Continuity of HYM connections with respect to metric variations

Rémi Delloque

Abstract

We investigate the set of (real Dolbeault classes of) balanced metrics Θ on a balanced manifold X with respect to which a torsion-free coherent sheaf $\mathcal{E}$ on X is slope stable. We prove that the set of all such $[\Theta] \in H^{n-1,n-1}(X, \mathbb{R})$ is an open convex cone locally defined by a finite number of linear inequalities.

When $\mathcal{E}$ is a Hermitian vector bundle, the Kobayashi–Hitchin correspondence provides associated Hermitian Yang–Mills connections, which we show depend continuously on the metric, even around classes with respect to which $\mathcal{E}$ is only semi-stable. In this case, the holomorphic structure induced by the connection is the holomorphic structure of the associated graded object. The method relies on semi-stable perturbation techniques for geometric PDEs with a moment map interpretation and is quite versatile, and we hope that it can be used in other similar problems.

1 Introduction

The famous Kobayashi–Hitchin correspondence [14, 18, 5, 27] gives the equivalence between the existence of Hermitian Yang–Mills (HYM) connections on a vector bundle $\mathcal{E} \to X$ and slope stability [20, 26]. HYM connections correspond to solutions of a differential equation while slope stability is a purely algebraic notion. These tools are used to construct and study the moduli space of vector bundles on a given complex manifold.

Both of these notions depend on the balanced metric on the balanced manifold X (balanced metrics are defined below). It is a natural question to ask how $\mathcal{E}$ may become stable or unstable when this metric varies, and to study the behaviour of the associated HYM connections. It is easy to see that stability is an open condition in the space of balanced metrics and that we can locally build a smooth family of associated HYM connections around a metric Θ_0 with respect to which $\mathcal{E}$ is stable. However, when $\mathcal{E}$ is only semi-stable with respect to Θ_0, problems occur and wall-crossing phenomena appear. The method used here is inspired from [25, 4, 3], and relies on semi-stable perturbation techniques for geometric PDEs with a moment map interpretation.

Let g be a metric on X and ω be its Kähler form. Then, by definition, X is Kähler if and only if $d\omega = 0$. However, slope stability of $\mathcal{E}$ is determined by the inequalities $\mu_{[\omega^{n-1}]}(\mathcal{S}) < \mu_{[\omega^{n-1}]}(\mathcal{E})$ for all coherent sub-sheaves $\mathcal{S}$ of $\mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{S}) < \mathrm{rk}(\mathcal{E})$. Here, $n = \dim(X)$ and $\mu_{[\omega^{n-1}]}(\mathcal{E}) = \frac{c_1(\mathcal{E}) \cup [\omega^{n-1}]}{\mathrm{rk}(\mathcal{E})}$ is the slope of $\mathcal{E}$. We see that these inequalities are linear in ω^{n-1} and ω is not required to be closed, only ω^{n-1} is. When ω^{n-1} is closed (but not necessarily ω), we say that X is balanced and this is enough to get the Kobayashi–Hitchin correspondence. This is inspired from the idea of Greb, Ross and Toma in [9] where wall-crossing phenomena are studied with respect to multipolarisations to make the equations linear instead of polynomial. We recall general definitions and results about balanced manifolds in Section 2.

From here, the inequalities that define slope stability are linear in $\Theta = \frac{\omega^{n-1}}{(n-1)!}$. As in the Kähler case, we define the balanced cone $\mathcal{B}_X$ as the set of all real Dolbeault classes $[\Theta] \in H^{n-1,n-1}(X,\mathbb{R}) = H^{n-1,n-1}(X,\mathbb{C}) \cap H^{2n-1}(X,\mathbb{R})$ of positive $(n-1,n-1)$ forms Θ. In Section 3, we prove that locally, there is only a finite number of these inequalities to verify in order to obtain all of them. This is a consequence of the following result (Corollary 3.4 below).

Proposition 1. Let X be a compact complex balanced manifold of balanced cone $\mathcal{B}_X$ and $\mathcal{E} \to X$ a torsion-free coherent sheaf. Then, for all compact $K \subset \mathcal{B}_X$, there is a finite family $\mathcal{S}_1, \dots, \mathcal{S}_p \subset \mathcal{E}$ with for all k, $0 < \mathrm{rk}(\mathcal{S}_k) < \mathrm{rk}(\mathcal{E})$ such that for all $[\Theta] \in K$, $\mathcal{E}$ is $[\Theta]$-stable (resp. $[\Theta]$-semi-stable) if and only if for all k, $\mu_{[\Theta]}(\mathcal{S}_k) < \mu_{[\Theta]}(\mathcal{E})$ (resp. $\mu_{[\Theta]}(\mathcal{S}_k) \leqslant \mu_{[\Theta]}(\mathcal{E})$).

In the smooth projective setting, stronger results of the same kind are already known, such as [10, Theorem 6.7]. The main results of this paper are the analytical ones. Proposition 1 gives the structure of the set of metrics with respect to which $\mathcal{E}$ is stable (resp. semi-stable).

Theorem 2. The set $\mathcal{C}_s^{\mathrm{D}}(\mathcal{E})$ (resp. $\mathcal{C}_{ss}^{\mathrm{D}}(\mathcal{E})$) of balanced classes of metrics $[\Theta]$ with respect to which $\mathcal{E}$ is stable (resp. semi-stable) is locally an open (resp. closed) convex polyhedral cone.

A more precise version of this theorem with topological properties is given by Theorem 3.5 below.

Let $\mathcal{E} = (E, \overline{\partial}_E, h)$ be a Hermitian holomorphic vector bundle. When $\mathcal{E}$ is $[\Theta]$-stable (*i.e.* stable with respect to $[\Theta]$), the Kobayashi–Hitchin correspondence tells us that we can find a Θ-HYM connection $\nabla = \partial + \overline{\partial}$ (*i.e.* HYM with respect to Θ) such that $(E, \overline{\partial})$ is biholomorphic to $\mathcal{E}$. In other words, the holomorphic structure is preserved when we replace $\overline{\partial}_E$ by $\overline{\partial}$. Concretely, it is characterised by the fact that $\overline{\partial}$ is in the $\mathcal{G}^{\mathbb{C}}$-orbit of $\overline{\partial}_E$, where $\mathcal{G}^{\mathbb{C}}$ is the complex gauge group, whose action on complex structures is defined below.

In Section 4, we investigate the behaviour of the HYM connections when Θ approaches a metric Θ_0 with respect to which $\mathcal{E}$ is only semi-stable. When the graded object of $\mathcal{E}$ with respect to Θ_0, namely $\mathrm{Gr}_{[\Theta_0]}(\mathcal{E})$, is locally free, we show a convergence result when $\Theta \to \Theta_0$ (Theorem 4.7 below).

Theorem 3. For all $\Theta_\varepsilon = \Theta_0 + \varepsilon$ L_1^2-close to Θ_0 such that $\mathcal{E}$ is $[\Theta_\varepsilon]$-semi-stable, there is an associated connection $\nabla_\varepsilon = \partial_\varepsilon + \overline{\partial}_\varepsilon$ that has the three following properties,

1. ∇_ε is HYM with respect to Θ_ε.
2. $(E, \overline{\partial}_\varepsilon)$ is biholomorphic to $\mathrm{Gr}_{[\Theta_\varepsilon]}(\mathcal{E})$, which is locally free. In particular, if $\mathcal{E}$ is Θ_ε-stable, then $(E, \overline{\partial}_\varepsilon) \cong \mathcal{E}$.
3. For all integer $d \geqslant 2$, the bound $\left\|\overline{\partial}_\varepsilon - \overline{\partial}_0\right\|_{L_d^2} = \mathrm{O}\left(\|\varepsilon\|_{L_{d-1}^2} + \sqrt{\|[\varepsilon]\|}\right)$ holds, where $\|\cdot\|_{L_d^2}$ is an L_d^2 Sobolev norm and $\|\cdot\|$ is an Euclidean norm on $H^{n-1,n-1}(X,\mathbb{R})$. Therefore, the same bound holds for $\|\partial_\varepsilon - \partial_0\|_{L_d^2}$ and $\|\nabla_\varepsilon - \nabla_0\|_{L_d^2}$. In particular, $\nabla_\varepsilon \to \nabla_0$ for the $\mathcal{C}^\infty$ topologies.

The HYM connection, when it exists in some gauge orbit, is not unique but it is exactly one $\mathcal{G}$-orbit where $\mathcal{G}$ is the group of smooth sections u of endomorphisms of E which verify $uu^* = \mathrm{Id}_E$. A consequence of the previous theorem is the following one (Theorem 4.10 below).

Theorem 4. The function that maps each balanced metric Θ with respect to which $\mathcal{E}$ is semi-stable and sufficiently smooth to the class modulo $\mathcal{G}$ of the Θ-HYM connections $\nabla = \partial + \overline{\partial}$ such that $(E, \overline{\partial}) \cong \mathrm{Gr}_{[\Theta]}(\mathcal{E})$, is a continuous function with respect to the $\mathcal{C}^\infty$ topologies.

Theorem 2 is a generalisation of [3, Proposition 1.1] and Theorem 3 is a generalisation of [3, Theorem 1.2], where in both cases, we remove the hypothesis that the group of automorphisms of the graded object is abelian. Theorem 4 is a natural consequence of Theorem 3 and may be useful to study the variations of the moduli space of vector bundles when the metric of the manifold varies. Proposition 1 may also be more generally useful to study wall-crossing phenomena involving the slope stability.

The method used in this article are classical semi-stable perturbation techniques. When $\mathcal{E}$ is a $[\Theta_0]$-semi-stable vector bundle, it can be viewed as a small deformation of its graded object $\mathrm{Gr}(\mathcal{E})$. Then, by Kuranishi theory [15], all the small deformations of $\mathrm{Gr}(\mathcal{E})$ belong, up to a gauge transformation, in the Kuranishi slice which is a germ of finite dimensional complex manifold, on which the group G of automorphisms of $\mathrm{Gr}(\mathcal{E})$ acts naturally. Moreover, for all Θ L^2_1-close to $[\Theta_0]$, there is a Kähler form on this germ depending of Θ such that the action of G is Hamiltonian with respect to this form. Additionally, the zeroes of the associated moment map correspond to connections ∇ such that the contraction of the curvature with respect to Θ is orthogonal to the Lie algebra $\mathfrak{k}$ of the maximal compact subgroup of G. We want this contraction to be a constant homothety in order to obtain a HYM connection.

From here, there are two methods. The first one is to find a zero of this moment map using GIT theory, and then perturb the induced connection to obtain a new connection whose contraction of the curvature is a constant homothety. This method is used in [4] in the case of Z-critical connections for example. The second method is to first perturb the Kuranishi slice so the holomorphic structures in the perturbed slice induce connections whose contraction of the curvature belongs to $\mathfrak{k}$. Then, we find a zero of the associated perturbed moment map, whose induced connection is directly HYM. The main issue with this second method is that the perturbed slice is no more a germ of complex manifold, it is only real symplectic. Nevertheless, this second method is used in [22] in the context of K-stability and in [3] with HYM equations. It is also used in [21] together with the moment map flow in the context of constant scalar curvature metrics. In this paper, we use the second method too with the moment map flow, which is versatile and can probably be used in similar settings whenever there is a PDE with a moment map interpretation and have to study wall-crossing phenomena.

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2 Generalities about balanced metrics and their Laplace-Beltrami operators

2.1 Operations on compact complex manifolds

Let (X, g) be a Hermitian compact complex manifold of dimension $n \geqslant 2$ and let ω be its Kähler form (not necessarily closed). Then $\Theta = \frac{\omega^{n-1}}{(n-1)!}$ is a positive $(n-1, n-1)$ form in the following sens.

In a local frame $(z_1, \dots, z_n)$, we can define $\widehat{dz_i \wedge d\overline{z_j}}$ for each $1 \leqslant i, j \leqslant n$ as $dz_1 \wedge d\overline{z_1} \wedge \cdots \wedge dz_n \wedge d\overline{z_n}$ where we removed the terms dz_i and $d\overline{z_j}$. Then, the local expression of Θ has the form,

$$\Theta = \sum_{1 \leqslant i,j \leqslant n} \Theta_{ij} \widehat{dz_i \wedge d\overline{z_j}},$$

Positivity of Θ means that the matrix formed of the Θ_{ij} is Hermitian positive definite at each point, and this does not depend on the choice of the local frame. By [19, Equation (4.8)], ω (thus g) is entirely determined by Θ and depends smoothly on it. In other words, any positive $(n-1, n-1)$ form Θ gives rise to a metric g on X which depends smoothly on Θ. Let $\mathrm{Vol}_\Theta = \frac{\omega^n}{n!}$ be the natural associated volume form. For all $\Theta > 0$, we can define the trace operator Λ_Θ on complex forms as $\star^{-1} \circ (\omega \wedge \cdot) \circ \star$ where $\star$ is the Hodge star. In particular, on $(1,1)$ forms α, it is characterised by the equality,

$$(\Lambda_\Theta \alpha)\mathrm{Vol}_\Theta = \alpha \wedge \Theta.$$

This operator naturally extends to forms with values in a complex vector bundle E on X. Let (E, h) be a complex Hermitian vector bundle on X. Following for example [11, Definition 4.1.11], let us recall the natural Hermitian product on $\Omega^{p,q}(X, E)$ with respect to the metric Θ,

$$\langle \alpha, \beta \rangle_\Theta = \int_X (\alpha|\beta)_{\Theta,h} \mathrm{Vol}_\Theta, \tag{1}$$

where $(\cdot|\cdot)_{\Theta,h}$ is the natural Hermitian product on $\Lambda^{p,q}T^*X \otimes E$ depending on Θ and h, and we see α and β as sections of this bundle. It makes the decomposition $\Omega^*(X, E) = \bigoplus_{0 \leqslant p,q \leqslant n} \Omega^{p,q}(X, E)$ orthogonal. In particular, on sections ξ, η and on 1-forms α, β, we have,

$$\langle \xi, \eta \rangle_\Theta = \int_X h(\xi, \eta)\mathrm{Vol}_\Theta, \qquad \langle \alpha, \beta \rangle_\Theta = \int_X h(\alpha, J\beta) \wedge \Theta.$$

Here, J is the natural complex structure on $\Omega^*(X, E)$ given by $J = \mathbf{i}^{p-q}$ on (p,q)-forms. It enables us to consider adjoints of operators with respect to these Hermitian products. Let $\nabla = \partial + \overline{\partial}$ be an integrable unitary connection on E (*i.e.* $\overline{\partial}^2 = 0$ and ∇ is the Chern connection associated to $(\overline{\partial}, h)$). Then, we can define the Laplacian operators with respect to Θ and ∇,

$$\Delta_{\Theta,\partial} = \partial\partial^* + \partial^*\partial, \qquad \Delta_{\Theta,\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}, \qquad \Delta_{\Theta,\nabla} = \nabla\nabla^* + \nabla^*\nabla,$$

The second one is called the Dolbeault Laplacian and the third one the Laplace-Dolbeault Laplacian. These three operators are real self-adjoint and elliptic of order 2. See for example [12, Chapter VI, Section 5.4].

2.2 When X is balanced

The Hermitian metric g on X is said to be a *balanced metric* if the associated Kähler form ω is such that ω^{n-1} is closed. Balanced metrics where introduced by Michelsohn in [19], see [6] for a reference. Gauduchon showed that $\Theta = \frac{\omega^{n-1}}{(n-1)!}$ is closed (*i.e.* ω is balanced) if and only if ω is co-closed [7, Proposition 1]. We also call abusively Θ a balanced metric in this case. When such a metric exists, we say that X is balanced. A characterisation of balanced manifolds is given by [19, Theorem A]. Similarly to the Kähler cone, we can introduce the balanced cone,

$$\mathcal{B}_X = \{[\Theta] \in H^{n-1,n-1}(X, \mathbb{R}) | \Theta \text{ positive } (n-1, n-1)\text{-form}\},$$

where the $H^{p,p}(X, \mathbb{R}) = H^{p,p}(X, \mathbb{C}) \cap H^{2p}(X, \mathbb{R})$ are the real Dolbeault cohomology groups.

Remark. In the literature about balanced manifolds, the balanced cone usually contains elements of the Bott-Chern cohomology groups instead of the real Dolbeault ones, where the boundary morphisms are $\mathbf{i}\partial\overline{\partial}$ instead of d. Here, we only need to work modulo d. Since $\partial\overline{\partial} = \frac{1}{2}(\partial - \overline{\partial})d$, the real Dolbeault cohomology groups are naturally quotients of the Bott-Chern ones. Therefore, this article's balanced cone can be seen as a projection of the classical balanced cone on a quotient space.

From now on, we assume that Θ is balanced.

