Free Sets in Planar Graphs: History and Applications*

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Abstract

A subset S of vertices in a planar graph G is a *free set* if, for every set P of $|S|$ points in the plane, there exists a straight-line crossing-free drawing of G in which vertices of S are mapped to distinct points in P. In this survey, we review

- several equivalent definitions of free sets,
- results on the existence of large free sets in planar graphs and subclasses of planar graphs,
- and applications of free sets in graph drawing.

The survey concludes with a list of open problems in this still very active research area.

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1 Introduction

In 2005, an online game called *Planarity* became wildly popular. In this single-player game, created by John Tantalo based on a concept by Mary Radcliffe, the player is presented with a straight-line drawing Γ of a planar graph G. Although the graph G is planar, the drawing Γ is not crossing-free; some pairs of edges cross each other and an edge may even contain a vertex in its interior. Since Γ draws the edges of G as straight line segments, the drawing is completely determined by the vertex locations. The job of the player is to move vertices in the drawing Γ in order to obtain a crossing-free drawing of G. Since G is planar, Fáry's Theorem guarantees that this always possible.

The act of moving some vertices in Γ to obtain a crossing-free drawing is referred to as *untangling*. For a combinatorially inclined player, this naturally leads to the question: What is the fewest number, $\mathrm{shift}(\Gamma)$, of vertices that need to be moved in order untangle Γ?[1] Equivalently, what is the maximum number, $\mathrm{fix}(\Gamma) := |V(G)| - \mathrm{shift}(\Gamma)$ of vertices that can be kept fixed while untangling G? For a planar graph G, let $\mathrm{fix}(G) := \min\{\mathrm{fix}(\Gamma) : \Gamma \text{ is a straight-line drawing of } G\}$. For a family $\mathcal{F}$ of planar graphs, this defines a function $\mathrm{fix}_{\mathcal{F}}(n) := \min\{\mathrm{fix}(G) : G \in \mathcal{F}, |V(G)| = n\}$.[2]

Determining the asymptotic growth of $\mathrm{fix}_{\mathcal{F}}(n)$ turns out to be a challenging problem, even for very simple classes $\mathcal{F}$. In 1998, Mamoru Watanabe had already asked this problem for the class $\mathcal{C}$ of cycles and we now know that $\mathrm{fix}_{\mathcal{C}}(n) \in \Omega(n^{2/3}) \cap O((n \log n)^{2/3})$ but neither the lower bound, due to Pach and Tardos [46], nor the upper bound, due to Cibulka [23], is easy to prove. For the class $\mathcal{G}$ of all planar graphs, it is only known that $\mathrm{fix}_{\mathcal{G}}(n) \in \Omega(n^{1/4}) \cap O(n^{0.4948})$ [17, 20, 38, 42], despite the problem being studied since Pach and Tardos [46] asked for a polynomial lower bound over 20 years ago.

In this survey, we study various kinds of vertex subsets of a planar graph G: proper-good sets, collinear sets, free-collinear sets, and free sets. Each of these definitions is, at first glance, more stringent than the one that precedes it. Indeed, it follows immediately from definitions that every free set is a free-collinear set, every free-collinear set is a collinear set, and every collinear set is a proper-good set. The definition of a proper-good set is the most relaxed, which makes it easy to find large proper-good sets, even by hand using a pencil and a paper that contains a crossing-free drawing of G. At the other extreme, free sets satisfy a very strong property that says we can place the vertices of a free set on any given pointset and then find locations for the other vertices of G that results in a non-crossing straight-line drawing of G. This makes free sets incredibly useful for many applications, including untangling.

A key result in this area, discovered over a period of roughly 15 years, is that all of these definitions are equivalent: A vertex subset in a planar graph is proper-good if and only if it is collinear if and only if it is free-collinear if and only if it is free. Meanwhile, proper-good sets, collinear sets, free-collinear sets, and free sets were being used to resolve problems in graph drawing and related areas. With the benefit of hindsight, we attempt to organize and present two decades of results on free sets and their applications in a way that makes them as easy to understand and as useful as possible.

The rest of this survey is structured as follows: In Section 2 we present four definitions of free sets and explain why these four definitions are equivalent. In Section 3 we present upper and lower bounds on the size of free sets in planar graphs and various subclasses of planar graphs. In Section 4 we describe applications of free sets to a variety of graph drawing problems. Section 6 concludes with a list of open problems and directions for further research.

[1] The Planarity game actually measures the amount of time the player takes to accomplish this task, not the number of vertices moved, but we won't let facts get in the way of a good story.

[2] Technically $\mathrm{fix}_{\mathcal{F}}(n)$ is undefined if there are no n-vertex graphs in the family $\mathcal{F}$. We will ignore this, since all the graph families we consider are infinite and have at least one n-vertex member for each $n \in \mathbb{N}$.

2 Four Definitions of Free Sets

For definitions of standard graph theoretic terms and notations used in this survey (such a treewidth, k-trees, independent set, etc.) the reader is referred to the textbook by Diestel [26].

A *drawing* of a graph is representation of the graph in which each vertex v is represented by a distinct point in the plane and each edge vw is represented by a simple open curve in the plane the closure of which has v and w as endpoints. When discussing a particular drawing, we do not distinguish between a vertex and the point that represents it, or an edge and the curve that represents it. A drawing is a *straight-line* drawing if each edge is an open line segment. In a drawing, two edges may have a non-empty intersection, and an edge may contain a vertex that is not one of its endpoints. A drawing is a *crossing-free drawing* if this does not occur: any two distinct edges are disjoint and no vertex is contained in any edge.[3]

Recall that for any n-vertex planar graph G_0, on at least 3 vertices, it is possible to add edges to G_0 to obtain a planar graph G with $3(n-2)$ edges. The resulting graph G is a *triangulation*: in any crossing-free drawing of this graph, each of the faces has exactly three edges on its boundary. A *near-triangulation* is a biconnected embedded graph whose inner faces are all *triangles*, i.e., bounded by three edges. The *dual* G^* of an embedded graph G is a graph whose vertices are the faces of G that contains an edge between two vertices f and g if the corresponding faces of G have a common edge on their boundary. For a planar graph G with outer face f_0, the *weak dual* G^+ of G is defined as $G+ := G^* - f_0$, the graph obtained from the dual G^+ by removing the vertex corresponds to the outer face f_0 of G. A graph is *outerplanar* if it has a crossing-free drawing with all its vertices on the outer face. A graph equipped with such a crossing-free drawing is an *outerplane* graph. A *chord* of a cycle C is an edge that is not part of C but both of its endpoints are vertices of C.

Under our definitions, a *planar* graph is a graph that has a crossing-free drawing. *Fáry's Theorem* [35, 53, 58] asserts that a graph is planar if and only if it has a straight-line crossing-free drawing. Today, we use Fáry's Theorem without thinking about it, but it is worthwhile spending a moment reflecting on what would happen if Fáry's Theorem were not true. There would be two kinds of planar graphs, topological and straight-line, with the latter being a special case of the former. Any result proven for straight-line planar graphs would need a separate proof for topological planar graphs. Edge contractions, commonly used in inductive proofs for planar graphs would be off-limits for proofs about straight-line planar graphs. In order to establish the existence of a straight-line planar graph with some property one would need to find a topological planar graph with that property and then show that it has a straight-line crossing-free drawing.

Let Γ be a crossing-free drawing of a planar graph G. A simple closed curve[4] $C : [0,1] \to \mathbb{R}^2$ is *good* with respect to Γ if $C(0)$ is not contained in any edge or vertex of Γ. C is *proper* with respect to Γ if, for each edge e of Γ, the intersection of C and the closure of e is either empty, consists of a single point (possibly an endpoint of e), or consists of the entire edge e. In essence, a curve that is proper-good with respect to Γ behaves the way a line would behave if Γ were a straight-line crossing-free drawing.

Let $S := (v_1,\dots,v_s)$ be an ordered subset of vertices in a planar graph G.

1. S is a *proper-good set* [24] if there exists a crossing-free drawing Γ of G, a curve $C : [0,1) \to \mathbb{R}^2$ that is proper-good with respect to Γ, and $0 < x_1 < \cdots < x_s < 1$ such that $C(x_i)$ is the location of v_i in Γ, for each $i \in \{1,\dots,s\}$. In other words, we encounter the vertices of S, in order, while traversing C.

