta]\gamma\alpha'} + 2g_{\gamma[\alpha}\mathsf{D}_{\beta]\alpha'}, \tag{4.8d}$$

$$(k-1)\mathsf{D}_{\alpha\alpha'} = -\overline{\nabla}^{\beta}\mathring{L}_{\beta\alpha\alpha'} - W_{\alpha\beta\alpha'}{}^{\beta}, \tag{4.8e}$$

$$W_{\alpha\beta\alpha'\beta'} = \overline{R}_{\alpha\beta\alpha'\beta'} - \mathring{L}^{\gamma}{}_{\alpha\alpha'}\mathring{L}_{\gamma\beta\beta'} + \mathring{L}^{\gamma}{}_{\alpha\beta'}\mathring{L}_{\gamma\beta\alpha'}, \tag{4.8f}$$

where

$$\mathsf{F}_{\alpha\beta} := \frac{1}{k-2}\left(\mathring{L}_{\alpha\gamma\alpha'}\mathring{L}_{\beta}{}^{\gamma\alpha'} - W_{\alpha\gamma\beta}{}^{\gamma} - \mathsf{G}g_{\alpha\beta}\right) \tag{4.9}$$

is the manifestly conformally invariant (of weight 0) **Fialkow tensor** [Fia44] and

$$\mathsf{G} := \mathsf{F}_{\alpha}{}^{\alpha} = \frac{1}{2(k-1)}\left(\mathring{L}_{\alpha\beta\alpha'}\mathring{L}^{\alpha\beta\alpha'} - W_{\alpha\beta}{}^{\alpha\beta}\right)$$

is its trace. Equation (4.8f) follows from the Gauss–Codazzi–Ricci equations and the fact that $R_{\alpha\beta\alpha'\beta'} = W_{\alpha\beta\alpha'\beta'}$. Notably, it recovers the fact [Che74] that the curvature of the normal bundle is conformally invariant. Of course, equations involving the trace of the Fialkow tensor, but not the Fialkow tensor itself, require only $k \geq 2$; and Equations (4.8d)–(4.8f) hold for $k = 1$, but are trivial.

4.3. **Conformal submanifolds.** Let $L^g\colon C^\infty(Y;T_1) \to C^\infty(Y;T_2)$ be a metric-dependent differential operator on sections of vector bundles T_1 and T_2 over Y, where g is a metric on X. We say that L is **homogeneous** if there is an $h \in \mathbb{R}$ so that $L^{c^2g} = c^hL^g$ for all $c > 0$. In this case we call h the **homogeneity** of L. Note that if L_1 and L_2 have homogeneities h_1 and h_2, respectively, then $L_1 \circ L_2$ has homogeneity $h_1 + h_2$, provided the composition makes sense. We say that L is **conformally covariant** if there are constants $a, b \in \mathbb{R}$ such that

$$L^{e^{2u}g}(v) = e^{-bi^*u}L^g(e^{ai^*u}v)$$

for all $u \in C^\infty(X)$ and all $v \in C^\infty(Y;T_1)$. Note that conformally covariant operators are necessarily homogeneous.

Branson showed [Bra85, Section 1] that homogeneous differential operators on X are conformally covariant if and only if their conformal linearizations are zero. We develop the analogous framework for metric-dependent differential operators L associated to an immersion. Suppose that L has homogeneity h. Fix $w \in \mathbb{R}$. Given an immersion $i\colon Y^k \to (X^n,g)$ and a function $\Upsilon \in C^\infty(X)$, define the **conformal linearization of *L***, regarded as an operator on densities of weight w, by

$$L^\bullet := \left.\frac{\partial}{\partial t}\right|_{t=0} e^{-t(w+h)i^*\Upsilon} \circ L^{e^{2t\Upsilon}g} \circ e^{twi^*\Upsilon},$$

where $e^{ci^*\Upsilon}$ acts as a multiplication operator and $\left.\frac{\partial}{\partial t}\right|_{t=0} A^t$ is the evaluation at $t=0$ of the derivative in t of the one-parameter family of operators A^t. (We suppress the dependence of $L^\bullet$ on g, Υ, and w to simplify our notation.) We define the conformal linearization of metric-dependent tensor fields by regarding them as zeroth-order metric-dependent differential operators; note that $L^\bullet$ is independent of w in this case. We use the same notation for the conformal linearizations of metric-dependent differential operators, including tensors, defined on X.

It is easy to see (cf. [Bra85, Corollary 1.14]) that $L^\bullet = 0$ if and only if

$$L^{e^{2\Upsilon}g} = e^{(w+h)i^*\Upsilon} \circ L^g \circ e^{-wi^*\Upsilon}$$

for all Riemannian metrics g on X and all $\Upsilon \in C^\infty(X)$. It is straightforward to check that $^\bullet$ satisfies a Leibniz rule: Suppose that $L_1\colon C^\infty(Y;T_1) \to C^\infty(Y;T_2)$ and $L_2\colon C^\infty(Y;T_2) \to C^\infty(Y;T_3)$ have homogeneities h_1 and h_2, respectively. Fix $w \in \mathbb{R}$. Then

$$(L_2 \circ L_1)^\bullet = L_2 \circ L_1^\bullet + L_2^\bullet \circ L_1,$$

where L_1 and $L_2 \circ L_1$ are regarded as operators on densities of weight w, and L_2 is regarded as an operator on densities of weight $w + h_1$.

Since the difference of two connections is a tensor, it makes sense to consider the conformal linearization $\nabla_a^\bullet\colon C^\infty(X;T^*X) \to C^\infty(X;(T^*X)^{\otimes 2})$ of the Levi-Civita connection. Indeed, given $w \in \mathbb{R}$, the Koszul formula implies that

$$\nabla_a^\bullet \tau_b = (w-1)\Upsilon_a\tau_b - \Upsilon_b\tau_a + \Upsilon^c\tau_c g_{ab}.$$

From this one recovers the formulas

$$\begin{aligned} W^\bullet_{abcd} &= 0, \\ \mathsf{P}^\bullet_{ab} &= -\Upsilon_{ab}, \\ C^\bullet_{abc} &= -\Upsilon^d W_{abcd}, \\ B^\bullet_{ab} &= 2(n-4)\Upsilon^c C_{c(ab)}, \end{aligned} \tag{4.10}$$

for the conformal linearizations of the Weyl, Schouten, Cotton, and Bach tensors, respectively, where round parentheses denote symmetrization. One also recovers the formulas

$$\begin{aligned} \overline{\nabla}^\bullet_\alpha \tau_\beta &= (w-1)\Upsilon_\alpha\tau_\beta - \Upsilon_\beta\tau_\alpha + \Upsilon^\gamma\tau_\gamma g_{\alpha\beta}, \\ \overline{\nabla}^\bullet_\alpha \tau_{\alpha'} &= (w-1)\Upsilon_\alpha\tau_{\alpha'}, \\ L^\bullet_{\alpha\beta\alpha'} &= -\Upsilon_{\alpha'} g_{\alpha\beta}, \end{aligned} \tag{4.11}$$

for the conformal linearizations of the connections $\overline{\nabla}_\alpha$ on TY and NY, and of the second fundamental form $L_{\alpha\beta\alpha'}$. In particular, the various projections of the Weyl tensor W_{abcd} and the trace-free part $\mathring{L}_{\alpha\beta\alpha'}$ of the second fundamental form are conformal submanifold invariants of weight 2, as defined in Section 2. Equations (4.11) yield formulas for the conformal linearization of the tangential divergence of tangential one-forms, the tangential Laplacian on functions, and the mean curvature:

$$\begin{aligned} (\overline{\nabla}^{\alpha})^{\bullet}\tau_{\alpha} &= (k+w-2)\Upsilon^{\alpha}\tau_{\alpha}, \\ \overline{\Delta}^{\bullet}u &= (k+2w-2)\Upsilon^{\alpha}u_{\alpha} + w(\overline{\Delta}\Upsilon)u, \\ H^{\bullet}_{\alpha'} &= -\Upsilon_{\alpha'}. \end{aligned} \tag{4.12}$$

We conclude this subsection by introducing four more tensors, the first of which is a variant of a tensor introduced by Blitz, Gover, and Waldron [BGW21, Lemma 6.1]:

$$\mathcal{P}_{\alpha\beta} := \mathsf{P}_{\alpha\beta} + H^{\alpha'}\mathring{L}_{\alpha\beta\alpha'} + \frac{1}{2}H^{\alpha'}H_{\alpha'}g_{\alpha\beta}, \tag{4.13}$$

$$\mathcal{C}_{abc} := C_{abc} - H^{\alpha'}W_{abc\alpha'}, \tag{4.14}$$

$$\mathcal{C}_{a} := \mathcal{C}_{\beta a}{}^{\beta}, \tag{4.15}$$

$$\mathcal{B}_{\alpha\beta} := B_{\alpha\beta} + 2(n-4)H^{\alpha'}C_{\alpha'(\alpha\beta)} + (n-4)H^{\alpha'}H^{\beta'}W_{\alpha\alpha'\beta\beta'}. \tag{4.16}$$

There are two key points. First, these tensors make sense in all submanifold dimensions $k \geq 1$; in particular, Equation (4.8b) implies that $\mathcal{P}_{\alpha\beta}$ generalizes $\overline{\mathsf{P}}_{\alpha\beta} + \mathsf{F}_{\alpha\beta}$ to all dimensions. Second, Lemma 4.1 below shows that, under conformal change, these tensors depend only on tangential derivatives of the conformal factor. In particular, they depend only on an immersion $i\colon Y^k \to (X^n, [g])$ and a choice of representative $h \in i^*[g]$, in the sense that they are independent of the choice of local extension of h to a metric in $[g]$ defined on a neighborhood of $i(Y)$.

Lemma 4.1. *Let $i\colon Y^k \to (X^n, g)$ be an immersion with $1 \leq k < n$ and $n \geq 3$. Then*

$$\begin{aligned} \mathcal{P}^{\bullet}_{\alpha\beta} &= -\overline{\nabla}_{\alpha}\overline{\nabla}_{\beta}\Upsilon, \\ \mathcal{C}^{\bullet}_{abc} &= -\Upsilon^{\alpha}W_{abc\alpha}, \\ \mathcal{C}^{\bullet}_{a} &= -\Upsilon^{\gamma}W_{\beta a}{}^{\beta}{}_{\gamma}, \\ \mathcal{B}^{\bullet}_{\alpha\beta} &= 2(n-4)\Upsilon^{\gamma}\mathcal{C}_{\gamma(\alpha\beta)}, \\ \mathsf{D}^{\bullet}_{\alpha\alpha'} &= -\Upsilon^{\beta}\mathring{L}_{\beta\alpha\alpha'}. \end{aligned}$$

Remark 4.2. If $k=1$, then each of $\mathring{L}_{\alpha\beta\alpha'}$, $W_{\alpha\beta ab}$, and $\mathcal{C}_{\alpha\beta a}$ vanishes. Lemma 4.1 thus implies that $\mathsf{D}_{\alpha\alpha'}$ and $\mathcal{B}_{\alpha\beta}$ are conformal invariants of immersed curves. Indeed, the condition $\mathsf{D}_{\alpha\alpha'} = 0$ characterizes unparameterized conformal circles [Bel15, Theorem 5.4; CGS23, Theorem 1.2 and Proposition 5.4].

Proof. Equations (4.10) and (4.11) imply that

$$\begin{aligned} \mathcal{P}^{\bullet}_{\alpha\beta} &= -\nabla_{\alpha}\nabla_{\beta}\Upsilon - \Upsilon^{\alpha'}L_{\alpha\beta\alpha'}, \\ \mathcal{C}^{\bullet}_{abc} &= -\Upsilon^{\alpha}W_{abc\alpha}, \\ \mathcal{B}^{\bullet}_{\alpha\beta} &= 2(n-4)\Upsilon^{\gamma}\mathcal{C}_{\gamma(\alpha\beta)}, \\ \mathsf{D}^{\bullet}_{\alpha\alpha'} &= -\nabla_{\alpha}\nabla_{\alpha'}\Upsilon + \overline{\nabla}_{\alpha}\nabla_{\alpha'}\Upsilon + H_{\alpha'}\nabla_{\alpha}\Upsilon. \end{aligned}$$

Combining these with Equations (4.5) and (4.6) yields the desired conclusion. □

5. New conformal submanifold invariants

In this section we identify and apply four new non-obvious scalar conformal submanifold invariants. To that end, we introduce some useful notation:

$$\begin{aligned} \mathring{L}^2_{\alpha\beta} &:= \mathring{L}^{\gamma}{}_{\alpha\alpha'}\mathring{L}_{\gamma\beta}{}^{\alpha'}, \qquad |\mathring{L}|^2 = \mathring{L}_{\alpha\beta\alpha'}\mathring{L}^{\alpha\beta\alpha'}, \\ \langle \mathring{L}^2, \overline{\mathsf{P}}\rangle &:= \mathring{L}^2_{\alpha\beta}\overline{\mathsf{P}}^{\alpha\beta}, \qquad \operatorname{tr}\mathring{L}^3_{\alpha'} := \mathring{L}_{\alpha}{}^{\beta}{}_{\beta'}\mathring{L}_{\beta}{}^{\gamma\beta'}\mathring{L}_{\gamma}{}^{\alpha}{}_{\alpha'}, \\ \overline{\Delta}\mathring{L}_{\alpha\beta\alpha'} &:= \overline{\nabla}^{\gamma}\overline{\nabla}_{\gamma}\mathring{L}_{\alpha\beta\alpha'}. \end{aligned} \tag{5.1}$$

We use similar notation to denote other inner products and squared lengths.

5.1. **Identification of invariants.** We begin by identifying four scalar conformal submanifold invariants of weight -4 in general dimension and codimension. Our first step is to compute some useful tangential divergences.

Lemma 5.1. *Let $i\colon Y^k \to (X^n, g)$ be an immersion with $1 \leq k < n$ and $n \geq 3$. Then*

$$\overline{\nabla}^{\beta}\mathcal{P}_{\alpha\beta} = \overline{\nabla}_{\alpha}\mathcal{P}_{\beta}{}^{\beta} + \mathcal{C}_{\alpha} - \mathsf{D}^{\beta\alpha'}\mathring{L}_{\beta\alpha\alpha'}, \tag{5.2}$$

$$\overline{\nabla}^{\beta}\mathring{L}^2_{\alpha\beta} = \frac{1}{2}\overline{\nabla}_{\alpha}|\mathring{L}|^2 - (k-2)\mathsf{D}^{\beta\alpha'}\mathring{L}_{\alpha\beta\alpha'} - W^{\beta\gamma\alpha'}{}_{\gamma}\mathring{L}_{\alpha\beta\alpha'} - \mathring{L}^{\beta\gamma\alpha'}W_{\beta\alpha\gamma\alpha'}, \tag{5.3}$$

$$\overline{\nabla}^{\beta}W_{\alpha\gamma\beta}{}^{\gamma} = \frac{1}{2}\overline{\nabla}_{\alpha}W_{\beta\gamma}{}^{\beta\gamma} - (k-2)\mathcal{C}_{\alpha} - \mathring{L}^{\beta\gamma\alpha'}W_{\beta\alpha\gamma\alpha'} - \mathring{L}_{\alpha}{}^{\beta\alpha'}W_{\beta\gamma\alpha'}{}^{\gamma}. \tag{5.4}$$

Proof. First we compute the tangential divergence of $\mathcal{P}_{\alpha\beta}$. Equation (4.6) implies that

$$\overline{\nabla}^{\beta}\mathsf{P}_{\alpha\beta} - \overline{\nabla}_{\alpha}\mathsf{P}_{\beta}{}^{\beta} = C_{\beta\alpha}{}^{\beta} - \mathsf{P}^{\beta\alpha'}\mathring{L}_{\alpha\beta\alpha'} + (k-1)H^{\alpha'}\mathsf{P}_{\alpha\alpha'}.$$

Combining this with the definitions (4.7) and (4.13) of $\mathsf{D}_{\alpha\alpha'}$ and $\mathcal{P}_{\alpha\beta}$, respectively, yields

$$\overline{\nabla}^{\beta}\mathcal{P}_{\alpha\beta} - \overline{\nabla}_{\alpha}\mathcal{P}_{\beta}{}^{\beta} = C_{\beta\alpha}{}^{\beta} - \mathsf{D}^{\beta\alpha'}\mathring{L}_{\alpha\beta\alpha'} + (k-1)H^{\alpha'}\mathsf{D}_{\alpha\alpha'} + H^{\alpha'}\overline{\nabla}^{\beta}\mathring{L}_{\alpha\beta\alpha'}.$$

Combining this with Equations (4.8e) and (4.15) yields Equation (5.2).

Second we compute the tangential divergence of $\mathring{L}^2_{\alpha\beta}$. Direct computation using Equation (4.8e) yields

$$\overline{\nabla}^{\beta}\mathring{L}^2_{\alpha\beta} = -(k-1)\mathsf{D}^{\beta\alpha'}\mathring{L}_{\alpha\beta\alpha'} - W^{\beta\gamma\alpha'}{}_{\gamma}\mathring{L}_{\alpha\beta\alpha'} + \mathring{L}^{\beta\gamma\alpha'}\overline{\nabla}_{\beta}\mathring{L}_{\alpha\gamma\alpha'}.$$

Rewriting the last summand using Equation (4.8d) yields Equation (5.3).

Third we compute the tangential divergence of $W_{\alpha\gamma\beta}{}^{\gamma}$. On the one hand, Equation (4.6) implies that

$$\begin{aligned} \overline{\nabla}^{\beta}W_{\alpha\gamma\beta}{}^{\gamma} &= \nabla^{\beta}W_{\alpha\gamma\beta}{}^{\gamma} + \mathring{L}_{\alpha}{}^{\beta\alpha'}W_{\beta\gamma\alpha'}{}^{\gamma} - \mathring{L}^{\beta\gamma\alpha'}W_{\beta\alpha\gamma\alpha'} + kH^{\alpha'}W_{\alpha\beta\alpha'}{}^{\beta}, \\ \overline{\nabla}_{\alpha}W_{\beta\gamma}{}^{\beta\gamma} &= \nabla_{\alpha}W_{\beta\gamma}{}^{\beta\gamma} + 4\mathring{L}_{\alpha}{}^{\beta\alpha'}W_{\beta\gamma\alpha'}{}^{\gamma} + 4H^{\alpha'}W_{\alpha\beta\alpha'}{}^{\beta}. \end{aligned}$$

On the other hand, the Weyl–Bianchi identity (4.3) implies that

$$2\nabla^{\beta}W_{\alpha\gamma\beta}{}^{\gamma} = \nabla^{\beta}W_{\alpha\gamma\beta}{}^{\gamma} + \nabla_{\gamma}W^{\beta}{}_{\alpha\beta}{}^{\gamma} = \nabla_{\alpha}W_{\beta\gamma}{}^{\beta\gamma} - 2(k-2)C_{\beta\alpha}{}^{\beta}. \tag{5.5}$$

Combining these three identities with Equation (4.15) yields Equation (5.4). □

Our first two scalar conformal submanifold invariants involve a single derivative of the trace-free second fundamental form or the Weyl tensor. These invariants are tangential divergences in the critical case $\dim Y = 4$. This is in contrast to the intrinsic case: there is no nonzero scalar conformal invariant of weight -4 that is a natural divergence.

Proposition 5.2. *Let $i\colon Y^k \to (X^n, g)$ be an immersion with $1 \leq k < n$ and $n \geq 3$. Then*

$$\mathcal{K}_1 := \overline{\nabla}^\alpha \left(\mathring{L}^{\beta\gamma\alpha'} W_{\alpha\beta\alpha'\gamma} \right) + (k-4)\mathring{L}^{\alpha\beta\alpha'}\mathcal{C}_{\alpha\alpha'\beta},$$
$$\mathcal{K}_2 := \overline{\nabla}^\alpha \left(\mathring{L}_\alpha{}^{\beta\alpha'} W_{\beta\gamma\alpha'}{}^\gamma \right) + (k-4)\mathsf{D}^{\alpha\alpha'} W_{\alpha\beta\alpha'}{}^\beta,$$

are conformally invariant of weight -4.

Remark 5.3. Remark 4.2 implies that if $k=1$, then $\mathcal{K}_1 = 0$ and $\mathcal{K}_2 = 0$.

Remark 5.4. Equations (4.8d) and (4.8e) imply that

$$\mathcal{K}_1 = \mathring{L}^{\beta\gamma\alpha'}\overline{\nabla}^\alpha W_{\alpha\beta\alpha'\gamma} + \frac{1}{2} W_{\alpha\beta\alpha'\gamma}W^{\alpha\beta\alpha'\gamma} + \mathsf{D}^{\alpha\alpha'}W_{\alpha\beta\alpha'}{}^\beta + (k-4)\mathring{L}^{\alpha\beta\alpha'}\mathcal{C}_{\alpha\alpha'\beta},$$
$$\mathcal{K}_2 = \mathring{L}^{\alpha\beta\alpha'}\overline{\nabla}_\alpha W_{\beta\gamma\alpha'}{}^\gamma - 3\mathsf{D}^{\alpha\alpha'}W_{\alpha\beta\alpha'}{}^\beta - W_{\alpha\beta\alpha'}{}^\beta W^{\alpha\gamma\alpha'}{}_\gamma.$$

Note that $\mathcal{K}_2 = 0$ on hypersurfaces. Also, $\mathcal{K}_1 = \mathring{L}^{\beta\gamma\alpha'}\overline{\nabla}^\alpha W_{\alpha\beta\alpha'\gamma} + \frac{1}{2}W_{\alpha\beta\alpha'\gamma}W^{\alpha\beta\alpha'\gamma}$ when $k=4$ and $n=5$; hence $\mathring{L}^{\beta\gamma\alpha'}\overline{\nabla}^\alpha W_{\alpha\beta\alpha'\gamma}$ is conformally invariant in this case.