Lemma 2.1. *The following Kähler identities hold on the space* $\Omega^1(X, E)$*,*

$$\begin{aligned}\Lambda_\Theta \partial &= \mathbf{i}\overline{\partial}^*,\\ \Lambda_\Theta \overline{\partial} &= -\mathbf{i}\partial^*,\end{aligned}$$

and the following hold on the space of smooth sections $\Omega^0(X, E)$*,*

$$\begin{aligned}\Delta_{\Theta,\partial} &= \mathbf{i}\Lambda_\Theta\overline{\partial}\partial,\\ \Delta_{\Theta,\overline{\partial}} &= -\mathbf{i}\Lambda_\Theta\partial\overline{\partial},\\ \Delta_{\Theta,\partial} + \Delta_{\Theta,\overline{\partial}} = \Delta_{\Theta,\nabla} &= \mathbf{i}\Lambda_\Theta(\overline{\partial}\partial - \partial\overline{\partial}),\\ \Delta_{\Theta,\partial} - \Delta_{\Theta,\overline{\partial}} &= \Lambda_\Theta \mathbf{i} F_\nabla,\end{aligned}$$

where $F_\nabla = \nabla^2 = \overline{\partial}\partial + \partial\overline{\partial}$ *is the curvature form of* ∇*.*

Proof. The proof is the same as the standard proof of Kähler identities, but only on sections and 1-forms. Indeed, if α is a $(0,1)$ form and s a 0-form with values in E,

$$\begin{aligned}\left\langle \Lambda_\Theta\overline{\partial}\alpha, s\right\rangle_\Theta &= \int_X h(\Lambda_\Theta\overline{\partial}\alpha, s)\mathrm{Vol}_\Theta\\ &= \int_X h(\overline{\partial}\alpha, s)\wedge\Theta\\ &= \int_X dh(\alpha, s)\wedge\Theta + \int_X h(\alpha, \partial s)\wedge\Theta\\ &= \int_X h(\alpha, -J\mathbf{i}\partial s)\wedge\Theta \text{ because } d\Theta = 0,\\ &= \langle\alpha, -\mathbf{i}\partial s\rangle_\Theta .\end{aligned}$$

We obtain the second identity. The first one is similar. We deduce that, on smooth sections,

$$\Delta_{\Theta,\partial} = \partial\partial^* + \partial^*\partial = \partial^*\partial = \mathbf{i}\Lambda_\Theta\overline{\partial}\partial,$$

and similarly,

$$\Delta_{\Theta,\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} = \overline{\partial}^*\overline{\partial} = -\mathbf{i}\Lambda_\Theta\partial\overline{\partial}.$$

The equality $\Delta_{\Theta,\nabla} = \Delta_{\Theta,\partial} + \Delta_{\Theta,\overline{\partial}}$ always hold by definition even if we don't assume $d\Theta = 0$. The third equality involving Laplacians is the sum of the two first and the fourth one is their difference. $\square$

Proposition 2.2. *On smooth sections,* $\Delta_{\Theta,\nabla}$ *verifies* $\ker(\Delta_{\Theta,\nabla}) = \ker(\nabla)$ *is the space of* ∇*-parallel sections and if* $\Lambda_\Theta \mathbf{i} F_\nabla = 0$*, then* $\ker(\Delta_{\Theta,\nabla}) = \ker(\overline{\partial})$ *is the space of* $\overline{\partial}$*-holomorphic sections.*

Proof. The first equality is a direct consequence of the fact that $\Delta_{\Theta,\nabla} = \nabla^*\nabla$ on smooth forms. The second one comes from Lemma 2.1, which implies that $\Delta_{\Theta,\nabla} = \Delta_{\Theta,\partial} + \Delta_{\Theta,\overline{\partial}}$ and if $\Lambda_\Theta \mathbf{i} F_\nabla = 0$, $\Delta_{\Theta,\partial} = \Delta_{\Theta,\overline{\partial}}$ hence $\Delta_{\Theta,\nabla} = 2\Delta_{\Theta,\overline{\partial}} = 2\overline{\partial}^*\overline{\partial}$. □

By abuse of notation, we still call $\nabla = \partial + \overline{\partial}$ the natural connection on $\mathrm{End}(E)$ that arises from ∇ on E.

2.3 Gauge action and its derivative

Let us to introduce the *complex gauge group* $\mathcal{G}^{\mathbb{C}}$ of smooth sections of $\mathrm{End}(E)$ which are isomorphisms on each fibre. It is an infinite dimensional complex Lie group and $\mathrm{Lie}(\mathcal{G}^{\mathbb{C}}) = \Omega^0(X, \mathrm{End}(E))$. This group acts on connections on E with,

$$f \cdot \partial = f^{-1*} \circ \partial \circ f^*, \quad f \cdot \overline{\partial} = f \circ \overline{\partial} \circ f^{-1}, \quad f \cdot \nabla = f \cdot \partial + f \cdot \overline{\partial}.$$

The adjoints are computed with respect to the metric h. Notice that this action preserves the integrability condition of the Dolbeault operator $\overline{\partial}^2 = 0$ and it preserves the compatibility with the metric h of the connection. When $\overline{\partial}_1 = f \cdot \overline{\partial}_2$, the holomorphic vector bundles $\mathcal{E}_1 = (E, \overline{\partial}_1)$ and $\mathcal{E}_2 = (E, \overline{\partial}_2)$ are isomorphic because $f : \mathcal{E}_2 \to \mathcal{E}_1$ is a biholomorphism. Conversely, such an isomorphism gives rise to a gauge equivalence between the Dolbeault operators of $\mathcal{E}_1$ and $\mathcal{E}_2$.

Similarly, we call $\mathcal{G} \subset \mathcal{G}^{\mathbb{C}}$ the *unitary gauge group* defined as the set of all $u \in \mathcal{G}^{\mathbb{C}}$ such that $uu^* = \mathrm{Id}_E$. It is an infinite dimensional real Lie group with $\mathrm{Lie}(\mathcal{G}) = \mathbf{i}\Omega^0(X, \mathrm{End}_H(E,h))$ where $\mathrm{End}_H(E,h) \subset \mathrm{End}(E)$ is the real smooth bundle of Hermitian endomorphisms with respect to h. Notice that the action of $\mathcal{G}$ on connections is given by $u \cdot \nabla = u \circ \nabla \circ u^*$, thus the associated curvature verifies $F_{u\cdot\nabla} = uF_\nabla u^*$.

We will be interested in finding a Dolbeault operator $\overline{\partial}$ on E which satisfy the property of being Hermitian Yang–Mills (see Definition 3.6 in Section 3) such that $(E, \overline{\partial})$ is isomorphic to a given holomorphic vector bundle $\mathcal{E} = (E, \overline{\partial}_E)$. Therefore, we shall search $\overline{\partial}$ in the $\mathcal{G}^{\mathbb{C}}$-orbit of $\overline{\partial}_E$. For this, it will be useful to compute infinitesimal variations of $f \mapsto f \cdot \overline{\partial}_E$.

Proposition 2.3. *For any* ∇*, the map*

$$\Psi : \left\{ \begin{array}{rcl} \Omega^0(X, \mathrm{End}_H(E,h)) & \to & \Omega^0(X, \mathrm{End}_H(E,h)) \\ s & \mapsto & \Lambda_\Theta \mathbf{i} F_{\mathbf{e}^s\cdot\nabla} \end{array} \right. ,$$

verifies $d\Psi(0) = \Delta_{\Theta,\nabla}$ *(on* $\mathrm{End}(E)$*). Notice that since* $\Delta_{\Theta,\nabla}$ *is a real operator, it preserves* $\mathrm{End}_H(E,h)$*.*

Proof. We have for all smooth Hermitian sections s of $\mathrm{End}(E)$,

$$\mathbf{e}^s \cdot \nabla = \nabla + \mathbf{e}^s \overline{\partial}(\mathbf{e}^{-s}) + \mathbf{e}^{-s}\partial(\mathbf{e}^s),$$

hence

$$F_{\mathbf{e}^s\cdot\nabla} = F_\nabla + \nabla\alpha_s + \alpha_s \wedge \alpha_s,$$

where $\alpha_s = \mathbf{e}^s\overline{\partial}(\mathbf{e}^{-s})+\mathbf{e}^{-s}\partial(\mathbf{e}^s) \in \mathbf{i}\Omega^1(X, \mathrm{End}_H(E,h))$. We have $\alpha_0 = 0$ and if $v \in T_s\Omega^0(X, \mathrm{End}_H(E,h)) = \Omega^0(X, \mathrm{End}_H(E,h))$,

$$\frac{\partial}{\partial s}|_{s=0}(\mathbf{e}^s\overline{\partial}(\mathbf{e}^{-s}))\cdot v = \frac{\partial}{\partial s}|_{s=0}(\mathbf{e}^s)\cdot v\overline{\partial}(\mathbf{e}^{-0}) + \mathbf{e}^0\overline{\partial}\left(\frac{\partial}{\partial s}|_{s=0}(\mathbf{e}^{-s})\cdot v\right) = -\overline{\partial}v.$$

Hence $\frac{\partial}{\partial s}|_{s=0}\alpha_s \cdot v = \partial v - \overline{\partial} v$ symmetrically. Since $\alpha_s \wedge \alpha_s$ is quadratic and $\alpha_0 = 0$, its derivative at 0 vanishes thus

$$\frac{\partial}{\partial s}|_{s=0}\mathbf{i}F_{\mathbf{e}^s\cdot\nabla} = \mathbf{i}\nabla(\partial v - \overline{\partial}v) = \mathbf{i}(\overline{\partial}\partial - \partial\overline{\partial})v.$$

After applying Λ_Θ, by Lemma 2.1, we deduce that $d\Psi(0) = \Delta_{\Theta,\nabla}$. □

3 Hermitian Yang–Mills equation and finiteness results about slope stability

3.1 Slope stability

Any holomorphic vector bundle can be seen as the locally free coherent sheaf of its holomorphic sections. In this subsection, we study more generally coherent sheaves on X and their slope stablity with respect to a positive real Dolbeault class $[\Theta]$.

Definition 3.1. We define the $[\Theta]$-slope $\mu_{[\Theta]}$ of a non-zero torsion-free coherent sheaf $\mathcal{E}$ as

$$\mu_{[\Theta]}(\mathcal{E}) = \frac{c_1(\mathcal{E}) \cup [\Theta]}{\mathrm{rk}(\mathcal{E})}.$$

Following Mumford [20], we say that,

- $\mathcal{E}$ is $[\Theta]$-stable if for all coherent sub-sheaf $\mathcal{S} \subset \mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{S}) < \mathrm{rk}(\mathcal{E})$, $\mu_{[\Theta]}(\mathcal{S}) < \mu_{[\Theta]}(\mathcal{E})$,
- $\mathcal{E}$ is $[\Theta]$-semi-stable if for all coherent sub-sheaf $\mathcal{S} \subset \mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{S}) < \mathrm{rk}(\mathcal{E})$, $\mu_{[\Theta]}(\mathcal{S}) \leqslant \mu_{[\Theta]}(\mathcal{E})$,
- $\mathcal{E}$ is $[\Theta]$-polystable if $\mathcal{E}$ is a direct sum of stable coherent sheaves of the same $[\Theta]$-slope.

Clearly, stability implies polystability. When $\mathcal{E}$ can be written as a direct sum $\bigoplus_{i=1}^m \mathcal{E}_i$ (or more generally as an extension of the $\mathcal{E}_i$), then $c_1(\mathcal{E}) = \sum_{i=1}^m c_1(\mathcal{E}_i)$ thus $\mu_{[\Theta]}(\mathcal{E})$ is a weighted average of the $\mu_{[\Theta]}(\mathcal{E}_i)$ where the weights are the $\mathrm{rk}(\mathcal{E}_i)$. Therefore, polystability implies semi-stability.

In the case where $\mathcal{E}$ is only $[\Theta]$-semi-stable, we may use a Jordan-Hölder filtration to reduce to the polystable case.

Proposition 3.2. *If $\mathcal{E}$ is a $[\Theta]$-semi-stable vector bundle, then it admits a Jordan-Hölder filtration, i.e. a filtration by coherent sub-sheaves $0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_m = \mathcal{E}$ such that for all k, $\mu_{[\Theta]}(\mathcal{F}_k) = \mu_{[\Theta]}(\mathcal{E})$ and each $\mathcal{G}_k = \mathcal{F}_k/\mathcal{F}_{k-1}$ is torsion-free and $[\Theta]$-stable. Moreover, the graded object,*

$$\mathrm{Gr}_{[\Theta]}(\mathcal{E}) = \bigoplus_{k=1}^m \mathcal{G}_k,$$

is unique up to isomorphism.

Proof. This is a standard categorical result. See for example [24, Lemma 12.9.7]. $\square$

When $\mathcal{E}$ is locally free *i.e.* a vector bundle, it is said to be *sufficiently smooth* if its graded object $\mathrm{Gr}_{[\Theta]}(\mathcal{E})$ is locally free too, or equivalently, each $\mathcal{G}_i$ is locally free.

Remark. When $F \subset E$ are smooth vector bundles, a Hermitian metric h on E gives rise to a diffeomorphism between E/F and $F^{\perp}$. In other words, any exact sequence of smooth vector bundles splits. In particular, $\mathcal{E}$ and $\mathrm{Gr}_{[\Theta]}(\mathcal{E})$ have the same smooth structure when $\mathcal{E}$ is a sufficiently smooth vector bundle.

Remark. $\mathcal{E}$ is $[\Theta]$-polystable if and only if it is $[\Theta]$-semi-stable and $\mathrm{Gr}_{[\Theta]}(\mathcal{E}) \cong \mathcal{E}$.

3.2 Stable and semi-stable polyhedral cones

Let $\mathcal{E}$ be a torsion-free coherent sheaf. We are interested here in the structure of the set of Θ such that $\mathcal{E}$ is $[\Theta]$-stable (resp. $[\Theta]$-semi-stable). Let us introduce,

$$\mathcal{C}_s(\mathcal{E}) = \{\Theta > 0 | d\Theta = 0 \text{ and } \mathcal{E} \text{ is } [\Theta]\text{-stable}\}, \qquad \mathcal{C}_{ss}(\mathcal{E}) = \{\Theta > 0 | d\Theta = 0 \text{ and } \mathcal{E} \text{ is } [\Theta]\text{-semi-stable}\},$$

and their real Dolbeault projection counterparts,

$$\mathcal{C}_s^{\mathrm{D}}(\mathcal{E}) = \{[\Theta] | \Theta \in \mathcal{C}_s(\mathcal{E})\} = \{[\Theta] \in \mathcal{B}_X | \mathcal{E} \text{ is } [\Theta]\text{-stable}\},$$

$$\mathcal{C}_{ss}^{\mathrm{D}}(\mathcal{E}) = \{[\Theta] | \Theta \in \mathcal{C}_{ss}(\mathcal{E})\} = \{[\Theta] \in \mathcal{B}_X | \mathcal{E} \text{ is } [\Theta]\text{-semi-stable}\}.$$

When $\mathcal{S} \subset \mathcal{E}$ verifies $0 < \mathrm{rk}(\mathcal{S}) < \mathrm{rk}(\mathcal{E})$, let,

$$l_{\mathcal{S}} : \begin{cases} H^{n-1,n-1}(X, \mathbb{R}) & \to & \mathbb{R} \\ [\Theta] & \mapsto & \mu_{[\Theta]}(\mathcal{E}/\mathcal{S}) - \mu_{[\Theta]}(\mathcal{S}) = \left(\dfrac{c_1(\mathcal{E}/\mathcal{S})}{\mathrm{rk}(\mathcal{E}/\mathcal{S})} - \dfrac{c_1(\mathcal{S})}{\mathrm{rk}(\mathcal{S})} \right) \wedge [\Theta] \end{cases}.$$

Each $l_{\mathcal{S}}$ is a linear form on $H^{n-1,n-1}(X, \mathbb{R})$ and the see-saw property of slopes implies that for each $[\Theta] \in \mathcal{B}_X$, $\mathcal{E}$ is $[\Theta]$-stable (resp $[\Theta]$-semi-stable) if and only if for all $\mathcal{S}$, $l_{\mathcal{S}}([\Theta]) > 0$ (resp. $\geqslant 0$). In other words, we have,

$$\mathcal{C}_s^{\mathrm{D}}(\mathcal{E}) = \mathcal{B}_X \cap \bigcap_{\mathcal{S} \subset \mathcal{E}} \{l_{\mathcal{S}} > 0\}, \qquad \mathcal{C}_{ss}^{\mathrm{D}}(\mathcal{E}) = \mathcal{B}_X \cap \bigcap_{\mathcal{S} \subset \mathcal{E}} \{l_{\mathcal{S}} \geqslant 0\},$$

In particular, these four sets all are convex cones. The purpose of the next subsection is to show that the number of linear inequalities $l_{\mathcal{S}} > 0$ to verify in order to obtain stability is locally finite, so $\mathcal{C}_{ss}^{\mathrm{D}}(\mathcal{E})$ and $\mathcal{C}_s^{\mathrm{D}}(\mathcal{E})$ are locally polyhedral convex cones.

3.3 Local finiteness

Lemma 3.3. *Let V be a finite dimensional real vector space and $D \subset V$ a non-empty discrete subset. Let $U \subset V^{\vee}$ be open and such that for all $\varphi \in U$, $\varphi(D) \subset \mathbb{R}$ is bounded from above. Then, for all compact $K \subset U$, there is a finite set $F \subset D$ that may depend on K such that for all $\varphi \in K$, $\sup(\varphi(D)) = \max(\varphi(F))$.*

Proof.

Step 1 : For all real number a, for all $K \subset U$ compact, the set $F_{K,a} = \bigcup_{\varphi \in K} \{\varphi \geqslant a\} \cap D$ is finite.