2. S is a *collinear set* [48] if there exists a straight-line crossing-free drawing Γ of G and $x_1 < \cdots <$

[3]What we call a crossing-free drawing is sometimes called a *plane* drawing or an embedding (in the plane). What we call a straight-line crossing-free drawing is sometimes called a *geometric embedding* or *Fáry embedding*.

[4]A *simple closed curve* C is a continuous function $C : [0,1] \to \mathbb{R}^2$ such that $C(0) = C(1)$ and $C(a) \neq C(b)$ for all $0 \le a < b < 1$.

x_s such that $(x_i, 0)$ is the location of v_i in Γ, for each $i \in \{1, \ldots, s\}$.

3. S is a *free-collinear set* [48] if, for any $x_1 < \cdots < x_s$, *there exists* a straight-line crossing-free drawing Γ of G such that $(x_i, 0)$ is the location of v_i in Γ, for each $i \in \{1, \ldots, s\}$.

4. S is a *free set* if, for any $x_1 < \cdots < x_s$ and *any* $y_1, \ldots, y_s$, there exists a straight-line crossing-free drawing Γ of G such that (x_i, y_i) is the location of v_i in Γ, for each $i \in \{1, \ldots, s\}$.

We note that our definitions deviate from much of the literature by treating S as an ordered set. Later, we may say that some (unordered) vertex subset S of G is a proper-good, collinear, free-collinear, or free set. In these cases, we mean that there is some permutation of S that defines an ordered set that satisfies the relevant definition. When we want to emphasize the difference between these two we will say that (an ordered set) S is an *ordered free set* or that (a set) S is an *unordered free set*. This distinction becomes important when discussing two or more graphs with the same vertex set. A set S of vertices may be an unordered free set in two graphs G_1 and G_2 but there may be no permutation of S that is an ordered free set in G_1 and in G_2.

A few properties of free sets are immediate from the definition and we will use them throughout. If S is an ordered free set in a graph G then:

- the reversal of S is an ordered free set in G;
- S is an ordered free set in any subgraph of G that spans S; and
- any subsequence of S is an ordered free set in G.

Unordered collinear sets and unordered free-collinear sets were defined first by Ravsky and Verbitsky [48] and were implicit in [17]. In fact, Ravsky and Verbitsky [48] posed the equivalence between (unordered) collinear sets and (unordered) free-collinear set as an open question, and conjectured a negative answer. Thus Theorem 1 below disproves that conjecture. Proper-good sets (as unordered sets) were introduced first by Da Lozzo et al. [24].

Before continuing, we make a remark about pointsets whose x-coordinates are not all distinct, since the definition of (ordered) free set appears to disallow these. However, we claim that for any unordered free set S in a planar graph G, and *any* set P of $|S|$ points in the plane, there exists a straight-line crossing-free drawing of G in which each vertex in S is mapped to some point in P. If all the points in P have distinct x-coordinates then the claim follows immediately from the definition of ordered free set. If this is not the case, then a slight rotation of P gives a pointset P' in which each point has a distinct x-coordinate. Now G has a straight-line crossing-free drawing Γ' in which each vertex in S maps to a point in P'. Applying the inverse rotation to Γ' gives a drawing Γ in which each vertex in S maps to a point in P.

The proof of Lemma 1, below, gives one example of why definitions based on ordered sets are useful. Another example, discussed in Section 4, is the problem of simultaneous embedding with mapping.

Lemma 1. *Let G be a planar graph that has a free set of size at least k. Then $\text{fix}(G) \geq \sqrt{k}$.*

Proof. Let S be an ordered free set in G of size k. Fix any straight-line drawing Γ of G and, without loss of generality (by a slight rotation, if necessary) assume that no two vertices of Γ have the same x-coordinate. Let $(S, \prec)$ be the partial order in which $v \prec w$ if and only if v appears before w in S and the x-coordinate of v in Γ is less than the x-coordinate of w in Γ.

Suppose $(S, \prec)$ contains a chain $v_1 \prec \cdots \prec v_\ell$ of length $\ell \geq \sqrt{k}$. Then $(v_1, \ldots, v_\ell)$ is a free set since it is a subsequence of S. For each $i \in \{1, \ldots, \ell\}$, let (x_i, y_i) be the coordinates given to v_i in Γ. By the definition of free set, G has a straight-line crossing-free drawing in which v_i is mapped to (x_i, y_i), for each $i \in \{1, \ldots, \ell\}$. Therefore, $\text{fix}(\Gamma) \geq \ell \geq \sqrt{k}$ for any straight-line drawing Γ of G. Therefore $\text{fix}(G) \geq \sqrt{k}$.

To deal with the case in which $\prec$ contains no chain of length at least $\sqrt{k}$ we use the fact that the ordered set $\mathcal{S}$ obtained by reversing the order of S is also a free set. Let $(S, \prec')$ be the partial order on the elements of $\mathcal{S}$ in which $v \prec w$ if and only if v appears before w in $\mathcal{S}$ and the x-coordinate of v in Γ is less than the x-coordinate of w in Γ. By Dilworth's Theorem, $(S, \prec)$ contains an antichain of length at least $\sqrt{k}$ and the elements of this antichain form a chain in $(\mathcal{S}, \prec')$. Now we proceed as in the previous paragraph. $\square$

For most classes $\mathcal{X}$ of planar graphs, including trees, outerplanar graphs, and all planar graphs, the best known bounds (and for some classes asymptotically optimal bounds) for $\mathrm{fix}_{\mathcal{X}}(n)$ can be obtained by an application of Lemma 1, along with a result on free sets in graphs in the class $\mathcal{X}$ [17, 24, 38, 48]. The sole exception is the class $\mathcal{C}$ of cycles, for which Cibulka [23] uses repeated applications of the Erdős-Szekeres Theorem to obtain a lower bound of $\Omega(n^{2/3})$.

As discussed in the introduction, the following theorem provides the main motivation for this survey.

Theorem 1. *Let G be a planar graph and let S be an ordered subset of $V(G)$. Then the following statements are equivalent:*

1. *S is a (ordered) proper-good set.*
2. *S is a (ordered) collinear set.*
3. *S is a (ordered) free-collinear set.*
4. *S is a (ordered) free set.*

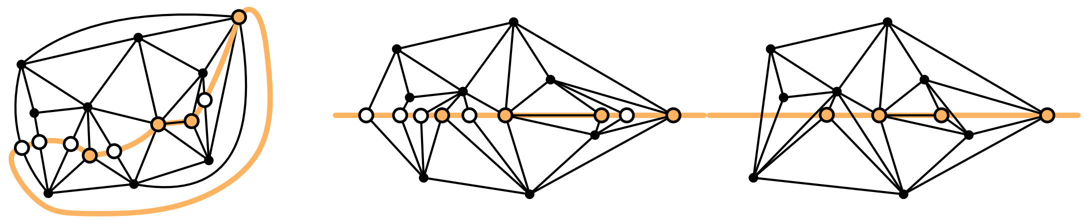
Note that, unlike the other three definitions, the definition of proper-good set is purely combinatorial/topological; it has no mention of straight lines or vertex coordinates. In this respect, Theorem 1 plays a role similar to Fáry's Theorem. Using the proper-good set definition allows purely combinatorial methods to be used in the search for free sets. For any proper-good set there is a short easily-verifiable certificate (a crossing-free drawing of G and a description of the curve C) that S is indeed a proper-good set. In contrast, in order to show directly that S is a free-collinear set (or a free set) involves proving a fact that must hold for all choices of $x_1, \dots, x_k$ (or $(x_1, y_1), \dots, (x_k, y_k)$, respectively). Working with this definition directly would involve (at the very least) a significant amount of linear algebra.

The proof of Theorem 1 is spread across a few different works [17, 24, 29], which we describe below. First we remark that it is immediate from the definitions that every free set is a free-collinear set and every free-collinear set is a free set. It is also easy to see that every collinear set is a proper-good set by taking a straight-line crossing-free drawing Γ of G with the vertices of the (ordered) collinear set S on the x-axis in increasing order of x-coordinate. Let s be a line segment that is contained in the x-axis and that contains every intersection between Γ and the x-axis in its interior. Then s is proper and good with respect to Γ. Finally, make s into a closed curve by joining the endpoints of s with a curve C that completely avoids Γ.

