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Action of the axial $U(1)$ non-invertible symmetry on the 't Hooft line operator: A lattice gauge theory study

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We study how the symmetry operator of the axial $U(1)$ non-invertible symmetry acts on the 't Hooft line operator in the $U(1)$ gauge theory by employing the modified Villain-type lattice formulation. We model the axial anomaly by a compact scalar boson, the "QED axion". For the gauge invariance, the simple 't Hooft line operator, which is defined by a line integral of the dual $U(1)$ gauge potential, must be "dressed" by the scalar and $U(1)$ gauge fields. A careful consideration on the basis of the anomalous Ward–Takahashi identity containing the 't Hooft operator with the dressing factor and a precise definition of the symmetry operator on the lattice shows that the symmetry operator leaves no effect when it sweeps out a 't Hooft loop operator. This result appears inequivalent with the phenomenon concluded in the continuum theory. The same result is obtained for the axion string operator.

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1 Introduction and summary

The idea that the existence of quantum anomaly does not necessarily implies the absence of symmetries [1, 2] from a modern perspective on the symmetry [3] (see Refs. [4–6] for reviews) is quite interesting and should be further studied from various point of view. In Refs. [1, 2], it is pointed out that, in $U(1)$ gauge theory, one can construct a topological gauge invariant symmetry operator which generates the axial rotation with discrete angles on the Dirac fermion, despite the axial $U(1)$ anomaly. This can be regarded as the existence of a symmetry from the modern perspective, but the symmetry operator generates the non-invertible symmetry, which is a subject of vigorous recent studies [1, 2, 7–26].

One of the interesting questions considered in Refs. [1, 2] is that how the symmetry operator acts on the 't Hooft line operator [27, 28]. In Refs. [1, 2], it is concluded that, when the symmetry operator sweeps out a 't Hooft line operator along a loop γ, it acquires a surface operator of the $U(1)$ field strength whose boundary is γ. Although this phenomenon might be understood as the Witten effect [29] on the 't Hooft line (i.e., the worldline of the monopole) in the presence of the axial anomaly $\epsilon_{\mu\nu\rho\sigma} f_{\mu\nu}(x) f_{\rho\sigma}(x)$, it is somewhat puzzling because, at least naively, the 't Hooft line operator does not receive the axial transformation.

Let us take the anomalous Ward–Takahashi (WT) identity in the $U(1)$ gauge theory:

$$\left\langle \exp\left\{\frac{i\alpha}{2}\int_{\mathcal{V}_4} d^4x \left[\partial_\mu j_{5\mu}(x) - \frac{\mathrm{e}^2}{16\pi^2}\epsilon_{\mu\nu\rho\sigma} f_{\mu\nu}(x) f_{\rho\sigma}(x)\right]\right\} \mathcal{O}\right\rangle = \langle \mathcal{O}^\alpha \rangle\,, \tag{1.1}$$

where α is the axial rotation angle, $j_{5\mu}(x)$ is the Noether current associated with the axial $U(1)$ transformation, $\mathrm{e} \in \mathbb{Z}$ is the $U(1)$ charge of the Dirac fermion, and $f_{\mu\nu}(x)$ is the field strength of the $U(1)$ gauge field. The superscript α on the right-hand side implies that the observable $\mathcal{O}$ within the 4-volume $\mathcal{V}_4$ is axially rotated, $\psi(x)^\alpha = e^{i(\alpha/2)\gamma_5}\psi(x)$ and $\bar{\psi}(x)^\alpha = \bar{\psi}(x)e^{i(\alpha/2)\gamma_5}$. This identity can be derived as usual by applying the axial rotation on the Dirac fermion and considering the resulting axial anomaly. The symmetry operator in Refs. [1, 2] precisely generates the transformation in Eq. (1.1) with $\alpha = 2\pi p/N$ (here, p and N are coprime integers). However, since the identity (1.1) is obtained by the axial rotation on the matter field, i.e., the fermion, it is not intuitively obvious why the 't Hooft operator has a non-trivial effect in the identity (1.1).

In this paper, we study this question by employing the lattice gauge theory which provides an unambiguous definition of gauge theory. This problem is challenging from the lattice gauge theory side, because it is difficult to introduce the 't Hooft line operator in lattice gauge theory with matter fields with the axial anomaly or with the topological θ term. So far, in lattice gauge theory, we have a control over the structure of the axial anomaly of the fermion and the topological charge in non-Abelian gauge theory only under the admissibility condition [30–32]. A consideration on the axial $U(1)$ non-invertible symmetry [1, 2] on the basis of this control is given in Ref. [33]. However, the admissibility implies that a vanishing monopole current (i.e., the Bianchi identity) and thus it prohibits the 't Hooft line operator. A possible way out from this dilemma is to "excise" lattice sites along the monopole worldline and this method actually works for a 2D compact scalar field on the lattice [34, 35]. This "excision method" however creates boundaries on the lattice and at the moment it is difficult to say something how the axial anomaly is affected by the presence of such lattice boundaries (see Ref. [36] for a related study).