Indeed, $F_{K,a} \subset D$ is discrete so if it is infinite, then it is unbounded. In this case, there is a sequence (v_m) of elements of $F_{K,a}$ such that $\|v_m\| \to +\infty$ where $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$ is any Euclidean norm on V. Up to extracting, $\frac{v_m}{\|v_m\|} \to v_\infty \in V$ is in the unit sphere. For all fixed m, $v_m \in \bigcup_{\varphi \in K} \{\varphi \geqslant a\}$ hence there is a $\varphi_m \in K$ such that $\varphi_m(v_m) \geqslant a$. Up to extracting again, $\varphi_m \to \varphi_\infty \in K$.

Let $\epsilon > 0$ and $\varphi = \varphi_\infty + \epsilon \langle v_\infty, \cdot \rangle$. Since U is open, for ϵ small enough, $\varphi \in U$. For all integer m,

$$\begin{aligned}
\varphi(v_m) &= \varphi_\infty(v_m) + \epsilon \langle v_\infty, v_m \rangle \\
&= \varphi_m(v_m) + \mathrm{O}(\|\varphi_m - \varphi_\infty\| \, \|v_m\|) + \epsilon \left\langle \frac{v_m}{\|v_m\|} + \mathrm{o}(1), v_m \right\rangle \\
&\geqslant a + \epsilon \|v_m\| + \mathrm{o}(\|v_m\|) \\
&\underset{m \to +\infty}{\longrightarrow} +\infty.
\end{aligned}$$

This contradicts the boundedness from above of $\varphi(D)$. It proves that $F_{K,a}$ is indeed finite. In particular, when $K = \{\varphi\}$ is a singleton and $a = \sup(\varphi(D)) - 1$, we obtain that for all $\varphi \in U$, $\{\varphi \geqslant \sup(\varphi(D)) - 1\} \cap D$ is finite. In particular, it proves that the sup is reached (and it is reached finitely many times).

Step 2 : The map $\varphi \mapsto \max(\varphi(D))$ is continuous on U.

Let $\varphi_0 \in U$ and $K \subset U$ be a compact set such that $\varphi_0 \in \overset{\circ}{K}$. Let $v \in D$ such that $\max(\varphi_0(D)) = \varphi_0(v)$ and $a = \max(\varphi_0(D)) - 1$. Clearly, $v \in F_{K,a}$ and when φ is close enough to φ_0, then $\varphi \in K$ and $\varphi(v) \geqslant a$. Thus $\max(\varphi(D)) \geqslant \varphi(v) \geqslant a$ so the $w \in D$ that verify $\max(\varphi(D)) = \varphi(w)$ belong to $F_{K,a}$. In other words, for all φ close enough to φ_0, $\max(\varphi(D)) = \max(\varphi(F))$.

Clearly, since $F_{K,a}$ is finite, $\varphi \mapsto \max(\varphi(F))$ is continuous at φ_0, proving the wanted result.

Step 3 : Conclusion.

Let $K \subset U$ be an arbitrary compact set. By the continuity result of step 2, $a = \min_{\varphi \in K}(\max(\varphi(D)))$ is well defined and by the result of step 1, $F = F_{K,a} = \bigcup_{\varphi \in K} \{\varphi \geqslant a\} \cap D$ is finite. For all $\varphi \in K$, if $v \in D$ is such that $\max(\varphi(D)) = \varphi(v)$, then $\varphi(v) \geqslant a$ so $v \in F$. We deduce that $\sup(\varphi(D))$ is reached by an element in the finite set F, which is the wanted result. $\square$

We can now prove Introduction's Proposition 1.

Corollary 3.4. *For all torsion-free vector bundle $\mathcal{E}$ and all compact $K \subset \mathcal{B}_X$, there is a finite tuple $\mathcal{S}_1, \ldots, \mathcal{S}_p$ of coherent sub-sheaves of $\mathcal{E}$ with for all k, $0 < \mathrm{rk}(\mathcal{S}_k) < \mathrm{rk}(\mathcal{E})$ that may depend on K such that for all $[\Theta] \in K$, $\mathcal{E}$ is $[\Theta]$-stable (resp. $[\Theta]$-semi-stable) if and only if for all k, $l_{\mathcal{S}_k}([\Theta]) > 0$ (resp. $l_{\mathcal{S}_k}([\Theta]) \geqslant 0$).*

Proof. Let $V = H^{1,1}(X, \mathbb{R})$. By Serre duality, $V^\vee$ is naturally identified with $H^{n-1,n-1}(X, \mathbb{R})$ thus we can see $\mathcal{B}_X$ as an open subset of $V^\vee$. Let $D = \left\{ \frac{c_1(\mathcal{S})}{\mathrm{rk}(\mathcal{S})} \middle| \mathcal{S} \subset \mathcal{E}, 0 < \mathrm{rk}(\mathcal{S}) < \mathrm{rk}(\mathcal{E}) \right\}$. D is discrete because the first Chern classes belong to the lattice $H^2(X, \mathbb{Z})$ hence $D \subset \frac{1}{\mathrm{rk}(\mathcal{E})!} H^2(X, \mathbb{Z})$. For each fixed $[\Theta] \in \mathcal{B}_X$, $\{c \cup [\Theta] | c \in D\} \subset \mathbb{R}$ is bounded from above. This is a standard fact uses to build Harder-Narasimhan filtrations (see for example [13, Lemma 5.7.16]).

Therefore, we can use Lemma 3.3 which tells us that for all compact $K \subset \mathcal{B}_X$, there is a finite set $F \subset D$ such that for all $[\Theta] \in K$, $\sup\{c \cup [\Theta] | c \in D\} = \max\{c \cup [\Theta] | c \in F\}$. Let $F = \{c^1, \ldots, c^p\}$

and for all k, $\mathcal{S}_k \subset \mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{S}_k) < \mathrm{rk}(\mathcal{E})$ such that $\frac{c_1(\mathcal{S}_k)}{\mathrm{rk}(\mathcal{S}_k)} = c^k$. This tuple fits. Indeed, when $[\Theta] \in K$,

$$\begin{aligned}
\mathcal{E} \text{ is } [\Theta]\text{-stable } &\Leftrightarrow \forall \mathcal{S} \subset \mathcal{E}, 0 < \mathrm{rk}(\mathcal{S}) < \mathrm{rk}(\mathcal{E}) \Rightarrow \frac{c_1(\mathcal{S})}{\mathrm{rk}(\mathcal{S})} \cup [\Theta] < \frac{c_1(\mathcal{E})}{\mathrm{rk}(\mathcal{E})} \cup [\Theta] \\
&\Leftrightarrow \sup\{c \cup [\Theta] | c \in D\} < \frac{c_1(\mathcal{E})}{\mathrm{rk}(\mathcal{E})} \cup [\Theta] \text{ because the sup is reached,} \\
&\Leftrightarrow \max\{c \cup [\Theta] | c \in F\} < \frac{c_1(\mathcal{E})}{\mathrm{rk}(\mathcal{E})} \cup [\Theta] \\
&\Leftrightarrow \forall 1 \leqslant k \leqslant p, l_{\mathcal{S}_k}([\Theta]) > 0.
\end{aligned}$$

The same equivalence holds with large inequalities. □

This last corollary enables us to understand better the geometric properties of the stable and semi-stable cones. For the topological structures, we endow the space of $(n-1, n-1)$ closed forms with the L^2_1 topology. We call $\|\cdot\|_{L^2_1}$ an L^2_1 norm (computed thanks to some background metric on X). In particular, the L^2_1 topology makes the subset of exact forms closed hence its projection on $H^{n-1,n-1}(X, \mathbb{R})$ continuous.

Clearly, $\mathcal{C}_s(\mathcal{E})$ is an L^2_1-open convex cone in the space of closed $(n-1, n-1)$ forms. $\mathcal{C}^{\mathrm{D}}_s(\mathcal{E})$ is an open convex cone in $H^{n-1,n-1}(X, \mathbb{R})$. $\mathcal{C}_{ss}(\mathcal{E})$ is an L^2_1-closed convex cone in the space of balanced metrics. $\mathcal{C}^{\mathrm{D}}_{ss}(\mathcal{E})$ is a convex cone in $H^{n-1,n-1}(X, \mathbb{R})$ and is closed in $\mathcal{B}_X$. All of these are an trivial consequences of the local finiteness result of Corollary 3.4 and of the continuity of the natural projection on $H^{n-1,n-1}(X, \mathbb{R})$. Moreover, we have the more precise version of Introduction's Theorem 2.

Theorem 3.5. *On a balanced manifold X, for all torsion-free coherent sheaf $\mathcal{E}$, the stable and semi-stable cones satisfy the following properties,*

- $\mathcal{C}^{\mathrm{D}}_s(\mathcal{E})$ *and* $\mathcal{C}^{\mathrm{D}}_{ss}(\mathcal{E})$ *are locally polyhedral convex cones.*
- *If there is a coherent $\mathcal{S} \subset \mathcal{E}$ such that $0 < \mathrm{rk}(\mathcal{S}) < \mathrm{rk}(\mathcal{E})$ and $\frac{c_1(\mathcal{S})}{\mathrm{rk}(\mathcal{S})} = \frac{c_1(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}$, then $\mathcal{C}_s(\mathcal{E}) = \mathcal{C}^{\mathrm{D}}_s(\mathcal{E}) = \emptyset$.*
- *If there is no coherent $\mathcal{S} \subset \mathcal{E}$ such that $0 < \mathrm{rk}(\mathcal{S}) < \mathrm{rk}(\mathcal{E})$ and $\frac{c_1(\mathcal{S})}{\mathrm{rk}(\mathcal{S})} = \frac{c_1(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}$, then $\overset{\circ}{\mathcal{C}_{ss}}(\mathcal{E}) = \mathcal{C}_s(\mathcal{E})$ and $\overset{\circ}{\mathcal{C}^{\mathrm{D}}_{ss}}(\mathcal{E}) = \mathcal{C}^{\mathrm{D}}_s(\mathcal{E})$.*
- *If $\mathcal{C}^{\mathrm{D}}_s(\mathcal{E}) \neq \emptyset$, $\overline{\mathcal{C}_s(\mathcal{E})} \cap \mathcal{B}_X = \mathcal{C}_{ss}(\mathcal{E})$ and $\overline{\mathcal{C}^{\mathrm{D}}_s(\mathcal{E})} \cap \mathcal{B}_X = \mathcal{C}^{\mathrm{D}}_{ss}(\mathcal{E})$.*
- *For all $[\Theta_0] \in \mathcal{C}_{ss}(\mathcal{E})$, if $[\Theta] \in \mathcal{B}_X$ is close enough to $[\Theta_0]$, $[\Theta] \in \mathcal{C}_s(\mathcal{E})$ (resp. $[\Theta] \in \mathcal{C}_{ss}(\mathcal{E})$) if and only if for all $\mathcal{S} \subset \mathcal{E}$ such that $l_{\mathcal{S}}([\Theta_0]) = 0$, $l_{\mathcal{S}}([\Theta]) > 0$ (resp. $l_{\mathcal{S}}([\Theta]) \geqslant 0$).*

Proof. By Corollary 3.4, if $K \subset \mathcal{B}_X$ is compact, there is an integer p and a finite p-tuple $\mathcal{S}_1, \dots, \mathcal{S}_p \subsetneq \mathcal{E}$ of coherent sheaves such that for all k, $0 < \mathrm{rk}(\mathcal{S}_k) < \mathrm{rk}(\mathcal{E})$ and,

$$\mathcal{C}^{\mathrm{D}}_s(\mathcal{E}) \cap K = \bigcap_{k=1}^{p} \{l_{\mathcal{S}_k} > 0\} \cap K, \qquad \mathcal{C}^{\mathrm{D}}_{ss}(\mathcal{E}) \cap K = \bigcap_{k=1}^{p} \{l_{\mathcal{S}_k} \geqslant 0\} \cap K.$$

It gives then local polyhedral structures.

Moreover, since the pairing $H^{n-1,n-1}(X,\mathbb{R}) \times H^{1,1}(X,\mathbb{R}) \to \mathbb{R}$ is non-degenerate by Serre duality, for each $\mathcal{S}$, $l_{\mathcal{S}} = 0 \Leftrightarrow \frac{c_1(\mathcal{S})}{\mathrm{rk}(\mathcal{S})} = \frac{c_1(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}$. From here, the results about the finite dimensional cones follow from basic topology and convex geometry, and the results about the infinite dimensional cones follow from the continuity of the projection onto $H^{n-1,n-1}(X,\mathbb{R})$. □

3.4 Hermitian Yang–Mills equation

Definition 3.6. Following [13, Section 4.1], we say that ∇ is a Θ-Hermitian Yang–Mills connection (or satisfies the Einstein condition) if

$$\Lambda_\Theta \mathbf{i} F_\nabla = c\mathrm{Id}_E,$$

for some constant c.

Remark. When it exists, c is entirely determined by the topology of E and the real Dolbeault class $[\Theta] \in H^{n-1,n-1}(X,\mathbb{R})$. Indeed, by Chern-Weil theory, we have $c_{[\Theta]} = \frac{2\pi c_1(E)\cup[\Theta]}{\mathrm{rk}(E)\mathrm{Vol}(X)}$ by integrating over X the above equality.

Since the Hermitian metric h on E is fixed, ∇ depends only on $\overline{\partial}$ as it is the Chern connection associated with $(\overline{\partial}, h)$. We write $F_{\overline{\partial}} = F_\nabla$ and we also say abusively that $\overline{\partial}$ is Θ-HYM.

A powerful analytic tool to study stability is the Kobayashi–Hitchin correspondence, which was originally proven for Kähler metrics [27] and extended to more general cases, including balanced metrics [16], and on Higgs bundles [17].

Proposition 3.7. *If $\mathcal{E} = (E, \overline{\partial}_E)$ is a holomorphic vector bundle of positive rank, then it is $[\Theta]$-polystable if and only if it admits a Θ-HYM operator $\overline{\partial}$ in the gauge orbit of $\overline{\partial}_E$. In this case, it is unique up to a unitary gauge transformation.*

Our purpose from now on is to extend locally the algebraic results of Theorem 3.5 to analytic results. Concretely, we want to show that when $\Theta \in \mathcal{C}_s(\mathcal{E})$ approaches a point $\Theta_0 \in \mathcal{C}_{ss}(\mathcal{E})$, we can build some Θ-HYM $\overline{\partial}$ in the gauge orbit of $\overline{\partial}_E$ that approaches some Θ_0-HYM $\overline{\partial}_0$, which is in the gauge orbit of the Dolbeault operator of $\mathrm{Gr}_{[\Theta_0]}(\mathcal{E})$ (under the additional condition that $\mathcal{E}$ is sufficiently smooth with respect to $[\Theta_0]$).

4 Local analytic results

Let Θ_0 be a balanced metric on X and $\mathcal{E} = (E, \overline{\partial}_E)$ a $[\Theta_0]$-semi-stable sufficiently smooth vector bundle. Assume that $\mathcal{C}_s(\mathcal{E}) \neq \emptyset$ so $\Theta_0 \in \overline{\mathcal{C}_s(\mathcal{E})}$ by Theorem 3.5 and $\mathcal{E}$ is simple. We associate to $\mathcal{E}$ a Jordan-Hölder filtration $0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_m = \mathcal{E}$ given by Proposition 3.2 and we set $\mathrm{Gr}(\mathcal{E}) = \mathrm{Gr}_{[\Theta_0]}(\mathcal{E}) = \bigoplus_{k=1}^m \mathcal{G}_k$ to be its graded object where the $\mathcal{G}_k = \mathcal{F}_k/\mathcal{F}_{k-1}$ are its stable components. We call $\nabla_0 = \partial_0 + \overline{\partial}_0$ the connection on $\mathrm{Gr}(\mathcal{E})$. We may assume it is Θ_0-HYM by Proposition 3.7.

Being a balanced metric is an open condition in the space of closed $(n-1, n-1)$ forms endowed with the L^2_1 topology. During all this article, U will be an L^2_1 open neighbourhood of 0 such that for all $\varepsilon \in U$, $\Theta_\varepsilon = \Theta_0 + \varepsilon$ is positive. With the L^2_1 topology, all the natural operations involving Θ like $\Theta \mapsto \mathrm{Vol}_\Theta$ or $\Theta \mapsto \Lambda_\Theta \alpha$ for some $(1,1)$-form α, are smooth. In particular, up to shrinking U, the L^2 norms $\|\cdot\|_\varepsilon$ are uniformly equivalent. From now on, when we consider objects that depend on the balanced metric, we replace the subscript $\Theta = \Theta_\varepsilon$ by ε for simplicity.