We now address the other directions.

Every proper-good set is a collinear set: The proof that every proper-good set is a collinear set is due to Da Lozzo et al. [24] and is illustrated in Fig. 1. For each edge of G that crosses C, the authors introduce a dummy vertex (white vertices in Fig. 1(a)) that splits the edge into two edges with a common vertex on C. The resulting graph G' then has three types of vertices: a set X of inner vertices (contained in the interior of C), a set Y of boundary vertices (contained in C) and a set Z of outer vertices (the vertices in the exterior of C). These steps are depicted in Fig. 1(a).

They then use Tutte's Convex Embedding Theorem [57] twice: Once to find a straight-line crossing-free drawing of $G'[X \cup Y]$ with all vertices of X below the x-axis and all vertices of Y on the x-axis, and a second time to find a straight-line crossing-free drawing of $G'[Y \cup Z]$ with the



(a) (b) (c)

Figure 1: Showing that every proper-good set is a collinear set. Dummy vertices are hollow circles and elements of S are filled circles.

vertices of Z above the x-axis. This gives a straight-line crossing-free drawing Γ' of G', as depicted in Fig. 1(b). The straight-line crossing-free drawing Γ' is almost a straight-line crossing-free drawing of G except that the edges that properly intersect C are represented by two line segments, one below and one above the x-axis. The authors then apply a result of Pach and Tóth [47] that shows the edges in this crossing-free drawing can be straightened without changing the y-coordinate of any vertex, as depicted in Fig. 1(c). In particular, in the resulting straight-line crossing-free drawing of G, the vertices of G intersected by C remain on the x-axis, as required.

A few notes on the proof are in order. First, we recall that the proof of Tutte's Convex Embedding Theorem is mostly algebraic: it involves setting up a system of linear equations whose solution determines the coordinates of the vertices in the straight-line crossing-free drawing. Second, we note that the preceding proof *almost* proves that every proper-good set is a *free*-collinear set. Tutte's theorem allows us to choose the x-coordinates of the vertices in Y, including the vertices in S. However, the straightening step, which converts Γ' into Γ by straightening the edges that cross the x-axis, changes the x-coordinates of the vertices.

Every collinear set is a free-collinear set: The proof that every collinear set is a free-collinear set is due to Dujmović et al. [29]. Prior to this, Da Lozzo et al. [24] had established the equivalence of collinear and free-collinear sets for the class of planar 3-trees and asked again if this equivalence was true in general. In subsequent years this repeatedly lead to the question (originally posed by Ravsky and Verbitsky [48]) "is every collinear set free?". The proof of Dujmović et al. [29] has two main parts, which we outline below. If $S := (v_1,\dots,v_s)$ is an ordered collinear set, then G has a straight-line crossing-free drawing Γ in which the vertices of S appear, in order, on the x-axis. The goal is to show that, for any $x_1 < \cdots < x_s$, the vertex v_i can be moved to $(x_i, 0)$, for each $i \in \{1,\dots,s\}$, such that the resulting drawing is a straight-line crossing-free drawing. By adding (straight-line) edges to Γ, we may assume that G is a triangulation. In the following, when we say that an edge or a face *properly intersects* the x-axis, we mean that the edge or the face contains points that are strictly above and points that are strictly below the x-axis.

The first part of the proof is combinatorial. In this part, a sequence of modifications is done to G with the end goal of obtaining a triangulation G' that has an (no longer straight-line) crossing-free drawing Γ' with following properties (see Fig. 2(c)):

(P1) Every vertex in S is a vertex of G' and is mapped to the same location (on the x-axis) in both Γ and Γ'.
(P2) The closure of each edge of Γ' intersects the x-axis in at most one point.
(P3) For each vertex $v \in S$ there are exactly two faces of G' incident on v that properly intersect the x-axis.
(P4) If an edge vw of G' has no endpoint in S then vw properly crosses the x-axis or vw is on the boundary of two faces vwa and vwb and the closure of each of these faces intersects the x-axis.

Note that (P1) and (P2) imply that no edge of G' has both endpoints in S. For reasons discussed

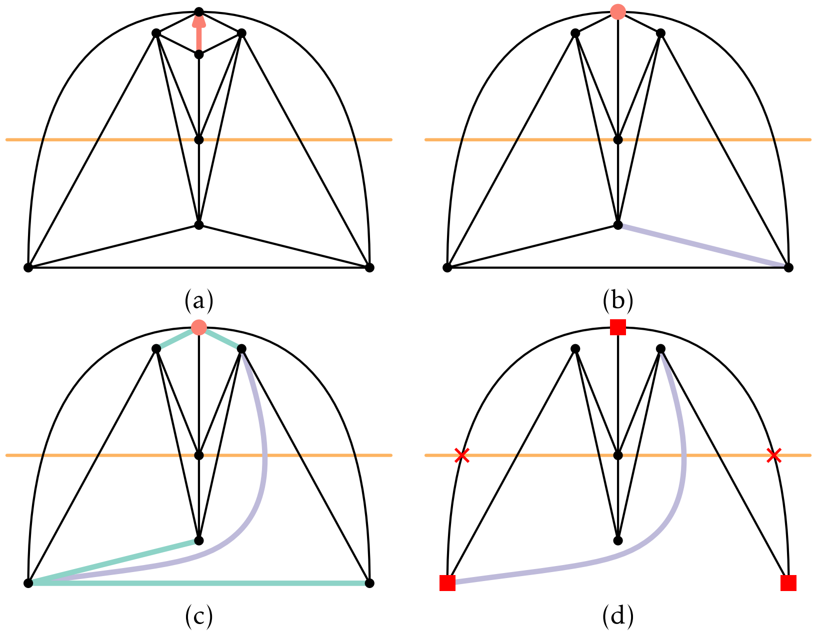


Figure 2: Modifying a triangulation to obtain an A-graph.

below, we may assume that G has no separating triangles and that no separating triangles are created during these operations.

To obtained G' we apply three types of operations, two of which are illustrated in Fig. 2. (1) If some edge with no endpoint in S is incident on two faces, neither of which properly intersect the x-axis, then this edge is contracted (the orange edge in Fig. 2(a) meets those conditions and its contraction results in an embedded graph depicted in Fig. 2(b)). This operation eliminates these two faces. (2) If some edge with no endpoints in S is incident on one face whose closure is disjoint from the x-axis, whose other incident face properly intersects the x-axis, and meets some other technical conditions,[5] then this edge is flipped (the lilac edge in Fig. 2(b) meets the necessary conditions and its flipping results in an embedded graph depicted in Fig. 2(c)). This edge flip replaces the two faces incident to the original edge with two faces that each properly intersect the x-axis. (3) Finally, any edge with both endpoints in S is flipped. After this flip, the flipped edge and the two faces incident to it properly intersect the x-axis. These operations (contractions and flips) are done exhaustively, one by one, in any order. The operations maintain that S is a proper-good set in G'. As discussed above, either of these operations may introduce a separating triangle in G'. How this is handled is explained below. Once these operations are done exhaustively, removing the edges of G' both of whose endpoints are either strictly above or strictly below x-axis (see the green edges in Fig. 2(c)) yields a structure, called an *A-graph* (see Fig. 2(d)), in which every face is either a quadrilateral whose four edges each intersect the x-axis or have an endpoint on the x-axis or a triangle with vertices above, on, and below the x-axis. The main task now is to find a straight-line crossing-free drawing of this A-graph so that the vertices of S are moved to the specified locations on the x-axis. Once we achieve that, the edge-removal, flipping, and contracting operations are easily undone without changing the locations of vertices in S, resulting in straight-line crossing-free drawing of G with the vertices of S at their specified locations on the x-axis.

Finding a straight-line crossing-free drawingof this A-graph so that the vertices of S are placed

[5]These technical conditions ensure that, after drawing G' the union of the two faces incident on the flipped edge is a convex quadrilateral.

to the specified locations on the x-axis is the goal of the second part of the proof, which involves a combination of linear algebra and graph theory. The authors prove a very strong statement about the A-graph G'. They show that it is possible to obtain a straight-line crossing-free drawing of G' in which the intersection point of the closure of each edge with the x-axis is specified, subject to the ordering constraints imposed by the crossing-free drawing of G'. Note that this implies that S is a free-collinear set in G' since each vertex in S is incident to an edge of G' (recall that no edge incident to a vertex of S is flipped or contracted unless both of its endpoints are on the x-axis). The vertex v_i has at least three incident edges in G'. By specifying that the closure of each of these edges must intersect the x-axis at $(x_i, 0)$ we ensure that v_i is mapped to $(x_i, 0)$.