In the present paper, to study the action of the non-invertible symmetry operator on the 't Hooft line operator while avoiding the above difficulties in lattice gauge theory, we model the axial anomaly by a 2π periodic real scalar field. The continuum Euclidean action is (here, f is a constant of the mass dimension 1),

$$\int d^4x \left[\frac{f^2}{2}\partial_\mu\phi(x)\partial_\mu\phi(x) + \frac{i\mathrm{e}^2}{32\pi^2}\phi(x)\epsilon_{\mu\nu\rho\sigma}f_{\mu\nu}(x)f_{\rho\sigma}(x)\right], \tag{1.2}$$

for which the axial $U(1)$ transformation is defined by

$$\phi(x) \to \phi(x) - \alpha, \tag{1.3}$$

and the Noether current is given by

$$j_{5\mu}(x) = -2if^2\partial_\mu\phi(x). \tag{1.4}$$

This can be regarded as a chiral Lagrangian for the $U(1)_A$ symmetry with the Wess–Zumino term or the system of the "QED axion" [37].

On the other hand, to introduce the 't Hooft line operator in $U(1)$ lattice gauge theory, we employ the modified Villain-type lattice formulation developed in Ref. [38]. See also Refs. [39, 40]. This lattice formulation, containing the dual $U(1)$ gauge field from the beginning, allows a straightforward introduction of the 't Hooft line operator. For the gauge invariance, however, we find that the simple 't Hooft line operator, which is defined by a line integral of the dual $U(1)$ gauge potential, must be "dressed" by the scalar and $U(1)$ gauge fields.[1] In the presence of the 't Hooft line operator, we also need a modification of the lattice action for the gauge invariance.[2] One can then write down the anomalous WT identity containing the 't Hooft line operator with the dressing factor on the lattice. The anomalous WT

[1] The necessity of this dressing of the 't Hooft line operator is discussed from a very general grounds of a 5D TQFT in Ref. [41]. See also Ref. [42] for a related analysis.

[2] This modification of the lattice action itself, guided by the construction of the θ term in the modified Villain formulation [40], appears highly non-trivial.

identity becomes non-trivial because of the dressing factor. A careful consideration of the anomalous WT identity in terms of the symmetry operator of the non-invertible symmetry on the lattice [33] shows however that the symmetry operator leaves no effect when it sweeps out a 't Hooft loop operator. This result, while it is consistent with the naive intuition, appears inequivalent with the phenomenon concluded in Refs. [1, 2]. A similar analysis for the axion string operator [37] shows that it is also not affected from the symmetry operator.

The extension of our analysis to the system containing fermions and to the system with the non-Abelian gauge symmetry is highly desirable, although we do not know how to do that at the moment.

2 Modified Villain-type lattice formulation

2.1 $U(1)$ gauge theory

Our lattice gauge theory is defined on a finite hypercubic lattice of size L:[3]

$$\Gamma := \left\{x \in \mathbb{Z}^4 \,\middle|\, 0 \leq x_\mu < L\right\} \tag{2.1}$$

and we assume periodic boundary conditions for all lattice fields.

Following the lattice formulation developed in Refs. [38, 40], as the lattice action for the pure $U(1)$ gauge theory, we adopt[4]

$$\frac{1}{4g_0^2}\sum_{x\in\Gamma} f_{\mu\nu}(x)f_{\mu\nu}(x) + \frac{i}{2}\sum_{x\in\Gamma}\epsilon_{\mu\nu\rho\sigma}\tilde{a}_\mu(x)\partial_\nu z_{\rho\sigma}(x+\hat{\mu}). \tag{2.2}$$

In this modified Villain-type lattice formulation, the degrees of freedom of the $U(1)$ gauge field is represented by real non-compact link variables $a_\mu(x) \in \mathbb{R}$ and integer plaquette variables $z_{\mu\nu}(x) = -z_{\nu\mu}(x) \in \mathbb{Z}$. The $U(1)$ field strength in Eq. (2.2) is defined from these variables by

$$f_{\mu\nu}(x) = \partial_\mu a_\nu(x) - \partial_\nu a_\mu(x) + 2\pi z_{\mu\nu}(x). \tag{2.3}$$

The basic principle of the formulation [38] is that the lattice action and observables are required to be invariant under the 1-form $\mathbb{Z}$ gauge transformation,

$$a_\mu(x) \to a_\mu(x) + 2\pi m_\mu(x), \qquad z_{\mu\nu}(x) \to z_{\mu\nu}(x) - \partial_\mu m_\nu(x) + \partial_\nu m_\mu(x), \tag{2.4}$$

where $m_\mu(x) \in \mathbb{Z}$. $z_{\mu\nu}(x)$ is thus a 2-form $\mathbb{Z}$ gauge field. Then, one may take a gauge such that $-\pi < a_\mu(x) \leq \pi$ and $z_{\mu\nu}(x) \in \mathbb{Z}$, which would be almost equivalent to a compact $U(1)$

[3] The Lorentz index is denoted by Greek letters μ, ν, ..., and runs over 0, 1, 2, and 3.
[4] $\hat{\mu}$ denotes the unit vector in μ direction. ∂_μ is the forward difference operator, $\partial_\mu f(x) := f(x+\hat{\mu}) - f(x)$.

lattice gauge formulation which well captures topological properties in continuum theory. The field strength (2.3) is invariant under the 1-form $\mathbb{Z}$ transformation (2.4) and the lattice action (2.2) is thus consistent with this 1-form gauge invariance.