4.1 Perturbed Kuranishi slice and moment map

We call $G = \mathrm{Aut}_0(\mathrm{Gr}(\mathcal{E})) \subset \mathcal{G}^{\mathbb{C}}$ the group of automorphisms of $\mathrm{Gr}(\mathcal{E})$ of determinant 1 and $K = \mathrm{Aut}_0(\mathrm{Gr}(\mathcal{E}), h) = \mathcal{G} \cap G$ the group of unitary automorphisms of $\mathrm{Gr}(\mathcal{E})$ of determinant 1. They are both finite dimensional, K is compact and $G = K^{\mathbb{C}}$, hence G is a reductive Lie group with maximal compact subgroup K. Let $\mathfrak{g} = \mathrm{Lie}(G) \subset \mathrm{Lie}(\mathcal{G}^{\mathbb{C}})$ and $\mathfrak{k} = \mathrm{Lie}(K) = \mathrm{Lie}(\mathcal{G}) \cap \mathfrak{g}$. Notice that the condition of having a determinant that equals 1 on G and K implies that all elements of $\mathfrak{g}$ and $\mathfrak{k}$ have a trace that equals 0. Each of these Lie algebras is identified with its dual space thanks to the scalar products $\langle \cdot, \cdot \rangle_\varepsilon$ on $\mathrm{Lie}(\mathcal{G}^{\mathbb{C}}) = \Omega^0(X, \mathrm{End}(E))$. We must be careful that this identification depends on ε. Notice also that $\mathrm{Aut}(\mathrm{Gr}(\mathcal{E})) = \mathbb{C}^* G$ and its Lie algebra is $\mathfrak{g} \oplus \mathbb{C}\mathrm{Id}_E$. Similarly, $\mathrm{Aut}(\mathrm{Gr}(\mathcal{E}), h) = \mathbb{S}^1 K$ and $\mathrm{Lie}(\mathrm{Aut}(\mathrm{Gr}(\mathcal{E}), h)) = \mathfrak{k} \oplus \mathbf{i}\mathbb{R}\mathrm{Id}_E$. The reason why we restrict ourselves to the automorphisms that have a determinant equal to 1 is that $\mathbb{C}^*\mathrm{Id}_E \subset \mathrm{Aut}(\mathrm{Gr}(\mathcal{E}))$ acts trivially on connections.

Define $\Omega^D_\varepsilon : (\alpha, \beta) \mapsto \langle J\alpha, \beta \rangle_\varepsilon$ to be the symplectic form associated to the Hermitian product introduced at (1) with respect to Θ_ε on $\mathrm{End}(E)$. We will mostly be interested in the case where α and β are $(0,1)$-forms. In this case,

$$\Omega^D_\varepsilon(\alpha, \beta) = \int_X \mathrm{tr}(\alpha \wedge \beta^*) \wedge \Theta_\varepsilon.$$

The set of (not necessarily integrable) Dobleault operators is an affine space of direction $\Omega^{0,1}(X, \mathrm{End}(E))$. Thus it is in particular an infinite dimensional manifold and Ω^D_ε is formally a Kähler form on it. Moreover, by [1, 5], the gauge group acts by Hamiltonian actions on this manifold and the associated equivariant moment map is $\nu_{\infty,\varepsilon} : \overline{\partial} \mapsto \Lambda_\varepsilon F_{\overline{\partial}}$. Concretely, it means that for all $\overline{\partial}$, for all v tangent to $\overline{\partial}$ and all $a \in \mathrm{Lie}(\mathcal{G}^{\mathbb{C}}) = \Omega^0(X, \mathrm{End}(E))$,

$$\left\langle d\nu_{\infty,\varepsilon}(\overline{\partial})v, a \right\rangle_\varepsilon = \Omega^D_\varepsilon(L_{\overline{\partial}}a, v), \tag{2}$$

where $\overline{\partial} \mapsto L_{\overline{\partial}}a = \frac{\partial}{\partial t}|_{t=0} \mathbf{e}^{ta} \cdot \overline{\partial}$ is the vector field induced by the infinitesimal action of a. We use here the same notation as in [8]. We want now to reduce ourselves to the same kind of moment map, but for finite dimensional Lie groups. For this, we use first the same method as in [4, 3, 25] involving the Kuranishi slice.

Let V be the finite dimensional complex space $H^{0,1}(X, \mathrm{End}(\mathrm{Gr}(\mathcal{E})), \Theta_0)$ of harmonic $(0,1)$-forms with values in $\mathrm{End}(\mathrm{Gr}(\mathcal{E}))$ with respect to Θ_0. Let $\Phi : B \to \Omega^{0,1}(X, \mathrm{End}(\mathrm{Gr}(\mathcal{E})))$ be the Kuranishi slice [15] where $B \subset V$ is a ball around 0. It is explicitly given by $\Phi : b \mapsto b + \overline{\partial}_0^* A(b \wedge b)$ where A is the Green operator associated to $\Delta_0 = \Delta_{\Theta_0, \nabla_0}$. In particular, it is holomorphic. Recall that Φ verifies the following properties,

1. $\Phi(0) = 0$ and $d\Phi(0) : v \mapsto v$ for $v \in T_0 V = V \subset \Omega^{0,1}(X, \mathrm{End}(E))$ is injective hence, up to shrinking B, Φ is an embedding.

2. B is K-invariant and Φ is G-equivariant where it is defined in the sens that for all $b \in B$ and all $g \in G$ such that $g \cdot b \in B$, $\overline{\partial}_{g \cdot b} = g \cdot \overline{\partial}_b$ where $\overline{\partial}_b$ is defined as $\overline{\partial}_0 + \Phi(b)$.

3. For any small enough deformation $\overline{\partial}$ of $\overline{\partial}_0$, there is a gauge transformation $f \in \mathcal{G}^{\mathbb{C}}$ such that $f \cdot \overline{\partial} \in \Phi(B)$.

4. $Z = \left\{ b \in B \middle| \overline{\partial}_b^2 = 0 \right\}$ is a complex subspace of B.

Since $\mathcal{E}$ is a small holomorphic deformation of $\mathrm{Gr}(\mathcal{E})$ (see for example [3, Section 3]), we can find some $b_0 \in B$ such that $\overline{\partial}_{b_0} = \overline{\partial}_0 + \Phi(b_0)$ is gauge equivalent to $\overline{\partial}_E$. From now on, we call $\mathcal{O} = G \cdot b_0 \cap B$ the G-orbit of b_0 in B. Any $b \in \mathcal{O}$ is such that $\overline{\partial}_b$ is gauge equivalent to $\overline{\partial}_E$.

Then, we deform Φ because we are interested in Dolbeault operators $\overline{\partial}$ such that $\nu_{\infty,\varepsilon}(\overline{\partial})$ belongs to the Lie algebra $\mathfrak{k}$ of K as in [3, Proposition 3.2]. In the following, we build functions that depend on $\varepsilon \in U$. To understand accurately the smoothness of these objects with respect to ε, let us introduce for all integer $d \geqslant 1$, an L^2_d Sobolev norm $\|\cdot\|_{L^2_d}$ on $\Omega^{n-1,n-1}(X, \mathbb{C})$. For example, we can compute them with respect to a background metric on X and the resulting norm doesn't depend on the choice of this metric up to equivalence.

Proposition 4.1. *Up to shrinking U and B, there exists a map $\sigma : U \times B \to \Omega^0_0(X, \mathrm{End}_H(E, h))$ such that for all integer $d \geqslant 1$, if U is endowed with an L^2_d norm and $\Omega^0_0(X, \mathrm{End}_H(E, h))$ with an L^2_{d+2} norm, σ is smooth and we have the following properties : $\sigma(0, 0) = 0$, $\frac{\partial}{\partial b}|_{b=0}\sigma(0, b) = 0$ and if we set $\tilde{\Phi} : (\varepsilon, b) \mapsto \mathbf{e}^{\sigma(\varepsilon,b)} \cdot \overline{\partial}_b - \overline{\partial}_0 \in \Omega^{0,1}(X, \mathrm{End}(E))$, $\overline{\partial}_{\varepsilon,b} = \overline{\partial}_0 + \tilde{\Phi}(\varepsilon, b)$ and $F_{\varepsilon,b} = F_{\overline{\partial}_{\varepsilon,b}}$, we have,*

1. *For all $b \in \mathcal{O}$, $\overline{\partial}_{\varepsilon,b}$ is gauge equivalent to $\overline{\partial}_E$.*
2. *For all $\varepsilon \in U$, $\tilde{\Phi}(\varepsilon, \cdot)$ is K-equivariant.*
3. *For all $(\varepsilon, b) \in U \times B$, $\Lambda_\varepsilon F_{\varepsilon,b} + \mathbf{i}c_\varepsilon \mathrm{Id}_E \in \mathfrak{k}$.*
4. *$\frac{\partial}{\partial b}|_{b=0}\tilde{\Phi}(0, b) : v \mapsto v$. In particular, up to shrinking $U \times B$, each $\tilde{\Phi}(\varepsilon, \cdot)$ is a smooth embedding.*

Proof. Let $\Pi_\perp : \Omega^0_0(X, \mathrm{End}_H(E, h)) \to \mathbf{i}\mathfrak{k}^\perp$ be the orthogonal projection on $\mathbf{i}\mathfrak{k}^\perp$ (thus parallel to $\mathbf{i}\mathfrak{k}$) with respect to $\langle \cdot, \cdot \rangle_0$ and,

$$\Psi : \left\{ \begin{array}{ccc} U \times B \times \mathbf{i}\mathfrak{k}^\perp & \to & \mathbf{i}\mathfrak{k}^\perp \\ (\varepsilon, b, s) & \mapsto & \Pi_\perp(\Lambda_\varepsilon \mathbf{i} F_{\mathbf{e}^s \cdot \overline{\partial}_b} - c_\varepsilon \mathrm{Id}_E) \end{array} \right. .$$

By Proposition 2.3, $\frac{\partial}{\partial s}_{|s=0}\Psi(0, 0, s) = \Pi_\perp \Delta_0$ and Δ_0 is a symmetric semi-definite positive operator and it is continuous from the L^2_3 completion of $\Omega^0_0(X, \mathrm{End}_H(E, h))$ to its L^2_1 completion. In particular, Ψ is smooth when we endow the starting $\mathbf{i}\mathfrak{k}^\perp$ with an L^2_3 norm, the arrival one with an L^2_1 norm and U with an L^2_1 norm. According to Proposition 2.2, its kernel (hence its cokernel) is $\mathbf{i}\mathfrak{k}$. In particular, $\Pi_\perp \Delta_0 = \Delta_0$ and $\frac{\partial}{\partial s}_{|s=0}\Psi(0, 0, s) = \Delta_0 : L^2_3(\mathbf{i}\mathfrak{k}^\perp) \to L^2_1(\mathbf{i}\mathfrak{k}^\perp)$ is an isomorphism. With the same argument, for any integer $d \geqslant 1$, $\Psi : L^2_d(U) \times B \times L^2_{d+2}(\mathbf{i}\mathfrak{k}^\perp) \to L^2_d(\mathbf{i}\mathfrak{k}^\perp)$ is smooth. Here, $L^2_d(W)$ is the completion of a space W endowed with an L^2_d norm.

By the implicit functions theorem, up to shrinking $U \times B$, there is a unique smooth $\sigma : U \times B \to L^2_3(\mathbf{i}\mathfrak{k}^\perp)$ such that $\sigma(0, 0) = 0$ and for all $(\varepsilon, b) \in U \times B$, $\Psi(\varepsilon, b, \sigma(\varepsilon, b)) = 0$. When we set $\tilde{\Phi}(\varepsilon, b) = \mathbf{e}^{\sigma(\varepsilon,b)} \cdot \overline{\partial}_b - \overline{\partial}_0$, the first point is immediate and the third point is verified. For the second one, notice that for all $u \in K$ and all $(\varepsilon, b) \in U \times B$

$$\begin{aligned} \Psi(\varepsilon, b, u^*\sigma(\varepsilon, u \cdot b)u) &= \Pi_\perp(\Lambda_\varepsilon \mathbf{i} F_{\mathbf{e}^{u^*\sigma(\varepsilon,u\cdot b)u} \cdot \overline{\partial}_b} - c_\varepsilon \mathrm{Id}_E) \\ &= \Pi_\perp(\Lambda_\varepsilon \mathbf{i} F_{u^*\mathbf{e}^{\sigma(\varepsilon,u\cdot b)} \cdot \overline{\partial}_{u\cdot b} u} - c_\varepsilon \mathrm{Id}_E) \\ &= \Pi_\perp(u^*(\Lambda_\varepsilon \mathbf{i} F_{\mathbf{e}^{\sigma(\varepsilon,u\cdot b)} \cdot \overline{\partial}_{u\cdot b}} - c_\varepsilon \mathrm{Id}_E)u) \\ &= 0. \end{aligned}$$

By uniqueness of σ, it implies that $u^*\sigma(\varepsilon, u\cdot b)u = \sigma(\varepsilon, b)$. Therefore,

$$\overline{\partial}_{\varepsilon,u\cdot b} = \overline{\partial}_0 + \tilde{\Phi}(\varepsilon, u\cdot b) = \mathbf{e}^{\sigma(\varepsilon,u\cdot b)}\cdot\overline{\partial}_{u\cdot b} = u\mathbf{e}^{\sigma(\varepsilon,b)}u^*u\overline{\partial}_b u^* = u(\overline{\partial}_0 + \tilde{\Phi}(\varepsilon,b))u^* = u\overline{\partial}_{\varepsilon,b}u^*.$$

It proves the second point. For the fourth point, we have for all b,

$$\begin{aligned}\Psi(0,b,0) &= \Pi_\perp(\Lambda_0 F_{\overline{\partial}_b} + \mathbf{i}c_0\mathrm{Id}_E)\\ &= -\mathbf{i}\Pi_\perp(\Lambda_0\mathbf{i}F_{\overline{\partial}_0} + \nabla_0(\Phi(b)-\Phi(b)^*) + (\Phi(b)-\Phi(b)^*)\wedge(\Phi(b)-\Phi(b)^*) - c_0\mathrm{Id}_E)\\ &= -\mathbf{i}\Pi_\perp\nabla_0(\Phi(b)-\Phi(b)^*) + \mathrm{o}(b).\end{aligned}$$

Therefore, $\frac{\partial}{\partial b}_{|b=0}\Psi(0,b,0) = -\mathbf{i}\Pi_\perp\nabla_0(d\Phi(0)-d\Phi(0)^*)$. Now, recall that the image of $d\Phi(0)$ is the space V of $(0,1)$-harmonic forms. In particular, it is included in the kernel of ∇_0. Same thing for $d\Phi(0)^*$ since ∇_0 is unitary. Thus $\frac{\partial}{\partial b}_{|b=0}\Psi(0,b,0) = 0$. Therefore, when we differentiate the equality $\Psi(0,b,\sigma(0,b))$ at $b=0$,

$$0 = \frac{\partial}{\partial b}|_{b=0}\Psi(0,b,0) + \frac{\partial}{\partial s}|_{s=0}\Psi(0,0,s)\frac{\partial}{\partial b}|_{b=0}\sigma(0,b) = \Delta_0\frac{\partial}{\partial b}|_{b=0}\sigma(0,b).$$

Since $\frac{\partial}{\partial b}|_{b=0}\sigma(0,b)$ belongs to $\mathbf{i}\mathfrak{k}^\perp$ where Δ_0 is injective, we have $\frac{\partial}{\partial b}|_{b=0}\sigma(0,b) = 0$ hence,

$$\tilde{\Phi}(0,b) - \overline{\partial}_0 = \mathbf{e}^{\sigma(0,b)}\cdot\overline{\partial}_b - \overline{\partial}_0 = \mathbf{e}^{\sigma(0,b)}\overline{\partial}_0(\mathbf{e}^{-\sigma(0,b)}) + \mathbf{e}^{\sigma(0,b)}\Phi(b)\mathbf{e}^{-\sigma(0,b)} = \Phi(b) + \mathrm{o}(b).$$

We deduce the fourth point and we showed along the way that $\frac{\partial}{\partial b}|_{b=0}\sigma(0,b) = 0$. All we have left to show is that the $\sigma(\varepsilon,b)$ are smooth as sections and that σ has the wanted regularity. The first part follows from elliptic regularity. For the second one let $d \geqslant 1$ be an integer. Then, $\Psi : L^2_d(U)\times B\times L^2_{d+2}(\mathbf{i}\mathfrak{k}^\perp) \to L^2_d(\mathbf{i}\mathfrak{k}^\perp)$ is smooth. When we differentiate m times the equality $\Psi(\varepsilon,b,\sigma(\varepsilon,b)) = 0$, we can, by the Faà di Bruno formula, express $\frac{\partial}{\partial s}\Psi(\varepsilon,b,s)d^m\sigma(\varepsilon,b)$ as a polynomial expression of the successive derivatives of Ψ and the lower order derivatives of σ, which are all smooth by induction on m. $\frac{\partial}{\partial s}|_{s=0}\Psi(0,0,s) = \Delta_0 : L^2_{d+2}(\mathbf{i}\mathfrak{k}^\perp) \to L^2_d(\mathbf{i}\mathfrak{k}^\perp)$ is an isomorphism and this condition is open, hence $\sigma : L^2_d(U)\times B \to L_{d+2}(\mathbf{i}\mathfrak{k}^\perp)$ is smooth. □

Following [3, Section 3.2], we define for all $\varepsilon \in U$, the closed 2-form $\Omega_\varepsilon = \tilde{\Phi}(\varepsilon,\cdot)^*\Omega^D_\varepsilon$ on B and,

$$\nu_\varepsilon : \left\{\begin{array}{ccl} B & \to & \mathfrak{k}\\ b & \mapsto & \nu_{\infty,\varepsilon}(\overline{\partial}_{\varepsilon,b}) = \Lambda_\varepsilon F_{\varepsilon,b} + \mathbf{i}c_\varepsilon\mathrm{Id}_E\end{array}\right. .$$

By Proposition 4.1, this function indeed takes values in $\mathfrak{k}$. It is easy to compute that since $\nu_{\infty,\varepsilon}$ is a K-equivariant moment map for Ω^D_ε and $\tilde{\Phi}(\varepsilon,\cdot)$ is a K-equivariant embedding, ν_ε is a K-equivariant moment map for Ω_ε, which is a simplectic form. We must be careful about the fact that Ω_ε is not compatible with the complex structures because $\tilde{\Phi}(\varepsilon,\cdot)$ is *a priori* not holomorphic, but we still have the following positivity condition.