In addition to specifying the location where each edge crosses the x-axis, the locations of the three vertices x, y, z on the convex hull of this crossing-free drawing (the vertices marked with $\blacksquare$ in Fig. 2(d)) can be chosen, provided that the choice agrees with the choice of intersection points chosen for the edges induced by x, y, z (the intersections marked with $\times$ in Fig. 2(d)). This latter requirement justifies the assumption that the graph does not contain separating triangles: The subgraph in the closure of the exterior of a separating triangle xyz can be drawn inductively. This determines the locations of x, y, and z which are then specified when inductively drawn the graph in the closure of the interior of the separating triangle.

The proof works by assigning slopes to the edges in G'. To avoid vertical edges, it is helpful to rotate the coordinate system by 90 degrees so that the roles of the x- and y-axes swap. To prove this stronger claim about A-graphs, the authors set up a system M of linear equations whose variables are the slopes of the edges of G' and whose coefficients are determined by the (given) intersection location of each edge with the y-axis. Most of the equations in M correspond to the fact that three or more edges incident to a common vertex v must intersect in a single point (the location of v).[6]

The hardest step in this proof is to show that the linear system M has a unique solution. Here, the authors leverage the fact that S is a proper-good set in G' and, therefore, a collinear set in G'. Therefore, G' has a straight-line crossing-free drawing Γ_0 in which the vertices of S appear, in order, on the y-axis. From this straight-line crossing-free drawing Γ_0, one can read off a system M_0 of linear equations that has the same structure as M but with different coefficients (that are determined by the y-coordinates of vertices in S in Γ_0). The system M_0 has a solution, namely the solution given by the slopes of the edges in Γ_0. The authors first show that this solution to M_0 is unique. The final step involves continuously modifying the coefficients in M_0 to obtain a continuum of linear systems M_t, $0 \le t \le 1$ where $M_1 = M$. Using the fact that M_0 has a unique solution as a starting point, they are then able to show that M_t has a unique solution for each $0 \le t \le 1$. In particular, $M_1 = M$ has a solution, which determines the desired straight-line crossing-free drawing of G'

Every free-collinear set is a free set: The equivalence between free-collinear sets and free sets has a disappointingly easy proof. The following argument appears in Bose et al. [17], but similar "perturb and scale" arguments are fairly common.

Let $S := (v_1, \dots, v_s)$ be a free-collinear set in G and let $x_1 < \cdots < x_s$ and $y_1, \dots, y_s$ be real numbers. Since S is a free-collinear set in G, G has a straight-line crossing-free drawing in Γ_0 where v_i is at $(x_i, 0)$ for each $i \in \{1, \dots, s\}$. It follows easily from the definition of straight-line crossing-free drawing that there exists some $\epsilon > 0$ such that perturbing each of the vertices in Γ_0 by at most ϵ results in another straight-line crossing-free drawing Γ_0'. Let $y := \max\{|y_1|, \dots, |y_s|\}$. Define Γ_0' by moving v_i to $(x_i, \epsilon y_i / y)$, for each $i \in \{1, \dots, s\}$ and leaving the other vertices of G fixed. Then each vertex in Γ_0' has been moved a distance of at most ϵ from its location in Γ_0, so Γ_0' is a straight-line crossing-free drawing of G. Finally, multiply the y-coordinate of each vertex by y/ϵ to obtain a

[6]Some additional constraints, call proportionality constraints, are placed on the slopes of edges incident to vertices on the y-axis.

final straight-line crossing-free drawing Γ. This y-scaling is an affine transformation that does not introduce any crossings, so Γ is a straight-line crossing-free drawing of G and, for each $i \in \{1,\dots,s\}$, Γ places v_i at (x_i, y_i).

3 Graph Classes with Large Free Sets

What is the maximum size of a free set that is guaranteed to exist in any n-vertex planar graph? Ravsky and Verbitsky [48] observed that upper bounds on this value can be obtained from existing work on the circumference of cubic triconnected planar graphs. They observe that if a triangulation G has a collinear set of size ℓ (and thus a large free set by Theorem 1), then its dual graph G^* (which is a cubic triconnected planar graph) has a cycle of length $\Omega(\ell)$. The length $c(G^*)$ of a longest cycle in G^* is called the *circumference* of G^*. The circumference of cubic triconnected planar graphs has a long and rich history dating back to at least 1884 when Tait [55] conjectured that every such graph is Hamiltonian.

Tait's Conjecture was famously disproved in 1946 by Tutte, who gave an example of a non-Hamiltonian cubic triconnected planar graph having 46 vertices [56]. Repeatedly replacing vertices of Tutte's graph with copies of itself gives a family of graphs, $\langle G_i : i \in \mathbb{N}\rangle$ in which G_i has $46 \cdot 45^i$ vertices and circumference at most $45 \cdot 44^i$. Stated another way, n-vertex members of the family have circumference $O(n^\sigma)$, for $\sigma = \log_{45}(44) < 0.9941$. The current best upper bound of this type is due to Grünbaum and Walther [40] who construct a family of cubic triconnected planar graphs in which n-vertex members have circumference $O(n^\sigma)$ for $\sigma = \log_{23}(22) < 0.9859$. The dual of such a graph is a triangulation having $\Theta(n)$ vertices whose largest free set has size $O(n^\sigma)$.

Since not all planar graphs have free sets of linear size, it is natural to ask which subclasses of planar graphs do. Two obvious candidates are planar graphs of maximum degree Δ and planar graphs of treewidth at most k. However, constructions like those described above can be used to rule out this possibility except for $\Delta < 7$ and $k < 5$, as we now explain.

Owens [45] constructs a family of n-vertex cubic triconnected planar graphs whose faces have size at most 7, and that contain no cycle of length $\Omega(n^{0.9976})$. The dual of such a graph is a triangulation having $\Theta(n)$ vertices and maximum-degree 7 whose largest free set has size $O(n^{0.9976})$.

Ravsky and Verbitsky [48] show that a construction based on the Barnette-Bosák-Lederberg graph produces triangulations of treewidth at most 8 whose largest free set has size $o(n)$. Da Lozzo et al. [24] observe that the recursive construction based on Tutte's counterexample to Tait's Conjecture [56] leads to a triangulation of treewidth at most 5 whose largest free set has size $O(n^{0.9941})$.

Cano et al. [20] also use upper bounds on the circumference of cubic triconnected planar graphs, along with the Erdős-Szekeres Theorem to show that $\mathrm{fix}_{\mathcal{G}}(n) \in O(n^{0.4948})$.

Since not all planar graphs have linear size free sets, we first consider the subclasses of planar graphs that do admit such large free sets (in Section 3.1). We then study general planar graphs (in Section 3.2). The relationship between these two key graph parameters (largest free set and circumference) is then used to obtain strong bounds for bounded-degree planar graphs (in Section 3.3).

3.1 Subclasses of planar graphs with linear-sized free sets

A *level planar drawing* of a graph is a straight-line crossing-free drawing in the plane, such that the vertices are placed on a sequence of parallel lines (called *levels*), where each edge joins vertices in two consecutive levels. For example, a natural straight-line crossing-free drawing of a tree T, with root r, places each vertex of T that is at distance i from r on the line $y = -i$. Thus, trees have level planar drawings.