Since the field strength (2.3) is also invariant under

$$a_\mu(x) \to a_\mu(x) + \partial_\mu \lambda(x), \qquad z_{\mu\nu}(x) \to z_{\mu\nu}(x), \tag{2.5}$$

where $\lambda(x) \in \mathbb{R}$, the lattice action (2.2) possesses this ordinary 0-form $\mathbb{R}$ gauge invariance.

The field $\tilde{a}_\mu(x) \in \mathbb{R}$ in Eq. (2.2) is an auxiliary field. Since $\partial_\nu z_{\rho\sigma}(x)$ in Eq. (2.2) are integers, there is another 1-form $\mathbb{Z}$ gauge symmetry, under which

$$\tilde{a}_\mu(x) \to \tilde{a}_\mu(x) + 2\pi\mathbb{Z}. \tag{2.6}$$

Then, by using this gauge symmetry, we may restrict $-\pi < \tilde{a}_\mu(x) \leq \pi$. Then, $\tilde{a}_\mu(x)$ acts as a Lagrange multiplier whose functional integral imposes the Bianchi identity,

$$\epsilon_{\mu\nu\rho\sigma}\partial_\nu f_{\rho\sigma}(x) = 2\pi\epsilon_{\mu\nu\rho\sigma}\partial_\nu z_{\rho\sigma}(x) = 0. \tag{2.7}$$

Now, one of the advantages of the present lattice formulation is that the auxiliary field $\tilde{a}_\mu(\tilde{x})$ provides the dual $U(1)$ gauge field. Using this, one may readily introduce the 't Hooft line operator as

$$T_{\mathrm{q}}(\gamma) \sim \exp\left[i\mathrm{q}\sum_{(x,\mu)\in\gamma} \tilde{a}_\mu(x)\right], \tag{2.8}$$

where γ is a curve on Γ and the link sum is taken along this curve. The magnetic charge q must be an integer for the invariance under Eq. (2.6). When $T_{\mathrm{q}}(\gamma)$ is inserted in the functional integral, the integral over the auxiliary field $\tilde{a}_\mu(x)$ enforces the breaking of the Bianchi identity along the curve γ, $(1/2)\epsilon_{\mu\nu\rho\sigma}\partial_\nu z_{\rho\sigma}(x) = \mathrm{q}$. This represents the monopole worldline in the μ direction, i.e., the 't Hooft line in the μ direction.

2.2 Gauge invariant lattice action of the periodic scalar field

Next, we introduce a 2π periodic scalar field $\phi(x)$ as a simple system which exhibits the axial $U(1)$ anomaly. For the kinetic term, we adopt (a 2D analogue can be found in Ref. [43], although we put the scalar field on sites of the original lattice Γ instead of the dual lattice)

$$\sum_{x\in\Gamma}\left[\frac{f^2}{2}\partial\phi(x,\mu)\partial\phi(x,\mu) + \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}\partial_\mu \ell_\nu(x)\chi_{\rho\sigma}(x+\hat{\mu}+\hat{\nu})\right], \tag{2.9}$$

where

$$\partial\phi(x,\mu) := \partial_\mu\phi(x) + 2\pi\ell_\mu(x), \tag{2.10}$$

and $\ell_\mu(x) \in \mathbb{Z}$. In this modified Villain-type formulation, the 2π periodic scalar field is represented by variables, $\phi(x) \in \mathbb{R}$ and $\ell_\mu(x) \in \mathbb{Z}$. Then, requiring the invariance under the 0-form $\mathbb{Z}$ gauge transformation,

$$\phi(x) \to \phi(x) + 2\pi k(x), \qquad \ell_\mu(x) \to \ell_\mu(x) - \partial_\mu k(x), \tag{2.11}$$

where $k(x) \in \mathbb{Z}$, we may take a gauge such that $-\pi < \phi(x) \leq \pi$ and $\ell_\mu(x) \in \mathbb{Z}$. The lattice action (2.9) is consistent with this gauge invariance.

The field $\chi_{\mu\nu}(x) \in \mathbb{Z}$ in Eq. (2.9) is an another auxiliary field. By using the gauge invariance under

$$\chi_{\mu\nu}(x) \to \chi_{\mu\nu}(x) + 2\pi\mathbb{Z}, \tag{2.12}$$

we can restrict $-\pi < \chi_{\mu\nu}(x) \leq \pi$. The functional integral over $\chi_{\mu\nu}(x)$ then imposes the Bianchi identity for the scalar field,

$$\epsilon_{\mu\nu\rho\sigma}\partial_\rho\partial\phi(x,\sigma) = 2\pi\epsilon_{\mu\nu\rho\sigma}\partial_\rho\ell_\sigma(x) = 0. \tag{2.13}$$

To the free part (2.9), we want to further add a lattice counterpart of $\phi\epsilon_{\mu\nu\rho\sigma}f_{\mu\nu}f_{\rho\sigma}$, which generates the axial anomaly under the shift $\phi \to \phi + \alpha$. To find and write down desired expressions, the notion of the cochain, coboundary operator and the cup products on the hypercubic lattice [40, 44] is quite helpful. Thus, we identify lattice fields, $a_\mu(x)$, $z_{\mu\nu}(x)$, $f_{\mu\nu}(x)$, $\tilde{a}(x)$, $\phi(x)$, $\ell_\mu(x)$, $\partial\phi(x,\mu)$, $\chi(x)$, with cochains, a, z, f, $\tilde{a}$, ϕ, ℓ, $\partial\phi$, χ with obvious ranks, respectively. In particular, corresponding to Eqs. (2.3) and (2.10), we have

$$f = \delta a + 2\pi z, \qquad \partial\phi = \delta\phi + 2\pi\ell, \tag{2.14}$$

where δ is the coboundary operator [40, 44]. The coboundary operator is nilpotent $\delta^2 = 0$. Also, the Bianchi identities, Eqs. (2.7) and (2.13), are simply expressed as

$$\delta f = 2\pi\delta z = 0, \qquad \delta\partial\phi = 2\pi\delta l = 0. \tag{2.15}$$