Lemma 4.2. *Up to shrinking U and B, for all $b \in B$ and $v \in T_bB\backslash\{0\}$, $\Omega_\varepsilon(v,\mathbf{i}v)$ has a positive real part.*

Proof. This is due to the fact that $d\tilde{\Phi}(0,0) : v \mapsto v$ is $\mathbb{C}$-linear, hence, for all $v \in T_0B = V$,

$$\Omega_0(v,\mathbf{i}v) = \Omega^D_0(d\tilde{\Phi}(0,0)v, d\tilde{\Phi}(0,0)(\mathbf{i}v)) = \Omega^D_0(v,\mathbf{i}v) > 0.$$

It means that $\Omega_0(\cdot,\mathbf{i}\cdot)$ at $b=0$ is a positive quadratic form. By openness of positivity, up to shrinking U and B, $\Re\Omega_\varepsilon$ is positive at any point b of B, hence the result. □

4.2 Bound on the norm of b

Now, when Θ_ε is in the stable cone of $\mathcal{E}$, we want to find a Θ_ε-HYM connection in the image of $\overline{\partial}_0 + \tilde{\Phi}(\varepsilon, \cdot)$ restricted to the orbit $\mathcal{O}$, because if $\overline{\partial} = \overline{\partial}_{\varepsilon,b} = \overline{\partial}_0 + \tilde{\Phi}(\varepsilon, b)$, then all the Sobolev norms of $\overline{\partial} - \overline{\partial}_0$ can be controlled by the norm of b for any Euclidean norm (see the later Proposition 4.3). The first thing to do is the following estimate on the norm of b in term of the norm of $\nu_\varepsilon(b)$. We denote by $\|\cdot\|$ Euclidean norms on the finite dimensional vector spaces V and $H^{n-1,n-1}(X,\mathbb{R})$.

Proposition 4.3. *Up to shrinking U and B, for all $\varepsilon \in U$ and $b \in \overline{\mathcal{O}}$, $\|b\|^2 \leqslant C(\|\nu_\varepsilon(b)\|_\varepsilon + \|\varepsilon\|^2_{L^2_1} + \|[\varepsilon]\|)$ for some positive constant C independent from ε and b.*

Proof. By density, it is enough to verify it for $b \in \mathcal{O}$.

Step 1 : Finding an orthogonal decomposition of E where $\Phi(b)$ is strictly upper triangular.

If $b \in \mathcal{O}$, $\overline{\partial}_b$ is gauge equivalent to $\overline{\partial}_E$ thus the holomorphic vector bundle $\mathcal{E}_b = (E, \overline{\partial}_b)$ admits a Jordan-Hölder filtration,

$$0 \subsetneq \mathcal{F}_{1,b} \subsetneq \cdots \subsetneq \mathcal{F}_{m,b} = \mathcal{E}_b,$$

For all k, the smooth structure of $\mathcal{F}_{k,b}$ is F_k and the smooth structure of $\mathcal{G}_{k,b} = \mathcal{F}_{k,b}/\mathcal{F}_{k-1,b}$ is G_k. The $(0,1)$-form $\Phi(b)$ can be decomposed as $\Phi(b) = \sum_{i,j=1}^m \gamma_{ij}$ with $\gamma_{ij} \in \Omega^{0,1}(X, \mathrm{Hom}(G_j, G_i))$. Since each $\mathcal{F}_{k,b}$ is a sub-bundle of $\mathcal{E}_b$ (isomorphic to $\mathcal{F}_k$), $\overline{\partial}_b$ preserves it, meaning that $\gamma_{ij} = 0$ whenever $i > j$. Therefore, $\Phi(b) = \sum_{i \leqslant j} \gamma_{ij}$ is upper triangular.

Now, let for all $t \geqslant 1$, $g_t = \sum_{k=1}^m a_t t^k \mathrm{Id}_{G_k}$ where $a_t \in \mathbb{R}_+^*$ is chosen such that g_t has determinant 1. In particular, $g_t \in G$ and for all t, when $g_t \cdot b \in B$, $g_t \cdot \overline{\partial}_0 = \overline{\partial}_0$ so,

$$\overline{\partial}_{g_t \cdot b} = g_t \cdot \overline{\partial}_b = \overline{\partial}_0 + g_t \Phi(b) g_t^{-1} = \overline{\partial}_0 + \sum_{i \leqslant j} t^{i-j} \gamma_{ij}.$$

Moreover, the decomposition $E = \bigoplus_{k=1}^m G_k$ is orthogonal, which implies that the decomposition $\Omega^{0,1}(X, \mathrm{End}(E)) = \bigoplus_{i,j=1}^m \Omega^{0,1}(X, \mathrm{Hom}(G_j, G_i))$ is orthogonal too for any $\langle \cdot, \cdot \rangle_\varepsilon$. In particular,

$$\|\Phi(g_t \cdot b)\|^2_\varepsilon = \sum_{i \leqslant j} t^{2i-2j} \|\gamma_{ij}\|^2_\varepsilon \leqslant \|\Phi(b)\|^2_\varepsilon,$$

because $t \geqslant 1$. Since Φ is a smooth embedding, there are, up to shrinking B, positive constants C_1, C_2 such that for all $b' \in B$, $\frac{1}{C_1}\|b\| \leqslant \|\Phi(b)\|_0 \leqslant C_2 \|b\|$. Let $r > 0$ such that the closed ball of centre 0 and radius r is included in B. Let B' be the open ball of centre 0 and radius $\frac{r}{C_1 C_2}$ and assume that $b \in B'$. Let $T = \sup\{t \geqslant 1 | g_t \cdot b \in B\}$. $g_1 = \mathrm{Id}_E$ so $T \geqslant 1$ is well defined. If T is finite, we have, for all $t < T$, $\|g_t \cdot b\| \leqslant C_1 \|\Phi(g_t \cdot b)\|^2_0 \leqslant C_1 \|\Phi(b)\|^2_0 \leqslant C_1 C_2 \|b\| \leqslant r$. We stay in the closed ball of radius r, included in B. When $t \to T$, we reach a point of B and we contradict the maximality of T by openness of B. It proves that $T = +\infty$ and,

$$\lim_{t \to +\infty} \Phi(g_t \cdot b) = \sum_{k=1}^m \gamma_{kk}.$$

This convergence implies that $g_t \cdot b \to b_\infty$ for some b_∞ in the closed ball of radius r because Φ is a closed embedding. In particular, $b_\infty \in B$. Notice that for all k, $(G_k, \overline{\partial}_{b_\infty|_{G_k}}) = (G_k, \overline{\partial}_0 + \gamma_{kk}) = \mathcal{G}_{k,b} \cong \mathcal{G}_k$ is $[\Theta_0]$-stable. It means that $\mathcal{E}' = (E, \overline{\partial}_{b_\infty})$ is $[\Theta_0]$-polystable and by [2, Theorem 1, 3, 4][1], $\overline{\partial}_{b_\infty}$ is in the G-orbit of $\overline{\partial}_0$ *i.e.* it equals $\overline{\partial}_0$. It means that for all k, $\gamma_{kk}=0$. Up to replacing B by B', it means that $\Phi(b)=\sum_{i<j}\gamma_{ij}$ is strictly upper diagonal in the orthogonal decomposition $E=\bigoplus_{k=1}^m G_k$ whenever $b\in B$.

Step 2 : Computing the scalar product between $\nu_{\infty,\varepsilon}(b)$ and each $a_S=\frac{\mathbf{i}}{\mathrm{rk}(S)}\mathrm{Id}_S-\frac{\mathbf{i}}{\mathrm{rk}(S^\perp)}\mathrm{Id}_{S^\perp}\in\mathfrak{k}$.

Let $\mathcal{S}\subset\mathcal{E}_b$ be a sub-bundle with $0<\mathrm{rk}(\mathcal{S})<\mathrm{rk}(\mathcal{E})$ and S its smooth structure. Then, in the decomposition $E=S\oplus S^\perp$, we can write $\overline{\partial}_b=\partial_{b|S}+\beta_{S,b}$ and by [13, Equation (1.6.12)],

$$\mathbf{i}F_b=\begin{pmatrix}\mathbf{i}F_S-\mathbf{i}\beta_{S,b}\wedge\beta_{S,b}^* & \mathbf{i}\partial_{\varepsilon,b}\beta_{S,b}\\ -\mathbf{i}\overline{\partial}_b\beta_{S,b}^* & \mathbf{i}F_{S^\perp}-\mathbf{i}\beta_{S,b}^*\wedge\beta_{S,b}\end{pmatrix},$$

where F_b is the curvature of the Chern connection associated to $\overline{\partial}_b$ on E, F_S is some curvature form on $\mathcal{S}$ and $F_{S^\perp}$ is some curvature form on $\mathcal{E}_b/\mathcal{S}$ (they depend on b). By Chern-Weil theory,

$$\int_X\mathrm{tr}(\mathbf{i}F_S)\wedge\Theta_\varepsilon=2\pi\mathrm{rk}(S)\mu_\varepsilon(S),\qquad\int_X\mathrm{tr}(\mathbf{i}F_{S^\perp})\wedge\Theta_\varepsilon=2\pi\mathrm{rk}(E/S)\mu_\varepsilon(E/S),$$

and for any $(0,1)$-form β, $\int_X\mathrm{tr}(\mathbf{i}\beta^*\wedge\beta)\wedge\Theta_\varepsilon=\|\beta\|_\varepsilon^2$ and $\int_X\mathrm{tr}(\mathbf{i}\beta\wedge\beta^*)\wedge\Theta_\varepsilon=-\|\beta\|_\varepsilon^2$. We can now compute that,

$$\begin{aligned}\left\langle\nu_{\infty,\varepsilon}(\overline{\partial}_b),a_S\right\rangle_\varepsilon&=\langle\Lambda_\varepsilon F_b+\mathbf{i}c_\varepsilon\mathrm{Id}_E,a_S\rangle_\varepsilon\\&=\int_X\mathrm{tr}\left(\Lambda_\varepsilon\mathbf{i}F_b\left(\frac{1}{\mathrm{rk}(S)}\mathrm{Id}_S-\frac{1}{\mathrm{rk}(S^\perp)}\mathrm{Id}_{S^\perp}\right)\right)\mathrm{Vol}_\varepsilon\\&=\frac{1}{\mathrm{rk}(S)}\int_X\mathrm{tr}(\mathbf{i}F_b\mathrm{Id}_S)\wedge\Theta_\varepsilon-\frac{1}{\mathrm{rk}(S^\perp)}\int_X\mathrm{tr}(\mathbf{i}F_b\mathrm{Id}_{S^\perp})\wedge\Theta_\varepsilon\\&=\frac{1}{\mathrm{rk}(S)}\int_X\mathrm{tr}(\mathbf{i}F_S-\mathbf{i}\beta_{S,b}\wedge\beta_{S,b}^*)\wedge\Theta_\varepsilon-\frac{1}{\mathrm{rk}(S^\perp)}\int_X\mathrm{tr}(\mathbf{i}F_{S^\perp}-\mathbf{i}\beta_{S,b}^*\wedge\beta_{S,b}\mathrm{Id}_{S^\perp})\wedge\Theta_\varepsilon\\&=2\pi\mu_\varepsilon(S)+\frac{1}{\mathrm{rk}(S)}\|\beta_{S,b}\|_\varepsilon^2-2\pi\mu_\varepsilon(E/S)+\frac{1}{\mathrm{rk}(S^\perp)}\|\beta_{S,b}\|_\varepsilon^2\\&=-2\pi l_{\mathcal{S}}(\varepsilon)+\left(\frac{1}{\mathrm{rk}(S)}+\frac{1}{\mathrm{rk}(S^\perp)}\right)\|\beta_{S,b}\|_\varepsilon^2.\end{aligned}$$

Step 3 : Inequality with $\nu_{\infty,\varepsilon}$.

Applying step 2 with the particular case $\mathcal{S}=\mathcal{F}_{k,b}\subset\mathcal{E}$ with $k<m$, we obtain $\beta_{S,b}=$

[1] In this paper, the results are proven in the Kähler case. However, in the third point of their conclusion and in the remark following [2, Proposition 3.2], the authors notice that the only obstruction for it to work on any compact complex manifold equipped with a Gauduchon metric is that in this case, the slope might not be of topological nature. In the balanced case, the slope is of topological nature since it only depends on the first Chern class of the bundle and not on the choice of its representative.

$\mathrm{Id}_{F_{k,b}}\overline{\partial}_{b|F_{k,b}^{\perp}} = \sum_{i\leqslant k\leqslant j}\gamma_{ij}$. Moreover, all the $l_{\mathcal{S}} : H^{n-1,n-1}(X,\mathbb{R}) \to \mathbb{R}$ are continuous thus,

$$\begin{aligned}
\sum_{k=1}^{m}\left\langle \nu_{\infty,\varepsilon}(\overline{\partial}_b), a_{F_{k,b}}\right\rangle_{\varepsilon} &= \sum_{k=1}^{m} -2\pi l_{F_{k,b}}(\varepsilon) + \left(\frac{1}{\mathrm{rk}(F_{k,b})}+\frac{1}{\mathrm{rk}(F_{k,b}^{\perp})}\right)\left\|\mathrm{Id}_{F_{k,b}}\Phi(b)_{|F_{k,b}^{\perp}}\right\|_{\varepsilon}^{2} \\
&\geqslant \sum_{k=1}^{m} C_3 \sum_{1\leqslant i\leqslant k\leqslant j\leqslant m}\|\gamma_{ij}\|_0^2 - C_4\|[\varepsilon]\| \\
&\geqslant C_3\|\Phi(b)\|_0^2 - C_4\|[\varepsilon]\| \\
&\geqslant \frac{C_3}{C_1}\|b\|^2 - C_4\|[\varepsilon]\|,
\end{aligned}$$

where C_3 and C_4 are positive constants that only depend on $\mathrm{rk}(\mathcal{E})$, $\max_{1\leqslant k\leqslant m}\left\{\left\|l_{\mathcal{F}_{k,b}}\right\|\right\}$, $\max_{\varepsilon\in U}\{\mathrm{Vol}_{\varepsilon}(X)\}$ (which is finite up to shrinking U) and the fact that the $\|\cdot\|_{\varepsilon}$ are uniformly equivalent. Clearly, it exists a constant C_5 independent from ε and b such that for all k and ε, $\left\|a_{F_{k,b}}\right\|_{\varepsilon}\leqslant C_5$. Let for all ε, Π_{ε} be the orthogonal projection on $\mathfrak{k}$ with respect to $\langle\cdot,\cdot\rangle_{\varepsilon}$. We have,

$$\sum_{k=1}^{m}\left\langle \nu_{\infty,\varepsilon}(\overline{\partial}_b), a_{F_{k,b}}\right\rangle_{\varepsilon} = \sum_{k=1}^{m}\left\langle \Pi_{\varepsilon}\nu_{\infty,\varepsilon}(\overline{\partial}_b), a_{F_{k,b}}\right\rangle_{\varepsilon} \leqslant C_5\sum_{k=1}^{m}\left\|\Pi_{\varepsilon}\nu_{\infty,\varepsilon}(\overline{\partial}_b)\right\|_{\varepsilon} = mC_5\left\|\Pi_{\varepsilon}\nu_{\infty,\varepsilon}(\overline{\partial}_b)\right\|.$$

We deduce that $\|b\|^2 = \mathrm{O}\left(\left\|\Pi_{\varepsilon}\nu_{\infty,\varepsilon}(\overline{\partial}_b)\right\|_{\varepsilon} + \|[\varepsilon]\|\right)$ where the O is independent from ε and b. From now on, all the asymptotic developments are with respect to $(\varepsilon,b)\to(0,0)$ with respect to the L_1^2 Sobolev norm for ε.

Step 4 : Conclusion.