The following easy argument shows that n-vertex graphs that have level planar drawings have

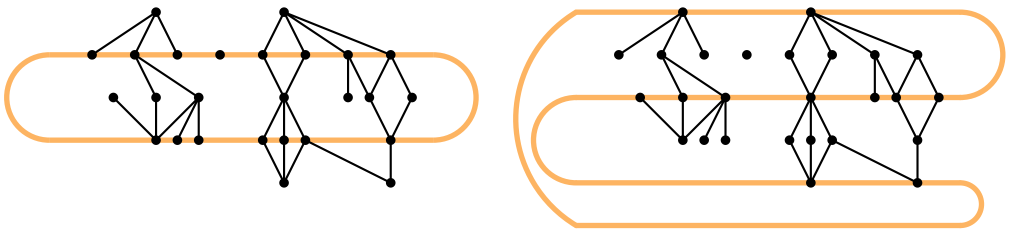


Figure 3: Two proper-good curves in a level planar graph.

free sets of size at least $n/2$. See Fig. 3. Starting with a level planar drawing Γ, create a simple closed curve C that coincides with the drawing in some set of levels. If C does not contain any two consecutive levels of Γ, then C intersects Γ only in the vertices of G that form an independent set. Such a curve is thus a proper-good curve, and by having it intersect every second level of G one can ensure that Γ contains at least $\lceil n/2 \rceil$ vertices of G.

Note that the preceding argument holds even if the definition of level planar drawings is relaxed to allow: consecutive vertices within a level to be adjacent; edges between consecutive levels to be strictly y-monotone instead of straight; and even edges to go between non-consecutive levels (as long as they are not too far). This leads to the following definition: An *s-span weakly level planar* drawing is a crossing-free drawing in the plane, such that the vertices are placed on a sequence of parallel lines, where each edge e is either a straight-line segment between two consecutive vertices on the same level (called, a *horizontal edge*, or a strictly y-monotone curve that intersects at most $s+1$ levels (called, a *vertical edge*). A graph is *s-span weakly level planar* if it has an s-span weakly level planar drawing. See [36] for example, for a similar definition. The following lemma formalizes the usefulness of this notion for obtaining free sets. This connection between 1-span weakly level planar level planar drawings have been observed by [48].

Lemma 2. *Every n-vertex s-span weakly level planar graph G has a free set of size at least $\lceil n/(s+1) \rceil$.*

Proof. Refer to Fig. 4. Number the levels in an s-span weakly level planar drawing Γ of G by $0, 1, 2 \dots$. Then for some $i \in \{0, 1, \dots, s\}$, the union of levels $j = i \mod (s+1)$ has at least $\lceil n/(s+1) \rceil$ vertices, S of G. Moreover $G[S]$ is a forest of paths (induced by the horizontal edges and the vertices on these levels). As in the case of level planar graphs, depict now a closed curve C such that C contains these levels of Γ (and thus the vertices of S) such that C intersects each vertical edge of Γ at most once. This is possible since vertical edges are y-monotone and their endpoints lie on the levels whose difference (the absolute value) is at most s. Such a curve C is a proper-good curve and thus by Theorem 1, S a proper-good set whose vertices are ordered by their appearance on C. □

Exactly the same proof used to show Lemma 2 shows the following "hereditary" variant:

Lemma 3. *For any s-span weakly level planar graph G and any subset X of vertices in G, G has a free set $S \subseteq X$ of size at least $\lceil |X|/(s+1) \rceil$.*

These lemmas have several immediate consequences. Firstly, as argued above, breath-first-search leveling of trees can be easily turned into level planar drawings (and thus, 1-span weakly level planar drawings). Similarly, breath-first-search leveling of outerplanar graphs can be turned into 1-span weakly level planar drawings of such graphs, as proved by Felsner, Liotta, and Wismath [32]. A natural way to draw the $n \times n$ grid graph is a 1-level planar drawing. More generally, Bannister, Devanny, Dujmović, Eppstein, and Wood [7] show that squaregraphs are 1-span weakly level planar. A *squaregraph* is a graph that has a crossing-free drawing in which each bounded

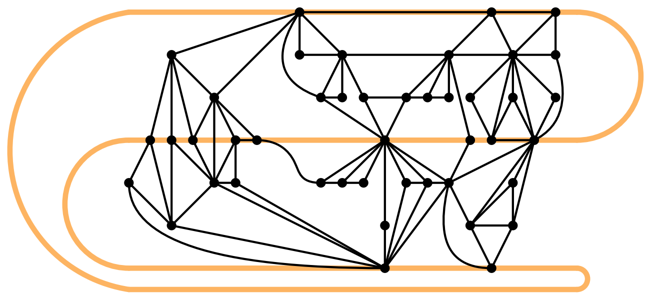
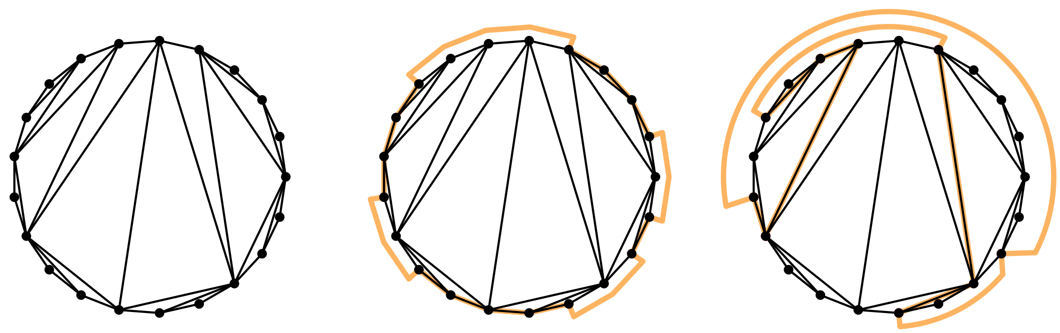


Figure 4: A 2-span weakly level planar drawing and a proper-good curve that contains every third level.



(a) (b) (c)

Figure 5: An edge-maximal outerplane graph G; (b) a proper good curve guaranteed by Lemma 4; and (c) a proper good curve guaranteed by Lemma 2.

face is a 4-cycle and each vertex either belongs to the unbounded face or has four or more incident edges. The same authors also show that Halin graphs are 1-span weakly level planar. Giacomo et al. [36] identified several classes of planar graphs that are s-span weakly level planar, for some constant s.

Corollary 1. *Let G be a n-vertex tree, outerplanar graph, Halin graph, or square graph and let X be any subset of vertices of G. The G has a free set $S \subseteq X$ of size at least $\lceil |X|/2 \rceil$. In particular, G has a free set of size at least $\lceil n/2 \rceil$.*

The $\lceil n/2 \rceil$ lower bound for trees appears in [17] in the context of untangling. The extension to outerplanar graphs appears in [38], also in the context of untangling. The proof for outerplanar graphs by Goaoc et al. [38] produces a proper-good curve that is qualitatively different than the one in Corollary 1. See Fig. 5. In particular, the curve obtained there is contained in the closure of the outer face of G and contains G in the closure of its interior. This distinction turns out to be important when studying general planar graphs. Goaoc et al. [38] prove the following result although it is not stated in this form. We include a variant of their proof.

Lemma 4. *Let G be an edge-maximal n-vertex outerplane graph for some $n \geq 4$. Then there exists a proper-good curve C that is contained in the closure of the outer face and C, that contains G in the closure of its interior, and that contains at least $\frac{n}{2} + 1$ vertices of G. Thus the vertices in C form a proper-good set of size at least $\frac{n}{2} + 1$.*

Proof. Since G is edge-maximal and $n \geq 4$, the boundary of the outer face is a cycle, denoted by O. Consider a set $S \subseteq V(G)$ that has the following two properties: $G[S]$ induces a forest of paths

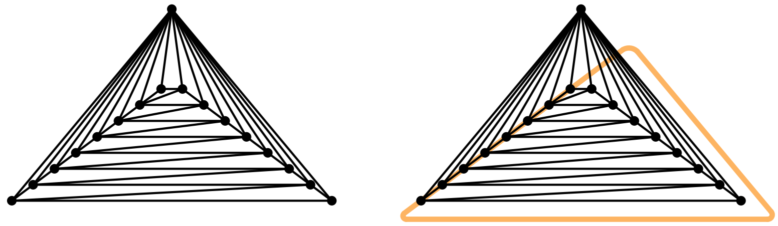


Figure 6: A graph of pathwidth-2 with no $o(n)$-span weakly level planar drawing that contains a free set of size $(n-1)/2$

and all the edges of $G[S]$ lie on O. It is easy to see that every such set S is a proper-good set of G. Moreover, it is not hard to produce a proper-good curve that is contained in the closure of the outer face of G and contains the vertices and edges of $G[S]$ by closely tracing the outer face of G.