Proof. Recall that $\mathring{L}_{\alpha\beta\alpha'}$ and $W_{\alpha\beta\gamma\alpha'}$ are conformal submanifold invariants of weight 2. We deduce from Equations (4.12) that

$$\left(\overline{\nabla}^\alpha(\mathring{L}^{\beta\gamma\alpha'}W_{\alpha\beta\alpha'\gamma})\right)^\bullet = (k-4)\Upsilon^\alpha \mathring{L}^{\beta\gamma\alpha'}W_{\alpha\beta\alpha'\gamma}, \tag{5.6}$$
$$\left(\overline{\nabla}^\alpha(\mathring{L}_\alpha{}^{\beta\alpha'}W_{\beta\gamma\alpha'}{}^\gamma)\right)^\bullet = (k-4)\Upsilon^\alpha \mathring{L}_\alpha{}^{\beta\alpha'}W_{\beta\gamma\alpha'}{}^\gamma. \tag{5.7}$$

The conclusion now follows from Lemma 4.1. ∎

Our third scalar conformal submanifold invariant is the following:

Proposition 5.5. *Let $i\colon Y^k \to (X^n, g)$ be an immersion with $1 \leq k < n$ and $n \geq 3$, with $n \neq 4$. Then*

$$\begin{aligned}\mathcal{I} := (k-1)\bigl(-\overline{\Delta}\mathsf{G} + 2\mathsf{G}\mathcal{P}_\alpha{}^\alpha\bigr) + (k-6)\Bigl[&\bigl(\mathring{L}^2_{\alpha\beta} - W_{\alpha\gamma\beta}{}^\gamma\bigr)\mathcal{P}^{\alpha\beta} \\ &+ \overline{\nabla}^\alpha\bigl(\mathcal{C}_\alpha - \mathsf{D}^{\beta\alpha'}\mathring{L}_{\alpha\beta\alpha'}\bigr) + \frac{k-3}{n-4}\mathcal{B}_\alpha{}^\alpha - (k-3)\lvert\mathsf{D}\rvert^2\Bigr]\end{aligned} \tag{5.8}$$

is conformally invariant of weight -4.

Remark 5.6. If $k=1$, then $\mathcal{I} = -10\lvert\mathsf{D}\rvert^2 + \frac{10}{n-4}\mathcal{B}_\alpha{}^\alpha$ is conformally invariant by Remark 4.2.

Remark 5.7. If $k=6$, then the operator $-\overline{\Delta} + 2\mathcal{P}_\alpha{}^\alpha$ is conformally invariant on densities of weight -2. This explains the invariance of $\mathcal{I}$ in this dimension. A similar phenomenon occurs for other conformal invariants below.

Remark 5.8. Interpreting $(k-3)/(n-4) = 1$ when $k=3$ and $n=4$ extends the definition of $\mathcal{I}$ to these dimensions. A straightforward modification of the proof of Proposition 5.5 shows that $\mathcal{I}$ remains conformally invariant of weight -4.

Remark 5.9. Since the Bach tensor is conformally invariant in dimension four, the residue of $\mathcal{I}$ at $n=4$ is conformally invariant. In other words, $(n-4)\mathcal{I}$ is conformally invariant and defined in all dimensions $1 \leq k < n$ and $n \geq 3$. Again, a similar phenomenon occurs for other conformal invariants below.

Remark 5.10. If $i\colon Y^k \to (X^n, g)$ is a minimal immersion into an Einstein manifold with $\mathrm{Ric}(g) = \lambda(n-1)g$, then

$$\mathcal{I} = (k-1)\left(-\overline{\Delta}\mathsf{G} + 2\lambda(k-3)\mathsf{G}\right).$$

On minimal four-dimensional hypersurfaces in Poincaré–Einstein manifolds, this realizes the last integrand in Tyrrell's formula [Tyr23, Equation (1.4)] for the renormalized area as a constant multiple of $\mathcal{I}$. In particular, the conformal invariance of $\mathcal{I}$ explains the finiteness of that integral.

Proof. Remark 5.6 implies that we can assume $k \geq 2$.

First, set

$$I_1 := (k-1)\left(-\overline{\Delta}\mathsf{G} + (k-4)\mathsf{G}\mathcal{P}_\alpha{}^\alpha\right). \tag{5.9}$$

Since G is a conformal submanifold invariant of weight -2, we conclude from Equations (4.12) and Lemma 4.1 that

$$I_1^\bullet = -(k-1)(k-6)\left(\overline{\nabla}^\alpha \circ \mathsf{G} \circ \overline{\nabla}_\alpha\right)\Upsilon. \tag{5.10}$$

Second, set

$$I_2 := \overline{\nabla}^\alpha\left(\mathcal{C}_\alpha - \mathsf{D}^{\beta\alpha'}\mathring{L}_{\alpha\beta\alpha'}\right) + \frac{k-4}{2(n-4)}\mathcal{B}_\alpha{}^\alpha - \frac{k-4}{2}|\mathsf{D}|^2. \tag{5.11}$$

Combining Equation (4.12) and Lemma 4.1 yields

$$I_2^\bullet = \left(\overline{\nabla}^\alpha \circ \left(\mathring{L}^2_{\alpha\beta} - W_{\alpha\gamma\beta}{}^\gamma\right) \circ \overline{\nabla}^\beta\right)\Upsilon. \tag{5.12}$$

Third, set

$$I_3 := \left(\mathring{L}^2_{\alpha\beta} - W_{\alpha\gamma\beta}{}^\gamma\right)\mathcal{P}^{\alpha\beta} - (k-1)\mathsf{G}\mathcal{P}_\alpha{}^\alpha + \frac{k-2}{2(n-4)}\mathcal{B}_\alpha{}^\alpha - \frac{k-2}{2}|\mathsf{D}|^2. \tag{5.13}$$

Lemma 4.1 implies that

$$\begin{aligned} I_3^\bullet = -\left(\mathring{L}^2_{\alpha\beta} - W_{\alpha\gamma\beta}{}^\gamma\right)\overline{\nabla}^\alpha\overline{\nabla}^\beta\Upsilon + (k-1)\mathsf{G}\overline{\Delta}\Upsilon \\ - (k-2)\mathcal{C}_\alpha\Upsilon^\alpha + (k-2)\Upsilon^\beta\mathsf{D}^{\alpha\alpha'}\mathring{L}_{\alpha\beta\alpha'}. \end{aligned}$$

Combining this with Lemma 5.1 yields

$$I_3^\bullet = -\left(\overline{\nabla}^\alpha \circ \left(\mathring{L}^2_{\alpha\beta} - W_{\alpha\gamma\beta}{}^\gamma - (k-1)\mathsf{G}g_{\alpha\beta}\right) \circ \overline{\nabla}^\beta\right)\Upsilon. \tag{5.14}$$

Finally, observe that

$$\mathcal{I} = I_1 + (k-6)(I_2 + I_3). \tag{5.15}$$

We conclude from Equations (5.10), (5.12), and (5.14) that $\mathcal{I}$ is conformally invariant of weight -4. $\square$

Unlike $\mathcal{I}$, our fourth scalar conformal submanifold invariant is trivial for codimension one submanifolds.

Proposition 5.11. *Let $i\colon Y^k \to (X^n, g)$ be an immersion with $1 \leq k < n$ and $n \geq 3$, with $n \neq 4$. Then*

$$\mathcal{J} := -\overline{\Delta} W_{\alpha\beta}{}^{\alpha\beta} + 2W_{\alpha\beta}{}^{\alpha\beta}\mathcal{P}_{\gamma}{}^{\gamma} - 2(k-6)\Big[\overline{\nabla}^{\alpha}\mathcal{C}_{\alpha} - \mathcal{P}^{\alpha\beta}W_{\alpha\gamma\beta}{}^{\gamma} + \mathsf{D}^{\alpha\alpha'}W_{\alpha\beta\alpha'}{}^{\beta} + \mathring{L}^{\alpha\beta\alpha'}\mathcal{C}_{\alpha\alpha'\beta} + \frac{k-3}{n-4}\mathcal{B}_{\alpha}{}^{\alpha}\Big] \tag{5.16}$$

is conformally invariant of weight -4.

Remark 5.12. The definition (4.5) of the second fundamental form and the definitions (4.13)–(4.16) of $\mathcal{P}_{\alpha\beta}$, $\mathcal{C}_{\alpha}$, and $\mathcal{B}_{\alpha\beta}$ imply that $\mathcal{J} = 0$ on hypersurfaces. The key observation is that, on hypersurfaces,

$$\overline{\nabla}^{\alpha}\mathcal{C}_{\alpha} = -\nabla^{\alpha}C_{\alpha\beta}{}^{\beta} - \mathring{L}^{\alpha\beta\alpha'}C_{\alpha\alpha'\beta} = -B_{\alpha}{}^{\alpha} + W_{\alpha\gamma\beta}{}^{\gamma}\mathsf{P}^{\alpha\beta} - \mathring{L}^{\alpha\beta\alpha'}C_{\alpha\alpha'\beta}.$$

This remark includes the case $k = 3$ and $n = 4$ upon replacing $(k-3)/(n-4)$ by 1 in the last term of Equation (5.16).

Remark 5.13. Note that

$$2\mathcal{I} + \mathcal{J} = -\overline{\Delta}|\mathring{L}|^2 + 2|\mathring{L}|^2\mathcal{P}_{\alpha}{}^{\alpha} + 2(k-6)\Big[\mathring{L}^2_{\alpha\beta}\mathcal{P}^{\alpha\beta} - \overline{\nabla}^{\alpha}(\mathsf{D}^{\beta\alpha'}\mathring{L}_{\alpha\beta\alpha'}) - (k-3)|\mathsf{D}|^2 - \mathsf{D}^{\alpha\alpha'}W_{\alpha\beta\alpha'}{}^{\beta} - \mathring{L}^{\alpha\beta\alpha'}\mathcal{C}_{\alpha\alpha'\beta}\Big]$$

is defined when $n = 4$. It is straightforward to adapt the proofs of Propositions 5.5 and 5.11 to conclude that it is conformally invariant for $1 \leq k < n$ and $n \geq 3$.

Remark 5.14. If $k = 1$, then $\mathcal{J} = -\frac{20}{n-4}\mathcal{B}_{\alpha}{}^{\alpha}$ is conformally invariant by Remark 4.2.

Remark 5.15. Analogous to Remark 5.10, if $i\colon Y^k \to (X^n, g)$ is a minimal immersion into an Einstein manifold with $\mathrm{Ric}(g) = \lambda(n-1)g$, then

$$\mathcal{J} = -\overline{\Delta}W_{\alpha\beta}{}^{\alpha\beta} + 2\lambda(k-3)W_{\alpha\beta}{}^{\alpha\beta}.$$

Proof. First, set

$$J_1 := -\overline{\Delta}W_{\alpha\beta}{}^{\alpha\beta} + (k-4)W_{\alpha\beta}{}^{\alpha\beta}\mathcal{P}_{\gamma}{}^{\gamma}.$$

Since $W_{\alpha\beta}{}^{\alpha\beta}$ is conformally invariant of weight -2, we deduce from Equation (4.12) and Lemma 4.1 that

$$J_1^{\bullet} = -(k-6)\left(\overline{\nabla}^{\alpha} \circ W_{\beta\gamma}{}^{\beta\gamma} \circ \overline{\nabla}_{\alpha}\right)\Upsilon. \tag{5.17}$$

Second, set

$$J_2 := \overline{\nabla}^{\alpha}\mathcal{C}_{\alpha} + \frac{k-4}{2(n-4)}\mathcal{B}_{\alpha}{}^{\alpha}.$$

Equation (4.12) and Lemma 4.1 imply that

$$J_2^{\bullet} = -\left(\overline{\nabla}^{\alpha} \circ W_{\alpha\gamma\beta}{}^{\gamma} \circ \overline{\nabla}^{\beta}\right)\Upsilon. \tag{5.18}$$

Third, set

$$J_3 := \left(W_{\alpha\gamma\beta}{}^{\gamma} - \frac{1}{2}W_{\gamma\delta}{}^{\gamma\delta}g_{\alpha\beta}\right)\mathcal{P}^{\alpha\beta} - \mathring{L}^{\alpha\beta\alpha'}\mathcal{C}_{\alpha\alpha'\beta} - \mathsf{D}^{\alpha\alpha'}W_{\alpha\beta\alpha'}{}^{\beta} - \frac{k-2}{2(n-4)}\mathcal{B}_{\alpha}{}^{\alpha}.$$

Combining Lemmas 4.1 and 5.1 yields

$$J_3^{\bullet} = -\left(\overline{\nabla}^{\alpha} \circ \left(W_{\alpha\gamma\beta}{}^{\gamma} - \frac{1}{2}W_{\gamma\delta}{}^{\gamma\delta}g_{\alpha\beta}\right) \circ \overline{\nabla}^{\beta}\right)\Upsilon. \tag{5.19}$$

The conclusion now follows from Equations (5.17), (5.18), and (5.19) and the observation

$$\mathcal{J} = J_1 - 2(k-6)(J_2 - J_3). \qquad \square$$

5.2. **The fourth-order extrinsic Q-curvature.** Our main goal in this subsection is to derive a decomposition of the form (1.3) for the critical extrinsic Q-curvature for submanifolds of dimension $k = 4$. This is easily accomplished using a representation of Q derived in [CGK23] together with our knowledge of the invariant $\mathcal{I}$ identified in Proposition 5.5. Our analysis applies more generally in dimensions $3 \leq k < n$ with $n \neq 4$ to the extrinsic scalar Q_4 of order 4 derived in [CGK23] and given by Equation (5.23) below, for which $Q_4 = Q$ in the critical case $k = 4$.

Recall the formula

$$\overline{Q}_4 = -\overline{\Delta}\,\overline{\mathsf{J}} - 2|\overline{\mathsf{P}}|^2 + \frac{k}{2}\overline{\mathsf{J}}^2 \tag{5.20}$$

for the intrinsic Q-curvature of order 4 in general dimension k. Define

$$\begin{aligned} Q_\dagger := (k-2)\overline{\Delta}\mathsf{G} - (k-6)\overline{\nabla}^\alpha\big(\mathcal{C}_\alpha - \mathsf{D}^{\beta\alpha'}\mathring{L}_{\alpha\beta\alpha'}\big) - 2(k-4)\mathsf{G}\mathcal{P}_\alpha{}^\alpha \\ - (k-4)^2\mathsf{F}_{\alpha\beta}\mathcal{P}^{\alpha\beta} - \frac{(k-4)(k-5)}{n-4}\mathcal{B}_\alpha{}^\alpha + (k-4)(k-5)|\mathsf{D}|^2. \end{aligned} \tag{5.21}$$

Observe that $Q_\dagger$ is a divergence in the critical case $k = 4$.

Proposition 5.16. *Let $i\colon Y^k \to (X^n, g)$ be an immersion with $3 \leq k < n$ and $n \neq 4$. Then*

$$Q_4 = \overline{Q}_4 + Q_\dagger + \mathcal{I} + 2|\mathsf{F}|^2 - \frac{k}{2}\mathsf{G}^2. \tag{5.22}$$

Remark 5.17. Since Q_4 depends only on a background conformal class $[g]$ on X and a representative $h \in i^*[g]$ on Y, we deduce from Equation (5.22) that the same is true of $Q_\dagger$.

Remark 5.18. The conformal transformation law of $Q_\dagger$ can be written in terms of the operator

$$P_\dagger := P_4 - \overline{P}_4 - \frac{k-4}{2}\left(\mathcal{I} + 2|\mathsf{F}|^2 - \frac{k}{2}\mathsf{G}^2\right),$$

where P_4 is the extrinsic Paneitz operator defined in [CGK23] and $\overline{P}_4$ is the intrinsic Paneitz operator. Indeed, Equation (5.22) and the formula [CGK23, Equation (5.12)] for $P_4 - \overline{P}_4$ imply that $P_\dagger$ is equivalently written

$$P_\dagger = \overline{\nabla}^\alpha \circ (4\mathsf{F}_{\alpha\beta} - (k-2)\mathsf{G}g_{\alpha\beta}) \circ \overline{\nabla}^\beta + \frac{k-4}{2}Q_\dagger,$$

and in particular is second-order. Since $\mathcal{I} + 2|\mathsf{F}|^2 - \frac{k}{2}\mathsf{G}^2$ is conformally invariant of weight -4, it follows from the conformal transformation laws of P_4 and $\overline{P}_4$ that if $\widehat{h} = e^{2\Upsilon}h$, then

$$\begin{aligned} P_\dagger^{\widehat{h}} &= e^{(-k/2-2)\Upsilon} \circ P_\dagger^h \circ e^{(k/2-2)\Upsilon}, && \text{for all } k \geq 3, \\ e^{4\Upsilon}Q_\dagger^{\widehat{h}} &= Q_\dagger^h + P_\dagger^h\Upsilon, && \text{if } k = 4. \end{aligned}$$

Proof. Case, Graham, and Kuo showed [CGK23, Theorem 5.3] that

$$Q_4 = \overline{Q}_4 + \widetilde{Q}_4, \tag{5.23}$$

where

$$\widetilde{Q}_4 := -\overline{\Delta}\mathsf{G} - 2|\mathsf{F}|^2 + \frac{k}{2}\mathsf{G}^2 - 4\mathsf{F}_{\alpha\beta}\overline{\mathsf{P}}^{\alpha\beta} + k\mathsf{G}\overline{\mathsf{J}} - \frac{2}{n-4}\mathcal{B}_\alpha{}^\alpha + 2|\mathsf{D}|^2.$$

Adding Equations (5.8) and (5.21) gives

$$\begin{aligned}\mathcal{I} + Q_\dagger = -\overline{\Delta}\mathsf{G} + 6\mathsf{G}\mathcal{P}_\alpha{}^\alpha + (k-6)\left(\mathring{L}^2_{\alpha\beta} - W_{\alpha\gamma\beta}{}^\gamma\right)\mathcal{P}^{\alpha\beta} \\ - (k-4)^2\mathsf{F}_{\alpha\beta}\mathcal{P}^{\alpha\beta} - \frac{2}{n-4}\mathcal{B}_\alpha{}^\alpha + 2|\mathsf{D}|^2.\end{aligned}$$

We deduce that

$$\begin{aligned}\widetilde{Q}_4 - \left(\mathcal{I} + Q_\dagger\right) = -2|\mathsf{F}|^2 + \frac{k}{2}\mathsf{G}^2 - 4\mathsf{F}_{\alpha\beta}\overline{\mathsf{P}}^{\alpha\beta} + k\mathsf{G}\overline{\mathsf{J}} - 6\mathsf{G}\mathcal{P}_\alpha{}^\alpha \\ - (k-6)\left(\mathring{L}^2_{\alpha\beta} - W_{\alpha\gamma\beta}{}^\gamma\right)\mathcal{P}^{\alpha\beta} + (k-4)^2\mathsf{F}_{\alpha\beta}\mathcal{P}^{\alpha\beta}.\end{aligned}$$

This reduces to $2|\mathsf{F}|^2 - \frac{k}{2}\mathsf{G}^2$ upon using Equations (4.8b), (4.9), and (4.13) to substitute

$$\overline{\mathsf{P}}^{\alpha\beta} = \mathcal{P}^{\alpha\beta} - \mathsf{F}^{\alpha\beta}, \qquad \overline{\mathsf{J}} = \mathcal{P}_\alpha{}^\alpha - \mathsf{G}, \qquad \mathring{L}^2_{\alpha\beta} - W_{\alpha\gamma\beta}{}^\gamma = (k-2)\mathsf{F}_{\alpha\beta} + \mathsf{G}g_{\alpha\beta}. \quad \square$$

The desired decomposition of Q when $k = 4$ is an immediate consequence:

Proposition 5.19. *Let $i\colon Y^4 \to (X^n, g)$ be an immersion with $n > 4$. Then*

$$Q = 2\overline{\mathrm{Pf}} + \mathcal{W}_Q - \overline{\nabla}^\alpha\left(\overline{\nabla}_\alpha\overline{\mathsf{J}} - 2\overline{\nabla}_\alpha\mathsf{G} - 2\mathcal{C}_\alpha + 2\mathsf{D}^{\beta\alpha'}\mathring{L}_{\alpha\beta\alpha'}\right) \tag{5.24}$$

with $\mathcal{W}_Q = -\frac{1}{4}|\overline{W}|^2 + \mathcal{I} + 2|\mathsf{F}|^2 - 2\mathsf{G}^2$.

Proof. Recall that $2\overline{\mathrm{Pf}} = \frac{1}{4}|\overline{W}|^2 - 2|\overline{\mathsf{P}}|^2 + 2\overline{\mathsf{J}}^2$ when $k = 4$. Substituting Equations (5.20) and (5.21) into Equation (5.22) gives Equation (5.24). $\square$

Our formula for the renormalized area of a minimal four-manifold in a Poincaré–Einstein manifold immediately follows.

Proof of Corollary 1.3. Apply Theorem 1.2 and Proposition 5.19. $\square$

5.3. **Comparison to other conformal submanifold invariants.** Our derivation of the invariants of Subsection 5.1 allows us to generalize to general dimension and codimension scalar conformal submanifold invariants discovered previously in special cases. In this subsection we identify these generalizations and defer to the appendix the proofs, which are long but straightforward computations using the formulas developed in Subsection 5.1.

Blitz, Gover, and Waldron derived [BGW21, Theorem 1.2] a conformal invariant Wm for four-dimensional hypersurfaces which involves second derivatives of $\mathring{L}_{\alpha\beta\alpha'}$. We extend their invariant to higher dimensions and higher codimensions.