Now, as a lattice counterpart of $\phi\epsilon_{\mu\nu\rho\sigma}f_{\mu\nu}f_{\rho\sigma}$, we start with the expression,

$$\sum_{\text{hypercube}\in\Gamma} \phi \cup f \cup f = \sum_{x\in\Gamma} \frac{1}{4}\phi(x)\epsilon_{\mu\nu\rho\sigma}f_{\mu\nu}(x)f_{\rho\sigma}(x+\hat{\mu}+\hat{\nu}). \tag{2.16}$$

The particular way of shifts of lattice sites on the right-hand side corresponds to the cup product [40, 44] on the left-hand side. This particular shift structure is well-known in the context of the axial anomaly in lattice $U(1)$ gauge theory [32, 45, 46]. Under the cup product, the coboundary operator δ satisfies the Leibniz rule.[5]

Starting from Eq. (2.16), we seek an appropriate combination which is invariant under all the following gauge transformations: The 0-form $\mathbb{Z}$ gauge transformation (2.11),

$$\phi \to \phi + 2\pi k, \qquad \ell \to \ell - \delta k, \tag{2.17}$$

where $k \in \mathbb{Z}$ is a 0-cochain, the 1-form $\mathbb{Z}$ gauge transformation (2.4) is

$$a \to a + 2\pi m, \qquad z \to z - \delta m, \tag{2.18}$$

where $m \in \mathbb{Z}$ is a 1-cochain and, finally, the 0-form $\mathbb{R}$ gauge transformation (2.5),

$$a \to a + \delta\lambda, \qquad z \to z, \tag{2.19}$$

where $\lambda \in \mathbb{R}$ is a 0-cochain.

While referring the construction of the θ term in Ref. [40], with some trial and error, we find that the following combination possesses desired invariant properties:[6]

$$\begin{aligned} I := &\sum_{\text{hypercube}\in\Gamma} \big(\phi \cup f \cup f - 2\pi\ell \cup a \cup f - 4\pi^2 \ell \cup z \cup a\big) \\ &+ \sum_{\text{hypercube}\in\Gamma} \big\{\phi \cup \big[-2\pi a \cup \delta z + 2\pi \delta z \cup a + 2\pi\delta(a \cup_1 \delta z) + 4\pi^2 z \cup_1 \delta z\big] \\ &\qquad - 4\pi^2 \ell \cup (a \cup_1 \delta z)\big\}, \end{aligned} \tag{2.21}$$

where $\cup_1$ is a higher cup product [40, 44]. We do not need the explicit form of $\cup_1$ in what follows but what is crucial to us is that $\cup_1$ enables one to exchange the ordering of the cup

[5] These notions has been also known as the non-commutative differential calculus [45].

[6] In Eq. (2.21), the first line can be regarded as a natural lattice transcription of $\phi \wedge f \wedge f$, which is consistent with the gauge symmetries (this line alone is gauge invariant if the Bianchi identities hold). Although the second line is of a somewhat complicated composition, it is proportional to δz and thus vanishes if the Bianchi identity $\delta z = 0$ holds. When the Bianchi identity $\delta z = 0$ holds, the explicit form of the lattice action is then

$$\begin{aligned} \frac{ie^2}{8\pi^2} I = \frac{ie^2}{32\pi^2} \sum_{x\in\Gamma} \Big[&\phi(x)\epsilon_{\mu\nu\rho\sigma} f_{\mu\nu}(x) f_{\rho\sigma}(x+\hat{\mu}+\hat{\nu}) \\ &- 4\pi\epsilon_{\mu\nu\rho\sigma}\ell_\mu(x) a_\nu(x+\hat{\mu}) f_{\rho\sigma}(x+\hat{\mu}+\hat{\nu}) \\ &- 8\pi^2\epsilon_{\mu\nu\rho\sigma}\ell_\mu(x) z_{\nu\rho}(x+\hat{\mu}) a_\sigma(x+\hat{\mu}+\hat{\nu}+\hat{\rho}) \Big]. \end{aligned} \tag{2.20}$$

product of two cochains (here α and β are p and q cochains, respectively):

$$(-1)^{pq}\beta \cup \alpha = \alpha \cup \beta + (-1)^{p+q}\left[\delta(\alpha \cup_1 \beta) - \delta\alpha \cup_1 \beta - (-1)^p \alpha \cup_1 \delta\beta\right]. \tag{2.22}$$