By Proposition 4.1, we have $\nu_{\varepsilon}(b) = \nu_{\infty,\varepsilon}(\overline{\partial}_{\varepsilon,b})$ with $\overline{\partial}_{\varepsilon,b} = \mathbf{e}^{\sigma(\varepsilon,b)}\cdot\overline{\partial}_b$ for some smooth $\sigma : L_1^2(U)\times B \to L_3^2(\Omega_0^0(X,\mathrm{End}_H(E,h)))$ which verifies $\sigma(0,0)=0$ and $\frac{\partial}{\partial b}|_{b=0}\sigma(0,b)=0$. Therefore $\|\sigma(\varepsilon,b)\|_{L_3^2} = \mathrm{O}(\|\varepsilon\|_{L_1^2}+\|b\|^2)$ and using the fact that ν_{ε} takes values in $\mathfrak{k}$,

$$\begin{aligned}
\left\|\nu_{\varepsilon}(b) - \Pi_{\varepsilon}\nu_{\infty,\varepsilon}(\overline{\partial}_b)\right\|_{\varepsilon} &= \left\|\Pi_{\varepsilon}(\nu_{\varepsilon}(b) - \nu_{\infty,\varepsilon}(\overline{\partial}_b))\right\|_{\varepsilon} \\
&= \left\|\Pi_{\varepsilon}(\nu_{\infty,\varepsilon}(\overline{\partial}_{\varepsilon,b}) - \nu_{\infty,\varepsilon}(\overline{\partial}_b))\right\|_{\varepsilon} \\
&= \left\|\Pi_{\varepsilon}(\nu_{\infty,\varepsilon}(\mathbf{e}^{\sigma(\varepsilon,b)}\cdot\overline{\partial}_b) - \nu_{\infty,\varepsilon}(\overline{\partial}_b))\right\|_{\varepsilon} \\
&= \left\|\Pi_{\varepsilon}\Delta_{\varepsilon,\overline{\partial}_b}\sigma(\varepsilon,b)\right\|_{\varepsilon} + \mathrm{O}(\|\sigma(\varepsilon,b)\|_{L_3^2}^2) \text{ by Proposition 2.3,} \\
&= \|\Pi_0\Delta_0\sigma(\varepsilon,b)\|_{\varepsilon} + \mathrm{O}((\|\varepsilon\|_{L_1^2}+\|b\|)\|\sigma(\varepsilon,b)\|_{L_3^2}^2) + \mathrm{O}(\|\sigma(\varepsilon,b)\|_{L_3^2}^2) \\
&= \mathrm{O}(\|\varepsilon\|_{L_1^2}^2 + \|b\|^3).
\end{aligned}$$

The last equality holds because Δ_0 takes values in $\mathfrak{k}^{\perp}$ (for $\langle\cdot,\cdot\rangle_0$) hence $\Pi_0\Delta_0 = 0$. Finally,

$$\|b\|^2 = \mathrm{O}\left(\left\|\Pi_{\varepsilon}\nu_{\infty,\varepsilon}(\overline{\partial}_b)\right\|_{\varepsilon} + \|[\varepsilon]\|\right) = \mathrm{O}(\|\nu_{\varepsilon}(b)\|_{\varepsilon} + \|\varepsilon\|_{L_1^2}^2 + \|b\|^3 + \|[\varepsilon]\|),$$

so $\|b\|^2 = \mathrm{O}(\|\nu_{\varepsilon}(b)\|_{\varepsilon} + \|\varepsilon\|_{L_1^2}^2 + \|[\varepsilon]\|)$, which is the wanted bound. $\square$

4.3 Finding a zero of the moment map

The method now consists in defining a vector field whose flow converges toward a zero of the moment map. When $b \in B$ and $a \in \mathfrak{g}$, we denote by $L_b a = \frac{\partial}{\partial t}|_{t=0} \mathbf{e}^{ta} \cdot b \in T_b B$ the infinitesimal action of a on b. Following [8, Chapter 3], we define the vector field,

$$V_\varepsilon : b \mapsto -\mathbf{i} L_b \nu_\varepsilon(b).$$

Let $t \mapsto b_\varepsilon(t)$ be the flow on V_ε whose starting point will be defined by the following proposition.

Proposition 4.4. *Up to shrinking U, for all $\varepsilon \in U$, if we choose $b_\varepsilon(0)$ close enough to 0, b_ε is defined for all non-negative times, $t \mapsto \|\nu_\varepsilon(b_\varepsilon(t))\|_\varepsilon$ decreases and $\|b_\varepsilon(t)\| = \mathrm{O}\left(\sqrt{\|\varepsilon\|_{L_1^2}}\right)$ uniformly with respect to t.*

Proof. For all t where the flow is defined, whatever is the choice of the starting point,

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\|\nu_\varepsilon(b_\varepsilon(t))\|_\varepsilon^2\right) &= \langle d\nu_\varepsilon(b_\varepsilon(t))(-\mathbf{i}L_b\nu_\varepsilon(b_\varepsilon(t))), \nu_\varepsilon(b_\varepsilon(t))\rangle_\varepsilon \\ &= -\Omega_\varepsilon(L_b\nu_\varepsilon(b_\varepsilon(t)), \mathbf{i}L_b\nu_\varepsilon(b_\varepsilon(t))) \text{ because } \nu_\varepsilon \text{ is a moment map,} \\ &\leqslant 0 \text{ by Lemma 4.2.}\end{aligned}$$

It proves that $t \mapsto \|\nu_\varepsilon(b_\varepsilon(t))\|_\varepsilon$ decreases while the flow is defined. In particular, by Proposition 4.3,

$$\|b_\varepsilon(t)\|^2 \leqslant C(\|\nu_\varepsilon(b_\varepsilon(t))\|_\varepsilon + \|\varepsilon\|_{L_1^2}^2 + \|[\varepsilon]\|) \leqslant C(\|\nu_\varepsilon(b_\varepsilon(0))\|_\varepsilon + \|\varepsilon\|_{L_1^2}^2 + \|[\varepsilon]\|) \tag{3}$$

Let $r > 0$ such that the closed ball of centre 0 and radius r is included in B. Assume, up to shrinking U, that for all ε, $\|\varepsilon\|_{L_1^2}^2 + \|[\varepsilon]\| \leqslant \frac{r^2}{2C}$ and $\|\nu_\varepsilon(0)\|_\varepsilon \leqslant \frac{r^2}{4C}$. Then choose $b_\varepsilon(0)$ close enough to 0 so $\|\nu_\varepsilon(b_\varepsilon(0))\|_\varepsilon \leqslant 2\|\nu_\varepsilon(0)\|_\varepsilon \leqslant \frac{r^2}{2C}$. It implies by (3) that $\|b_\varepsilon(t)\| \leqslant r$ hence b_ε stays in a compact set included in B. By the finite time explosion theorem, the flow is defined at all time. Moreover,

$$\|b_\varepsilon(t)\|^2 \leqslant C(\|\nu_\varepsilon(b_\varepsilon(0))\|_\varepsilon + \|\varepsilon\|_{L_1^2}^2 + \|[\varepsilon]\|) \leqslant C(2\|\nu_\varepsilon(0)\|_\varepsilon + \|\varepsilon\|_{L_1^2}^2 + \|[\varepsilon]\|) = \mathrm{O}(\|\varepsilon\|_{L_1^2}).$$

□

Let $(g_\varepsilon(t))$ in G be the smooth function defined by,

$$g_\varepsilon(0) = \mathrm{Id}_E, \qquad g_\varepsilon(t)^{-1} g'_\varepsilon(t) = \mathbf{i}\nu_\varepsilon(b_\varepsilon(t)).$$

Similarly to [8, Lemma 3.2], we have for all t,

$$\frac{\partial}{\partial t}(g_\varepsilon(t)^{-1} \cdot b_\varepsilon(0)) = -g_\varepsilon(t)^{-1} g'_\varepsilon(t) g_\varepsilon^{-1}(t) \cdot b_\varepsilon(0) = -\mathbf{i} L_{g_\varepsilon(t)^{-1} \cdot b_\varepsilon(0)} \nu_\varepsilon(b_\varepsilon(t)).$$

By uniqueness of the flow, it proves that for all t, $g_\varepsilon(t)^{-1} \cdot b_\varepsilon(0) = b_\varepsilon(t)$. In particular, $\mathcal{O}$ is preserved by the flow. Let for all t, $\tilde{g}_\varepsilon(t) = \mathbf{e}^{\sigma(\varepsilon, b_\varepsilon(0))} g_\varepsilon(t) \mathbf{e}^{-\sigma(\varepsilon, b_\varepsilon(t))} \in \mathcal{G}^{\mathbb{C}}$. We have,

$$\begin{aligned}\overline{\partial}_{\varepsilon, b_\varepsilon(t)} &= \mathbf{e}^{\sigma(\varepsilon, b_\varepsilon(t))} \cdot \overline{\partial}_{b_\varepsilon(t)} \\ &= \mathbf{e}^{\sigma(\varepsilon, b_\varepsilon(t))} g_\varepsilon(t)^{-1} \cdot \overline{\partial}_{b_\varepsilon(0)} \\ &= \mathbf{e}^{\sigma(\varepsilon, b_\varepsilon(t))} g_\varepsilon(t)^{-1} \mathbf{e}^{-\sigma(\varepsilon, b_\varepsilon(0))} \cdot \overline{\partial}_{\varepsilon, b_\varepsilon(0)} \\ &= \tilde{g}_\varepsilon(t)^{-1} \cdot \overline{\partial}_{\varepsilon, b_\varepsilon(0)}.\end{aligned}$$

Now, when $\overline{\partial}$ is a Dolbeault operator on E, we call $M_{\varepsilon,\overline{\partial}}$ the Donaldson functional [5, Proposition 6] associated with Θ_ε and $\overline{\partial}$. It takes two Hermitian metrics on E as argument and is characterised by,

$$\forall k_1, k_2, k_3, M_{\varepsilon,\overline{\partial}}(k_1, k_2) + M_{\varepsilon,\overline{\partial}}(k_2, k_3) = M_{\varepsilon,\overline{\partial}}(k_1, k_3), \tag{4}$$

$$\frac{\partial}{\partial s}|_{s=0} M_{\varepsilon,\overline{\partial}}(k, \mathbf{e}^{-s/2} \cdot k) = \int_X \operatorname{tr}\left(v(\Lambda_\varepsilon \mathbf{i} F_{\overline{\partial},k} - c_\varepsilon \mathrm{Id}_E)\right) \mathrm{Vol}_\varepsilon. \tag{5}$$

Here, $s \in \Omega^0(X, \mathrm{End}_H(E,k))$ and $v \in T_s\Omega^0(X, \mathrm{End}_H(E,k))$ are k-Hermitian, $F_{\overline{\partial},k}$ is the curvature associated to the Chern connection of $(\overline{\partial}, k)$ and $\mathcal{G}^{\mathbb{C}}$ acts on metrics by $(f \cdot k)(\xi, \eta) = k(f^{-1}\xi, f^{-1}\eta)$[2]. Ulhenbeck and Yau showed that on a stable bundle, $M(h, \mathbf{e}^{-s/2} \cdot h)$ uniformly bounds s [27]. We refer the reader to Simpson's proof of this result [23, Proposition 5.3][3]. Concretely, when $(E, \overline{\partial})$ is $[\Theta_\varepsilon]$-stable, there exists positive constants $C_{1,\varepsilon}, C_{2,\varepsilon}$ that depend on ε (and $\overline{\partial}$) such that for all $s \in \Omega^0_0(X, \mathrm{End}_H(E,h))$,

$$\|s\|_\varepsilon \leqslant C_{1,\varepsilon} + C_{2,\varepsilon} M_{\varepsilon,\overline{\partial}}(h, \mathbf{e}^{-s/2} \cdot h). \tag{6}$$

We could even get this inequality with a $\mathcal{C}^0$ norm instead of an L^2 norm on s. Now, let,

$$\varphi_\varepsilon : b \mapsto M_{\varepsilon,\overline{\partial}_E}(h, f_{\varepsilon,b} \cdot h),$$

where $f_{\varepsilon,b}$ is defined as the gauge transformation up to constant homothety such that $\overline{\partial}_{\varepsilon,b} = f_{\varepsilon,b}^{-1} \cdot \overline{\partial}_E$. It is unique by simplicity of $\mathcal{E}$. We want now to show that φ decreases with the flow and use the inequality (6) to show that g_ε is bounded, thus converges up to extraction.

In the classical case where we consider the heat equation flow, the derivative of $t \mapsto \varphi(b_\varepsilon(t))$ is given by $-2\|\nu_\varepsilon(b_\varepsilon(t))\|_\varepsilon^2$ (the constant 2 may disapear in function of the conventions). See for reference [13, Proposition 6.9.1], [23, Lemma 7.1], [5, Section 1.2]. It is a natural consequence of (5). Here, the fact that $\tilde{g}_\varepsilon(t)$ is not exactly $g_\varepsilon(t)$ makes this equality false in general. However, since these two gauge transformations are close when $\varepsilon \to 0$ uniformly with respect to t, we are still able to show the wanted decrease. The issue is that we fix the metric and we make the complex structure vary to find a HYM connection. However, Donaldson's functional works better with variations of the metric when the holomorphic structure is fixed. This issue however, is only technical and we can move from one to the other. The proof of the next proposition is the only part of this article where we need to consider an other metric on $\mathcal{E}$.

Proposition 4.5. *We have for all t,*

$$\frac{\partial}{\partial t}\varphi_\varepsilon(b_\varepsilon(t)) = -2\|\nu_\varepsilon(b_\varepsilon(t))\|_\varepsilon^2 + \mathrm{o}(\|\nu_\varepsilon(b_\varepsilon(t))\|_\varepsilon^2),$$

where the o is when $\varepsilon \to 0$ and is uniform with respect to t. In particular, up to shrinking U, for all ε, $t \mapsto \varphi_\varepsilon(b_\varepsilon(t))$ decreases.

[2]In literature, it is more common to only consider the action of a positive definite Hermitian $\mathbf{e}^s$ with respect to the metric k. In this case, the notation used is $k\mathbf{e}^s : (\xi, \eta) \mapsto k(\mathbf{e}^s\xi, \eta)$. By symmetry, we obtain that $\mathbf{e}^{-s/2} \cdot k = k\mathbf{e}^s$. To avoid confusion, we only the use in this case the notation $\mathbf{e}^{-s/2} \cdot k$.

[3]Simpson proves it in the Kähler case but the closedness of ω is not used in the proof of [23, Proposition 5.3]. Only the closedness of ω^{n-1} is necessary for [23, Proposition 5.1]. He also only proves it when $\mathrm{tr}(s) = 0$ identically, but this hypothesis is only useful to get that the u_∞ built in [23, Lemma 5.4] is not a homothety (see [23, Lemma 5.5]). For this, it is enough to know that $\int_X \mathrm{tr}(s)\mathrm{Vol}_0 = 0$ so the inequality holds in this more general case.

Proof.
Step 1 : Derivative of the Hermitian part of a gauge transformation.

φ_ε is defined intrinsically in the sens that it doesn't depend on the choice of the starting point $b_\varepsilon(0)$. Since we have moreover a uniform bound on the norm of $b_\varepsilon(t)$ given by Proposition 4.4, it is enough to verify the wanted equality at $t=0$ and the uniform bound follows.

For all t near 0, we have $\overline{\partial}_{\varepsilon,b_\varepsilon(t)} = \tilde{g}_\varepsilon(t)^{-1}\cdot\overline{\partial}_{\varepsilon,b_\varepsilon(0)} = \tilde{g}_\varepsilon(t)^{-1}f^{-1}_{\varepsilon,b_\varepsilon(0)}\cdot\overline{\partial}_E$ hence $f_{\varepsilon,b_\varepsilon(t)} = f_{\varepsilon,b_\varepsilon(0)}\tilde{g}_\varepsilon(t)$. In particular, if we set $k = f_{\varepsilon,b_\varepsilon(0)}\cdot h$, we have,

$$f_{\varepsilon,b_\varepsilon(t)}\cdot h = f_{\varepsilon,b_\varepsilon(0)}\tilde{g}_\varepsilon(t)f^{-1}_{\varepsilon,b_\varepsilon(0)}\cdot k.$$