Proposition 5.20. *Let $i\colon Y^k \to (X^n, g)$ be an immersion with $3 \leq k < n$ and $k \neq 6$. Define the weight -4 scalar conformal submanifold invariant*

$$\begin{aligned} Wm := {} & \frac{k(k-1)}{4(k-6)}(2\mathcal{I} + \mathcal{J}) + \frac{k-3}{2}\mathcal{K}_1 - \frac{k^2-2k+3}{2(k-1)}\mathcal{K}_2 - \frac{k-3}{2}W_{\alpha\beta\alpha'}{}^{\beta}W^{\alpha\gamma\alpha'}{}_{\gamma} \\ & - \frac{k-3}{4}W_{\alpha\beta\alpha'\gamma}W^{\alpha\beta\alpha'\gamma} - \frac{k-3}{2}W_{\alpha\beta\gamma\delta}\mathring{L}^{\alpha\gamma\alpha'}\mathring{L}^{\beta\delta}{}_{\alpha'} - \frac{k-3}{2}W_{\alpha\beta\alpha'\beta'}\mathring{L}^{\gamma\alpha\alpha'}\mathring{L}_{\gamma}{}^{\beta\beta'} \\ & - \frac{k^2-3k+6}{2}\mathring{L}^2_{\alpha\beta}\mathsf{F}^{\alpha\beta} - \frac{k(k-1)}{2(k-6)}\mathsf{G}|\mathring{L}|^2 - \frac{k-3}{2}\mathring{L}^{\alpha\beta\alpha'}\mathring{L}_{\alpha\beta\beta'}\mathring{L}^{\gamma\delta\beta'}\mathring{L}_{\gamma\delta\alpha'} \\ & - \frac{k-3}{2}|\mathring{L}^2|^2 + (k-3)\mathring{L}^{\alpha\beta\alpha'}\mathring{L}^{\gamma\delta}{}_{\alpha'}\mathring{L}_{\alpha\gamma\beta'}\mathring{L}_{\beta\delta}{}^{\beta'}. \end{aligned}$$

If $k = 4$ and $n = 5$, then

$$\begin{aligned} Wm = {} & \frac{1}{2}\mathring{L}^{\alpha\beta\alpha'}\overline{\Delta}\mathring{L}_{\alpha\beta\alpha'} + \frac{4}{3}\overline{\nabla}^{\alpha}(\mathring{L}_{\alpha}{}^{\beta\alpha'}\overline{\nabla}^{\gamma}\mathring{L}_{\gamma\beta\alpha'}) + \frac{3}{2}\overline{\Delta}|\mathring{L}|^2 - \frac{7}{2}\overline{\mathsf{J}}|\mathring{L}|^2 \\ & - 6\mathring{L}^{\alpha\beta\alpha'}C_{\alpha\alpha'\beta} + 4\mathring{L}^2_{\alpha\beta}\overline{\mathsf{P}}^{\alpha\beta} - 6H^{\alpha'}\operatorname{tr}\mathring{L}^3_{\alpha'} + 12H^{\alpha'}\mathring{L}^{\alpha\beta}{}_{\alpha'}\mathsf{F}_{\alpha\beta} \end{aligned}$$

equals the invariant of the same name derived in [BGW21, Theorem 1.2]*, where we recall the conventions* (5.1)*.*

Remark 5.21. One can add any multiple of $\mathcal{J}$ and $\mathcal{K}_2$ to Wm without losing the fact that Wm equals the invariant derived in [BGW21] when evaluated at four-dimensional hypersurfaces. By Remark 5.13, our choice of multiple of $\mathcal{J}$ removes the pole at $n = 4$. Our choice of multiple of $\mathcal{K}_2$ arises from our proof of Proposition 5.20 in the appendix.

Juhl derived [Juh23, Proposition 8.1] two conformal invariants $\mathcal{J}_1$ and $\mathcal{J}_2$ for hypersurfaces which involve a transverse derivative of R_{abcd}. On compact four-dimensional hypersurfaces, $\int \mathcal{J}_1$ equals the global conformal invariant identified by Astaneh and Solodukhin [AS21, Equation (29)]. We extend Juhl's invariants to higher codimension.

Proposition 5.22. *Let $i\colon Y^k \to (X^n, g)$ be an immersion with $4 \leq k < n$ and $k \neq 6$. Define the weight -4 scalar conformal submanifold invariant*

$$\begin{aligned} \mathcal{J}_1 := {} & -\frac{k-2}{2(k-3)(k-6)}(2\mathcal{I} + \mathcal{J}) + \frac{k-2}{k-3}(\mathcal{K}_1 + \mathcal{K}_2 + \mathring{L}^2_{\alpha\beta}\mathsf{F}^{\alpha\beta}) + \mathring{L}^2_{\alpha\beta}W^{\alpha\gamma\beta}{}_{\gamma} \\ & + \frac{k-2}{(k-3)(k-6)}\mathsf{G}|\mathring{L}|^2 + \mathring{L}^{\alpha\gamma\alpha'}\mathring{L}^{\beta\delta}{}_{\alpha'}W_{\alpha\beta\gamma\delta} - \mathring{L}^{\alpha\beta\alpha'}\mathring{L}_{\alpha\beta}{}^{\beta'}W_{\alpha'\gamma\beta'}{}^{\gamma} \\ & + \mathring{L}^{\gamma\alpha\alpha'}\mathring{L}_{\gamma}{}^{\beta\beta'}(2W_{\alpha\alpha'\beta\beta'} - W_{\alpha\beta\alpha'\beta'}) - \frac{1}{2}W_{\alpha\beta\alpha'\gamma}W^{\alpha\beta\alpha'\gamma} + W_{\alpha\beta\alpha'}{}^{\beta}W^{\alpha\gamma\alpha'}{}_{\gamma}. \end{aligned}$$

If $n = k + 1$, then

$$\begin{aligned} \mathcal{J}_1 = {} & \frac{k-4}{(k-3)(k-6)}\left(\overline{\Delta}|\mathring{L}|^2 - \frac{k-2}{k-4}\overline{\mathsf{J}}|\mathring{L}|^2\right) - \frac{1}{k-3}\overline{\nabla}^{\alpha}\overline{\nabla}^{\beta}\mathring{L}^2_{\alpha\beta} + \mathring{L}^{\alpha\beta\alpha'}\nabla_{\alpha'}W_{\alpha\gamma\beta}{}^{\gamma} \\ & + \frac{k-2}{(k-1)^2}(\overline{\nabla}^{\beta}\mathring{L}_{\beta\alpha\alpha'})(\overline{\nabla}_{\gamma}\mathring{L}^{\gamma\alpha\alpha'}) - \frac{k-2}{k-3}\mathring{L}^2_{\alpha\beta}\overline{\mathsf{P}}^{\alpha\beta} - 2H_{\alpha'}\mathring{L}^{\alpha\beta\alpha'}W_{\alpha\gamma\beta}{}^{\gamma} \end{aligned}$$

equals the invariant derived in [Juh23, Equation (8.1)].

Proposition 5.23. *Let $i\colon Y^k \to (X^n, g)$ be an immersion with $4 \leq k < n$ and $k \neq 6$. Define the weight -4 scalar conformal submanifold invariant*

$$\mathcal{J}_2 := -\frac{1}{2(k-3)(k-6)}(2\mathcal{I} + \mathcal{J}) + \frac{1}{k-3}(\mathcal{K}_1 + \mathcal{K}_2) - \frac{k-4}{k-3}\mathring{L}^2_{\alpha\beta}\mathsf{F}^{\alpha\beta} + \frac{1}{(k-3)(k-6)}\mathsf{G}|\mathring{L}|^2.$$

If $n = k+1$, then

$$\begin{aligned}\mathcal{J}_2 = &-\mathring{L}^{\alpha\beta\alpha'}\nabla_{\alpha'}\mathsf{P}_{\alpha\beta} - \mathring{L}^{\alpha\beta\alpha'}\mathring{L}_{\alpha\beta}{}^{\beta'}\mathsf{P}_{\alpha'\beta'} + \mathring{L}^{\alpha\beta\alpha'}\overline{\nabla}_\alpha\overline{\nabla}_\beta H_{\alpha'} + H^{\alpha'}\mathring{L}^{\alpha\beta}{}_{\alpha'}\overline{\mathsf{P}}_{\alpha\beta} \\ &- \frac{1}{k-3}\overline{\nabla}^\alpha\overline{\nabla}^\beta\mathring{L}^2_{\alpha\beta} + \frac{k-5}{2(k-3)(k-6)}\overline{\Delta}|\mathring{L}|^2 - \frac{1}{(k-3)(k-6)}\overline{\mathsf{J}}|\mathring{L}|^2 \\ &+ \frac{k-4}{k-3}\mathring{L}^2_{\alpha\beta}\overline{\mathsf{P}}^{\alpha\beta} - \frac{k-3}{k-2}H^{\alpha'}\operatorname{tr}\mathring{L}^3_{\alpha'} - \frac{k-1}{k-2}H^{\alpha'}\mathring{L}^{\alpha\beta}{}_{\alpha'}W_{\alpha\gamma\beta}{}^\gamma - \frac{3}{2}|H|^2|\mathring{L}|^2 \\ &+ \frac{k}{(k-1)^2}(\overline{\nabla}^\beta\mathring{L}_{\beta\alpha\alpha'})(\overline{\nabla}_\gamma\mathring{L}^{\gamma\alpha\alpha'})\end{aligned}$$

equals the invariant in [Juh23, Equation (8.2)].

Remark 5.24. As in Remark 5.21, one can add multiples of $\mathcal{J}$ or $\mathcal{K}_2$ to these invariants without losing the fact that our invariants generalize Juhl's invariants. With our choices, we see that $\mathcal{J}_1 \equiv (k-2)\mathcal{J}_2$ modulo polynomials in the Weyl tensor and the trace-free second fundamental form, as was observed already by Juhl [Juh23, Equation (1.19)] in the case of hypersurfaces.

Chalabi et al. derived [CHO+22, Equations (3.2) and (3.3)] natural submanifold scalars whose integrals are conformal invariants of compact four-dimensional submanifolds in arbitrary codimension. We find two pointwise conformal invariants for general dimensional submanifolds that, when restricted to dimension four, equal their scalars modulo divergences, and hence explain the conformal invariance of their integrals.

Proposition 5.25. *Let $i\colon Y^k \to (X^n, g)$ be an immersion with $2 \leq k < n$. Define the weight -4 scalar conformal submanifold invariant*

$$\begin{aligned}\mathcal{N}_1 := &\frac{1}{2}(2\mathcal{I} + \mathcal{J}) - \frac{k-6}{2}\Big(\mathcal{K}_1 + \mathcal{K}_2\Big) + \frac{k-6}{2}\Big(\mathring{L}^{\gamma\alpha\alpha'}\mathring{L}_\gamma{}^{\beta\beta'}W_{\alpha\beta\alpha'\beta'} \\ &- W_{\alpha\beta\alpha'}{}^\beta W^{\alpha\gamma\alpha'}{}_\gamma + \frac{1}{2}W_{\alpha\beta\gamma\alpha'}W^{\alpha\beta\gamma\alpha'} - \mathring{L}^2_{\alpha\beta}W^{\alpha\gamma\beta}{}_\gamma \\ &- \mathring{L}^{\alpha\gamma\alpha'}\mathring{L}^{\beta\delta}{}_{\alpha'}W_{\alpha\beta\gamma\delta} + \mathring{L}^{\alpha\beta\alpha'}\mathring{L}_{\alpha\beta}{}^{\beta'}W_{\alpha'\gamma\beta'}{}^\gamma - 2\mathring{L}^{\gamma\alpha\alpha'}\mathring{L}_\gamma{}^{\beta\beta'}W_{\alpha\alpha'\beta\beta'}\Big).\end{aligned}$$

If $k = 4$, then

$$\mathcal{N}_1 = -\frac{1}{2}\overline{\Delta}|\mathring{L}|^2 + 2\overline{\nabla}^\beta(\mathsf{D}^{\alpha\alpha'}\mathring{L}_{\alpha\beta\alpha'}) + \mathcal{J}_1^{\mathrm{CHO}^+},$$

where

$$\begin{aligned}\mathcal{J}_1^{\mathrm{CHO}^+} := &2|\mathsf{D}|^2 + \mathring{L}^{\alpha\beta\alpha'}\nabla_{\alpha'}W_{\alpha\gamma\beta}{}^\gamma + \frac{1}{n-1}R|\mathring{L}|^2 - \frac{1}{n-2}|\mathring{L}|^2R_{\alpha'}{}^{\alpha'} \\ &- \frac{2}{n-2}\mathring{L}^2_{\alpha\beta}R^{\alpha\beta} + |H|^2|\mathring{L}|^2 - 2H^{\alpha'}\operatorname{tr}\mathring{L}^3_{\alpha'} - 2H^{\alpha'}\mathring{L}^{\alpha\beta}{}_{\alpha'}W_{\alpha\gamma\beta}{}^\gamma\end{aligned}$$

equals the scalar in [CHO+22, Equation (3.2)].

Proposition 5.26. *Let $i\colon Y^k \to (X^n, g)$ be an immersion with $2 \leq k < n$ and $k \notin \{3,6\}$. Define the weight -4 scalar conformal submanifold invariant*

$$\begin{aligned}\mathcal{N}_2 := {}& \frac{n-4}{(k-3)(k-6)}\mathcal{J} - \frac{2}{k-1}\left(W_{\alpha\beta cd}W^{\alpha\beta cd} - W_{\alpha c\beta d}W^{\alpha c\beta d} + W_{c\alpha d}{}^{\alpha}W^{c\beta d}{}_{\beta}\right) \\ & - \frac{2(n-k-1)}{(k-1)(k-3)}\mathcal{K}_1 - \frac{2(n-5k+11)}{(k-1)(k-3)}\mathcal{K}_2 - \frac{2(n-3k+5)}{(k-1)(k-3)}W_{\alpha\beta\alpha'}{}^{\beta}W^{\alpha\gamma\alpha'}{}_{\gamma} \\ & + \frac{n-k-1}{(k-1)(k-3)}W_{\alpha\beta\alpha'\gamma}W^{\alpha\beta\alpha'\gamma} - \frac{2(n-k-1)}{(k-1)(k-3)}\mathring{L}^{\alpha\gamma\alpha'}\mathring{L}^{\beta\delta}{}_{\alpha'}W_{\alpha\beta\gamma\delta} \\ & - \frac{2(n-3k+5)}{(k-1)(k-3)}\mathring{L}^2_{\alpha\beta}W^{\alpha\gamma\beta}{}_{\gamma} + \frac{2(n-5k+11)}{(k-1)(k-3)}\mathring{L}^{\gamma\alpha\alpha'}\mathring{L}_{\gamma}{}^{\beta\beta'}W_{\alpha\beta\alpha'\beta'} \\ & + \frac{2(n-3k+5)}{(k-1)(k-3)}\mathring{L}^{\alpha\beta\alpha'}\mathring{L}_{\alpha\beta}{}^{\beta'}W_{\alpha'\gamma\beta'}{}^{\gamma} - \frac{4(n-2k+2)}{(k-1)(k-3)}\mathring{L}^{\gamma\alpha\alpha'}\mathring{L}_{\gamma}{}^{\beta\beta'}W_{\alpha\alpha'\beta\beta'}.\end{aligned}$$

If $k=4$, then

$$\mathcal{N}_2 = \frac{3n-10}{6}\overline{\Delta}W_{\alpha\beta}{}^{\alpha\beta} - \frac{4(n-5)}{3}\overline{\nabla}^{\alpha}\mathcal{C}_{\alpha} + \mathcal{J}_2^{\mathrm{CHO}^+},$$

where

$$\begin{aligned}\mathcal{J}_2^{\mathrm{CHO}^+} := {}& \frac{1}{3}\nabla^{\alpha'}\nabla_{\alpha'}W_{\alpha\beta}{}^{\alpha\beta} + \frac{n-10}{3}H^{\alpha'}\nabla_{\alpha'}W_{\alpha\beta}{}^{\alpha\beta} - \frac{n-4}{n-1}RW_{\alpha\beta}{}^{\alpha\beta} \\ & + \frac{n-4}{n-2}R_{\alpha'}{}^{\alpha'}W_{\alpha\beta}{}^{\alpha\beta} + \frac{4(n-5)}{3(n-2)}R^{\alpha\beta}W_{\alpha\gamma\beta}{}^{\gamma} - \frac{4}{3}W_{\alpha\beta\alpha'}{}^{\beta}\overline{\nabla}^{\alpha}H^{\alpha'} \\ & - \frac{2(n-5)}{3}\mathring{L}^{\alpha\beta\alpha'}\nabla_{\alpha'}W_{\alpha\gamma\beta}{}^{\gamma} + \frac{8(n-5)}{3}H^{\alpha'}\mathring{L}^{\alpha\beta}{}_{\alpha'}W_{\alpha\gamma\beta}{}^{\gamma} \\ & - \frac{4(n+1)}{3}\mathsf{D}^{\alpha\alpha'}W_{\alpha\beta\alpha'}{}^{\beta} - \frac{5(n-4)}{3}|H|^2W_{\alpha\beta}{}^{\alpha\beta}\end{aligned} \tag{5.25}$$

equals the scalar in [CHO+22, Equation (3.3)].

Remark 5.27. Equation (5.25) corrects two typos in [CHO+22, Equation (3.3)].

6. Proof of Conjecture 1.1 in dimensions two and four

We conclude this paper by proving Conjecture 1.1 in dimensions two and four. First we identify spanning sets for the spaces of natural submanifold scalars of weight -2 and -4 modulo tangential divergences and scalar conformal submanifold invariants. Then we compute the conformal linearization of the integral of an arbitrary linear combination of elements of our spanning sets in order to prove Conjecture 1.1.

We begin with some terminology. Let $r,s,t,u \in \mathbb{N}_0$ and let I be a natural submanifold tensor of weight w taking values in $(T^*Y)^{\otimes r} \otimes (TY)^{\otimes s} \otimes (N^*Y)^{\otimes t} \otimes (NY)^{\otimes u}$. The **tensor weight** of I is defined to be $w-r-t+s+u$. In particular,

(1) the tensor weight and the weight coincide for scalars;
(2) contraction between an upper and a lower index preserves the tensor weight;
(3) each of $g_{\alpha\beta}$, $g^{\alpha\beta}$, $g_{\alpha'\beta'}$, and $g^{\alpha'\beta'}$ has tensor weight 0, so we can raise and lower indices without changing the tensor weight;
(4) $\mathring{L}_{\alpha\beta\alpha'}$ and $H_{\alpha'}$ have tensor weight -1;
(5) the various projections of R_{abcd} have tensor weight -2; and
(6) if T has tensor weight w, then $\overline{\nabla}T$ has tensor weight $w-1$.

6.1. **The two-dimensional case.** The proof of the two-dimensional case of Conjecture 1.1 is straightforward.

Proof of Theorem 1.5 when $k=2$. Let I be a natural scalar on 2-dimensional submanifolds Y of n-dimensional Riemannian manifolds (X,g) whose integral over compact Y is invariant under conformal rescaling of g. Then I has weight -2. Evaluation of the tensor weight of tensors of the form (2.1) shows that I is a linear combination of complete contractions of $L\otimes L$, $\overline{\nabla}L$, and projections of Rm. The tensor $\overline{\nabla}L$ has exactly one normal index, and hence cannot be completely contracted, while a projection of Rm can only be completely contracted if it has an even number of tangential indices. Since the Weyl tensor is trace-free, we see that

$$W_{a\alpha'b}{}^{\alpha'} = -W_{a\alpha b}{}^{\alpha}, \tag{6.1}$$

and thus $W_{\alpha\beta}{}^{\alpha\beta} = -W_{\alpha\alpha'}{}^{\alpha\alpha'} = W_{\alpha'\beta'}{}^{\alpha'\beta'}$. Equation (4.8c) now implies that

$$I \in \operatorname{span}\{W_{\alpha\beta}{}^{\alpha\beta}, \mathsf{P}_{\alpha'}{}^{\alpha'}, \overline{\mathsf{J}}, |H|^2, |\mathring{L}|^2\}.$$

Clearly $W_{\alpha\beta}{}^{\alpha\beta}$ and $|\mathring{L}|^2$ are conformally invariant of weight -2. In dimension two, $\overline{\mathrm{Pf}} = \overline{\mathsf{J}}$. Therefore it suffices to show that if $I = a_1|H|^2 + a_2\mathsf{P}_{\alpha'}{}^{\alpha'}$, then $a_1=a_2=0$. The assumption of conformal invariance implies that

$$0 = \int_Y (I\,\mathrm{dA})^\bullet = -\int_Y \left(2a_1H^{\alpha'}\nabla_{\alpha'}\Upsilon + a_2\nabla^{\alpha'}\nabla_{\alpha'}\Upsilon\right)\mathrm{dA}$$

for all $\Upsilon\in C^\infty(Y)$. Choosing first Υ such that $\nabla_{\alpha'}\Upsilon=0$ along Y implies that $a_2=0$; then taking a non-minimal immersion implies that $a_1=0$. $\square$

6.2. **The four-dimensional case.** We begin by finding a spanning set for the space of natural submanifold scalars of weight -4 modulo tangential divergences and scalar conformal submanifold invariants. As a means to organize this set, we say that $I^\bullet$ **depends only on the transverse j-jet of the conformal factor** if it vanishes for all $\Upsilon\in C^\infty(Y)$ such that $\nabla_{\alpha'_1}\cdots\nabla_{\alpha'_k}\Upsilon=0$ along Y for all nonnegative integers $k\le j$.