Using Eq. (2.22), under Eqs. (2.17)–(2.19), we see that the combination (2.21) changes into

$$\begin{aligned}
I &\overset{\text{0-form }\mathbb{Z}}{\to} I + 8\pi^3\mathbb{Z} \\
&\overset{\text{1-form }\mathbb{Z}}{\to} I + \sum_{x\in\Gamma}\left(-8\pi^2\phi \cup m \cup \delta z + 4\pi^2 \delta\ell \cup m \cup a\right) + 8\pi^3\mathbb{Z} \\
&\overset{\text{0-form }\mathbb{R}}{\to} I + \sum_{x\in\Gamma}\left\{\left(-4\pi\phi \cup \delta\lambda + 8\pi^2 \ell \cup \lambda\right) \cup \delta z + \delta\ell \cup \left[-2\pi\lambda\delta \cup a - 4\pi^2\left(\lambda \cup z + z \cup \lambda\right)\right]\right\},
\end{aligned} \tag{2.23}$$

where we have used $\delta^2 z = 0$, which always holds because the field z is single-valued on Γ. Note that all the gauge breakings in Eq. (2.23) are proportional to the Bianchi identities in Eq. (2.15). Therefore, setting the lattice action as

$$S := \sum_{\text{hypercube}\in\Gamma}\left(\frac{1}{g_0^2} f \cup \star f + i\tilde{a} \cup \delta z\right) + \sum_{\text{hypercube}\in\Gamma}\left(\frac{f^2}{2}\partial\phi \cup \star\partial\phi + i\delta\ell \cup \chi\right) + \frac{i\mathrm{e}^2}{8\pi^2} I, \tag{2.24}$$

where the first two terms are Eqs. (2.2) and (2.9), respectively, by endowing the auxiliary fields with gauge transformations,

$$\begin{aligned}
\tilde{a} &\to \tilde{a} + \mathrm{e}^2 \phi \cup m + \frac{\mathrm{e}^2}{2\pi}(\phi \cup \delta\lambda - 2\pi\ell \cup \lambda), \\
\chi &\to \chi - \frac{\mathrm{e}^2}{2} m \cup a + \frac{\mathrm{e}^2}{4\pi}\left[\lambda \cup \delta a + 2\pi\left(\lambda \cup z + z \cup \lambda\right)\right],
\end{aligned} \tag{2.25}$$

the Boltzmann factor e^{-S} is invariant under all the gauge transformations if e is an even integer.[7]

In this way, we have obtained a gauge invariant lattice action and gauge transformations of auxiliary fields. This knowledge then enables us to find a gauge invariant 't Hooft line operator. Note that the simple definition (2.8) is not invariant under Eq. (2.25). Therefore, the expectation value of Eq. (2.8) is not sensible in this lattice formulation with the scalar field. The operator must be "dressed" by the scalar and gauge fields for the gauge invariance. We find that the following dressed one is a gauge invariant combination when $\delta\ell = 0$:

$$T_{\mathrm{q}}(\gamma) := \exp\left[i\mathrm{q}\left\{\sum_{\text{link}\in\gamma}\left(\tilde{a} - \frac{\mathrm{e}^2}{2\pi}\phi \cup a\right) + \sum_{\text{plaquette}\in\mathcal{R}} \frac{\mathrm{e}^2}{2\pi}\left(-2\pi\ell \cup a + \phi \cup f\right)\right]\right\}, \tag{2.26}$$

where $\mathcal{R}$ is a 2-dimensional surface whose boundary is γ, $\partial(\mathcal{R}) = \gamma$.

[7] Throughout this paper, we assume this condition.

Similarly, we can obtain a gauge invariant axion string operator (it is basically given by a 2-dimensional surface integral of the auxiliary field χ) as

$$S_{\mathrm{q}}(\sigma):=\exp\left[i\mathrm{q}\left\{\sum_{\text{plaquette}\in\sigma}\left(\chi+\frac{\mathrm{e}^2}{4\pi}a\cup a\right)-\sum_{\text{cube}\in\mathcal{V}_3}\frac{\mathrm{e}^2}{4\pi}\left(f\cup a+2\pi a\cup z\right)\right]\right\}, \quad (2.27)$$

where $\mathcal{V}_3$ is a 3-volume whose boundary is σ, $\partial(\mathcal{V}_3)=\sigma$. This is gauge invariant when $\delta z=0$. We use these expressions to investigate how these operators are affected by the symmetry operator of the axial $U(1)$ non-invertible symmetry.

3 Action of the symmetry operator on the gauge invariant 't Hooft line

Now, in the above lattice formulation, we derive an anomalous WT identity containing the gauge invariant 't Hooft line operator (2.26). We first take the expectation value of the operators

$$\langle T_{\mathrm{q}}(\gamma)\cdots\rangle_{\mathrm{B}}\,, \quad (3.1)$$

where the subscript B stands for only the functional integral over the scalar field sector (ϕ, ℓ, χ) and the auxiliary field $\tilde{a}$ is carried out. We then consider the change of integration variables of the form of a localized infinitesimal axial transformation, $\phi(x)\to\phi(x)+\alpha(x)$ and $\ell_\mu(x)\to\ell_\mu(x)$. We assume that the abbreviated term $\cdots$ in Eq. (3.1) is invariant under this transformation. The lattice action (2.24) then changes into (see also Eqs. (2.9) and (2.20))[8]

$$\begin{aligned}S&-i\sum_{x\in\Gamma}\frac{\alpha(x)}{2}\left[\partial_\mu^* j_{5\mu}(x)-\frac{\mathrm{e}^2}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}f_{\mu\nu}(x)f_{\rho\sigma}(x+\hat{\mu}+\hat{\nu})\right]\\&-\frac{i\mathrm{e}^2}{2\pi}\sum_{\text{hypercube}\in\Gamma}\alpha\cup\left(a\cup\delta z-\frac{1}{2}f\cup_1\delta z\right),\end{aligned} \quad (3.2)$$

where the axial vector current is given by

$$j_{5\mu}(x):=-2if^2\partial\phi(x,\mu), \quad (3.3)$$

where $\partial\phi(x,\mu)$ is defined by Eq. (2.10); note that this is invariant under the gauge transformation (2.11).