Let us write the polar decomposition of $f_{\varepsilon,b_\varepsilon(0)}\tilde{g}_\varepsilon(t)f^{-1}_{\varepsilon,b_\varepsilon(0)}$ with respect to k as $\mathbf{e}^{-s(t)/2}u(t)$. In particular,

$$\mathbf{e}^{-s(t)} = f_{\varepsilon,b_\varepsilon(0)}\tilde{g}_\varepsilon(t)f^{-1}_{\varepsilon,b_\varepsilon(0)}(f_{\varepsilon,b_\varepsilon(0)}\tilde{g}_\varepsilon(t)f^{-1}_{\varepsilon,b_\varepsilon(0)})^\dagger,$$

where $\dagger$ is the adjoint with respect to k. s is smooth with respect to any Sobolev norm and we have $s(0)=0$ because $\tilde{g}_\varepsilon(0)=0$ so,

$$\begin{aligned}
-s(t) &= \mathbf{e}^{-s(t)} - \mathrm{Id}_E + \mathrm{o}(t)\\
&= f_{\varepsilon,b_\varepsilon(0)}\tilde{g}_\varepsilon(t)f^{-1}_{\varepsilon,b_\varepsilon(0)}f^{-1\dagger}_{\varepsilon,b_\varepsilon(0)}\tilde{g}_\varepsilon(t)^\dagger f^\dagger_{\varepsilon,b_\varepsilon(0)} - \mathrm{Id}_E + \mathrm{o}(t)\\
&= (f_{\varepsilon,b_\varepsilon(0)}\tilde{g}'_\varepsilon(0)f^{-1}_{\varepsilon,b_\varepsilon(0)} + f^{-1\dagger}_{\varepsilon,b_\varepsilon(0)}\tilde{g}'_\varepsilon(0)^\dagger f^\dagger_{\varepsilon,b_\varepsilon(0)})t + \mathrm{o}(t)\\
&= 2\Re_k(f_{\varepsilon,b_\varepsilon(0)}\tilde{g}'_\varepsilon(0)f^{-1}_{\varepsilon,b_\varepsilon(0)})t + \mathrm{o}(t),
\end{aligned}$$

where $\Re_k$ is the Hermitian part with respect to k. We deduce that $s'(0) = -2\Re_k(f_{\varepsilon,b_\varepsilon(0)}\tilde{g}'_\varepsilon(0)f^{-1}_{\varepsilon,b_\varepsilon(0)})$. Now, recall that for all t, $\tilde{g}_\varepsilon(t) = \mathbf{e}^{\sigma(\varepsilon,b_\varepsilon(0))}g_\varepsilon(t)\mathbf{e}^{-\sigma(\varepsilon,b_\varepsilon(t))}$. Therefore, since $\|b_\varepsilon(0)\| = \mathrm{o}(1)$ when $\varepsilon\to 0$,

$$\begin{aligned}
\tilde{g}'_\varepsilon(0) &= d\exp(\sigma(\varepsilon,b_\varepsilon(0)))\frac{\partial}{\partial b}|_{b=b_\varepsilon(0)}\sigma(\varepsilon,b)b'_\varepsilon(0) + \mathbf{e}^{\sigma(\varepsilon,b_\varepsilon(0))}g'_\varepsilon(0)\mathbf{e}^{-\sigma(\varepsilon,b_\varepsilon(0))}\\
&= g'_\varepsilon(0) + \mathrm{o}(b'_\varepsilon(0)) + \mathrm{o}(g'_\varepsilon(0)) \text{ because } \sigma(0,0)=0 \text{ and } \frac{\partial}{\partial b}|_{b=0}\sigma(0,b)=0,\\
&= \mathbf{i}\nu_\varepsilon(b_\varepsilon(0)) + \mathrm{o}(L_{b_\varepsilon(0)}\nu_\varepsilon(b_\varepsilon(0))) + \mathrm{o}(\nu_\varepsilon(b_\varepsilon(0)))\\
&= \mathbf{i}\nu_\varepsilon(b_\varepsilon(0)) + \mathrm{o}(\nu_\varepsilon(b_\varepsilon(0))).
\end{aligned}$$

$\mathbf{i}\nu_\varepsilon(b_\varepsilon(0))$ is Hermitian with respect to h and $k = f_{\varepsilon,b_\varepsilon(0)}\cdot h$ so $\mathbf{i}f_{\varepsilon,b_\varepsilon(0)}\nu_\varepsilon(b_\varepsilon(0))f^{-1}_{\varepsilon,b_\varepsilon(0)}$ is Hermitian with respect to k. We deduce that $s'(0) = -2\mathbf{i}f_{\varepsilon,b_\varepsilon(0)}\nu_\varepsilon(b_\varepsilon(0))f^{-1}_{\varepsilon,b_\varepsilon(0)} + \mathrm{o}(f_{\varepsilon,b_\varepsilon(0)}\nu_\varepsilon(b_\varepsilon(0))f^{-1}_{\varepsilon,b_\varepsilon(0)})$.
Step 2 : Relation between curvatures.

Since $f_{\varepsilon,b_\varepsilon(t)}\cdot h = \mathbf{e}^{-s(t)/2}u(t)\cdot k = \mathbf{e}^{-s(t)/2}\cdot k$, we have, by (4),

$$\varphi_\varepsilon(b_\varepsilon(t)) - \varphi_\varepsilon(b_\varepsilon(0)) = M_{\varepsilon,\overline{\partial}_E}(h,\mathbf{e}^{-s(t)/2}\cdot k) - M_{\varepsilon,\overline{\partial}_E}(h,k) = M_{\varepsilon,\overline{\partial}_E}(k,\mathbf{e}^{-s(t)/2}\cdot k).$$

Moreover, if we call $\partial_{E,k}$ the $(1,0)$ part of the Chern connection associated with $(\overline{\partial}_E,k)$ and we set $\partial_{\varepsilon,b_\varepsilon(0)} = f^{-1}_{\varepsilon,b_\varepsilon(0)}\circ\partial_{E,k}\circ f_{\varepsilon,b_\varepsilon(0)}$, we can compute that the connection $\nabla_{\varepsilon,b_\varepsilon(0)} = \partial_{\varepsilon,b_\varepsilon(0)} + \overline{\partial}_{\varepsilon,b_\varepsilon(0)}$

is unitary with respect to h. Thus, by uniqueness, $\nabla_{\varepsilon,b_\varepsilon(0)}$ is the Chern connection associated with $(\overline{\partial}_{\varepsilon,b_\varepsilon(0)},h)$. Therefore,

$$\begin{aligned}
F_{\overline{\partial}_{E,k}} &= \partial_{E,k}\overline{\partial}_{E,k} + \overline{\partial}_{E,k}\partial_{E,k} \\
&= f_{\varepsilon,b_\varepsilon(0)} \circ \partial_{\varepsilon,b_\varepsilon(0)} \circ f^{-1}_{\varepsilon,b_\varepsilon(0)} \circ f_{\varepsilon,b_\varepsilon(0)} \circ \overline{\partial}_{\varepsilon,b_\varepsilon(0)} \circ f^{-1}_{\varepsilon,b_\varepsilon(0)} + f_{\varepsilon,b_\varepsilon(0)} \circ \overline{\partial}_{\varepsilon,b_\varepsilon(0)} \circ f^{-1}_{\varepsilon,b_\varepsilon(0)} \circ f_{\varepsilon,b_\varepsilon(0)} \circ \partial_{\varepsilon,b_\varepsilon(0)} \circ f^{-1}_{\varepsilon,b_\varepsilon(0)} \\
&= f_{\varepsilon,b_\varepsilon(0)} F_{\overline{\partial}_{\varepsilon,b_\varepsilon(0)},h} f^{-1}_{\varepsilon,b_\varepsilon(0)}.
\end{aligned}$$

Thus, $\Lambda_\varepsilon \mathbf{i} F_{\overline{\partial}_{E,k}} - c_\varepsilon \mathrm{Id}_E = \mathbf{i} f_{\varepsilon,b_\varepsilon(0)} \nu_\varepsilon(b_\varepsilon(0)) f^{-1}_{\varepsilon,b_\varepsilon(0)}$.

Step 3 : Conclusion.

Finally, by (5),

$$\begin{aligned}
\frac{\partial}{\partial t}\varphi_\varepsilon(b_\varepsilon(t)) &= \int_X \mathrm{tr}\left(s'(0)(\Lambda_\varepsilon \mathbf{i} F_{\overline{\partial}_{E,k}} - c_\varepsilon \mathrm{Id}_E)\right)\mathrm{Vol}_\varepsilon \\
&= -2\int_X \mathrm{tr}\left((f_{\varepsilon,b_\varepsilon(0)}\mathbf{i}\nu_\varepsilon(b_\varepsilon(0))f^{-1}_{\varepsilon,b_\varepsilon(0)} + \mathrm{o}(f_{\varepsilon,b_\varepsilon(0)}\mathbf{i}\nu_\varepsilon(b_\varepsilon(0))f^{-1}_{\varepsilon,b_\varepsilon(0)}))f_{\varepsilon,b_\varepsilon(0)}\mathbf{i}\nu_\varepsilon(b_\varepsilon(0))f^{-1}_{\varepsilon,b_\varepsilon(0)}\right)\mathrm{Vol}_\varepsilon \\
&= -2\left\|\nu_\varepsilon(b_\varepsilon(0)\right\|^2_\varepsilon + \mathrm{o}(\left\|\nu_\varepsilon(b_\varepsilon(0)\right\|^2_\varepsilon).
\end{aligned}$$

□

We are now ready to conclude this subsection.

Proposition 4.6. *If $\mathcal{E}$ is $[\Theta_\varepsilon]$-semi-stable, there exists a $b_{\infty,\varepsilon} \in B \cap \overline{\mathcal{O}}$ such that $\nu_\varepsilon(b_{\infty,\varepsilon}) = 0$. If moreover, $\mathcal{E}$ is $[\Theta_\varepsilon]$-stable, then $b_{\infty,\varepsilon} \in \mathcal{O}$. Moreover, $\|b_{\infty,\varepsilon}\| = \mathrm{O}\left(\|\varepsilon\|_{L^2_1} + \sqrt{\|[\varepsilon]\|}\right)$.*

Proof. Assume first of all that $\mathcal{E}$ is $[\Theta_\varepsilon]$-stable. Let for all t, $f_{\varepsilon,b_\varepsilon(t)} \cdot h = \mathbf{e}^{-s(t)/2} \cdot h$ for some $s(t)$ Hermitian with respect to h. We may assume that for all t, $\int_X \mathrm{tr}(s(t))\mathrm{Vol}_0 = 0$. Then, by the inequality (6) and Proposition 4.5, for all t,

$$\left\|s(t)\right\|_\varepsilon \leqslant C_{1,\varepsilon} + C_{2,\varepsilon} M(h, \mathbf{e}^{-s(t)/2} \cdot h) = C_{1,\varepsilon} + C_{2,\varepsilon}\varphi_\varepsilon(b_\varepsilon(t)) \leqslant C_{1,\varepsilon} + C_{2,\varepsilon}\varphi_\varepsilon(b_\varepsilon(0)).$$

It means that $(s(t))$ is L^2 bounded. We have by definition $\mathbf{e}^{-s(t)} = f_{\varepsilon,b_\varepsilon(0)}\tilde{g}_\varepsilon(t) f^{-1}_{\varepsilon,b_\varepsilon(0)}$ hence $(\tilde{g}_\varepsilon(t))$ and $\left(\tilde{g}_\varepsilon(t)^{-1}\right)$ are L^2 bounded. We also have $g_\varepsilon(t) = \mathbf{e}^{-\sigma(\varepsilon,b_\varepsilon(0))}\tilde{g}_\varepsilon(t)\mathbf{e}^{\sigma(\varepsilon,b_\varepsilon(t))}$ thus $(g_\varepsilon(t))$ and $(g_\varepsilon(t)^{-1})$ are bounded too (for any norm since they live in a finite dimensional vector space). Therefore, we can find a sequence $t_m \to +\infty$ such that $g_\varepsilon(t_m) \to g_\varepsilon \in G$ is invertible and in particular, $b_\varepsilon(t_m) \to b_{\infty,\varepsilon} = g_\varepsilon \cdot b_\varepsilon(0) \in \mathcal{O}$. Let $l = \nu_\varepsilon(b_{\infty,\varepsilon}) \in \mathfrak{k}$. Since the norm of the moment map decreases by Proposition 4.4, $\left\|\nu_\varepsilon(b_\varepsilon(t))\right\|_\varepsilon \to \left\|l\right\|_\varepsilon$. By Proposition 4.5, up to shrinking U, $\frac{\partial}{\partial t}\varphi_\varepsilon(b_\varepsilon(t)) \leqslant -\left\|l\right\|^2_\varepsilon + \mathrm{o}(t)$ and inequality (6) implies that φ is bounded from below hence $l = 0$.

In the semi-stable case, we have by Theorem 3.5 that $\varepsilon = \lim_{m\to+\infty}\varepsilon_m$ for some ε_m such that $\mathcal{E}$ is $[\Theta_{\varepsilon_m}]$-stable. The b_{∞,ε_m} remain in a compact set included in B so, up to extraction, we may build $b_{\infty,\varepsilon} \in B \cap \overline{\mathcal{O}}$ as $\lim_{m\to+\infty} b_{\infty,\varepsilon_m}$. The bound on the norm of $b_{\infty,\varepsilon}$ is given by Proposition 4.3. □

4.4 Convergence when $\varepsilon \to 0$ and continuity results

A consequence of Proposition 4.6 is Introduction's Theorem 3.

Theorem 4.7. *Let (X, Θ_0) be a compact balanced manifold and $\mathcal{E}$ a $[\Theta_0]$-semi-stable sufficiently smooth holomorphic vector bundle. There is an L^2_1 open neighbourhood U of 0 in the space of closed $(n-1, n-1)$ forms and a family $(\overline{\partial}_\varepsilon)_{\varepsilon \in U | \Theta_\varepsilon \in \mathcal{C}_{ss}(\mathcal{E})}$ of integrable Dolbeault operators on E such that for all ε, $\overline{\partial}_\varepsilon$ is Θ_ε-HYM and $(E, \overline{\partial}_\varepsilon) \cong \mathrm{Gr}_{[\Theta_\varepsilon]}(\mathcal{E})$. Moreover, for all integer $d \geqslant 2$,*

$$\left\| \overline{\partial}_\varepsilon - \overline{\partial}_0 \right\|_{L^2_d} = \mathrm{O}\left(\|\varepsilon\|_{L^2_{d-1}} + \sqrt{\|[\varepsilon]\|} \right)$$

In particular, $\overline{\partial}_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} \overline{\partial}_0$ for the $\mathcal{C}^\infty$ topology on U and on the space of Dolbeault operators.

Proof. Let for all $\varepsilon \in U$ such that $\mathcal{E}$ is $[\Theta_\varepsilon]$-semi-stable, $\overline{\partial}_\varepsilon = \overline{\partial}_0 + \tilde{\Phi}(\varepsilon, b_{\infty,\varepsilon})$ where $b_{\infty,\varepsilon}$ is given by Proposition 4.6. The HYM condition is fulfilled and when Θ_ε is in the stable cone, $b_{\infty,\varepsilon} \in \mathcal{O}$ so $(E, \overline{\partial}_\varepsilon) \cong \mathcal{E} \cong \mathrm{Gr}_{[\Theta_\varepsilon]}(\mathcal{E})$. When Θ_ε is only in the semi-stable cone, by [2, Theorem 4], $(E, \overline{\partial}_\varepsilon) \cong (E, \overline{\partial}_0 + \Phi(b_{\infty,\varepsilon})) \cong \mathrm{Gr}_{[\Theta_\varepsilon]}(\mathcal{E})$.

By smoothness of σ given by Proposition 4.1 and the bound on the norm of b given by Proposition 4.6, we have for all $d \geqslant 2$,

$$\begin{aligned}
\left\| \overline{\partial}_\varepsilon - \overline{\partial}_0 \right\|_{L^2_d} &= \left\| \mathbf{e}^{\sigma(\varepsilon, b_{\infty,\varepsilon})} \cdot (\overline{\partial}_0 + \Phi(b_{\infty,\varepsilon})) - \overline{\partial}_0 \right\|_{L^2_d} \\
&= \left\| \mathbf{e}^{\sigma(\varepsilon, b_{\infty,\varepsilon})} \overline{\partial}_0 (\mathbf{e}^{-\sigma(\varepsilon, b_{\infty,\varepsilon})}) + \mathbf{e}^{\sigma(\varepsilon, b_{\infty,\varepsilon})} \Phi(b_{\infty,\varepsilon}) \mathbf{e}^{-\sigma(\varepsilon, b_{\infty,\varepsilon})} \right\|_{L^2_d} \\
&= \mathrm{O}(\|\sigma(\varepsilon, b_{\infty,\varepsilon})\|_{L^2_{d+1}}) + \mathrm{O}(\|\Phi(b_{\infty,\varepsilon})\|_{L^2_d}) \\
&= \mathrm{O}(\|\varepsilon\|_{L^2_{d-1}} + \|b_{\infty,\varepsilon}\|^2) + \mathrm{O}(\|b_{\infty,\varepsilon}\|) \\
&= \mathrm{O}\left(\|\varepsilon\|_{L^2_{d-1}} + \|\varepsilon\|_{L^2_1} + \sqrt{\|[\varepsilon]\|} \right) \\
&= \mathrm{O}\left(\|\varepsilon\|_{L^2_{d-1}} + \sqrt{\|[\varepsilon]\|} \right).
\end{aligned}$$

□

Remark. Clearly, the same bound holds for $\|\partial_\varepsilon - \partial_0\|_{L^2_d}$ and $\|\nabla_\varepsilon - \nabla_0\|_{L^2_d}$, where $\nabla_\varepsilon = \partial_\varepsilon + \overline{\partial}_\varepsilon$ is the Chern connection associated to $(\overline{\partial}_\varepsilon, h)$.