Lemma 6.1. *Let $n>4$. Denote by $\mathscr{I}_4$ the space of natural submanifold scalars of weight -4 for immersions $i\colon Y^4\to(X^n,g)$. Let $\mathscr{I}_{4,\mathrm{div}}\subseteq\mathscr{I}_4$ be the subspace of natural tangential divergences and let $\mathscr{I}_{4,\mathrm{c}}\subseteq\mathscr{I}_4$ be the subspace of conformal submanifold invariants. Set*

$$\begin{aligned}
\mathscr{I}_4^0 &:= \{\langle\mathsf{F},\overline{\mathsf{P}}\rangle, W_{\alpha\beta\alpha'}{}^{\beta}\mathsf{D}^{\alpha\alpha'}, |\mathring{L}|^2\overline{\mathsf{J}}, \langle\mathring{L}^2,\overline{\mathsf{P}}\rangle, \overline{\mathsf{J}}^2, |\overline{\mathsf{P}}|^2, \mathsf{G}\overline{\mathsf{J}}, |\mathsf{D}|^2\},\\
\mathscr{I}_4^1 &:= \{H^{\alpha'}\overline{\Delta}H_{\alpha'}, H^{\alpha'}\overline{\nabla}^{\alpha}\mathsf{D}_{\alpha\alpha'}, H^{\alpha'}\overline{\nabla}^{\alpha}W_{\alpha\beta\alpha'}{}^{\beta}, |H|^4, |H|^2|\mathring{L}|^2,\\
&\qquad H^{\alpha'}\mathring{L}_{\alpha\beta\alpha'}\mathring{L}^{\alpha\beta\beta'}H_{\beta'}, |H|^2\overline{\mathsf{J}}, \mathsf{G}|H|^2, H^{\alpha'}\operatorname{tr}\mathring{L}^3_{\alpha'}, H^{\alpha'}\mathring{L}_{\alpha\beta\alpha'}\mathsf{F}^{\alpha\beta},\\
&\qquad H^{\alpha'}\mathring{L}_{\alpha\beta\alpha'}\overline{\mathsf{P}}^{\alpha\beta}, H^{\alpha'}H^{\beta'}W_{\alpha'\alpha\beta'}{}^{\alpha}, H^{\alpha'}\mathring{L}^{\alpha\beta\beta'}W_{\alpha\alpha'\beta\beta'}, H^{\alpha'}\mathcal{C}_{\alpha'}\},\\
\mathscr{I}_4^2 &:= \{\mathsf{G}\mathsf{P}_{\alpha'}{}^{\alpha'}, \mathsf{P}^{\alpha'\beta'}W_{\alpha'\alpha\beta'}{}^{\alpha}, |H|^2\mathsf{P}_{\alpha'}{}^{\alpha'}, |\mathring{L}|^2\mathsf{P}_{\alpha'}{}^{\alpha'}, H^{\alpha'}H^{\beta'}\mathsf{P}_{\alpha'\beta'},\\
&\qquad \mathring{L}_{\alpha\beta}{}^{\alpha'}\mathring{L}^{\alpha\beta\beta'}\mathsf{P}_{\alpha'\beta'}, \overline{\mathsf{J}}\mathsf{P}_{\alpha'}{}^{\alpha'}, \mathsf{P}_{\alpha'}{}^{\alpha'}\mathsf{P}_{\beta'}{}^{\beta'}, \mathsf{P}_{\alpha'\beta'}\mathsf{P}^{\alpha'\beta'}\},\\
\mathscr{I}_4^3 &:= \{H^{\alpha'}\nabla_{\alpha'}\mathsf{J}\},\\
\mathscr{I}_4^4 &:= \{\Delta\mathsf{J}\},
\end{aligned}$$

where we recall our conventions (5.1)*. Then*

$$\mathscr{I}_4 = \mathscr{I}_{4,\mathrm{div}} + \mathscr{I}_{4,c} + \operatorname{span} \bigcup_{j=0}^{4} \mathscr{I}_4^j .$$

Moreover, for each integer $0 \leq j \leq 4$, the conformal linearizations of the elements of $\mathscr{I}_4^j$ depend only on the transverse j-jet of the conformal factor.

Proof. It is clear from the discussion of Subsection 4.3 that if $I \in \mathscr{I}_4^j$, then $I^\bullet$ depends only on the transverse j-jet of the conformal factor.

Evaluation of the tensor weight of tensors of the form (2.1) shows that if I is a natural submanifold scalar of weight -4, then

$$I \in \operatorname{Contr}(\nabla^2 \operatorname{Rm}, L \otimes \overline{\nabla}^2 L, \overline{\nabla} L \otimes \overline{\nabla} L, L \otimes \nabla \operatorname{Rm}, \overline{\nabla} L \otimes \operatorname{Rm}, L^{\otimes 4}, L^{\otimes 2} \otimes \operatorname{Rm}, \operatorname{Rm}^{\otimes 2}),$$

where $\operatorname{Contr}(\cdots)$ denotes the space of linear combinations of complete contractions of the various projections of its arguments. We consider such contractions in decreasing order of the total number of derivatives of L and Rm taken.

Step 1: Analyze $\operatorname{Contr}(\nabla^2 \operatorname{Rm}, L \otimes \overline{\nabla}^2 L, \overline{\nabla} L \otimes \overline{\nabla} L)$. Given two natural submanifold scalars A and B of weight -4, write $A \sim B$ if

$$A - B \in \mathscr{I}_{4,\mathrm{div}} + \mathscr{I}_{4,c} + \operatorname{Contr}(L \otimes \nabla \operatorname{Rm}, \overline{\nabla} L \otimes \operatorname{Rm}, L^{\otimes 4}, L^{\otimes 2} \otimes \operatorname{Rm}, \operatorname{Rm}^{\otimes 2}).$$

We first consider $\operatorname{Contr}(\nabla^2 \operatorname{Rm})$. The definition (4.5) of the second fundamental form implies that if V_α is a partial contraction of a projection of $\nabla \operatorname{Rm}$ with values in T^*Y, then $\nabla^\alpha V_\alpha \sim \overline{\nabla}^\alpha V_\alpha \sim 0$. By combining this with the Ricci identity, we see that if I is a complete contraction of a projection of $\nabla^2 \operatorname{Rm}$ with at least one tangential index on a covariant derivative, then $I \sim 0$. Therefore

$$\begin{aligned} \operatorname{Contr}(\nabla^2 \operatorname{Rm}) \subseteq \operatorname{Contr}(\nabla_{\alpha'} \nabla_{\beta'} R_{abcd}) + \mathscr{I}_{4,\mathrm{div}} + \mathscr{I}_{4,c} \\ + \operatorname{Contr}(L \otimes \nabla \operatorname{Rm}, \overline{\nabla} L \otimes \operatorname{Rm}, L^{\otimes 4}, L^{\otimes 2} \otimes \operatorname{Rm}, \operatorname{Rm}^{\otimes 2}). \end{aligned}$$

Proposition 5.5 implies that $\mathcal{B}_\alpha{}^\alpha \sim (n-4)|\mathsf{D}|^2$. Therefore

$$\begin{aligned} \nabla^{\alpha'} \nabla_{\alpha'} \mathsf{J} &\sim \Delta \mathsf{J}, \\ \nabla^{\alpha'} C_{\alpha'\alpha}{}^\alpha &\sim \mathcal{B}_\alpha{}^\alpha \sim (n-4)|\mathsf{D}|^2. \end{aligned} \tag{6.2}$$

Equations (6.2) and direct computation yield

$$\begin{aligned} \nabla^{\alpha'} \nabla_{\alpha'} \mathsf{P}_\alpha{}^\alpha &= \nabla^{\alpha'} C_{\alpha'\alpha}{}^\alpha + \nabla^{\alpha'} \nabla_\alpha \mathsf{P}^\alpha{}_{\alpha'} \sim (n-4)|\mathsf{D}|^2, \\ \nabla^{\alpha'} \nabla_{\alpha'} \mathsf{P}_{\beta'}{}^{\beta'} &= \nabla^{\alpha'} \nabla_{\alpha'} \mathsf{J} - \nabla^{\alpha'} \nabla_{\alpha'} \mathsf{P}_\alpha{}^\alpha \sim \Delta \mathsf{J} - (n-4)|\mathsf{D}|^2, \\ \nabla^{\alpha'} \nabla^{\beta'} \mathsf{P}_{\alpha'\beta'} &= \nabla^{\alpha'} \nabla_{\alpha'} \mathsf{J} - \nabla^{\alpha'} \nabla^\alpha \mathsf{P}_{\alpha\alpha'} \sim \Delta \mathsf{J}. \end{aligned}$$

Similarly, Equations (4.3), (4.4), and (6.2) yield

$$\begin{aligned} \nabla^{\alpha'} \nabla_{\alpha'} W_{\alpha\beta}{}^{\alpha\beta} &= 2\nabla^{\alpha'} \nabla_\alpha W_{\alpha'\beta}{}^{\alpha\beta} + 6\nabla^{\alpha'} C_{\alpha\alpha'}{}^\alpha \sim -6(n-4)|\mathsf{D}|^2, \\ \nabla^{\alpha'} \nabla^{\beta'} W_{\alpha'\alpha\beta'}{}^\alpha &\sim (n-3)\nabla^{\alpha'} C_{\alpha'\alpha}{}^\alpha \sim (n-3)(n-4)|\mathsf{D}|^2. \end{aligned}$$

We conclude from Equation (6.1) that

$$\begin{aligned} \operatorname{Contr}(\nabla^2 \operatorname{Rm}) \subseteq \operatorname{span}\{\Delta \mathsf{J}, |\mathsf{D}|^2\} + \mathscr{I}_{4,\mathrm{div}} + \mathscr{I}_{4,c} \\ + \operatorname{Contr}(L \otimes \nabla \operatorname{Rm}, \overline{\nabla} L \otimes \operatorname{Rm}, L^{\otimes 4}, L^{\otimes 2} \otimes \operatorname{Rm}, \operatorname{Rm}^{\otimes 2}). \end{aligned} \tag{6.3}$$

We now consider $\mathrm{Contr}(L\otimes\overline{\nabla}^2L,\overline{\nabla}L\otimes\overline{\nabla}L)$. Since $\mathrm{Contr}(\overline{\nabla}(L\otimes\overline{\nabla}L))\subseteq\mathscr{I}_{4,\mathrm{div}}$, we see that

$$\mathrm{Contr}(L\otimes\overline{\nabla}^2L,\overline{\nabla}L\otimes\overline{\nabla}L)\subseteq\mathrm{Contr}(L\otimes\overline{\nabla}^2L)+\mathscr{I}_{4,\mathrm{div}}.$$

The Ricci identity and Gauss equation imply that any element of $\mathrm{Contr}(L\otimes\overline{\nabla}^2L)$ is a linear combination of

$$H^{\alpha'}\overline{\Delta}H_{\alpha'},\mathring{L}^{\alpha\beta\alpha'}\overline{\Delta}\mathring{L}_{\alpha\beta\alpha'},H^{\alpha'}\overline{\nabla}^{\alpha}\overline{\nabla}^{\beta}\mathring{L}_{\alpha\beta\alpha'},\mathring{L}^{\alpha\beta\alpha'}\overline{\nabla}_{\alpha}\overline{\nabla}_{\beta}H_{\alpha'},\mathring{L}^{\alpha\beta\alpha'}\overline{\nabla}_{\alpha}\overline{\nabla}^{\gamma}\mathring{L}_{\gamma\beta\alpha'},$$

and elements of $\mathrm{Contr}(L^{\otimes4},L^{\otimes2}\otimes\mathrm{Rm})$. Equations (4.7) and (4.8e) imply that

$$\begin{aligned}\mathring{L}^{\alpha\beta\alpha'}\overline{\nabla}_{\alpha}\overline{\nabla}_{\beta}H_{\alpha'}&\sim H^{\alpha'}\overline{\nabla}^{\alpha}\overline{\nabla}^{\beta}\mathring{L}_{\alpha\beta\alpha'}\sim-3H^{\alpha'}\overline{\nabla}^{\alpha}\mathsf{D}_{\alpha\alpha'}\sim3H^{\alpha'}\overline{\Delta}H_{\alpha'},\\ \mathring{L}^{\alpha\beta\alpha'}\overline{\nabla}_{\alpha}\overline{\nabla}^{\gamma}\mathring{L}_{\gamma\beta\alpha'}&\sim3\mathring{L}^{\alpha\beta\alpha'}\overline{\nabla}_{\alpha}\overline{\nabla}_{\beta}H_{\alpha'}\sim9H^{\alpha'}\overline{\Delta}H_{\alpha'}.\end{aligned}\tag{6.4}$$

Therefore

$$\begin{aligned}\mathrm{Contr}(L\otimes\overline{\nabla}^2L,\overline{\nabla}L\otimes\overline{\nabla}L)&\subseteq\mathrm{span}\{H^{\alpha'}\overline{\Delta}H_{\alpha'},\mathring{L}^{\alpha\beta\alpha'}\overline{\Delta}\mathring{L}_{\alpha\beta\alpha'}\}\\&\quad+\mathscr{I}_{4,\mathrm{div}}+\mathscr{I}_{4,c}+\mathrm{Contr}(L\otimes\nabla\mathrm{Rm},\overline{\nabla}L\otimes\mathrm{Rm},L^{\otimes4},L^{\otimes2}\otimes\mathrm{Rm},\mathrm{Rm}^{\otimes2}).\end{aligned}$$

Similarly, Equation (4.7) implies that $|\mathsf{D}|^2\sim-H^{\alpha'}\overline{\Delta}H_{\alpha'}$, and Equations (4.7), (4.8d), and (6.4) imply that

$$\begin{aligned}\mathring{L}^{\alpha\beta\alpha'}\overline{\Delta}\mathring{L}_{\alpha\beta\alpha'}&=\mathring{L}^{\alpha\beta\alpha'}\overline{\nabla}^{\gamma}(\overline{\nabla}_{\alpha}\mathring{L}_{\gamma\beta\alpha'}+2\overline{\nabla}_{[\gamma}\mathring{L}_{\alpha]\beta\alpha'})\\&\sim\mathring{L}^{\alpha\beta\alpha'}(\overline{\nabla}_{\alpha}\overline{\nabla}^{\gamma}\mathring{L}_{\gamma\beta\alpha'}-\overline{\nabla}_{\alpha}\overline{\nabla}_{\beta}H_{\alpha'})\\&\sim6H^{\alpha'}\overline{\Delta}H_{\alpha'}.\end{aligned}$$

We conclude from Statement (6.3) that

$$\begin{aligned}\mathrm{Contr}(\nabla^2\mathrm{Rm},L\otimes\overline{\nabla}^2L,\overline{\nabla}L\otimes\overline{\nabla}L)&\subseteq\mathrm{span}\{\Delta\mathsf{J},H^{\alpha'}\overline{\Delta}H_{\alpha'}\}+\mathscr{I}_{4,\mathrm{div}}\\&\quad+\mathscr{I}_{4,c}+\mathrm{Contr}(L\otimes\nabla\mathrm{Rm},\overline{\nabla}L\otimes\mathrm{Rm},L^{\otimes4},L^{\otimes2}\otimes\mathrm{Rm},\mathrm{Rm}^{\otimes2}).\end{aligned}\tag{6.5}$$

Step 2: Analyze $\mathrm{Contr}(L\otimes\nabla\mathrm{Rm},\overline{\nabla}L\otimes\mathrm{Rm})$. Given two natural submanifold scalars A and B of weight -4, write $A\approx B$ if

$$A-B\in\mathscr{I}_{4,\mathrm{div}}+\mathscr{I}_{4,c}+\mathrm{Contr}(L^{\otimes4},L^{\otimes2}\otimes\mathrm{Rm},\mathrm{Rm}^{\otimes2}).$$

Since $\mathrm{Contr}(\overline{\nabla}(L\otimes\mathrm{Rm}))\subseteq\mathscr{I}_{4,\mathrm{div}}$, we deduce from Equation (4.5) that

$$\mathrm{Contr}(L\otimes\nabla\mathrm{Rm},\overline{\nabla}L\otimes\mathrm{Rm})\subseteq\mathrm{Contr}(L\otimes\nabla\mathrm{Rm})+\mathscr{I}_{4,\mathrm{div}}+\mathrm{Contr}(L^{\otimes2}\otimes\mathrm{Rm}).$$

In particular, it suffices to analyze $\mathrm{Contr}(L\otimes\nabla\mathrm{Rm})$.

First, Equations (4.3) and (4.5)–(4.7) imply that

$$\begin{aligned}\nabla_{\alpha}\mathsf{P}_{\beta\alpha'}&\equiv\overline{\nabla}_{\alpha}\overline{\nabla}_{\beta}H_{\alpha'}+\overline{\nabla}_{\alpha}\mathsf{D}_{\beta\alpha'},\\ \nabla_{\alpha'}\mathsf{P}_{\alpha\beta}&\equiv-\mathcal{C}_{\alpha\alpha'\beta}+\overline{\nabla}_{\alpha}\overline{\nabla}_{\beta}H_{\alpha'}+\overline{\nabla}_{\alpha}\mathsf{D}_{\beta\alpha'},\\ \nabla^{\gamma}W_{\alpha\gamma\beta\alpha'}&\equiv\overline{\nabla}^{\gamma}W_{\alpha\gamma\beta\alpha'},\\ \nabla_{\alpha'}W_{\alpha\gamma\beta}{}^{\gamma}&\equiv\overline{\nabla}_{\alpha}W_{\alpha'\gamma\beta}{}^{\gamma}+\overline{\nabla}_{\gamma}W_{\alpha\alpha'\beta}{}^{\gamma}+2\mathcal{C}_{\alpha\alpha'\beta}+\mathcal{C}_{\alpha'}g_{\alpha\beta},\end{aligned}$$

where $\equiv$ denotes equivalence modulo partial contractions of $L\otimes\mathrm{Rm}\otimes g$. Recalling Equation (4.8e), we deduce that any complete contraction of a projection of

$L_{\alpha\beta\alpha'}\nabla^{\alpha'}R_{abcd}$ is a linear combination of

$$\begin{aligned}
H^{\alpha'}\nabla_{\alpha'}\mathsf{J}&,\\
H^{\alpha'}\nabla_{\alpha'}\mathsf{P}_{\alpha}{}^{\alpha} &\approx -H^{\alpha'}\mathcal{C}_{\alpha'} + H^{\alpha'}\overline{\Delta}H_{\alpha'} + H^{\alpha'}\overline{\nabla}^{\alpha}\mathsf{D}_{\alpha\alpha'},\\
H^{\alpha'}\nabla_{\alpha'}W_{\alpha\beta}{}^{\alpha\beta} &\approx 2H^{\alpha'}\overline{\nabla}^{\alpha}W_{\alpha\beta\alpha'}{}^{\beta} + 6H^{\alpha'}\mathcal{C}_{\alpha'},\\
\mathring{L}^{\alpha\beta\alpha'}\nabla_{\alpha'}\mathsf{P}_{\alpha\beta} &\approx -\mathring{L}^{\alpha\beta\alpha'}\mathcal{C}_{\alpha\alpha'\beta} + 3|\mathsf{D}|^2 + \mathsf{D}^{\alpha\alpha'}W_{\alpha\beta\alpha'}{}^{\beta}\\
&\quad - 3H^{\alpha'}\overline{\nabla}^{\alpha}\mathsf{D}_{\alpha\alpha'} - H^{\alpha'}\overline{\nabla}^{\alpha}W_{\alpha\beta\alpha'}{}^{\beta},\\
\mathring{L}^{\alpha\beta\alpha'}\nabla_{\alpha'}W_{\alpha\gamma\beta}{}^{\gamma} &\approx 3\mathsf{D}^{\alpha\alpha'}W_{\alpha\beta\alpha'}{}^{\beta} + \mathring{L}^{\alpha\beta\alpha'}\overline{\nabla}^{\gamma}W_{\alpha\alpha'\beta\gamma} + 2\mathring{L}^{\alpha\beta\alpha'}\mathcal{C}_{\alpha\alpha'\beta}.
\end{aligned}$$

Similarly, any complete contraction of a projection of $L_{\alpha\beta\alpha'}\nabla_a R_{bcd}{}^{\alpha'}$ is a linear combination of

$$\begin{aligned}
H^{\alpha'}\nabla^{\alpha}W_{\alpha\beta\alpha'}{}^{\beta} &\approx H^{\alpha'}\overline{\nabla}^{\alpha}W_{\alpha\beta\alpha'}{}^{\beta},\\
H^{\alpha'}\nabla^{\alpha}\mathsf{P}_{\alpha\alpha'} &\approx H^{\alpha'}\overline{\Delta}H_{\alpha'} + H^{\alpha'}\overline{\nabla}^{\alpha}\mathsf{D}_{\alpha\alpha'},\\
\mathring{L}^{\alpha\beta\alpha'}\nabla_{\alpha}\mathsf{P}_{\beta\alpha'} &\approx 3|\mathsf{D}|^2 + \mathsf{D}^{\alpha\alpha'}W_{\alpha\beta\alpha'}{}^{\beta} - 3H^{\alpha'}\overline{\nabla}^{\alpha}\mathsf{D}_{\alpha\alpha'} - H^{\alpha'}\overline{\nabla}^{\alpha}W_{\alpha\beta\alpha'}{}^{\beta},\\
\mathring{L}^{\alpha\beta\alpha'}\nabla_{\alpha}W_{\beta\gamma\alpha'}{}^{\gamma} &\approx 3\mathsf{D}^{\alpha\alpha'}W_{\alpha\beta\alpha'}{}^{\beta},\\
\mathring{L}^{\alpha\beta\alpha'}\nabla^{\gamma}W_{\alpha\gamma\beta\alpha'} &\approx \mathring{L}^{\alpha\beta\alpha'}\overline{\nabla}^{\gamma}W_{\alpha\gamma\beta\alpha'},
\end{aligned}$$

and

$$\begin{aligned}
H^{\alpha'}\nabla^{\beta'}W_{\alpha'\alpha\beta'}{}^{\alpha} &\approx -(n-3)H^{\alpha'}\mathcal{C}_{\alpha'} - H^{\alpha'}\nabla^{\alpha}W_{\alpha\beta\alpha'}{}^{\beta},\\
H^{\alpha'}\nabla^{\beta'}\mathsf{P}_{\alpha'\beta'} &= H^{\alpha'}\nabla_{\alpha'}\mathsf{J} - H^{\alpha'}\nabla^{\alpha}\mathsf{P}_{\alpha\alpha'},\\
\mathring{L}^{\alpha\beta\alpha'}\nabla^{\beta'}W_{\alpha\alpha'\beta\beta'} &\approx -(n-3)\mathring{L}^{\alpha\beta\alpha'}\mathcal{C}_{\alpha\alpha'\beta} - \mathring{L}^{\alpha\beta\alpha'}\nabla^{\gamma}W_{\alpha\gamma\beta\alpha'}.
\end{aligned}$$