We now prove that G has such a set S of claimed size. Let T denote the weak dual of G. T is a tree on at least two vertices since $n \geq 4$. Each vertex of T has a degree 1, 2 or 3, so T is a binary tree. Let t_1 and t_3 denote the number of degree 1 and degree 3 vertices in T, respectively. It is well known that in every binary tree $t_1 = t_3 + 2$.

Consider now the graph G' obtained from G by removing the edges with both endpoints on the outer face (in other words, G' contains only the chords of O). Every independent set in G' meets the two conditions imposed on S earlier. Thus it remains to prove that G' has independent set S of size $\frac{n}{2}+1$. Construct S greedily in G' as follows: put in S the vertex of G' of minimum degree; remove that vertex and its neighbors from G' to obtain a new G'; and, repeat. S is clearly an independent set in G'. It remains to show that it has the claimed size. In the moment a vertex is placed in S its degree in current G' was 0, 1 or 2. Let n_i denote the number of vertices that had degree i when they were placed in S. Thus $|S| = n_0 + n_1 + n_2$. From the description of the algorithm, it follows that $n = n_0 + 2n_1 + 3n_2$. Finally, each leaf of T contributes 1 to n_0. Thus $n_0 \geq t_1$. Each vertex of G' that contributes to n_2 corresponds to a unique face whose dual vertex has degree 3 in T. Thus $n_2 \leq t_3$. Combining that inequality with the earlier obtained equality, $t_1 = t_3 + 2$, we get that $n_0 \geq n_2 + 2$. To summarize, we have the two equations and one inequality:

$$|S| = n_0 + n_1 + n_2 \qquad n = n_0 + 2n_1 + 3n_2 \qquad n_0 - n_2 \geq 2 \ .$$

Replacing n_1 in the first equality with $n_1 = (n - n_0 - 3n_2)/2$ obtained from the second equality gives $|S| = \frac{n}{2} + \frac{n_0 - n_2}{2}$. Combined with the last inequality, we obtain the claimed result. □

Ravsky and Verbitsky [48] extend this result even further to show that all graphs of treewidth 2 have free collinear sets of linear size. This extension does not follow from the argument used to prove Lemma 2, since there is a class of planar graphs of treewidth at most 2 whose graphs do not have s-span weakly level planar drawings for any fixed s [11].

Having a weakly level planar drawing of small span is a too strong of a condition to provide a general tool for finding large free sets in wider classes of planar graphs. In addition to the example of 2-trees, there are planar graphs of bounded pathwidth, illustrated in Fig. 6 that do not have $o(n)$-span weakly level planar drawings (a result that can be derived from Biedl [11, Theorem 5]) and yet each graph in the class has an induced path of length $(n-1)/2$ that can be covered by a proper-good curve and thus the vertices on that path form a free set of size at least $(n-1)/2$, by Theorem 1.

Da Lozzo et al. [24] show that the linear bound for treewidth 2 graphs [48] extends to planar graphs of treewidth at most 3:

Theorem 2 ([24])**.** *Every n-vertex planar graph of treewidth at most three has a free set of size at least $\lceil \frac{n-3}{8} \rceil$.*

As noted in the introduction to this section, Theorem 2 cannot be generalized to all n-vertex planar graphs of treewidth at most 5. This leaves open the question of whether a linear bound is possible for planar graphs of treewidth at most 4.

For planar graphs of *large* treewidth, Da Lozzo et al. [24] use the fact that any planar graph of treewidth k contains a $k \times k$ grid-minor to show that planar graphs with large treewidth have large free sets:

Theorem 3 ([24])**.** *Every planar graph of treewidth k has a free set of size $\Omega(k^2)$.*

Theorem 3 implies that all n-vertex planar graphs of treewidth $\Omega(\sqrt{n})$ have free set of linear size (a vast generalization of the above observation that square grids have linear size free sets).

Rather than consider planar graphs of small treewidth, one can also consider planar graphs of small (maximum) degree. Da Lozzo et al. [24] prove the following result in this vein:

Theorem 4 ([24])**.** *Every n-vertex planar triconnected cubic graph has a free set of size at least $\lceil \frac{n}{4} \rceil$*

Da Lozzo et al. [24] suggest the possibility of extending Theorem 4 to show the existence of a linear-sized free set in any planar graph of maximum degree 3. As discussed in the introduction to this section, no such result is possible for all planar graphs of maximum degree 7. For $\Delta \in \{3,4,5,6\}$ it is still open whether a linear bound is possible for all planar graphs of maximum degree Δ.

3.2 Free Sets in Planar Graphs

Theorem 5. *Every n-vertex planar graph G has a free set of size at least $\sqrt{n/2}$.*

A version of Theorem 5 with an $\Omega(\sqrt{n})$ bound is attributed to Bose et al. [17] though the authors at the time were working on the untangling problem discussed in the introduction, so their result is never stated in terms of free sets. They prove that $\text{fix}_G(n) = \Omega(n^{1/4})$ by proving that a triangulation G either contains an induced outerplane graph of size $\Omega(\sqrt{n})$ or a free-collinear set of size $\Omega(\sqrt{n})$. At the time, the equivalence between proper-good sets, collinear sets, and free-collinear sets was not known, so their proof includes both combinatorial and geometric elements (including a proof that free-collinear and free sets are equivalent). In the following, we extract these combinatorial elements from [17] and use Lemma 4 (proven in [38]) to give a self-contained proof of Theorem 5 with the best currently known bound of $\sqrt{n/2}$.

The *interior*, $\text{int}(G)$, of a near-triangulation G is the interior of the cycle that bounds the outer face of G.

Lemma 5. *Let G be a near-triangulation, and let v and w be two points (possibly vertices) on the boundary of the outer face of G such that vw is not an edge of the outer face of G. Then there exists a simple open curve $C : (0,1) \to \text{int}(G)$ with endpoints v and w that is proper with respect to G.*

Proof. (Da Lozzo et al. [24] give a 2-line proof of Lemma 5 using Tutte's convex embedding theorem.) If vw if a (internal) edge of $E(G)$ then the curve C consists of the edge vw. Then C contains the edge vw, intersects each other edge incident to v or w in one point, and does not intersect any other edge of G, so C is proper with respect to G.

Since vw is not an internal edge of G and since G is a near-triangulation, there is no internal face of G that contains both v and w on its boundary. Refer to Fig. 7. Let f_v be some internal face that contains v and f_w some internal face that contains w. Let G^+ be the weak dual of G. Let

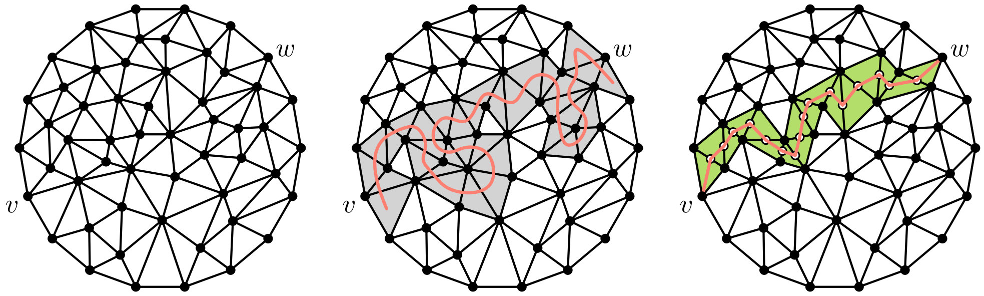


Figure 7: The near-triangulation G, the curve D from f_v to f_w, and the proper-good curve C from v to w.

$D : [0,1] \to \mathrm{int}(G)$ be a simple open curve with one endpoint in f_v and one in f_w such that D does not contain any vertices of G, as illustrated in Fig. 7(b). The sequence of faces of G intersected by D defines a walk W in G^+ with endpoints f_v and f_w. The walk W contains a path P in G^+ from f_v to f_w. When traversing the path P from f_v to f_w, let f'_v be the last face on P that contains v on its boundary; and, let f'_w be the first face on P that contains w on its boundary. Let the resulting sub-path between f'_v and f'_w be denoted by P'.