[8] Here, ∂_μ^* is the backward difference operator, $\partial_\mu^* f(x):=f(x)-f(x-\hat{\mu})$.

On the other hand, owing to the dressing factor in Eq. (2.26), under the infinitesimal shift, $\phi(x) \to \phi(x) + \alpha(x)$,

$$T_{\rm q}(\gamma) \to T_{\rm q}(\gamma) \exp\left[-\frac{i{\rm q}{\rm e}^2}{2\pi}\left(\sum_{{\rm link}\in\gamma} \alpha \cup a - \sum_{{\rm plaquette}\in\mathcal{R}} \alpha \cup f\right)\right]. \tag{3.4}$$

We then repeat this change of variables to make a finite axial rotation, setting $\alpha(x) = \alpha$ within a 4-volume $\mathcal{V}_4$, $x \in \mathcal{V}_4 \subset \Gamma$, and $\alpha(x) = 0$ otherwise. We assume that the curve γ and the surface $\mathcal{R}$ are completely within $\mathcal{V}_4$. With the insertion of Eq. (2.26), the integration over the auxiliary field $\tilde{a}$ enforces $\delta z = {\rm q}\delta_3[\gamma]$, where $\delta_3[\gamma]$ is the delta function 3-cochain along the curve γ. If this is substituted in Eq. (3.2), we see that the $\sum_{{\rm hypercube}\in\Gamma} \alpha \cup a \cup \delta z$ term in Eq. (3.2) and the $\sum_{{\rm link}\in\gamma} \alpha \cup a$ term in Eq. (3.4) cancel to each other.

In this way, we have the identity,

$$\begin{aligned}
&\left\langle \exp\left\{\frac{i\alpha}{2}\sum_{x\in\mathcal{V}_4}\left[\partial_\mu^* j_{5\mu}(x) - \frac{{\rm e}^2}{16\pi^2}\epsilon_{\mu\nu\rho\sigma} f_{\mu\nu}(x) f_{\rho\sigma}(x+\hat{\mu}+\hat{\nu})\right]\right\} T_{\rm q}(\gamma)\cdots\right\rangle_{\rm B} \\
&= \exp\left\{-\frac{i{\rm q}{\rm e}^2\alpha}{2\pi}\left[\sum_{{\rm plaquette}\in\mathcal{R}} f - \sum_{{\rm hypercube}\in\Gamma} \frac{1}{2} f \cup_1 \delta_3[\gamma]\right]\right\} \langle T_{\rm q}(\gamma)\cdots\rangle_{\rm B},
\end{aligned} \tag{3.5}$$

or

$$\begin{aligned}
&\left\langle \exp\left[\frac{i\alpha}{2}\sum_{{\rm hypercube}\in\mathcal{V}_4}\left(\delta \star j_5 - \frac{{\rm e}^2}{4\pi^2} f \cup f\right)\right] T_{\rm q}(\gamma)\cdots\right\rangle_{\rm B} \\
&= \exp\left\{-\frac{i{\rm q}{\rm e}^2\alpha}{2\pi}\left[\sum_{{\rm plaquette}\in\mathcal{R}} f - \sum_{{\rm hypercube}\in\Gamma} \frac{1}{2} f \cup_1 \delta_3[\gamma]\right]\right\} \langle T_{\rm q}(\gamma)\cdots\rangle_{\rm B}.
\end{aligned} \tag{3.6}$$

This is the anomalous WT identity containing the gauge invariant 't Hooft line operator.

Now, as shown in Appendix C of Ref. [33], when

$$\alpha = \frac{2\pi p}{N}, \tag{3.7}$$

and p and N are coprime integers and $p \in 2\mathbb{Z}$, we can represent the symmetry generator of the axial $U(1)$ non-invertible symmetry [1, 2] for an arbitrary closed oriented 3-surface $\mathcal{M}_3$

as[9]

$$U_{2\pi p/N}(\mathcal{M}_3) = \exp\left\{\frac{i\pi p}{N}\sum_{\text{cube}\in\mathcal{M}_3}\left[\star j_5 - \frac{\mathrm{e}^2}{4\pi^2}\left(a\cup f + 2\pi z\cup a\right)\right]\right\}\mathcal{Z}_{\mathcal{M}_3}[z], \tag{3.8}$$

where $\mathcal{Z}_{\mathcal{M}_3}[z]$ is the partition function of a lattice 3D TQFT (the level N BF theory) (s denotes the number of sites) [33],

$$\mathcal{Z}_{\mathcal{M}_3}[z] := \frac{1}{N^s}\int \mathrm{D}[b]\mathrm{D}[c]\,\exp\left\{\frac{ip\pi}{N}\sum_{\text{cube}\in\mathcal{M}_3}\left[b\left(\delta c - \mathrm{e}z\right) - \mathrm{e}z\cup c\right]\right\}, \tag{3.9}$$

where ($b_\mu(\tilde{x})$ is residing on dual links),

$$\mathrm{D}[b] := \prod_{(x,\mu)\in\mathcal{M}_3}\left[\frac{1}{N}\sum_{b_\mu(\tilde{x})=0}^{N-1}\right], \qquad \mathrm{D}[c] := \prod_{(x,\mu)\in\mathcal{M}_3}\left[\sum_{c_\mu(x)=0}^{N-1}\right]. \tag{3.10}$$