Combining these computations with Remark 5.4, Propositions 5.5 and 5.11, and Statement (6.5) yields

$$\begin{aligned}
&\operatorname{Contr}(\nabla^2\operatorname{Rm}, L\otimes\overline{\nabla}^2L, \overline{\nabla}L\otimes\overline{\nabla}L, L\otimes\nabla\operatorname{Rm}, \overline{\nabla}L\otimes\operatorname{Rm})\\
&\quad\subseteq \operatorname{span}\{\Delta\mathsf{J}, H^{\alpha'}\overline{\Delta}H_{\alpha'}, H^{\alpha'}\nabla_{\alpha'}\mathsf{J}, H^{\alpha'}\overline{\nabla}^{\alpha}\mathsf{D}_{\alpha\alpha'}, |\mathsf{D}|^2,\\
&\qquad H^{\alpha'}\mathcal{C}_{\alpha'}, H^{\alpha'}\overline{\nabla}^{\alpha}W_{\alpha\beta\alpha'}{}^{\beta}, \mathsf{D}^{\alpha\alpha'}W_{\alpha\beta\alpha'}{}^{\beta}\}\\
&\qquad + \mathscr{I}_{4,\mathrm{div}} + \mathscr{I}_{4,c} + \operatorname{Contr}(L^{\otimes 4}, L^{\otimes 2}\otimes\operatorname{Rm}, \operatorname{Rm}^{\otimes 2}).
\end{aligned}\tag{6.6}$$

<u>Step 3: Analyze $\operatorname{Contr}(L^{\otimes 4}, L^{\otimes 2}\otimes\operatorname{Rm}, \operatorname{Rm}^{\otimes 2})$</u>. It is clear that

$$\operatorname{Contr}(\mathring{L}^{\otimes 4}, \mathring{L}^{\otimes 2}\otimes W, W^{\otimes 2}) \subseteq \mathscr{I}_{4,c}. \tag{6.7}$$

Denote by

$$\mathscr{I}_{(0)} := \operatorname{Contr}(L^{\otimes 4}, L^{\otimes 2}\otimes\operatorname{Rm}, \operatorname{Rm}^{\otimes 2}) \cap \bigcup_{j=0}^{4}\mathscr{I}_4^j$$

the set of undifferentiated scalars in $\bigcup\mathscr{I}_4^j$. We will show that

$$\operatorname{Contr}(L^{\otimes 4}, L^{\otimes 2}\otimes\operatorname{Rm}, \operatorname{Rm}^{\otimes 2}) \subseteq \mathscr{I}_{4,\mathrm{div}} + \mathscr{I}_{4,c} + \operatorname{span}\mathscr{I}_{(0)}. \tag{6.8}$$

This together with Statement (6.6) then completes the proof.

We first show that

$$
\begin{aligned}
\mathrm{Contr}(\mathrm{Rm}^{\otimes 2}) \subseteq \operatorname{span}\Big\{ & W_{\alpha'\alpha\beta'}{}^{\alpha}\mathsf{P}^{\alpha'\beta'}, \mathsf{P}_{\alpha'}{}^{\alpha'}\mathsf{P}_{\beta'}{}^{\beta'}, \mathsf{P}^{\alpha'\beta'}\mathsf{P}_{\alpha'\beta'}, \mathsf{G}\mathsf{P}_{\alpha'}{}^{\alpha'}, \overline{\mathsf{J}}\mathsf{P}_{\alpha'}{}^{\alpha'}, \\
& \mathsf{D}^{\alpha\alpha'}W_{\alpha\beta\alpha'}{}^{\beta}, H^{\alpha'}\overline{\nabla}^{\alpha}W_{\alpha\beta\alpha'}{}^{\beta}, |\mathsf{D}|^2, H^{\alpha'}\overline{\nabla}^{\alpha}\mathsf{D}_{\alpha\alpha'}, H^{\alpha'}\overline{\Delta}H_{\alpha'}, \\
& \overline{\mathsf{J}}^2, |\overline{\mathsf{P}}|^2, \mathsf{G}\overline{\mathsf{J}}, \langle \mathsf{F}, \overline{\mathsf{P}}\rangle \Big\} + \mathrm{Contr}(L^{\otimes 4}, L^{\otimes 2}\otimes \mathrm{Rm}) + \mathscr{I}_{4,\mathrm{div}} + \mathscr{I}_{4,c}.
\end{aligned}
\tag{6.9}
$$

To that end, for each nonnegative integer j, let $\mathrm{Contr}_j(\mathrm{Rm}^{\otimes 2})$ be the set of elements of $\mathrm{Contr}(\mathrm{Rm}^{\otimes 2})$ that can be expressed as a linear combination of complete contractions of projections of $W\otimes\mathsf{P}$ and $\mathsf{P}^{\otimes 2}$ with at least j tangential contractions. Statement (6.7) and the decomposition (4.2) of Rm imply that

$$
\mathrm{Contr}(\mathrm{Rm}^{\otimes 2}) \subseteq \mathrm{Contr}_0(\mathrm{Rm}^{\otimes 2}) + \mathscr{I}_{4,c}, \tag{6.10}
$$

while an index count implies that $\mathrm{Contr}_j(\mathrm{Rm}^{\otimes 2}) = \{0\}$ for all $j>3$.

Using Equation (6.1) yields

$$
\mathrm{Contr}_0(\mathrm{Rm}^{\otimes 2}) \subseteq \mathrm{Contr}(\mathsf{P}_{\alpha'\beta'}\mathsf{P}_{\gamma'\delta'}) + \mathrm{Contr}_1(\mathrm{Rm}^{\otimes 2}).
$$

Therefore

$$
\mathrm{Contr}_0(\mathrm{Rm}^{\otimes 2}) \subseteq \operatorname{span}\{\mathsf{P}_{\alpha'}{}^{\alpha'}\mathsf{P}_{\beta'}{}^{\beta'}, \mathsf{P}^{\alpha'\beta'}\mathsf{P}_{\alpha'\beta'}\} + \mathrm{Contr}_1(\mathrm{Rm}^{\otimes 2}). \tag{6.11}
$$

Using Equation (6.1) again yields

$$
\mathrm{Contr}_1(\mathrm{Rm}^{\otimes 2}) \subseteq \mathrm{Contr}(W_{\alpha\alpha'\beta\beta'}\mathsf{P}_{\gamma'\delta'}, \mathsf{P}_{\alpha\beta}\mathsf{P}_{\alpha'\beta'}, \mathsf{P}_{\alpha\alpha'}\mathsf{P}_{\beta\beta'}) + \mathrm{Contr}_2(\mathrm{Rm}^{\otimes 2}).
$$

Equation (4.9) implies that the Fialkow tensor F can be written as a linear combination of partial contractions of $L^2_{\alpha\beta}g_{\gamma\delta}$ and $W_{\alpha\beta\gamma\delta}g_{\epsilon\zeta}$. Combining this observation with Equations (4.7), (4.8b), and (6.1) yields

$$
\begin{aligned}
\mathrm{Contr}_1(\mathrm{Rm}^{\otimes 2}) \subseteq{} & \mathrm{Contr}_2(\mathrm{Rm}^{\otimes 2}) + \mathrm{Contr}(L^{\otimes 2}\otimes\mathrm{Rm}) + \mathscr{I}_{4,\mathrm{div}} \\
& + \operatorname{span}\{W_{\alpha'\alpha\beta'}{}^{\alpha}\mathsf{P}^{\alpha'\beta'}, \overline{\mathsf{J}}\mathsf{P}_{\alpha'}{}^{\alpha'}, |\mathsf{D}|^2, H^{\alpha'}\overline{\nabla}^{\alpha}\mathsf{D}_{\alpha\alpha'}, H^{\alpha'}\overline{\Delta}H_{\alpha'}\}.
\end{aligned}
\tag{6.12}
$$

Applying Equation (6.1) yet again yields

$$
\mathrm{Contr}_2(\mathrm{Rm}^{\otimes 2}) \subseteq \mathrm{Contr}(W_{\alpha\beta\gamma\alpha'}\mathsf{P}_{\delta\beta'}, W_{\alpha\beta\gamma\delta}\mathsf{P}_{\alpha'\beta'}, \mathsf{P}_{\alpha\beta}\mathsf{P}_{\gamma\delta}) + \mathrm{Contr}_3(\mathrm{Rm}^{\otimes 2}).
$$

Similar to the previous paragraph, Equations (4.7), (4.8b), and (4.9) yield

$$
\begin{aligned}
\mathrm{Contr}_2(\mathrm{Rm}^{\otimes 2}) \subseteq{} & \operatorname{span}\{\mathsf{D}^{\alpha\alpha'}W_{\alpha\beta\alpha'}{}^{\beta}, H^{\alpha'}\overline{\nabla}^{\alpha}W_{\alpha\beta\alpha'}{}^{\beta}, \mathsf{G}\mathsf{P}_{\alpha'}{}^{\alpha'}, \overline{\mathsf{J}}^2, |\overline{\mathsf{P}}|^2\} \\
& + \mathrm{Contr}_3(\mathrm{Rm}^{\otimes 2}) + \mathrm{Contr}(L^{\otimes 4}, L^{\otimes 2}\otimes\mathrm{Rm}) + \mathscr{I}_{4,\mathrm{div}} + \mathscr{I}_{4,c}.
\end{aligned}
\tag{6.13}
$$

An index count and Equation (4.9) imply that

$$
\mathrm{Contr}_3(\mathrm{Rm}^{\otimes 2}) \subseteq \mathrm{Contr}(W_{\alpha\beta\gamma\delta}\mathsf{P}_{\epsilon\zeta}) \subseteq \mathrm{Contr}(\mathsf{F}_{\alpha\beta}\mathsf{P}_{\gamma\delta}) + \mathrm{Contr}(L^{\otimes 2}\otimes\mathrm{Rm}).
$$

Combining this with Equations (4.8b) and (4.9) yields

$$
\mathrm{Contr}_3(\mathrm{Rm}^{\otimes 2}) \subseteq \operatorname{span}\{\mathsf{G}\overline{\mathsf{J}}, \langle\mathsf{F},\overline{\mathsf{P}}\rangle\} + \mathrm{Contr}(L^{\otimes 4}, L^{\otimes 2}\otimes\mathrm{Rm}) + \mathscr{I}_{4,c}. \tag{6.14}
$$

Statements (6.10)–(6.14) together yield Statement (6.9).

We now show that

$$
\begin{aligned}
\mathrm{Contr}(L^{\otimes 2}\otimes\mathrm{Rm}) \subseteq \operatorname{span}\Big\{ & |H|^2\overline{\mathsf{J}}, |\mathring{L}|^2\overline{\mathsf{J}}, \langle\mathring{L}^2,\overline{\mathsf{P}}\rangle, H^{\alpha'}H^{\beta'}W_{\alpha'\alpha\beta'}{}^{\alpha}, \\
& \mathsf{G}|H|^2, H^{\alpha'}\mathring{L}_{\alpha\beta\alpha'}\mathsf{F}^{\alpha\beta}, H^{\alpha'}\mathring{L}_{\alpha\beta\alpha'}\overline{\mathsf{P}}^{\alpha\beta}, H^{\alpha'}\mathring{L}^{\alpha\beta\beta'}W_{\alpha\alpha'\beta\beta'}, H^{\alpha'}H^{\beta'}\mathsf{P}_{\alpha'\beta'}, \\
& |H|^2\mathsf{P}_{\alpha'}{}^{\alpha'}, |\mathring{L}|^2\mathsf{P}_{\alpha'}{}^{\alpha'}, \mathring{L}_{\alpha\beta}{}^{\alpha'}\mathring{L}^{\alpha\beta\beta'}\mathsf{P}_{\alpha'\beta'}\Big\} + \mathrm{Contr}(L^{\otimes 4}) + \mathscr{I}_{4,c}.
\end{aligned}
\tag{6.15}
$$

Equations (4.2) and (6.1) imply that

$$\begin{aligned}\operatorname{Contr}(L^{\otimes 2}\otimes \mathrm{Rm}) \subseteq \operatorname{Contr}(&L^{\otimes 2}\otimes W_{\alpha\beta\alpha'\beta'}, L^{\otimes 2}\otimes W_{\alpha\alpha'\beta\beta'}, L^{\otimes 2}\otimes W_{\alpha\beta\gamma\delta},\\ &L^{\otimes 2}\otimes \mathsf{P}_{\alpha'\beta'}, L^{\otimes 2}\otimes \mathsf{P}_{\alpha\beta}).\end{aligned}$$

On the one hand, combining Equations (4.9) and (6.1) with Statement (6.7) yields

$$\begin{aligned}\operatorname{Contr}(L^{\otimes 2}\otimes W_{\alpha\beta\alpha'\beta'}) &\subseteq \mathscr{I}_{4,c},\\ \operatorname{Contr}(L^{\otimes 2}\otimes W_{\alpha\alpha'\beta\beta'}) &\subseteq \operatorname{span}\{H^{\alpha'}H^{\beta'}W_{\alpha'\alpha\beta'}{}^{\alpha}, H^{\alpha'}\mathring{L}^{\alpha\beta\beta'}W_{\alpha\alpha'\beta\beta'}\}\\ &\qquad + \operatorname{Contr}(L^{\otimes 2}\otimes W_{\alpha\beta\gamma\delta}) + \mathscr{I}_{4,c},\\ \operatorname{Contr}(L^{\otimes 2}\otimes W_{\alpha\beta\gamma\delta}) &\subseteq \operatorname{Contr}(L^{\otimes 2}\otimes \mathsf{F}) + \operatorname{Contr}(L^{\otimes 4}) + \mathscr{I}_{4,c}.\end{aligned}$$

On the other hand, Equation (4.8b) implies that

$$\operatorname{Contr}(L^{\otimes 2}\otimes \mathsf{P}_{\alpha\beta}) \subseteq \operatorname{Contr}(L^{\otimes 2}\otimes \overline{\mathsf{P}}, L^{\otimes 2}\otimes \mathsf{F}, L^{\otimes 4}),$$

while direct computation yields

$$\begin{aligned}\operatorname{Contr}(L^{\otimes 2}\otimes \mathsf{P}_{\alpha'\beta'}) &\subseteq \operatorname{span}\{|H|^2\mathsf{P}_{\alpha'}{}^{\alpha'}, |\mathring{L}|^2\mathsf{P}_{\alpha'}{}^{\alpha'}, H^{\alpha'}H^{\beta'}\mathsf{P}_{\alpha'\beta'}, \mathring{L}_{\alpha\beta}{}^{\alpha'}\mathring{L}^{\alpha\beta\beta'}\mathsf{P}_{\alpha'\beta'}\},\\ \operatorname{Contr}(L^{\otimes 2}\otimes \overline{\mathsf{P}}) &\subseteq \operatorname{span}\{|H|^2\overline{\mathsf{J}}, |\mathring{L}|^2\overline{\mathsf{J}}, H^{\alpha'}\mathring{L}_{\alpha\beta\alpha'}\overline{\mathsf{P}}^{\alpha\beta}, \langle \mathring{L}^2, \overline{\mathsf{P}}\rangle\},\\ \operatorname{Contr}(L^{\otimes 2}\otimes \mathsf{F}) &\subseteq \operatorname{span}\{\mathsf{G}|H|^2, H^{\alpha'}\mathring{L}_{\alpha\beta\alpha'}\mathsf{F}^{\alpha\beta}\} + \mathscr{I}_{4,c}.\end{aligned}$$

Combining these observations yields Statement (6.15).

Finally, one directly checks using Equation (6.7) that

$$\operatorname{Contr}(L^{\otimes 4}) \subseteq \operatorname{span}\{|H|^4, |H|^2|\mathring{L}|^2, H^{\alpha'}\mathring{L}_{\alpha\beta\alpha'}\mathring{L}^{\alpha\beta\beta'}H_{\beta'}, H^{\alpha'}\operatorname{tr}\mathring{L}^3_{\alpha'}\} + \mathscr{I}_{4,c}. \tag{6.16}$$

Combining Statements (6.9), (6.15), and (6.16) yields Statement (6.8). □

Lemma 6.1 reduces the verification of Conjecture 1.1 for four-dimensional submanifolds to an analysis of the conformal linearization of the integral of linear combinations of elements of $\bigcup_{j=0}^{4}\mathscr{I}_4^j$.

Proof of Theorem 1.5 when $k=4$. Let I be a natural scalar on 4-dimensional submanifolds Y of n-dimensional Riemannian manifolds (X,g) whose integral over compact Y is invariant under conformal rescaling of g. Then I has weight -4. By Lemma 6.1, we may assume that $I \in \operatorname{span}\bigcup_{j=0}^{4}\mathscr{I}_4^j$. We will show that $I = a(\overline{\mathsf{J}}^2 - |\overline{\mathsf{P}}|^2)$ for some constant $a\in\mathbb{R}$. The final conclusion then follows from the fact that

$$\overline{\mathrm{Pf}} = \frac{1}{8}|\overline{W}|^2 + \overline{\mathsf{J}}^2 - |\overline{\mathsf{P}}|^2$$

in dimension four.

Step 1: Show no dependence on $\mathscr{I}_4^4$. By definition, there is a constant $a\in\mathbb{R}$ such that $I \equiv a\Delta\mathsf{J} \mod \operatorname{span}\bigcup_{j=0}^{3}\mathscr{I}_4^j$. Let $\Upsilon\in C^\infty(X)$. Suppose that Υ, $\Upsilon_{\alpha'}$, $\Upsilon_{\alpha'\beta'}$, and $\Upsilon_{\alpha'\beta'\gamma'}$ all vanish along Y. Then

$$0 = \int_Y (I\,\mathrm{dA})^\bullet = -a\int_Y \Upsilon_{\alpha'}{}^{\alpha'}{}_{\beta'}{}^{\beta'}\,\mathrm{dA}\,.$$

Since Υ is otherwise arbitrary, we conclude that $a=0$. Therefore $I \in \operatorname{span}\bigcup_{j=0}^{3}\mathscr{I}_4^j$.

Step 2: Show no dependence on $\mathscr{I}_4^3$. Step 1 implies that there is a constant $a \in \mathbb{R}$ such that $I \equiv aH^{\alpha'}\nabla_{\alpha'}\mathsf{J} \mod \operatorname{span}\bigcup_{j=0}^{2}\mathscr{I}_4^j$. Suppose that $\Upsilon \in C^\infty(X)$ is such that Υ, $\Upsilon_{\alpha'}$, and $\Upsilon_{\alpha'\beta'}$ all vanish along Y. Then

$$0 = \int_Y (I\,\mathrm{dA})^\bullet = -a\int_Y H^{\alpha'}\Upsilon_{\beta'}{}^{\beta'}{}_{\alpha'}\,\mathrm{dA}\,.$$

Since Υ is otherwise arbitrary, we conclude that $a = 0$. Therefore $I \in \operatorname{span}\bigcup_{j=0}^{2}\mathscr{I}_4^j$.

Step 3: Show no dependence on $\mathscr{I}_4^2$. Step 2 implies that there are constants $a_1, \dots, a_9 \in \mathbb{R}$ such that

$$\begin{aligned} I &\equiv a_1|H|^2\mathsf{P}_{\alpha'}{}^{\alpha'} + a_2H^{\alpha'}H^{\beta'}\mathsf{P}_{\alpha'\beta'} + a_3\mathsf{G}\mathsf{P}_{\alpha'}{}^{\alpha'} \\ &\quad + a_4W_{\alpha'\alpha\beta'}{}^{\alpha}\mathsf{P}^{\alpha'\beta'} + a_5|\mathring{L}|^2\mathsf{P}_{\alpha'}{}^{\alpha'} + a_6\mathring{L}^{\alpha\beta}{}_{\alpha'}\mathring{L}_{\alpha\beta\beta'}\mathsf{P}^{\alpha'\beta'} \\ &\quad + a_7\overline{\mathsf{J}}\mathsf{P}_{\alpha'}{}^{\alpha'} + a_8\mathsf{P}_{\alpha'}{}^{\alpha'}\mathsf{P}_{\beta'}{}^{\beta'} + a_9\mathsf{P}_{\alpha'\beta'}\mathsf{P}^{\alpha'\beta'} \mod \operatorname{span}\mathscr{I}_4^0 \cup \mathscr{I}_4^1. \end{aligned} \tag{6.17}$$

Suppose that $\Upsilon \in C^\infty(X)$ is such that Υ and $\Upsilon_{\alpha'}$ vanish along Y. Then

$$\begin{aligned} 0 &= \int_Y (I\,\mathrm{dA})^\bullet \\ &= -\int_Y \Big((a_1|H|^2 + a_3\mathsf{G} + a_5|\mathring{L}|^2 + a_7\overline{\mathsf{J}} + 2a_8\mathsf{P}_{\gamma'}{}^{\gamma'})g_{\alpha'\beta'} + a_2H_{\alpha'}H_{\beta'} \\ &\qquad + a_4W_{\alpha'\alpha\beta'}{}^{\alpha} + a_6\mathring{L}^{\alpha\beta}{}_{\alpha'}\mathring{L}_{\alpha\beta\beta'} + 2a_9\mathsf{P}_{\alpha'\beta'}\Big)\Upsilon^{\alpha'\beta'}\,\mathrm{dA}\,. \end{aligned} \tag{6.18}$$

Since Υ is otherwise arbitrary, we deduce that

$$\begin{aligned} 0 = (a_1|H|^2 &+ a_3\mathsf{G} + a_5|\mathring{L}|^2 + a_7\overline{\mathsf{J}} + 2a_8\mathsf{P}_{\gamma'}{}^{\gamma'})g_{\alpha'\beta'} \\ &+ a_2H_{\alpha'}H_{\beta'} + a_4W_{\alpha'\alpha\beta'}{}^{\alpha} + a_6\mathring{L}^{\alpha\beta}{}_{\alpha'}\mathring{L}_{\alpha\beta\beta'} + 2a_9\mathsf{P}_{\alpha'\beta'}. \end{aligned}$$

Contracting this with $\mathsf{P}^{\alpha'\beta'}$ and comparing with Equation (6.17) yields

$$I \equiv -a_8\mathsf{P}_{\alpha'}{}^{\alpha'}\mathsf{P}_{\beta'}{}^{\beta'} - a_9\mathsf{P}_{\alpha'\beta'}\mathsf{P}^{\alpha'\beta'} \mod \operatorname{span}\mathscr{I}_4^0 \cup \mathscr{I}_4^1.$$

Repeating the variational computation (6.18) yields $a_8\mathsf{P}_{\gamma'}{}^{\gamma'}g_{\alpha'\beta'} + a_9\mathsf{P}_{\alpha'\beta'} = 0$. Therefore $I \in \operatorname{span}\mathscr{I}_4^0 \cup \mathscr{I}_4^1$.