Let $e^+_1,\dots,e^+_t$ be the sequence of edges in P'. For each $i \in \{1,\dots,t\}$, the edge e^+_i in G^+ corresponds to some inner edge e_i of G. For each $i \in \{1,\dots,t\}$, let p_i be any point in the interior of e_i, let $p_0 := v$, and let $p_{t+1} = w$. Then, for each $i \in \{1,\dots,t+1\}$ there is an inner face f_i of G with the points p_{i-1} and p_i on the boundary of f_i. Construct an open simple curve C that visits $p_0,\dots,p_{t+1}$ in order in such a way that the portion of C between p_{i-1} and p_i is contained in the interior of f_i, for each $i \in \{1,\dots,t+1\}$. By the choice of f'_v, no edge incident to v intersects C other than in v. Similarly, by the choice of f'_w, no edge incident to w intersects C other than in v. Thus, the curve C satisfies the requirements of the lemma. □

Proof of Theorem 5. If G_0 is a triangulation such that G is a spanning subgraph of G_0, then clearly, any free set in G_0 is also a free set in G. Thus we may assume, without loss of generality, that G is a triangulation.

The steps in this proof are illustrated in Fig. 8. Fix a crossing-free drawing of G and let v_1, v_2, and v_n be the three vertices on the outer face of G.

We use a *canonical ordering* $v_1,\dots,v_n$ of $V(G)$, which has the following property: For each $i \in \{3,\dots,n\}$, the induced graph $G_i := G[\{v_1,\dots,v_i\}]$ is a near-triangulation that contains the vertex v_i and the edge v_1v_2 on its outer face. The existence of this ordering is proven by, for example, de Fraysseix, Pach, and Pollack [34]. In this ordering, the introduction of v_i introduces $d_i \geq 2$ new edges $v_iw_{i,1},\dots,v_iw_{i,d_i}$ to G_i that do not appear in G_{i-1}. The edges $e_i := v_iw_{i,1}$ and $e'_i := v_iw_{i,d_i}$ appear on the outer face of G_i and the vertices $w_{i,2},\dots,w_{i,d_i-1}$ on the outer face of G_{i-1} no longer appear on the outer face of G_i, as illustrated in Fig. 8(a). This defines a sequence of *frames* $F_1,\dots,F_n$, where $V(F_i) := \{v_1,\dots,v_i\}$ and $E(F_i) := \bigcup_{j=1}^{i}\{e_i, e'_i\}$. We treat each frame F_i as a directed acyclic graph with a single source v_1 and a single sink v_2 where the direction of edges e_i and e'_i are the same in each frame $F_i,\dots,F_n$ that includes these two edges.[7] Since the final frame $F := F_n$ is a directed acyclic graph, its transitive closure defines a partially ordered set $(V(G), \prec)$ in which $v \prec w$ if and only if F contains a directed path from v to w.

[7] For readers familiar with Schnyder Woods [52], the final frame F_n can be obtained by taking the union of two trees T_1 and T_2 (rooted at v_1 and v_2, respectively) in a Schnyder Woods, directing each edge of T_1 away from its root and directing each edge of T_2 towards its root, as illustrated in Fig. 8(b).

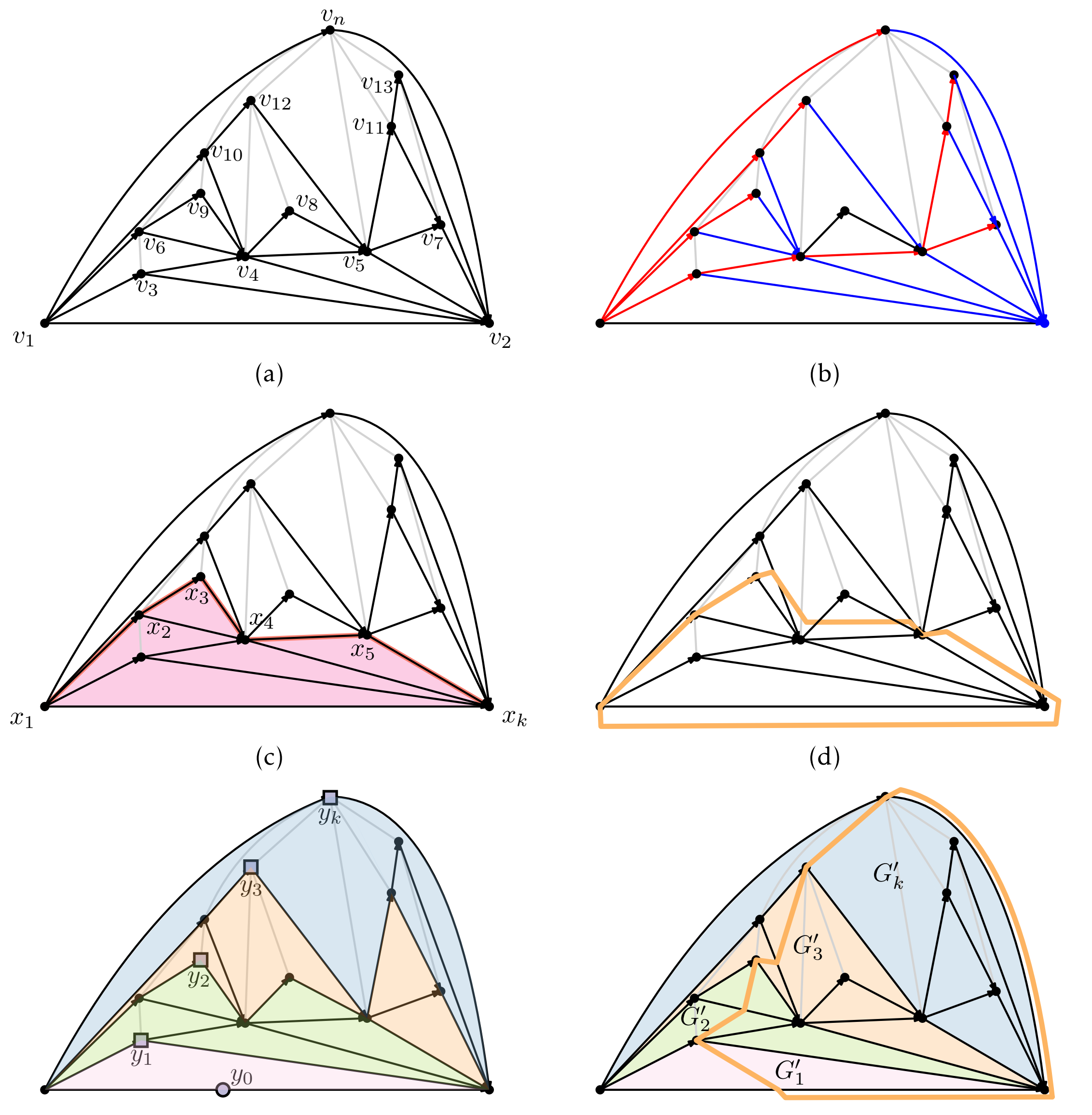


Figure 8: (a) and (b) A canonical ordering $v_1, \ldots, v_n$ of a triangulation G and the resulting frame F.
(c) and (d): a chain of $(V(G), \prec)$ and the resulting proper-good curve C
(e) and (f): An antichain of $(V(G), \prec)$ and the resulting proper-good curve C.

First, consider some maximal chain $x_1 \prec \cdots \prec x_k$ in this partial order. Since F is maximal, $x_i x_{i+1}$ is an edge of F for each $i \in \{1,\ldots,k-1\}$. Since F has a single source v_1 and a single sink v_2, $x_1 = v_1$ and $x_k = v_2$. Thus, $C_x := x_1,\ldots,x_k$ is a cycle in G, as illustrated in Fig. 8(c).