$U_{2\pi p/N}(\mathcal{M}_3)$ is topological by construction and it can be seen that $U_{2\pi p/N}(\mathcal{M}_3)$ is invariant under the lattice gauge transformation (2.5) [33]. Thus, this generates a physical symmetry transformation. In Ref. [33], fusion rules among the symmetry operators are computed by employing the lattice representation (3.8).

We now consider the situation depicted in Fig. 1. That is, we consider a 't Hooft line γ in a 4-volume $\mathcal{V}_4$ and $\partial(\mathcal{V}_4) = \mathcal{M}_3' \cup (-\mathcal{M}_3)$. We assume that γ is *not* contained in $\partial(\mathcal{V}_4) = \mathcal{M}_3' \cup (-\mathcal{M}_3)$.

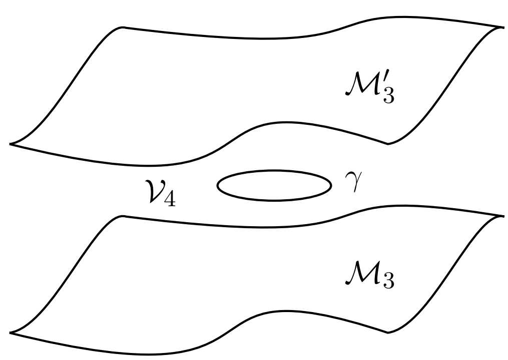

Fig. 1

[9] This was pointed out to us by Yuya Tanizaki.

It can be seen that, from the expression (3.9), $U_{2\pi p/N}(\mathcal{M}_3) = 0$ if $z \bmod N \neq 0$ in $H^2(\mathcal{M}_3; \mathbb{Z}_N)$ [33]. Since $\delta z = 0$ on both $\mathcal{M}_3'$ and $\mathcal{M}_3$, for symmetry operators $U_{2\pi p/N}(\mathcal{M}_3')$ and $U_{2\pi p/N}(\mathcal{M}_3)$ be sensible, we can assume

$$z = \delta\nu \tag{3.11}$$

on $\partial(\mathcal{V}_4) = \mathcal{M}_3' \cup (-\mathcal{M}_3)$. This relation is broken in $\mathcal{V}_4$ due to γ. Since, for $\gamma = \partial(\mathcal{R})$,

$$\delta_3[\gamma] = \delta\delta_2[\mathcal{R}], \tag{3.12}$$

where $\delta_2[\mathcal{R}]$ is the delta function 2-cochain on $\mathcal{R}$, we can represent z in $\mathcal{V}_4$ as

$$z = \delta\nu + \mathrm{q}\delta_2[\mathcal{R}]. \tag{3.13}$$

Going back to the anomalous WT identity (3.6), let us compute the exponent on the left-hand side of Eq. (3.6) by using Eqs. (2.14) and (3.13). After some calculation, we find

$$f \cup f = \delta\left(a \cup f + 2\pi z \cup a + 4\pi^2 \delta\nu \cup \nu\right) + 4\pi\mathrm{q} f \cup \delta_2[\mathcal{R}] - 2\pi\mathrm{q} f \cup_1 \delta_3[\gamma]. \tag{3.14}$$

Here, we have neglected $\delta_3[\gamma]$ and $\delta_2[\mathcal{R}]$ inside $\delta(\cdots)$, because under the integration over $\mathcal{V}_4$, those do not contribute. Then,

$$\begin{aligned} -\frac{i\alpha}{2}\frac{\mathrm{e}^2}{4\pi^2} \sum_{\mathrm{hypercube}\in\mathcal{V}_4} f \cup f = -\frac{i\alpha}{2}\frac{\mathrm{e}^2}{4\pi^2} \sum_{\mathrm{cube}\in\mathcal{M}_3'\cup(-\mathcal{M}_3)} \left(a \cup f + 2\pi z \cup a + 4\pi^2 \delta\nu \cup \nu\right) \\ - \frac{i\mathrm{q}\mathrm{e}^2\alpha}{2\pi} \sum_{\mathrm{hypercube}\in\mathcal{V}_4} \left(f \cup \delta_2[\mathcal{R}] - \frac{1}{2} f \cup_1 \delta_3[\gamma]\right). \end{aligned} \tag{3.15}$$