Step 4: Show no dependence on $\mathscr{I}_4^1$. Step 3 implies that there are constants $a_1, \dots, a_{14} \in \mathbb{R}$ such that

$$\begin{aligned} I &\equiv a_1H^{\alpha'}\overline{\Delta}H_{\alpha'} + a_2H^{\alpha'}\overline{\nabla}^{\alpha}\mathsf{D}_{\alpha\alpha'} + a_3H^{\alpha'}\overline{\nabla}^{\alpha}W_{\alpha\beta\alpha'}{}^{\beta} + a_4|H|^4 \\ &\quad + a_5|H|^2|\mathring{L}|^2 + a_6H^{\alpha'}\mathring{L}_{\alpha\beta\alpha'}\mathring{L}^{\alpha\beta\beta'}H_{\beta'} + a_7|H|^2\overline{\mathsf{J}} + a_8\mathsf{G}|H|^2 \\ &\quad + a_9H^{\alpha'}\operatorname{tr}\mathring{L}^3_{\alpha'} + a_{10}H^{\alpha'}\mathring{L}^{\alpha\beta\beta'}W_{\alpha\alpha'\beta\beta'} + a_{11}H^{\alpha'}\mathring{L}_{\alpha\beta\alpha'}\mathsf{F}^{\alpha\beta} \\ &\quad + a_{12}H^{\alpha'}\mathring{L}_{\alpha\beta\alpha'}\overline{\mathsf{P}}^{\alpha\beta} + a_{13}H^{\alpha'}H^{\beta'}W_{\alpha'\alpha\beta'}{}^{\alpha} + a_{14}H^{\alpha'}\mathcal{C}_{\alpha'} \mod \operatorname{span}\mathscr{I}_4^0. \end{aligned}$$

Suppose that $\Upsilon \in C^\infty(X)$ is such that Υ vanishes along Y. Then

$$\begin{aligned} 0 &= \int_Y (I\,\mathrm{dA})^\bullet \\ &= -\int_Y \Big(2a_1 \overline{\Delta} H_{\alpha'} + a_2 \overline{\nabla}^\alpha \mathsf{D}_{\alpha\alpha'} + a_3 \overline{\nabla}^\alpha W_{\alpha\beta\alpha'}{}^\beta + 4a_4 |H|^2 H_{\alpha'} + 2a_5 |\mathring{L}|^2 H_{\alpha'} \\ &\qquad + 2a_6 H^{\beta'} \mathring{L}_{\alpha\beta\beta'} \mathring{L}^{\alpha\beta}{}_{\alpha'} + 2a_7 \overline{\mathsf{J}} H_{\alpha'} + 2a_8 \mathsf{G} H_{\alpha'} + a_9 \operatorname{tr} \mathring{L}^3_{\alpha'} \\ &\qquad + a_{10} \mathring{L}^{\alpha\beta\beta'} W_{\alpha\alpha'\beta\beta'} + a_{11} \mathsf{F}^{\alpha\beta} \mathring{L}_{\alpha\beta\alpha'} + a_{12} \overline{\mathsf{P}}^{\alpha\beta} \mathring{L}_{\alpha\beta\alpha'} \\ &\qquad + 2a_{13} H^{\beta'} W_{\alpha'\alpha\beta'}{}^\alpha + a_{14} \mathcal{C}_{\alpha'} \Big) \Upsilon^{\alpha'}\,\mathrm{dA}\,. \end{aligned}$$

Since Υ is otherwise arbitrary, we deduce that

$$\begin{aligned} 0 &= 2a_1 \overline{\Delta} H_{\alpha'} + a_2 \overline{\nabla}^\alpha \mathsf{D}_{\alpha\alpha'} + a_3 \overline{\nabla}^\alpha W_{\alpha\beta\alpha'}{}^\beta + 4a_4 |H|^2 H_{\alpha'} + 2a_5 |\mathring{L}|^2 H_{\alpha'} \\ &\quad + 2a_6 H^{\beta'} \mathring{L}_{\alpha\beta\beta'} \mathring{L}^{\alpha\beta}{}_{\alpha'} + 2a_7 \overline{\mathsf{J}} H_{\alpha'} + 2a_8 \mathsf{G} H_{\alpha'} + a_9 \operatorname{tr} \mathring{L}^3_{\alpha'} + a_{10} \mathring{L}^{\alpha\beta\beta'} W_{\alpha\alpha'\beta\beta'} \\ &\quad + a_{11} \mathsf{F}_{\alpha\beta} \mathring{L}^{\alpha\beta}{}_{\alpha'} + a_{12} \overline{\mathsf{P}}^{\alpha\beta} \mathring{L}_{\alpha\beta\alpha'} + 2a_{13} H^{\beta'} W_{\alpha'\alpha\beta'}{}^\alpha + a_{14} \mathcal{C}_{\alpha'}. \end{aligned}$$

Contracting with $H^{\alpha'}$ and comparing this with our initial formula for I implies that

$$\begin{aligned} I \equiv -a_1 H^{\alpha'} \overline{\Delta} H_{\alpha'} - 3a_4 |H|^4 - a_5 |H|^2 |\mathring{L}|^2 - a_6 H^{\alpha'} \mathring{L}_{\alpha\beta\alpha'} \mathring{L}^{\alpha\beta\beta'} H_{\beta'} \\ - a_7 |H|^2 \overline{\mathsf{J}} - a_8 \mathsf{G} |H|^2 - a_{13} H^{\alpha'} H^{\beta'} W_{\alpha'\alpha\beta'}{}^\beta \mod \operatorname{span} \mathscr{I}^0_4. \end{aligned}$$

Repeating this argument twice more yields $I \in \operatorname{span} \mathscr{I}^0_4$.

Step 5: Show I is proportional to $\overline{\mathsf{J}}^2 - |\overline{\mathsf{P}}|^2$. Step 4 implies that there are constants $a_1, \dots, a_8 \in \mathbb{R}$ such that

$$\begin{aligned} I = a_1 (\overline{\mathsf{J}}^2 - |\overline{\mathsf{P}}|^2) + a_2 \overline{\mathsf{J}}^2 + a_3 \langle \mathsf{F}, \overline{\mathsf{P}} \rangle + a_4 |\mathsf{D}|^2 \\ + a_5 W_{\alpha\beta\alpha'}{}^\beta \mathsf{D}^{\alpha\alpha'} + a_6 |\mathring{L}|^2 \overline{\mathsf{J}} + a_7 \langle \mathring{L}^2, \overline{\mathsf{P}} \rangle + a_8 \mathsf{G} \overline{\mathsf{J}}. \end{aligned} \tag{6.19}$$

Let $\Upsilon \in C^\infty(X)$. By computing the conformal variation and integrating by parts, we deduce that

$$\begin{aligned} 0 &= \int_Y (I\,\mathrm{dA})^\bullet \\ &= -\int_Y \Big(2a_2 \overline{\Delta \mathsf{J}} + a_3 \overline{\nabla}^\alpha \overline{\nabla}^\beta \mathsf{F}_{\alpha\beta} - 2a_4 \overline{\nabla}^\alpha (\mathring{L}_{\alpha\beta\alpha'} \mathsf{D}^{\beta\alpha'}) - a_5 \overline{\nabla}^\alpha (\mathring{L}_\alpha{}^{\beta\alpha'} W_{\beta\gamma\alpha'}{}^\gamma) \\ &\qquad + a_6 \overline{\Delta} |\mathring{L}|^2 + a_7 \overline{\nabla}^\alpha \overline{\nabla}^\beta \mathring{L}^2_{\alpha\beta} + a_8 \overline{\Delta} \mathsf{G} \Big) \Upsilon\,\mathrm{dA}\,. \end{aligned}$$

Since Υ is arbitrary, we deduce from this and Equation (5.3) that

$$\begin{aligned} 0 = 2a_2 \overline{\Delta \mathsf{J}} + a_3 \overline{\nabla}^\alpha \overline{\nabla}^\beta \mathsf{F}_{\alpha\beta} - 2(a_4 + a_7) \overline{\nabla}^\alpha (\mathring{L}_{\alpha\beta\alpha'} \mathsf{D}^{\beta\alpha'}) + a_8 \overline{\Delta} \mathsf{G} \\ + \left(a_6 + \frac{a_7}{2} \right) \overline{\Delta} |\mathring{L}|^2 - (a_5 + a_7) \overline{\nabla}^\alpha (\mathring{L}_\alpha{}^{\beta\alpha'} W_{\beta\gamma\alpha'}{}^\gamma) - a_7 \overline{\nabla}^\alpha (\mathring{L}^{\beta\gamma\alpha'} W_{\beta\alpha\gamma\alpha'}). \end{aligned} \tag{6.20}$$

In dimension four, the conformal linearizations of the terms in this expression are

$$\begin{aligned}
\big(\overline{\Delta}\mathsf{J}\big)^{\bullet} &= -\overline{\Delta}^{2}\Upsilon - 2\overline{\nabla}^{\alpha}(\bar{\mathsf{J}}\Upsilon_{\alpha}),\\
\big(\overline{\nabla}^{\alpha}\overline{\nabla}^{\beta}\mathsf{F}_{\alpha\beta}\big)^{\bullet} &= \overline{\nabla}^{\alpha}(2\mathsf{F}_{\alpha\beta}\Upsilon^{\beta} - \mathsf{G}\Upsilon_{\alpha}),\\
\big(\overline{\nabla}^{\alpha}(\mathring{L}_{\alpha\beta\alpha'}\mathsf{D}^{\beta\alpha'})\big)^{\bullet} &= -\overline{\nabla}^{\alpha}(\mathring{L}^{2}_{\alpha\beta}\Upsilon^{\beta}),\\
\big(\overline{\Delta}|\mathring{L}|^{2}\big)^{\bullet} &= -2\overline{\nabla}^{\alpha}(|\mathring{L}|^{2}\Upsilon_{\alpha}),\\
\big(\overline{\Delta}\mathsf{G}\big)^{\bullet} &= -2\overline{\nabla}^{\alpha}(\mathsf{G}\Upsilon_{\alpha}),\\
\big(\overline{\nabla}^{\alpha}(\mathring{L}^{\beta\gamma\alpha'}W_{\beta\alpha\gamma\alpha'})\big)^{\bullet} &= 0,\\
\big(\overline{\nabla}^{\alpha}(\mathring{L}_{\alpha\beta\alpha'}W^{\beta\gamma\alpha'}{}_{\gamma})\big)^{\bullet} &= 0.
\end{aligned}$$

Since only $(\overline{\Delta}\mathsf{J})^{\bullet}$ depends on the full tangential four-jet of Υ, we see that $a_2 = 0$. Taking arbitrary Υ and applying Lemma 6.2 below yields

$$\begin{aligned}
&a_3 = 0, \qquad a_4 = -a_7, \qquad a_8 = 0, \qquad a_6 = -\frac{a_7}{2}, &&\text{if } n > 5,\\
&a_3 = 0, \qquad a_4 = -a_7, \qquad a_6 = -\frac{a_7}{2} - \frac{a_8}{6}, &&\text{if } n = 5.
\end{aligned}$$

Applying Lemma 6.2 below to Equation (6.20) yields

$$\begin{aligned}
&a_7 = 0, \qquad a_5 = 0, &&\text{if } n > 5,\\
&a_7 = 0, &&\text{if } n = 5.
\end{aligned}$$

Inserting these back into Equation (6.19) yields $I = a_1(\bar{\mathsf{J}}^{2} - |\overline{\mathsf{P}}|^{2})$. □

We conclude with a technical lemma used above which establishes the linear independence of certain sets of natural submanifold tensors.

Lemma 6.2. *Each of the sets*

$$\Big\{\mathsf{F}_{\alpha\beta}, \mathring{L}^{2}_{\alpha\beta}, |\mathring{L}|^{2}g_{\alpha\beta}\Big\} \quad \textit{and} \quad \Big\{\overline{\nabla}^{\alpha}(\mathring{L}^{\beta\gamma\alpha'}W_{\alpha\beta\gamma\alpha'})\Big\}$$

of natural tensors on 4-dimensional submanifolds of 5-dimensional Riemannian manifolds is linearly independent.

If $n \geq 6$, then each of the sets

$$\Big\{\mathsf{F}_{\alpha\beta}, \mathring{L}^{2}_{\alpha\beta}, |\mathring{L}|^{2}g_{\alpha\beta}, \mathsf{G}g_{\alpha\beta}\Big\} \quad \textit{and} \quad \Big\{\overline{\nabla}^{\alpha}(\mathring{L}^{\beta\gamma\alpha'}W_{\alpha\beta\gamma\alpha'}), \overline{\nabla}^{\alpha}(\mathring{L}_{\alpha}{}^{\beta\alpha'}W_{\beta\gamma\alpha'}{}^{\gamma})\Big\}$$

of natural tensors on 4-dimensional submanifolds of n-dimensional Riemannian manifolds is linearly independent.

Proof. First observe that the embedding $\mathbb{R}\times S^{3}\times\{0\} \hookrightarrow \mathbb{R}\times\mathbb{R}^{4}\times\mathbb{R}^{n-5}$ of a cylinder into flat Euclidean space is such that $\mathring{L}^{2}_{\alpha\beta}$ is not proportional to the induced metric. Hence, by Equation (4.9), it suffices to prove the following claim: If $n = 5$, then

$$W_{\alpha\gamma\beta}{}^{\gamma} \quad \text{and} \quad \overline{\nabla}^{\alpha}(\mathring{L}^{\beta\gamma\alpha'}W_{\alpha\beta\gamma\alpha'})$$

are nonzero natural submanifold tensors, and if $n \geq 6$, then each of the sets

$$\big\{W_{\alpha\gamma\beta}{}^{\gamma}, W_{\gamma\delta}{}^{\gamma\delta}g_{\alpha\beta}\big\} \quad \text{and} \quad \Big\{\overline{\nabla}^{\alpha}(\mathring{L}^{\beta\gamma\alpha'}W_{\alpha\beta\gamma\alpha'}), \overline{\nabla}^{\alpha}(\mathring{L}_{\alpha}{}^{\beta\alpha'}W_{\beta\gamma\alpha'}{}^{\gamma})\Big\}$$

is linearly independent. We prove this claim by computing some local examples. To that end, we do not use Einstein summation notation in the rest of this proof.

Set $Y := \{(x,0) \in \mathbb{R}^4 \times \mathbb{R}^{n-4}\} \subseteq \mathbb{R}^n$ and denote $p := (0,0) \in Y$. Equip $\mathbb{R}^n$ with the metric

$$g = \sum_{a=1}^{n} e^{2f_a}(dx^a)^2$$

for $f_a \colon \mathbb{R}^n \to \mathbb{R}$ functions satisfying $f_a(p) = 0$ and $df_a(p) = 0$. Mirroring our abstract index notation, we let $a \in \{1,\dots,n\}$ and $\alpha \in \{1,\dots,4\}$ and $\alpha' \in \{5,\dots,n\}$. Unless otherwise indicated, we use the convention that distinct characters are not equal; e.g. $\mathsf{P}_{\alpha\beta}$ denotes a component with $\alpha,\beta \in \{1,\dots,4\}$ and $\alpha \neq \beta$.

The nonvanishing Christoffel symbols of g are

$$\Gamma^a_{aa} = \partial_a f_a, \qquad \Gamma^a_{ab} = \partial_b f_a, \qquad \Gamma^b_{aa} = -e^{2(f_a - f_b)}\partial_b f_a.$$

Thus the only nonvanishing components of the second fundamental form are

$$L_{\alpha\alpha\alpha'} = -e^{2f_\alpha}\partial_{\alpha'} f_\alpha.$$

In particular, $L = 0$ at p. Since the Christoffel symbols all vanish at p, we compute that

$$R_{ab}{}^{c}{}_{d} = \partial_a \Gamma^c_{bd} - \partial_b \Gamma^c_{ad}$$

at p, where distinct indices can be equal in the above display. Thus the only nonvanishing components of Rm at p are

$$R_{abab} = -\partial^2_{aa} f_b - \partial^2_{bb} f_a, \qquad R_{acbc} = -\partial^2_{ab} f_c,$$

and those that can be obtained from the symmetries of Rm. Therefore

$$\begin{aligned} W_{abab} &= -\partial^2_{aa} f_b - \partial^2_{bb} f_a - \mathsf{P}_{aa} - \mathsf{P}_{bb}, \\ W_{acbc} &= -\partial^2_{ab} f_c - \mathsf{P}_{ab}, \end{aligned}$$

and

$$\begin{aligned} \mathsf{P}_{aa} &= -\frac{1}{n-1}\sum_{b\neq a}(\partial^2_{aa} f_b + \partial^2_{bb} f_a) + \frac{1}{(n-1)(n-2)}\sum_{\substack{b,c\neq a \\ b\neq c}} \partial^2_{bb} f_c, \\ \mathsf{P}_{ab} &= -\frac{1}{n-2}\sum_{c\notin\{a,b\}} \partial^2_{ab} f_c, \end{aligned}$$

at p. Additionally, the only nonvanishing components of $\overline{\nabla} L$ at p are

$$\overline{\nabla}_\alpha L_{\alpha\alpha\alpha'} = -\partial^2_{\alpha\alpha'} f_\alpha, \qquad \overline{\nabla}_\beta L_{\alpha\alpha\alpha'} = -\partial^2_{\beta\alpha'} f_\alpha.$$

We now simplify our computation by taking $f_{\alpha'} = 0$ and assuming at p that

$$\sum_{\beta=1}^{4} \partial^2_{\alpha\alpha'} f_\beta = 0 \quad \text{and} \quad \sum_{\beta'=5}^{n} \partial^2_{\beta'\beta'} f_\alpha = 0 \tag{6.21}$$

for each $\alpha \in \{1,\dots,4\}$ and $\alpha' \in \{5,\dots,n\}$. This assumption and our formula for P_{ab} yield

$$\begin{aligned} \mathsf{P}_{\alpha\alpha'} &= \frac{1}{n-2}\partial^2_{\alpha\alpha'} f_\alpha, \\ \overline{\nabla}_\alpha \mathring{L}_{\alpha\alpha\alpha'} &= -\partial^2_{\alpha\alpha'} f_\alpha, \\ \overline{\nabla}_\beta \mathring{L}_{\alpha\alpha\alpha'} &= -\partial^2_{\beta\alpha'} f_\alpha, \end{aligned}$$

at p. It follows that, at p,

$$\sum_{\gamma=1}^{4} W_{\alpha\gamma\beta}{}^{\gamma} = -\frac{n-4}{n-2} \sum_{\gamma\notin\{\alpha,\beta\}} \partial^2_{\alpha\beta} f_\gamma,$$
$$\sum_{\alpha,\beta=1}^{4} W_{\alpha\beta}{}^{\alpha\beta} = -\frac{2(n-4)(n-5)}{(n-1)(n-2)} \sum_{\alpha\neq\beta} \partial^2_{\alpha\alpha} f_\beta,$$

and

$$\sum_{\alpha,\beta,\gamma,\alpha'} \overline{\nabla}^{\alpha}(\mathring{L}^{\beta\gamma\alpha'} W_{\alpha\beta\gamma\alpha'}) = -\sum_{\alpha\neq\beta}\sum_{\alpha'}(\partial^2_{\beta\alpha'} f_\alpha)^2 + \frac{1}{n-2}\sum_{\alpha}\sum_{\alpha'}(\partial^2_{\alpha\alpha'} f_\alpha)^2,$$
$$\sum_{\alpha,\beta,\gamma,\alpha'} \overline{\nabla}^{\alpha}(\mathring{L}_{\alpha}{}^{\beta\alpha'} W_{\beta\gamma\alpha'}{}^{\gamma}) = -\frac{n-5}{n-2}\sum_{\alpha}\sum_{\alpha'}(\partial^2_{\alpha\alpha'} f_\alpha)^2.$$

Taking

$$f_1 = s(x^2)^2 + tx^2x^3 + ux^1x^5, \qquad f_2 = vx^1x^5,$$
$$f_3 = -(u+v)x^1x^5, \qquad f_4 = 0,$$

with $s,t,u,v \in \mathbb{R}$ yields our claim. □

Appendix A. Computations of conformal submanifold invariants

In this appendix we prove the five propositions in Subsection 5.3 relating our scalar conformal submanifold invariants of Subsection 5.1 to previously-known invariants. To that end, we first establish a useful formula for the inner product $\langle \mathring{L}, \overline{\Delta}\mathring{L}\rangle$ which is closely related to Simons' formula [Sim68, Theorem 4.2.1] for the Laplacian of the second fundamental form of a minimal submanifold.