Lemma 4.8. *Let $\mathcal{E}$ be a torsion-free coherent sheaf and $K \subset \mathcal{E}$ compact and convex. By Theorem 3.5, $C = \mathcal{C}_{ss}(\mathcal{E}) \cap K$ is a convex polyhedral cone. Thus for all $[\Theta] \in C$, there is a unique face $F_{[\Theta]} \subset C$ such that $[\Theta]$ belongs to the relative interior of $F_{[\Theta]}$. If $[\Theta_1]$ and $[\Theta_2]$ are in C and $F_{[\Theta_1]} \subset F_{[\Theta_2]}$, then $\mathrm{Gr}_{[\Theta_2]}(\mathcal{E})$ is $[\Theta_1]$-semi-stable and $\mathrm{Gr}_{[\Theta_1]}(\mathrm{Gr}_{[\Theta_2]}(\mathcal{E})) \cong \mathrm{Gr}_{[\Theta_1]}(\mathcal{E})$. In particular, if $F_{[\Theta_1]} = F_{[\Theta_2]}$, $\mathrm{Gr}_{[\Theta_1]}(\mathcal{E}) \cong \mathrm{Gr}_{[\Theta_2]}(\mathcal{E})$.*

Proof. Let $\mathcal{S}_1, \ldots, \mathcal{S}_p \subset \mathcal{E}$ be a finite family of coherent sheaves given by 3.4 such that $C = \bigcap_{k=1}^p \{l_{\mathcal{S}_k} > 0\} \cap K$. We may assume that the $l_{\mathcal{S}_k}$ are pairwise distinct. Up to rearranging the $\mathcal{S}_k$, we may also assume that the faces $F_{[\Theta_2]}$ and $F_{[\Theta_1]} \subset F_{[\Theta_2]}$ are defined by the equations,

$$F_{[\Theta_1]} = \bigcap_{k=1}^{j} \{l_{\mathcal{S}_k} = 0\} \cap \bigcap_{k=j+1}^{p} \{l_{\mathcal{S}_k} \geqslant 0\} \cap K, \qquad F_{[\Theta_2]} = \bigcap_{k=1}^{i} \{l_{\mathcal{S}_k} = 0\} \cap \bigcap_{k=i+1}^{p} \{l_{\mathcal{S}_k} \geqslant 0\} \cap K,$$

for some $0 \leqslant i \leqslant j \leqslant p$. Moreover, since $[\Theta_1]$ (resp. $[\Theta_2]$) belongs to the relative interior of $F_{[\Theta_1]}$ (resp. $F_{[\Theta_2]}$), we have $l_{\mathcal{S}_k}([\Theta_1]) > 0$ when $k > j$ (resp. $l_{\mathcal{S}_k}([\Theta_2]) > 0$ when $k > i$).

Let $0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_m = \mathcal{E}$ be a Jordan-Hölder filtration for $\mathcal{E}$ with respect to $[\Theta_2]$. We have $\mathrm{Gr}_{[\Theta_2]}(\mathcal{E}) = \bigoplus_{k=1}^m \mathcal{G}_k$ where $\mathcal{G}_k = \mathcal{F}_k/\mathcal{F}_{k-1}$. By definition, for all k, $l_{\mathcal{F}_k}([\Theta_2]) = 0$. Since the $\mathcal{F}_k$ destabilise $\mathcal{E}$ with respect to $[\Theta_2] \in K$, there must exist an integer $1 \leqslant q \leqslant p$ such that $\frac{c_1(\mathcal{F}_k)}{\mathrm{rk}(\mathcal{F}_k)} = \frac{c_1(\mathcal{S}_q)}{\mathrm{rk}(\mathcal{S}_q)}$. The equality $l_{\mathcal{F}_k}([\Theta_2]) = l_{\mathcal{S}_q}([\Theta_2]) = 0$ implies that $q \leqslant i$. In particular, $q \leqslant j$ so for all k, $l_{\mathcal{F}_k}([\Theta_1]) = 0$. It implies that each $\mathcal{F}_k$ is $[\Theta_1]$-semi-stable by $[\Theta_1]$-stability of $\mathcal{E}$. Therefore, for all k, $l_{\mathcal{G}_k}([\Theta_1]) = 0$ and $\mathcal{G}_k$ is $[\Theta_1]$-semi-stable as a quotient of $[\Theta_1]$-semi-stable sheaves of the same $[\Theta_1]$-slope.

It means that the $\mathcal{F}_k$ form a filtration of $\mathcal{E}$ by semi-stable sub-sheaves with torsion-free semi-stable quotients with respect to $[\Theta_1]$. Let us consider a maximal refinement of this filtration with for all $1 \leqslant k \leqslant m$,

$$\mathcal{F}_{k-1} = \mathcal{F}_{k-1,0} \subsetneq \mathcal{F}_{k-1,1} \subsetneq \cdots \subsetneq \mathcal{F}_{k-1,p_k} = \mathcal{F}_k = \mathcal{F}_{k,0}.$$

Then, the $\mathcal{F}_{k,q}$ form a Jordan-Hölder filtration of $\mathcal{E}$ with respect to $[\Theta_1]$ hence on the one hand,

$$\mathrm{Gr}_{[\Theta_1]}(\mathcal{E}) = \bigoplus_{k=1}^{m} \bigoplus_{q=1}^{p_k} \mathcal{F}_{k-1,q}/\mathcal{F}_{k-1,q-1},$$

and on the other hand, for each k, the $\mathcal{F}_{k-1,q}/\mathcal{F}_{k-1}$ form a Jordan-Hölder filtration of $\mathcal{G}_k = \mathcal{F}_k/\mathcal{F}_{k-1}$ with respect to $[\Theta_1]$ so,

$$\mathrm{Gr}_{[\Theta_1]}(\mathrm{Gr}_{[\Theta_2]}(\mathcal{E})) = \bigoplus_{k=1}^{m} \mathrm{Gr}_{[\Theta_1]}(\mathcal{G}_k) = \bigoplus_{k=1}^{m} \bigoplus_{q=1}^{p_k} \mathcal{F}_{k-1,q}/\mathcal{F}_{k-1,q-1}.$$

It proves the lemma. □

Let $\underline{\mathcal{C}}_{ss}(\mathcal{E}) \subset \mathcal{C}_{ss}(\mathcal{E})$ be the set of all Θ with respect to which $\mathcal{E}$ is semi-stable and sufficiently smooth. Let $\underline{\mathcal{C}}_{ss}^{\mathrm{D}}(\mathcal{E}) \subset \mathcal{C}_{ss}^{\mathrm{D}}(\mathcal{E})$ be its projection on the real Dolbeault cohomology group. Lemma 4.8 has the following consequence,

Corollary 4.9. *$\underline{\mathcal{C}}_{ss}^{\mathrm{D}}(\mathcal{E})$ is an open subset of $\mathcal{C}_{ss}^{\mathrm{D}}(\mathcal{E})$ whose complement is locally formed of a union of closed faces of $\mathcal{C}_{ss}^{\mathrm{D}}(\mathcal{E})$. Similarly, $\underline{\mathcal{C}}_{ss}(\mathcal{E})$ is an open subset of $\mathcal{C}_{ss}(\mathcal{E})$.*

Proof. This is a direct consequence of Lemma 4.8 and the fact that the graded object of a sheaf which is not locally free is itself not locally free (in particular, if $\mathcal{E}$ is not locally free, these cones are empty). □

We come back to the case where $\mathcal{E}$ is locally free. Let $\mathcal{H}(E)$ be the set of holomorphic structures on E *i.e.* the set of the integrable Dolbeault operators. We endow it with the $\mathcal{C}^\infty$ topology. By [13, Corollary 7.1.15], the action of $\mathcal{G}$ on $\mathcal{H}(E)$ is proper hence the space $\mathcal{H}(E)/\mathcal{G}$ is Hausdorff. We endow $\underline{\mathcal{C}}_{ss}(\mathcal{E}) \subset \Omega^{n-1,n-1}(X,\mathbb{R})$ with the $\mathcal{C}^\infty$ topology too. We are now ready to prove Introduction's Theorem 4.

Theorem 4.10. *The function,*

$$\Gamma_{\mathcal{E}} : \left\{ \begin{array}{ccl} \underline{\mathcal{C}}_{ss}(\mathcal{E}) & \to & \mathcal{H}(E)/\mathcal{G} \\ \Theta & \mapsto & \{\overline{\partial} \in \mathcal{H}(E) | (E,\overline{\partial}) \cong \mathrm{Gr}_{[\Theta]}(\mathcal{E}) \text{ and } \overline{\partial} \text{ is } \Theta\text{-HYM}\} \end{array} \right. ,$$

is well-defined and continuous.

Proof. For all $\Theta \in \underline{\mathcal{C}}_{ss}(\mathcal{E})$, $\mathrm{Gr}_{[\Theta]}(\mathcal{E})$ is a $[\Theta]$-polystable vector bundle so Proposition 3.7 tells us that the set $\{\overline{\partial} \in \mathcal{H}(E) | (\mathcal{E}, \overline{\partial}) \cong \mathrm{Gr}_{[\Theta]}(\mathcal{E})$ and $\overline{\partial}$ is Θ-HYM$\}$ is exactly one $\mathcal{G}$-orbit hence $\Gamma_{\mathcal{E}}$ is well defined and indeed takes values in $\mathcal{H}(E)/\mathcal{G}$. Let $K \subset \mathcal{B}_X$ be a convex compact subset. It is enough to show continuity on the polyhedral cone $C = \underline{\mathcal{C}}_{ss}(\mathcal{E}) \cap K$ for all such K and we may assume that $C \neq \emptyset$. We have to consider three cases.

First case : $\mathcal{C}_s(\mathcal{E}) \cap C \neq \emptyset$. In this case, continuity of Γ at each point Θ_0 is a consequence of Theorem 4.7.

Second case : It exists $\Theta_0 \in C$ such that $F_{[\Theta_0]}$ is the whole cone C and $\mathcal{E}$ is $[\Theta_0]$-polystable. In this case, let us write $\mathcal{E} = \bigoplus_{k=1}^m \mathcal{E}_k$ with the $\mathcal{E}_k$ $[\Theta_0]$-stable with the same $[\Theta_0]$-slope. The unitary gauge group of E is the direct product of the unitary gauge groups of the smooth structures E_k of $\mathcal{E}_k$. For all $\Theta \in \underline{\mathcal{C}}_{ss}(\mathcal{E})$, each $\mathcal{E}_k$ is $[\Theta]$-semi-stable and have the same $[\Theta]$-slope as $\mathcal{E}$ because $[\Theta] \in F_{[\Theta_0]}$. It implies that $\mathrm{Gr}_{[\Theta]}(\mathcal{E}) = \bigoplus_{k=1}^m \mathrm{Gr}_{[\Theta]}(\mathcal{E}_k)$. Moreover, any $[\Theta]$-HYM connection on $\mathrm{Gr}_{[\Theta]}(\mathcal{E})$ makes this decomposition holomorphic and induces a $[\Theta]$-HYM connection on each $\mathrm{Gr}_{[\Theta]}(\mathcal{E}_k)$. Therefore, it makes sens to write that $\Gamma_{\mathcal{E}} = \sum_{k=1}^m \Gamma_{\mathcal{E}_k}$. The fact that each $\Gamma_{\mathcal{E}_k}$ is continuous is the first case because for all k, $\Theta_0 \in \mathcal{C}_s(\mathcal{E}_k) \cap C$.

Third case : General case. C is assumed to be non-empty so there is a Θ_0 such that $F_{[\Theta_0]} = C$. Thus, for all $\Theta \in \underline{\mathcal{C}_{ss}}(\mathcal{E})$, $F_{[\Theta]} \subset F_{[\Theta_0]}$. Therefore, a consequence of Lemma 4.8 is that $\Gamma_{\mathcal{E}} = \Gamma_{\mathrm{Gr}_{[\Theta_0]}(\mathcal{E})}$ and $\mathrm{Gr}_{[\Theta_0]}(\mathcal{E})$ is a $[\Theta_0]$-polystable vector bundle. We conclude thanks to the second case. $\square$

References

[1] Michael Francis Atiyah and Raoul Bott. The yang-mills equations over riemann surfaces. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 308:523 – 615, 1983. URL: `https://api.semanticscholar.org/CorpusID:13601126`.

[2] Nicholas Buchdahl and Georg Schumacher. Polystable bundles and representations of their automorphisms. *Complex Manifolds*, 9(1):78–113, 2022. URL: `https://doi.org/10.1515/coma-2021-0131` [cited 2024-03-03], `doi:doi:10.1515/coma-2021-0131`.

[3] Andrew Clarke and Carl Tipler. Semi-stability and local wall-crossing for hermitian yang-mills connections. *Journal of Symplectic Geometry (to appear)*, 2023. `arXiv:2304.05245`.

[4] Ruadhai Dervan, John Benjamin McCarthy, and Lars Martin Sektnan. z-critical connections and bridgeland stability conditions. *Cambridge Journal of Mathematics*, 2023. `arXiv:2012.10426`.

[5] Simon K. Donaldson. Anti self-dual yang-mills connections over complex algebraic surfaces and stable vector bundles. *Proceedings of The London Mathematical Society*, 50:1–26, 1985. URL: `https://api.semanticscholar.org/CorpusID:123224158`.

[6] Jixiang Fu. A survey on balanced metrics. In Akito Futaki, Reiko Miyaoka, Zizhou Tang, and Weiping Zhang, editors, *Geometry and Topology of Manifolds*, pages 127–138, Tokyo, 2016. Springer Japan.

[7] Paul Gauduchon. Fibrés hermitiens à endomorphisme de ricci non négatif. *Bulletin de la Société Mathématique de France*, 105:113–140, 1977. URL: https://api.semanticscholar.org/CorpusID:125532227.

[8] V. Georgoulas, J.W. Robbin, and D.A. Salamon. *The Moment-Weight Inequality and the Hilbert–Mumford Criterion: GIT from the Differential Geometric Viewpoint*. Lecture Notes in Mathematics. Springer International Publishing, 2022. URL: https://books.google.fr/books?id=v2VXEAAAQBAJ.

[9] Daniel Greb, Julius Ross, and Matei Toma. Moduli of vector bundles on higher-dimensional base manifolds — construction and variation. *International Journal of Mathematics*, 27:1640007, 2016. URL: https://api.semanticscholar.org/CorpusID:199678021.

[10] Daniel Greb and Matei Toma. Compact moduli spaces for slope-semistable sheaves. *arXiv: Algebraic Geometry*, 4:40–78, 2013. URL: https://api.semanticscholar.org/CorpusID:55819018.

[11] D. Huybrechts. *Complex Geometry: An Introduction*. Universitext (Berlin. Print). Springer, 2005. URL: https://books.google.fr/books?id=sWbd0rE3mhIC.

[12] Hans Grauert Klaus Fritzsche. *From Holomorphic Functions to Complex Manifolds*. Springer New York, NY, 2002. doi:10.1007/978-1-4684-9273-6.

[13] S. Kobayashi. *Differential Geometry of Complex Vector Bundles*. Princeton Legacy Library. Princeton University Press, 2014. URL: https://books.google.fr/books?id=ePf_AwAAQBAJ.

[14] Shoshichi Kobayashi. Curvature and stability of vector bundles. *Proceedings of the Japan Academy, Series A, Mathematical Sciences*, 58(4):158 – 162, 1982. doi:10.3792/pjaa.58.158.

[15] M. Kuranishi. New proof for the existence of locally complete families of complex structures. In Alfred Aeppli, Eugenio Calabi, and Helmut Röhrl, editors, *Proceedings of the Conference on Complex Analysis*, pages 142–154, Berlin, Heidelberg, 1965. Springer Berlin Heidelberg.

[16] Jun Li and Shing Tung Yau. *Hermitian-Yang-Mills Connection on Non-Kähler Manifolds*, pages 560–573. World Scientific Publishing Company, 1987. URL: https://worldscientific.com/doi/abs/10.1142/9789812798411_0027, arXiv:https://worldscientific.com/doi/pdf/10.1142/9789812798411_0027, doi:10.1142/9789812798411_0027.

[17] M. Lübke and A. Teleman. *The Universal Kobayashi-Hitchin Correspondence on Hermitian Manifolds*. Number vol. 13 in Memoirs of the American Mathematical Society. American Mathematical Society, 2006. URL: https://books.google.fr/books?id=N_DUCQAAQBAJ.

[18] Martin Lübke. Stability of einstein-hermitian vector bundles. *manuscripta mathematica*, 42:245–257, 1983. URL: https://api.semanticscholar.org/CorpusID:121338200.

[19] M. L. Michelsohn. On the existence of special metrics in complex geometry. *Acta Mathematica*, 149(none):261 – 295, 1982. doi:10.1007/BF02392356.

[20] David Mumford. Projective invariants of projective structures and applications. *Proc. Internat. Congr. Mathematicians*, pages 526–530, 01 1963. doi:10.1007/978-1-4757-4265-7_2.

[21] Annamaria Ortu and Lars Martin Sektnan. Constant scalar curvature kähler metrics and semistable vector bundles, 2024 (to appear).

[22] Lars Martin Sektnan and Carl Tipler. Analytic k-semistability and wall-crossing, 2023. arXiv:2212.08383.

[23] Carlos Simpson. Constructing variations of hodge structure using yang-mills theory and applications to uniformization. *Journal of the American Mathematical Society*, 1:867–918, 1988. URL: https://api.semanticscholar.org/CorpusID:27955313.

[24] The Stacks project authors. The stacks project. https://stacks.math.columbia.edu, 2024.

[25] Gábor Székelyhidi. The kahler-ricci flow and k-stability. *American Journal of Mathematics*, 2008. URL: https://api.semanticscholar.org/CorpusID:123131100.

[26] Fumio Takemoto. Stable vector bundles on algebraic surfaces. *Nagoya Mathematical Journal*, 47:29–48, 1972. doi:10.1017/S0027763000014896.

[27] Karen K. Uhlenbeck and Shing-Tung Yau. On the existence of hermitian-yang-mills connections in stable vector bundles. *Communications on Pure and Applied Mathematics*, 39:257–293, 1986. URL: https://api.semanticscholar.org/CorpusID:121782589.