The last term precisely cancels the exponential on the right-hand side of Eq. (3.6) and thus the anomalous WT identity yields, for the rotation angle (3.7),

$$\begin{aligned} &\left\langle \exp\left\{ \frac{i\pi p}{N} \sum_{\mathrm{cube}\in\mathcal{M}_3'\cup(-\mathcal{M}_3)} \left[\star j_5 - \frac{\mathrm{e}^2}{4\pi^2}\left(a \cup f + 2\pi z \cup a + 4\pi^2 \delta\nu \cup \nu\right)\right]\right\} T_{\mathrm{q}}(\gamma) \cdots \right\rangle_{\mathrm{B}} \\ &= \langle T_{\mathrm{q}}(\gamma) \cdots \rangle_{\mathrm{B}}. \end{aligned} \tag{3.16}$$

On the other hand, in the above situation, from Eq. (C14) of Ref. [33],

$$\begin{aligned} \mathcal{Z}_{\mathcal{M}_3'}[z] &= \exp\left(-\frac{i\pi p \mathrm{e}^2}{N} \sum_{\mathrm{cube}\in\mathcal{M}_3'} \delta\nu \cup \nu\right) \mathcal{Z}_{\mathcal{M}_3'}[0] \\ &= \exp\left(-\frac{i\pi p \mathrm{e}^2}{N} \sum_{\mathrm{cube}\in\mathcal{M}_3'} \delta\nu \cup \nu\right) \mathcal{Z}_{\mathcal{M}_3}[0] \end{aligned}$$

$$= \exp\left[-\frac{i\pi p e^2}{N}\left(\sum_{\text{cube}\in\mathcal{M}_3'}\delta\nu\cup\nu - \sum_{\text{cube}\in\mathcal{M}_3}\delta\nu\cup\nu\right)\right]\mathcal{Z}_{\mathcal{M}_3}[z], \tag{3.17}$$

where $\mathcal{Z}_{\mathcal{M}_3}[0] = N^{b_2-1}$ (b_2 is the second Betti number of $\mathcal{M}_3$). Here, we have noted $\mathcal{Z}_{\mathcal{M}_3'}[0] = \mathcal{Z}_{\mathcal{M}_3}[0]$ because $\mathcal{M}_3'$ and $\mathcal{M}_3$ are homologically equivalent, $\partial(\mathcal{V}_4) = \mathcal{M}_3' \cup (-\mathcal{M}_3)$. Note that this relation holds irrespective of the existence of the 't Hooft line in $\mathcal{V}_4$.[10] Eqs. (3.17) and (3.8) show that the relation

$$\begin{aligned} &U_{2\pi p/N}(\mathcal{M}_3') \\ &= \exp\left\{\frac{i\pi p}{N}\sum_{\text{cube}\in\mathcal{M}_3'\cup(-\mathcal{M}_3)}\left[\star j_5 - \frac{e^2}{4\pi^2}\left(a\cup f + 2\pi z\cup a + 4\pi^2\delta\nu\cup\nu\right)\right]\right\}U_{2\pi p/N}(\mathcal{M}_3). \end{aligned} \tag{3.19}$$

Finally, we substitute $\cdots$ in Eq. (3.16) by $U_{2\pi p/N}(\mathcal{M}_3)$. We can do this because it can be seen that $U_{2\pi p/N}(\mathcal{M}_3)$ (3.8) is invariant under the shift of the scalar field within $\mathcal{V}_4$. Then, using Eq. (3.19), we have

$$\left\langle U_{2\pi p/N}(\mathcal{M}_3')T_{\mathrm{q}}(\gamma)\right\rangle_{\mathrm{B}} = \left\langle U_{2\pi p/N}(\mathcal{M}_3)T_{\mathrm{q}}(\gamma)\right\rangle_{\mathrm{B}}. \tag{3.20}$$

This is our main result. Schematically speaking, when the symmetry operator sweeps out a 't Hooft line operator as Fig. 1, it does not leave any factor. Note that Eq. (3.20) has been obtained without specifying what happens when the symmetry operator collides with the 't Hooft line operator. Therefore, Eq. (3.20) holds irrespective of possible modifications of the symmetry operator $U_{2\pi p/N}(\mathcal{M}_3)$ for the situation $\gamma \subset \mathcal{M}_3$. This our result appears inequivalent with the phenomenon concluded in Refs. [1, 2] in the continuum theory.

For the axion string operator (2.27), since its dressing factor does not contain the scalar field and the change of the action (2.24) under the axial transformation $\phi(x) \to \phi(x) + \alpha(x)$ does not contain $\delta\ell$, we infer that the axion string does not receive any effect under the

[10] Also, this relation should hold for any choice of TQFT to represent the symmetry operator because, when $\delta z = 0$ within $\mathcal{V}_4$, from Eq. (3.11), we should have [33],

$$\begin{aligned} \mathcal{Z}_{\mathcal{M}_3'}[z] &= \exp\left[-\frac{i\pi p e^2}{N}\sum_{\text{hypercube}\in\mathcal{V}_4} z\cup z\right]\mathcal{Z}_{\mathcal{M}_3}[z] \\ &= \exp\left[-\frac{i\pi p e^2}{N}\left(\sum_{\text{cube}\in\mathcal{M}_3'}\delta\nu\cup\nu - \sum_{\text{cube}\in\mathcal{M}_3}\delta\nu\cup\nu\right)\right]\mathcal{Z}_{\mathcal{M}_3}[z]. \end{aligned} \tag{3.18}$$

sweep by the symmetry operator. It might be interesting to consider the situation in which the 't Hooft line and the axion string coexist; the dressing factors in Eqs. (2.26) and (2.27) should then be further modified for the gauge invariance.

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