Lemma A.1. *Let $i\colon Y^k \to (X^n,g)$ be an immersion with $3 \leq k < n$. Then*

$$\begin{aligned} \mathring{L}^{\alpha\beta\alpha'}\overline{\Delta}\mathring{L}_{\alpha\beta\alpha'} = \mathring{L}^{\alpha\beta\alpha'}\Big(&-k\overline{\nabla}_\alpha \mathsf{D}_{\beta\alpha'} - \overline{\nabla}_\alpha W_{\beta\gamma\alpha'}{}^{\gamma} + \overline{\nabla}^{\gamma} W_{\gamma\alpha\alpha'\beta} + \overline{\mathsf{J}}\mathring{L}_{\alpha\beta\alpha'} \\ &+ k\overline{\mathsf{P}}_\alpha{}^{\gamma}\mathring{L}_{\gamma\beta\alpha'} - W_\alpha{}^{\gamma}{}_\beta{}^{\delta}\mathring{L}_{\gamma\delta\alpha'} - W_\alpha{}^{\gamma}{}_{\alpha'}{}^{\beta'}\mathring{L}_{\gamma\beta\beta'} + 2\mathsf{F}_\alpha{}^{\gamma}\mathring{L}_{\gamma\beta\alpha'} \\ &- \mathring{L}_{\alpha\beta\beta'}\mathring{L}^{\gamma\delta\beta'}\mathring{L}_{\gamma\delta\alpha'} - \mathring{L}_{\alpha\delta\alpha'}\mathring{L}^{\gamma\delta\beta'}\mathring{L}_{\gamma\beta\beta'} + 2\mathring{L}^{\gamma\delta}{}_{\alpha'}\mathring{L}_{\alpha\delta\beta'}\mathring{L}_{\beta\gamma}{}^{\beta'}\Big). \end{aligned} \tag{A.1}$$

Proof. Using Equations (4.8d) and (4.8e), we compute that

$$\begin{aligned} \mathring{L}^{\alpha\beta\alpha'}\overline{\Delta}\mathring{L}_{\alpha\beta\alpha'} &= \mathring{L}^{\alpha\beta\alpha'}\overline{\nabla}^{\gamma}\left(\overline{\nabla}_\alpha\mathring{L}_{\gamma\beta\alpha'} + W_{\gamma\alpha\alpha'\beta} - g_{\gamma\beta}\mathsf{D}_{\alpha\alpha'}\right) \\ &= \mathring{L}^{\alpha\beta\alpha'}\Big(\overline{\nabla}_\alpha\overline{\nabla}^{\gamma}\mathring{L}_{\gamma\beta\alpha'} + (k-2)\overline{\mathsf{P}}_\alpha{}^{\gamma}\mathring{L}_{\gamma\beta\alpha'} + \overline{\mathsf{J}}\mathring{L}_{\alpha\beta\alpha'} - \overline{R}_\alpha{}^{\gamma}{}_\beta{}^{\delta}\mathring{L}_{\gamma\delta\alpha'} \\ &\qquad - \overline{R}_\alpha{}^{\gamma}{}_{\alpha'}{}^{\beta'}\mathring{L}_{\gamma\beta\beta'} + \overline{\nabla}^{\gamma}W_{\gamma\alpha\alpha'\beta} - \overline{\nabla}_\beta\mathsf{D}_{\alpha\alpha'}\Big) \\ &= \mathring{L}^{\alpha\beta\alpha'}\Big(-k\overline{\nabla}_\beta\mathsf{D}_{\alpha\alpha'} - \overline{\nabla}_\alpha W_{\beta\gamma\alpha'}{}^{\gamma} + \overline{\nabla}^{\gamma}W_{\gamma\alpha\alpha'\beta} \\ &\qquad - \overline{W}_\alpha{}^{\gamma}{}_\beta{}^{\delta}\mathring{L}_{\gamma\delta\alpha'} + k\overline{\mathsf{P}}_\alpha{}^{\gamma}\mathring{L}_{\gamma\beta\alpha'} + \overline{\mathsf{J}}\mathring{L}_{\alpha\beta\alpha'} - \overline{R}_\alpha{}^{\gamma}{}_{\alpha'}{}^{\beta'}\mathring{L}_{\gamma\beta\beta'}\Big). \end{aligned}$$

Combining this with Equations (4.8a) and (4.8f) yields Equation (A.1). □

We now prove the results in Subsection 5.3.

Proof of Proposition 5.20. The definitions (4.9) and (4.14) of $\mathsf{F}_{\alpha\beta}$ and $\mathcal{C}_{\alpha\alpha'\beta}$, respectively, imply that

$$\mathring{L}^{\alpha\beta\alpha'}\mathcal{C}_{\alpha\alpha'\beta} = \mathring{L}^{\alpha\beta\alpha'}C_{\alpha\alpha'\beta} + H^{\alpha'}\operatorname{tr}\mathring{L}^3_{\alpha'} - (k-2)H^{\alpha'}\mathring{L}^{\alpha\beta}{}_{\alpha'}\mathsf{F}_{\alpha\beta} - H^{\alpha'}\mathring{L}^{\alpha\beta\beta'}W_{\alpha\alpha'\beta\beta'} - H^{\alpha'}\mathring{L}^{\alpha\beta}{}_{\alpha'}W_{\alpha\gamma\beta}{}^{\gamma}. \tag{A.2}$$

First set

$$\widetilde{\mathcal{I}}_1 := -\overline{\Delta}|\mathring{L}|^2 + 2\overline{\mathsf{J}}|\mathring{L}|^2 + 2(k-6)\Big[\mathring{L}^2_{\alpha\beta}\overline{\mathsf{P}}^{\alpha\beta} - \overline{\nabla}^{\alpha}(\mathsf{D}^{\beta\alpha'}\mathring{L}_{\alpha\beta\alpha'}) - (k-3)|\mathsf{D}|^2 - \mathsf{D}^{\alpha\alpha'}W_{\alpha\beta\alpha'}{}^{\beta} - \mathring{L}^{\alpha\beta\alpha'}\mathcal{C}_{\alpha\alpha'\beta}\Big]. \tag{A.3}$$

Equation (4.8b) and Remark 5.13 imply that

$$\widetilde{\mathcal{I}}_1 = 2\mathcal{I} + \mathcal{J} - 2\mathsf{G}|\mathring{L}|^2 - 2(k-6)\mathring{L}^2_{\alpha\beta}\mathsf{F}^{\alpha\beta}. \tag{A.4}$$

Second set

$$\widetilde{\mathcal{I}}_2 := \mathring{L}^{\alpha\beta\alpha'}\overline{\Delta}\mathring{L}_{\alpha\beta\alpha'} + k\overline{\nabla}^{\alpha}(\mathsf{D}^{\beta\alpha'}\mathring{L}_{\alpha\beta\alpha'}) + k(k-1)|\mathsf{D}|^2 + (k+4)\mathsf{D}^{\alpha\alpha'}W_{\alpha\beta\alpha'}{}^{\beta} - k\mathring{L}^2_{\alpha\beta}\overline{\mathsf{P}}^{\alpha\beta} - \overline{\mathsf{J}}|\mathring{L}|^2 + (k-4)\mathring{L}^{\alpha\beta\alpha'}\mathcal{C}_{\alpha\alpha'\beta}.$$

Combining the formulas in Remark 5.4 with Lemma A.1 and Equation (4.8e) yields

$$\begin{aligned}\widetilde{\mathcal{I}}_2 &= \mathcal{K}_1 - \mathcal{K}_2 - W_{\alpha\beta\alpha'}{}^{\beta}W^{\alpha\gamma\alpha'}{}_{\gamma} - \frac{1}{2}W_{\alpha\beta\alpha'\gamma}W^{\alpha\beta\alpha'\gamma} - W_{\alpha}{}^{\gamma}{}_{\beta}{}^{\delta}\mathring{L}^{\alpha\beta\alpha'}\mathring{L}_{\gamma\delta\alpha'} \\ &\quad - W_{\alpha}{}^{\beta}{}_{\alpha'}{}^{\beta'}\mathring{L}^{\gamma\alpha\alpha'}\mathring{L}_{\gamma\beta\beta'} + 2\mathring{L}^2_{\alpha\beta}\mathsf{F}^{\alpha\beta} - \mathring{L}^{\alpha\beta\alpha'}\mathring{L}_{\alpha\beta\beta'}\mathring{L}^{\gamma\delta\beta'}\mathring{L}_{\gamma\delta\alpha'} \\ &\quad - |\mathring{L}^2|^2 + 2\mathring{L}^{\alpha\beta\alpha'}\mathring{L}^{\gamma\delta}{}_{\alpha'}\mathring{L}_{\alpha\gamma\beta'}\mathring{L}_{\beta\delta}{}^{\beta'}.\end{aligned} \tag{A.5}$$

Third set

$$\widetilde{\mathcal{I}}_3 := \overline{\nabla}^{\beta}(\mathsf{D}^{\alpha\alpha'}\mathring{L}_{\alpha\beta\alpha'}) + \frac{1}{k-1}\overline{\nabla}^{\beta}(\mathring{L}_{\beta}{}^{\alpha\alpha'}\overline{\nabla}^{\gamma}\mathring{L}_{\gamma\alpha\alpha'}) - \frac{k-4}{k-1}\mathsf{D}^{\alpha\alpha'}W_{\alpha\beta\alpha'}{}^{\beta}.$$

Equation (4.8e) implies that

$$\widetilde{\mathcal{I}}_3 = -\frac{1}{k-1}\mathcal{K}_2. \tag{A.6}$$

Equations (A.4)–(A.6) imply that

$$Wm = \frac{k(k-1)}{4(k-6)}\widetilde{\mathcal{I}}_1 + \frac{k-3}{2}\widetilde{\mathcal{I}}_2 + k\widetilde{\mathcal{I}}_3.$$

Applying the definitions of $\widetilde{\mathcal{I}}_1$, $\widetilde{\mathcal{I}}_2$, and $\widetilde{\mathcal{I}}_3$ yields

$$\begin{aligned}Wm &= \frac{k-3}{2}\mathring{L}^{\alpha\beta\alpha'}\overline{\Delta}\mathring{L}_{\alpha\beta\alpha'} + \frac{k}{k-1}\overline{\nabla}^{\alpha}(\mathring{L}_{\alpha}{}^{\beta\alpha'}\overline{\nabla}^{\gamma}\mathring{L}_{\gamma\beta\alpha'}) \\ &\quad + \frac{1}{k-6}\left(-\frac{k(k-1)}{4}\overline{\Delta}|\mathring{L}|^2 + (4k-9)\overline{\mathsf{J}}|\mathring{L}|^2\right) - 3(k-2)\mathring{L}^{\alpha\beta\alpha'}C_{\alpha\alpha'\beta} + k\mathring{L}^2_{\alpha\beta}\overline{\mathsf{P}}^{\alpha\beta} \\ &\quad - \frac{3(k-2)}{k-1}\mathsf{D}^{\alpha\alpha'}W_{\alpha\beta\alpha'}{}^{\beta} - 3(k-2)H^{\alpha'}\operatorname{tr}\mathring{L}^3_{\alpha'} + 3(k-2)^2H^{\alpha'}\mathring{L}^{\alpha\beta}{}_{\alpha'}\mathsf{F}_{\alpha\beta} \\ &\quad + 3(k-2)H^{\alpha'}\mathring{L}^{\alpha\beta\beta'}W_{\alpha\alpha'\beta\beta'} + 3(k-2)H^{\alpha'}\mathring{L}^{\alpha\beta}{}_{\alpha'}W_{\alpha\gamma\beta}{}^{\gamma}.\end{aligned}$$

Specializing to the case $k = 4$ and $n = 5$ yields the final conclusion. $\square$

Proof of Proposition 5.22. Set

$$\begin{aligned}\widetilde{\mathcal{J}}_1 &:= \mathring{L}^{\alpha\beta\alpha'}\nabla_{\alpha'}W_{\alpha\gamma\beta}{}^{\gamma} - 2\mathsf{D}^{\alpha\alpha'}W_{\alpha\beta\alpha'}{}^{\beta} - 2H^{\alpha'}\mathring{L}^{\alpha\beta}{}_{\alpha'}W_{\alpha\gamma\beta}{}^{\gamma} - 2\mathring{L}^{\alpha\beta\alpha'}\mathcal{C}_{\alpha\alpha'\beta},\\ \widetilde{\mathcal{J}}_2 &:= \overline{\nabla}^{\alpha}\overline{\nabla}^{\beta}\mathring{L}^2_{\alpha\beta} - \frac{1}{2}\overline{\Delta}|\mathring{L}|^2 + (k-2)\overline{\nabla}^{\alpha}(\mathsf{D}^{\beta\alpha'}\mathring{L}_{\alpha\beta\alpha'})\\ &\quad - (k-4)\mathring{L}^{\alpha\beta\alpha'}\mathcal{C}_{\alpha\alpha'\beta} - (k-4)\mathsf{D}^{\alpha\alpha'}W_{\alpha\beta\alpha'}{}^{\beta}.\end{aligned}$$

On the one hand, combining Equations (4.3), (4.5), and (4.6) yields

$$\begin{aligned}\mathring{L}^{\alpha\beta\alpha'}\nabla_{\alpha'}W_{\alpha\gamma\beta}{}^{\gamma} = \mathring{L}^{\alpha\beta\alpha'}\Big(&\overline{\nabla}_{\alpha}W_{\alpha'\gamma\beta}{}^{\gamma} + \overline{\nabla}_{\gamma}W_{\alpha\alpha'\beta}{}^{\gamma} + (k-2)\mathcal{C}_{\alpha\alpha'\beta}\\ &+ \mathring{L}_{\alpha}{}^{\delta}{}_{\alpha'}W_{\delta\gamma\beta}{}^{\gamma} + \mathring{L}_{\gamma}{}^{\delta}{}_{\alpha'}W_{\alpha\delta\beta}{}^{\gamma} - \mathring{L}_{\alpha\beta}{}^{\beta'}W_{\alpha'\gamma\beta'}{}^{\gamma}\\ &+ \mathring{L}_{\alpha}{}^{\gamma\beta'}W_{\gamma\alpha'\beta\beta'} + \mathring{L}_{\beta}{}^{\gamma\beta'}W_{\alpha\alpha'\gamma\beta'} + 2H_{\alpha'}W_{\alpha\gamma\beta}{}^{\gamma}\Big).\end{aligned}$$

Combining this with the formulas in Remark 5.4 yields

$$\begin{aligned}\widetilde{\mathcal{J}}_1 &= \mathcal{K}_1 + \mathcal{K}_2 + W_{\alpha\beta\alpha'}{}^{\beta}W^{\alpha\gamma\alpha'}{}_{\gamma} - \frac{1}{2}W_{\alpha\beta\alpha'\gamma}W^{\alpha\beta\alpha'\gamma} + \mathring{L}^2_{\alpha\beta}W^{\alpha\gamma\beta}{}_{\gamma}\\ &\quad + \mathring{L}^{\alpha\gamma\alpha'}\mathring{L}^{\beta\delta}{}_{\alpha'}W_{\alpha\beta\gamma\delta} - \mathring{L}^{\alpha\beta\alpha'}\mathring{L}_{\alpha\beta}{}^{\beta'}W_{\alpha'\gamma\beta'}{}^{\gamma}\\ &\quad + 2\mathring{L}^{\gamma\alpha\alpha'}\mathring{L}_{\gamma}{}^{\beta\beta'}W_{\alpha\alpha'\beta\beta'} - \mathring{L}^{\gamma\alpha\alpha'}\mathring{L}_{\gamma}{}^{\beta\beta'}W_{\alpha\beta\alpha'\beta'}.\end{aligned} \tag{A.7}$$

On the other hand, taking the tangential divergence of Equation (5.3) yields

$$\widetilde{\mathcal{J}}_2 = -\mathcal{K}_1 - \mathcal{K}_2. \tag{A.8}$$

Equations (A.4), (A.7), and (A.8) imply that

$$\mathcal{J}_1 = -\frac{k-2}{2(k-3)(k-6)}\widetilde{\mathcal{I}}_1 + \widetilde{\mathcal{J}}_1 - \frac{1}{k-3}\widetilde{\mathcal{J}}_2.$$

Applying the definitions of $\widetilde{\mathcal{I}}_1$, $\widetilde{\mathcal{J}}_1$, and $\widetilde{\mathcal{J}}_2$ yields

$$\begin{aligned}\mathcal{J}_1 &= \frac{k-4}{(k-3)(k-6)}\overline{\Delta}|\mathring{L}|^2 - \frac{1}{k-3}\overline{\nabla}^{\alpha}\overline{\nabla}^{\beta}\mathring{L}^2_{\alpha\beta} + \mathring{L}^{\alpha\beta\alpha'}\nabla_{\alpha'}W_{\alpha\gamma\beta}{}^{\gamma}\\ &\quad + (k-2)|\mathsf{D}|^2 - \frac{k-2}{(k-3)(k-6)}\overline{\mathsf{J}}|\mathring{L}|^2 - \frac{k-2}{k-3}\mathring{L}^2_{\alpha\beta}\overline{\mathsf{P}}^{\alpha\beta} - 2H_{\alpha'}\mathring{L}^{\alpha\beta\alpha'}W_{\alpha\gamma\beta}{}^{\gamma}.\end{aligned}$$

Specializing to the case $n = k+1$ and using Equation (4.8e) yields the final conclusion. $\square$

Proof of Proposition 5.23. Set

$$\begin{aligned}\widetilde{\mathcal{J}}_3 &:= \mathring{L}^{\alpha\beta\alpha'}\nabla_{\alpha'}\mathsf{P}_{\alpha\beta} + \mathring{L}^{\alpha\beta\alpha'}\mathring{L}_{\alpha\beta}{}^{\beta'}\mathsf{P}_{\alpha'\beta'} - \mathring{L}^{\alpha\beta\alpha'}\overline{\nabla}_{\alpha}\overline{\nabla}_{\beta}H_{\alpha'} - \overline{\nabla}^{\beta}(\mathsf{D}^{\alpha\alpha'}\mathring{L}_{\alpha\beta\alpha'})\\ &\quad - (k-1)|\mathsf{D}|^2 - \mathsf{D}^{\alpha\alpha'}W_{\alpha\beta\alpha'}{}^{\beta} - \mathring{L}^2_{\alpha\beta}\overline{\mathsf{P}}^{\alpha\beta} + \frac{k-3}{k-2}H^{\alpha'}\operatorname{tr}\mathring{L}^3_{\alpha'}\\ &\quad - H^{\alpha'}\mathring{L}^{\alpha\beta}{}_{\alpha'}\overline{\mathsf{P}}_{\alpha\beta} + H^{\alpha'}\mathring{L}_{\alpha\beta\alpha'}\mathring{L}^{\alpha\beta\beta'}H_{\beta'} + \frac{1}{2}|H|^2|\mathring{L}|^2 + \mathring{L}^{\alpha\beta\alpha'}\mathcal{C}_{\alpha\alpha'\beta}\\ &\quad + H^{\alpha'}\mathring{L}^{\alpha\beta\beta'}W_{\alpha\alpha'\beta\beta'} + \frac{1}{k-2}H^{\alpha'}\mathring{L}^{\alpha\beta}{}_{\alpha'}W_{\alpha\gamma\beta}{}^{\gamma}.\end{aligned}$$

Using Equations (4.5), (4.6), (4.7), and (4.8b), we first compute that

$$\begin{aligned}\mathring{L}^{\alpha\beta\alpha'}\nabla_{\alpha'}\mathsf{P}_{\alpha\beta} &= \mathring{L}^{\alpha\beta\alpha'}(\nabla_\alpha\mathsf{P}_{\alpha'\beta} - C_{\alpha\alpha'\beta}) \\ &= \mathring{L}^{\alpha\beta\alpha'}(\overline{\nabla}_\alpha\mathsf{P}_{\alpha'\beta} + \mathring{L}^{\gamma}{}_{\alpha\alpha'}\mathsf{P}_{\beta\gamma} - \mathring{L}_{\alpha\beta}{}^{\beta'}\mathsf{P}_{\alpha'\beta'} + H_{\alpha'}\mathsf{P}_{\alpha\beta} - C_{\alpha\alpha'\beta}) \\ &= \mathring{L}^{\alpha\beta\alpha'}\big(\overline{\nabla}_\alpha\overline{\nabla}_\beta H_{\alpha'} + \overline{\nabla}_\alpha\mathsf{D}_{\beta\alpha'} + \overline{\mathsf{P}}_\alpha{}^\gamma\mathring{L}_{\gamma\beta\alpha'} - H^{\beta'}\mathring{L}^{\gamma}{}_{\alpha\beta'}\mathring{L}_{\gamma\beta\alpha'} \\ &\quad + H_{\alpha'}\overline{\mathsf{P}}_{\alpha\beta} - H_{\alpha'}H_{\beta'}\mathring{L}_{\alpha\beta}{}^{\beta'} + H_{\alpha'}\mathsf{F}_{\alpha\beta} - \frac{1}{2}|H|^2\mathring{L}_{\alpha\beta\alpha'} \\ &\quad + \mathsf{F}_\alpha{}^\gamma\mathring{L}_{\gamma\beta\alpha'} - \mathring{L}_{\alpha\beta}{}^{\beta'}\mathsf{P}_{\alpha'\beta'} - \mathcal{C}_{\alpha\alpha'\beta} - H^{\beta'}W_{\alpha\alpha'\beta\beta'}\big).\end{aligned}$$

Combining this with Equation (4.8e) and the definition (4.9) of $\mathsf{F}_{\alpha\beta}$ yields

$$\widetilde{\mathcal{J}}_3 = \mathring{L}^2_{\alpha\beta}\mathsf{F}^{\alpha\beta}. \tag{A.9}$$

Equations (A.4), (A.7), and (A.9) imply that

$$\mathcal{J}_2 = -\frac{1}{2(k-3)(k-6)}\widetilde{\mathcal{I}}_1 - \frac{1}{k-3}\widetilde{\mathcal{J}}_2 - \widetilde{\mathcal{J}}_3.$$

Combining Equation (5.3) with the definitions of $\widetilde{\mathcal{I}}_1$, $\widetilde{\mathcal{J}}_2$, and $\widetilde{\mathcal{J}}_3$ yields

$$\begin{aligned}\mathcal{J}_2 = &-\mathring{L}^{\alpha\beta\alpha'}\nabla_{\alpha'}\mathsf{P}_{\alpha\beta} - \mathring{L}^{\alpha\beta\alpha'}\mathring{L}_{\alpha\beta}{}^{\beta'}\mathsf{P}_{\alpha'\beta'} + \mathring{L}^{\alpha\beta\alpha'}\overline{\nabla}_\alpha\overline{\nabla}_\beta H_{\alpha'} + H^{\alpha'}\mathring{L}^{\alpha\beta}{}_{\alpha'}\overline{\mathsf{P}}_{\alpha\beta} \\ &- \frac{1}{k-3}\overline{\nabla}^\alpha\overline{\nabla}^\beta\mathring{L}^2_{\alpha\beta} + \frac{k-5}{2(k-3)(k-6)}\overline{\Delta}|\mathring{L}|^2 - \frac{1}{(k-3)(k-6)}\overline{\mathsf{J}}|\mathring{L}|^2 \\ &+ \frac{k-4}{k-3}\mathring{L}^2_{\alpha\beta}\overline{\mathsf{P}}^{\alpha\beta} - \frac{k-3}{k-2}H^{\alpha'}\operatorname{tr}\mathring{L}^3_{\alpha'} - H^{\alpha'}\mathring{L}^{\alpha\beta\beta'}W_{\alpha\alpha'\beta\beta'} \\ &- \frac{1}{k-2}H^{\alpha'}\mathring{L}^{\alpha\beta}{}_{\alpha'}W_{\alpha\gamma\beta}{}^\gamma - H^{\alpha'}\mathring{L}_{\alpha\beta\alpha'}\mathring{L}^{\alpha\beta\beta'}H_{\beta'} - \frac{1}{2}|H|^2|\mathring{L}|^2 \\ &+ k|\mathsf{D}|^2 + 2\mathsf{D}^{\alpha\alpha'}W_{\alpha\beta\alpha'}{}^\beta.\end{aligned}$$

Specializing to the case $n = k+1$ yields the final conclusion. □

Proof of Proposition 5.25. Equation (A.7) implies that

$$\mathcal{N}_1 = \mathcal{I} + \frac{1}{2}\mathcal{J} - \frac{k-6}{2}\widetilde{\mathcal{J}}_1.$$

The definition (4.13) of $\mathcal{P}_{\alpha\beta}$ yields

$$\begin{aligned}(k-6)\mathring{L}^2_{\alpha\beta}\mathcal{P}^{\alpha\beta} + |\mathring{L}|^2\mathcal{P}_\alpha{}^\alpha - (k-3)|H|^2|\mathring{L}|^2 - (k-6)H^{\alpha'}\operatorname{tr}\mathring{L}^3_{\alpha'} \\ = \frac{n-k+2}{(n-1)(n-2)}|\mathring{L}|^2R - \frac{1}{n-2}|\mathring{L}|^2R_{\alpha'}{}^{\alpha'} + \frac{k-6}{n-2}\mathring{L}^2_{\alpha\beta}R^{\alpha\beta}.\end{aligned}$$

Combining this with the definitions of $\widetilde{\mathcal{I}}_1$ and $\widetilde{\mathcal{J}}_1$ yields

$$\begin{aligned}\mathcal{N}_1 = &-\frac{1}{2}\overline{\Delta}|\mathring{L}|^2 - (k-6)\overline{\nabla}^\beta(\mathsf{D}^{\alpha\alpha'}\mathring{L}_{\alpha\beta\alpha'}) - \frac{k-6}{2}\mathring{L}^{\alpha\beta\alpha'}\nabla_{\alpha'}W_{\alpha\gamma\beta}{}^\gamma \\ &+ \frac{n-k+2}{(n-1)(n-2)}|\mathring{L}|^2R - \frac{1}{n-2}|\mathring{L}|^2R_{\alpha'}{}^{\alpha'} + \frac{k-6}{n-2}\mathring{L}^2_{\alpha\beta}R^{\alpha\beta} + (k-3)|H|^2|\mathring{L}|^2 \\ &+ (k-6)H^{\alpha'}\operatorname{tr}\mathring{L}^3_{\alpha'} + (k-6)H_{\alpha'}\mathring{L}^{\alpha\beta\alpha'}W_{\alpha\gamma\beta}{}^\gamma - (k-3)(k-6)|\mathsf{D}|^2.\end{aligned}$$

Specializing to the case $k = 4$ yields the final conclusion. □

Proof of Proposition 5.26. Set

$$\begin{aligned}\widetilde{\mathcal{N}} &:= \nabla^{\alpha'}\nabla_{\alpha'}W_{\alpha\beta}{}^{\alpha\beta} + (n-k-6)H^{\alpha'}\nabla_{\alpha'}W_{\alpha\beta}{}^{\alpha\beta} + 2(n-2)\overline{\nabla}^{\alpha}\mathcal{C}_{\alpha} + \overline{\Delta}W_{\alpha\beta}{}^{\alpha\beta} \\ &\quad - 8\overline{\nabla}^{\gamma}(\mathring{L}_{\gamma}{}^{\alpha\alpha'}W_{\alpha\beta\alpha'}{}^{\beta}) + 2(n-2)\mathring{L}^{\alpha\beta\alpha'}\mathcal{C}_{\alpha\alpha'\beta} + 2(k-1)\mathcal{B}_{\alpha}{}^{\alpha} - 2\mathsf{J}W_{\alpha\beta}{}^{\alpha\beta} \\ &\quad - 2(n-2)\mathsf{P}^{\alpha\beta}W_{\alpha\gamma\beta}{}^{\gamma} - 4W_{\alpha\beta\alpha'}{}^{\beta}\overline{\nabla}^{\alpha}H^{\alpha'} - 2(n-4)|H|^2W_{\alpha\beta}{}^{\alpha\beta} \\ &\quad + 4W_{\alpha\beta\alpha'}{}^{\beta}\overline{\nabla}_{\gamma}\mathring{L}^{\gamma\alpha\alpha'} - 2(n-2)H^{\alpha'}\mathring{L}^{\alpha\beta}{}_{\alpha'}W_{\alpha\gamma\beta}{}^{\gamma} - 2(n-2)\mathsf{D}^{\alpha\alpha'}W_{\alpha\beta\alpha'}{}^{\beta}.\end{aligned}$$

Observe that the Weyl–Bianchi identity (4.3) and the definitions (4.5) and (4.14) of $\mathring{L}_{\alpha\beta\alpha'}$ and $\mathcal{C}_{abc}$, respectively, imply that

$$\begin{aligned}H^{\alpha'}\nabla_{\alpha'}W_{\alpha\beta}{}^{\alpha\beta} &= 2H^{\alpha'}\overline{\nabla}^{\alpha}W_{\alpha\beta\alpha'}{}^{\beta} + 2(k-1)H^{\alpha'}\mathcal{C}_{\alpha'} + 2|H|^2W_{\alpha\beta}{}^{\alpha\beta} \\ &\quad + 2H^{\alpha'}\mathring{L}^{\alpha\beta\beta'}W_{\alpha\alpha'\beta\beta'} + 2H^{\alpha'}\mathring{L}^{\alpha\beta}{}_{\alpha'}W_{\alpha\gamma\beta}{}^{\gamma}.\end{aligned} \tag{A.10}$$

The Weyl–Bianchi identity also implies that

$$\begin{aligned}\Delta W_{abcd} &= 2(n-3)\nabla_{[a}C_{|cd|b]} + 2\nabla_{[c}C_{|ab|d]} - 2B_{a[c}g_{d]b} + 2B_{b[c}g_{d]a} \\ &\quad - W_{ab}{}^{ef}W_{efcd} - 2W_{aecf}W_{b}{}^{e}{}_{d}{}^{f} + 2W_{aedf}W_{b}{}^{e}{}_{c}{}^{f} + 2\mathsf{J}W_{abcd} \\ &\quad - 2(n-3)\mathsf{P}^{e}{}_{[a}W_{b]ecd} - 2\mathsf{P}^{e}{}_{[c}W_{d]eab},\end{aligned} \tag{A.11}$$

where indices enclosed in vertical bars are not included in the skew symmetrization; e.g.

$$2\nabla_{[a}C_{|cd|b]} := \nabla_a C_{cdb} - \nabla_b C_{cda}.$$

On the one hand, taking the complete tangential trace of Equation (A.11) yields

$$\begin{aligned}\Delta W_{\alpha\beta}{}^{\alpha\beta} &= -2(n-2)\nabla^{\alpha}C_{\beta\alpha}{}^{\beta} - 2(k-1)B_{\alpha}{}^{\alpha} + 2\mathsf{J}W_{\alpha\beta}{}^{\alpha\beta} \\ &\quad + 2(n-2)\mathsf{P}^{\alpha c}W_{c\beta\alpha}{}^{\beta} - 2\left(W_{\alpha\beta cd}W^{\alpha\beta cd} + W_{c\alpha d}{}^{\alpha}W^{c\beta d}{}_{\beta} - W_{\alpha c\beta d}W^{\alpha c\beta d}\right).\end{aligned}$$

Using the definitions (4.6), (4.7), (4.14), and (4.16) of $\mathring{L}_{\alpha\beta\alpha'}$, $\mathsf{D}_{\alpha\alpha'}$, $\mathcal{C}_{abc}$, and $\mathcal{B}_{\alpha\beta}$, respectively, yields

$$\begin{aligned}\Delta W_{\alpha\beta}{}^{\alpha\beta} &= -2(n-2)\overline{\nabla}^{\alpha}\left(\mathcal{C}_{\alpha} + H^{\alpha'}W_{\alpha\beta\alpha'}{}^{\beta}\right) - 2(n-2)\mathring{L}^{\alpha\beta\alpha'}\mathcal{C}_{\alpha\alpha'\beta} \\ &\quad - 2(k-1)\mathcal{B}_{\alpha}{}^{\alpha} - 2(n-2)H^{\alpha'}\mathring{L}^{\alpha\beta\beta'}W_{\alpha\alpha'\beta\beta'} + 2(n-2)\mathsf{P}^{\alpha\beta}W_{\alpha\gamma\beta}{}^{\gamma} \\ &\quad - 2(k-1)(n-6)H^{\alpha'}\mathcal{C}_{\alpha'} + 4(k-1)H^{\alpha'}H^{\beta'}W_{\alpha'\alpha\beta'}{}^{\alpha} + 2\mathsf{J}W_{\alpha\beta}{}^{\alpha\beta} \\ &\quad + 2(n-2)\mathsf{D}^{\alpha\alpha'}W_{\alpha\beta\alpha'}{}^{\beta} + 2(n-2)W_{\alpha\beta\alpha'}{}^{\beta}\overline{\nabla}^{\alpha}H^{\alpha'} \\ &\quad - 2\left(W_{\alpha\beta cd}W^{\alpha\beta cd} + W_{c\alpha d}{}^{\alpha}W^{c\beta d}{}_{\beta} - W_{\alpha c\beta d}W^{\alpha c\beta d}\right).\end{aligned}$$

Using Equation (A.10) to eliminate $H^{\alpha'}\mathcal{C}_{\alpha'}$ yields

$$\begin{aligned}\Delta W_{\alpha\beta}{}^{\alpha\beta} &= -2(n-2)\overline{\nabla}^{\alpha}\mathcal{C}_{\alpha} - 8\overline{\nabla}^{\alpha}(H^{\alpha'}W_{\alpha\beta\alpha'}{}^{\beta}) - 2(n-2)\mathring{L}^{\alpha\beta\alpha'}\mathcal{C}_{\alpha\alpha'\beta} \\ &\quad - (n-6)H^{\alpha'}\nabla_{\alpha'}W_{\alpha\beta}{}^{\alpha\beta} - 2(k-1)\mathcal{B}_{\alpha}{}^{\alpha} + 8W_{\alpha\beta\alpha'}{}^{\beta}\overline{\nabla}^{\alpha}H^{\alpha'} \\ &\quad + 2(n-6)|H|^2W_{\alpha\beta}{}^{\alpha\beta} + 4(k-1)H^{\alpha'}H^{\beta'}W_{\alpha'\alpha\beta'}{}^{\alpha} \\ &\quad - 8H^{\alpha'}\mathring{L}^{\alpha\beta\beta'}W_{\alpha\alpha'\beta\beta'} + 2(n-6)H^{\alpha'}\mathring{L}^{\alpha\beta}{}_{\alpha'}W_{\alpha\gamma\beta}{}^{\gamma} \\ &\quad + 2\mathsf{J}W_{\alpha\beta}{}^{\alpha\beta} + 2(n-2)\mathsf{P}^{\alpha\beta}W_{\alpha\gamma\beta}{}^{\gamma} + 2(n-2)\mathsf{D}^{\alpha\alpha'}W_{\alpha\beta\alpha'}{}^{\beta} \\ &\quad - 2\left(W_{\alpha\beta cd}W^{\alpha\beta cd} + W_{c\alpha d}{}^{\alpha}W^{c\beta d}{}_{\beta} - W_{\alpha c\beta d}W^{\alpha c\beta d}\right).\end{aligned} \tag{A.12}$$

On the other hand, the definition (4.6) of the second fundamental form yields

$$\begin{aligned}
\nabla^\gamma\nabla_\gamma W_{\alpha\beta}{}^{\alpha\beta} &= \overline{\Delta} W_{\alpha\beta}{}^{\alpha\beta} + 4W_{\alpha\beta\alpha'}{}^{\beta}\overline{\nabla}_\gamma \mathring{L}^{\gamma\alpha\alpha'} + 4W_{\alpha\beta\alpha'}{}^{\beta}\overline{\nabla}^\alpha H^{\alpha'} \\
&\quad - kH^{\alpha'}\nabla_{\alpha'}W_{\alpha\beta}{}^{\alpha\beta} - 4\mathring{L}^2_{\alpha\beta}W^{\alpha\gamma\beta}{}_\gamma + 8\mathring{L}^{\gamma\alpha\alpha'}\mathring{L}_\gamma{}^{\beta\beta'}W_{\alpha\beta\alpha'\beta'} \\
&\quad + 4\mathring{L}^{\alpha\beta\alpha'}\mathring{L}_{\alpha\beta}{}^{\beta'}W_{\alpha'\gamma\beta'}{}^\gamma - 4\mathring{L}^{\gamma\alpha\alpha'}\mathring{L}_\gamma{}^{\beta\beta'}W_{\alpha\alpha'\beta\beta'} \\
&\quad - 8H^{\alpha'}\mathring{L}^{\alpha\beta}{}_{\alpha'}W_{\alpha\gamma\beta}{}^\gamma - 8H^{\alpha'}\mathring{L}^{\alpha\beta\beta'}W_{\alpha\alpha'\beta\beta'} - 8\overline{\nabla}^\alpha(H^{\alpha'}W_{\alpha\beta\alpha'}{}^\beta) \\
&\quad - 4|H|^2W_{\alpha\beta}{}^{\alpha\beta} + 4(k-1)H^{\alpha'}H^{\beta'}W_{\alpha'\alpha\beta'}{}^\alpha - 8\overline{\nabla}^\gamma(\mathring{L}_\gamma{}^{\alpha\alpha'}W_{\alpha\beta\alpha'}{}^\beta).
\end{aligned} \tag{A.13}$$

Recall that $\nabla^{\alpha'}\nabla_{\alpha'}W_{\alpha\beta}{}^{\alpha\beta} = \Delta W_{\alpha\beta}{}^{\alpha\beta} - \nabla^\gamma\nabla_\gamma W_{\alpha\beta}{}^{\alpha\beta}$. Combining this with Equations (A.12) and (A.13) yields

$$\begin{aligned}
\widetilde{\mathcal{N}} &= -2\left(W_{\alpha\beta cd}W^{\alpha\beta cd} + W_{c\alpha d}{}^\alpha W^{c\beta d}{}_\beta - W_{\alpha c\beta d}W^{\alpha c\beta d}\right) + 4\mathring{L}^2_{\alpha\beta}W^{\alpha\gamma\beta}{}_\gamma \\
&\quad - 8\mathring{L}^{\gamma\alpha\alpha'}\mathring{L}_\gamma{}^{\beta\beta'}W_{\alpha\beta\alpha'\beta'} - 4\mathring{L}^{\alpha\beta\alpha'}\mathring{L}_{\alpha\beta}{}^{\beta'}W_{\alpha'\gamma\beta'}{}^\gamma + 4\mathring{L}^{\gamma\alpha\alpha'}\mathring{L}_\gamma{}^{\beta\beta'}W_{\alpha\alpha'\beta\beta'}.
\end{aligned} \tag{A.14}$$

Combining Equations (A.7) and (A.14) yields

$$\begin{aligned}
\mathcal{N}_2 = \frac{1}{k-1}\widetilde{\mathcal{N}} + \frac{n-4}{(k-3)(k-6)}\mathcal{J} - \frac{2(n-k-1)}{(k-1)(k-3)}\widetilde{\mathcal{J}}_1 \\
+ \frac{8}{k-1}\mathcal{K}_2 + \frac{4}{k-1}W_{\alpha\beta\alpha'}{}^\beta W^{\alpha\gamma\alpha'}{}_\gamma.
\end{aligned}$$

Now observe that

$$\begin{aligned}
&\frac{4(n-k-1)}{(k-1)(k-3)}\mathsf{P}^{\alpha\beta}W_{\alpha\gamma\beta}{}^\gamma - \frac{2}{k-1}\mathsf{J}W_{\alpha\beta}{}^{\alpha\beta} + \frac{2(n-4)}{(k-3)(k-6)}\mathsf{P}_\alpha{}^\alpha W_{\beta\gamma}{}^{\beta\gamma} \\
&\quad = \frac{2(n-4)(n-k+2)}{(k-3)(k-6)(n-1)(n-2)}RW_{\alpha\beta}{}^{\alpha\beta} \\
&\qquad - \frac{2(n-4)}{(k-3)(k-6)(n-2)}R_{\alpha'}{}^{\alpha'}W_{\alpha\beta}{}^{\alpha\beta} + \frac{4(n-k-1)}{(k-1)(k-3)(n-2)}R^{\alpha\beta}W_{\alpha\gamma\beta}{}^\gamma.
\end{aligned}$$

Combining this with the formula for $\mathcal{K}_2$ in Remark 5.4, the definitions of $\widetilde{\mathcal{N}}$, $\mathcal{J}$, and $\widetilde{\mathcal{J}}_1$, and Equations (4.8e) and (4.13) yields

$$\begin{aligned}
\mathcal{N}_2 &= \frac{1}{k-1}\nabla^{\alpha'}\nabla_{\alpha'}W_{\alpha\beta}{}^{\alpha\beta} + \frac{n-k-6}{k-1}H^{\alpha'}\nabla_{\alpha'}W_{\alpha\beta}{}^{\alpha\beta} \\
&\quad - \frac{4(n-k-1)}{(k-1)(k-3)}\overline{\nabla}^\alpha\mathcal{C}_\alpha + \frac{k^2-5k+14-(k-1)n}{(k-1)(k-3)(k-6)}\overline{\Delta}W_{\alpha\beta}{}^{\alpha\beta} \\
&\quad + \frac{2(n-4)(n-k+2)}{(k-3)(k-6)(n-1)(n-2)}RW_{\alpha\beta}{}^{\alpha\beta} - \frac{2(n-4)}{(k-3)(k-6)(n-2)}R_{\alpha'}{}^{\alpha'}W_{\alpha\beta}{}^{\alpha\beta} \\
&\quad + \frac{4(n-k-1)}{(k-1)(k-3)(n-2)}R^{\alpha\beta}W_{\alpha\gamma\beta}{}^\gamma - \frac{2(n-k-1)}{(k-1)(k-3)}\mathring{L}^{\alpha\beta\alpha'}\nabla_{\alpha'}W_{\alpha\gamma\beta}{}^\gamma \\
&\quad - \frac{4}{k-1}W_{\alpha\beta\alpha'}{}^\beta\overline{\nabla}^\alpha H^{\alpha'} + \frac{8(n-k-1)}{(k-1)(k-3)}H^{\alpha'}\mathring{L}^{\alpha\beta}{}_{\alpha'}W_{\alpha\gamma\beta}{}^\gamma \\
&\quad - \frac{4(n-k+5)}{k-1}\mathsf{D}^{\alpha\alpha'}W_{\alpha\beta\alpha'}{}^\beta + \frac{10(n-4)}{(k-1)(k-6)}|H|^2W_{\alpha\beta}{}^{\alpha\beta}.
\end{aligned}$$

Specializing to the case $k=4$ yields the final conclusion. □

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Department of Mathematics, Penn State University, University Park, PA 16802, USA
Email address: jscase@psu.edu

Department of Mathematics, University of Washington, Box 354350, Seattle, WA 98195-4350, USA
Email address: robin@math.washington.edu

Department of Mathematics, University of California, Santa Cruz, CA 95064, USA
Email address: tkuo6@ucsc.edu

Department of Mathematics and Statistics, Texas Tech University, Lubbock, TX 79409-1042, USA
Email address: aatyrrel@ttu.edu

Center for Quantum Mathematics and Physics (QMAP), Department of Mathematics, University of California, Davis, CA 95616, USA
Email address: wally@math.ucdavis.edu