Suppose that G contains some edge $x_i x_j$ that is a chord of C_x. We now argue that the edge $x_i x_j$ is embedded in the interior of the cycle C_x. Without loss of generality, suppose x_i appears after x_j in the canonical order, so $x_i = v_a$ and $x_j = v_b$ for some $a > b$. Then the edge $x_i x_j = v_a v_b$ is in the graph $G_a = G[\{v_1,\ldots,v_a\}]$, so $v_b \in \{w_{a,1},\ldots,w_{a,d_a}\}$. The two neighbours x_{i-1} and x_{i+1} of x_i in C_x are not in the set $w_{a,2},\ldots,w_{a,d_a-1}$, so the interior of the cycle C_x contains the interior of the cycle $v_a, w_{a,1},\ldots,w_{a,d_a}$. Therefore the edge $v_a v_b = x_i x_j$ is in the interior of C_x. (Indeed, the only possibility is that $v_b \in \{w_{a,1}, w_{a,d_a}\}$)

Therefore, $G_x := G[\{x_1,\ldots,x_k\}]$ is an induced outerplane subgraph of G whose outer face is bounded by the cycle C_x. By Lemma 4, there is simple closed curve C (which closely follows C_x) that contains a set S of at least $k/2$ vertices of G_x, that is contained in the closure of the outer face of G_x, that contains G_x in the closure of its interior, and that is proper and good with respect to G_x. Since each edge of G with two endpoints in G_x is contained in the closure of the interior of C_x, C (if drawn sufficiently close to C_x) intersects each edge in $E(G) \setminus E(G_x)$ in at most one point, as illustrated in Fig. 8(d). Thus C is a proper-good curve for G that contains the vertices in S. By Theorem 1, S is a free set in G of size at least $k/2$.

Next, consider some maximal antichain $S := y_1,\ldots,y_k := v_{i_1},\ldots,v_{i_k}$ of $(V(G),\prec)$ of length $k > 1$ ordered by canonical ordering so that $i_1 < \cdots < i_k$, as illustrated in Fig. 8(e). Since $k > 1$, S does not contain v_1 or v_2. Since S is maximal and does not contain v_1 or v_2, $y_k = v_n$. Consider the sequence of cycles $C_1,\ldots,C_k$ where C_j is the cycle that bounds the outer face of G_{i_j}. By definition of canonical ordering, the interior of C_j contains the interior of C_{j-1} for each $j \in \{2,\ldots,k\}$. The nesting of these cycles is illustrated in Fig. 8(e), where each new colour shows the interior of C_j that is not contained in the interior of C_{j-1}. Furthermore, the interior of C_j must contain the vertex y_{j-1} since, otherwise both y_{j-1} and y_j are on the outer face of G_{i_j}, which would mean that y_{j-1} and y_j are comparable. Consider now the union of two cycles C_{j-1} and C_j (or rather the union of the two closed curves that represent these two cycles in the crossing-free drawing of G). This union has one (bounded) faces, f, that contains both y_{j-1} and y_j on its boundary. The boundary of f is a cycle D_j in G comprised of two paths: a path in C_{j-1} containing y_{j-1} and a path in C_j containing y_j. Note that the union of all the cycles $D_1,\ldots D_k$, (or rather the union of the k closed curves that represent these k cycles in the crossing-free drawing of G) defines $k+1$ faces in the plane, the weak dual of which is a path. These faces are illustrated in Fig. 8(f), where the interior of each face is assigned its own colour.

We now construct a proper-good curve C for G that contains the vertices of S. Let y_0 be a point in the interior of the edge $v_1 v_2$. Let $G'_1 := G_{i_1}$ and, for each $j \in \{2,\ldots,k\}$, let G'_j be the near-triangulation whose outer face is bounded by the cycle D_j described in the previous paragraph, and whose inner faces are faces of G. Graphs G'_j are illustrated in Fig. 8(f) by having their interiors shaded. Observe that $G'_1,\ldots,G'_k$ have pairwise disjoint interiors. By Lemma 5, there is a simple open curve $I_j{:}(0,1) \to \mathrm{int}(G'_j)$ with endpoints y_{j-1} and y_j that is proper with respect to G'_j, for each $j \in \{1,\ldots,k\}$. Since $G'_1,\ldots,G'_k$ have pairwise disjoint interiors, and since the union of the closures of $\mathrm{int}(G'_1)\ldots,\mathrm{int}(G'_k)$ has a path for its week dual, the open curve obtained by taking the union of the closures of $I_1,\ldots,I_k$ is simple and it is proper with respect to G. Finally, let I_{k+1} be a simple open curve with endpoints y_0 and $y_k = v_n$ whose interior is contained in the outer face of G, as illustrated in Fig. 8(f). Then the simple closed curve C obtained from the union of the closure of open curves $I_1,\ldots,I_{k+1}$ is a proper-good curve for G that contains S. By Theorem 1, S is a free set of size k.

To complete the proof we use Dilworth's Theorem, which guarantees that the poset $(V(G),\prec)$ contains a chain of size at least $\sqrt{2n}$ or an antichain of size at least $\sqrt{n/2}$. In either case we obtain a free set of size at least $\sqrt{n/2}$. $\square$

The following generalization of Theorem 5, observed by Dujmović [27], has an almost identical proof, except that one considers the induced poset $(X, \prec)$ rather than $(V(G), \prec)$:

Theorem 6. *For every planar graph G and every $X \subseteq V(G)$, X contains a free set of size at least $\Omega(\sqrt{|X|})$.*

3.3 Free Sets in Max-Degree-Δ Planar Graphs

Da Lozzo et al. [24] suggest the possibility that, since upper bounds on the circumference of dual graphs can be used to obtain upper bounds on the size of free sets, maybe lower bounds on circumference can be used to prove the existence of large collinear sets. Dujmović and Morin [28] show that, for planar graphs of bounded degree, this is indeed the case. In short, they show that a triangulation G of maximum-degree Δ and whose dual has circumference $c(G^*)$ has a free set of size $\Omega(c(G^*)/\Delta^4)$. A series of results has steadily improved the lower bounds on the circumference of n-vertex (not necessarily planar) cubic triconnected graphs [10, 12, 14, 41, 44]. The current record is held by Liu et al. [44] who show that, for any n-vertex cubic triconnected graph G^*, $c(G^*) \in \Omega(n^{0.8})$.

In the remainder of this subsection we describe some of the techniques used in [28] to establish the $\Omega(c(G^*)/\Delta^4)$ result. Cycles in G^* are relevant to free sets because every cycle in G^* corresponds to a proper-good curve in G. The resulting curve C does not contain any vertices of G, but it is natural to try reroute C to obtain a new curve G that goes through some vertices of G. Fig. 9 (a) depicts a situation where this rerouting fails because the resulting curve is no longer proper because it intersects the red edge in Fig. 9(b) in two points. This leads to the following definition that lays out conditions under which such rerouting of curve C is safe. We say that a vertex v of G is *caressed* by C if the edges of G that are incident to v and intersected by C appear consecutively around v, as depicted in Fig. 9(c) (the thick grey edges crossed by C are consecutive around v unlike in Fig. 9(a) where the set of thick grey edges crossed by c is not consecutive around v). If C caresses v and intersects the edges $vv_1, \ldots, vv_r$ (see Fig. 9(c)) then there is a sequence of faces $f_0, \ldots, f_r$ where vv_i is the edge shared between f_{i-1} and f_i. Then the portion of C that intersects $f_0, \ldots, f_r$ can be replaced with a curve C' that enters f_0, proceeds directly to v, and immediate exits f_r (see Fig. 9(d)). Because the original curve C does not intersect any other edges incident to v, the modified curve C' is also a proper-good curve. This operation can be repeated on any set of caressed vertices that form an independent set:

Lemma 6. *Let C be a proper-good curve in a triangulation G and let S be an independent set of at least two vertices in G that are each caressed by C. Then S is a free set in G.*

Proof. For each $v \in S$, reroute C as described above so that C contains v. Since v is caressed by C, the rerouting that takes place at $v \in S$ causes C to intersect each edge incident to v in exactly one point, namely v, but does not change the intersection of C with any edges not incident to v. Since the vertices in S form an independent set, all of these rerouting operations do not cause C to intersect any edge of G in more than one point. Thus C is a proper-good curve that contains S so, by Theorem 1 S is a free set in G. (The condition that S have at least two vertices avoids the case in which C crosses all the edges incident to a single vertex v.) □

The requirement that a curve C caresses a vertex v of G is equivalent to requiring that the intersection of C with the face f_v of G^* that contains v is a path. When this happens, we say that C *caresses* f_v. Thus, finding a large collinear set in G is equivalent to finding a cycle in G^* that caresses many faces of G^*.

Lemma 7. *Let G be a triangulation and let G^* be the dual of G. If some cycle C in G^* caresses k faces of G^* then G contains a free set of size at least $k/4$.*

Proof. By the Four Colour Theorem [49], the faces of G^* can be coloured with four colours so that no two faces that share an edge are assigned the same colour. One of the resulting colour classes

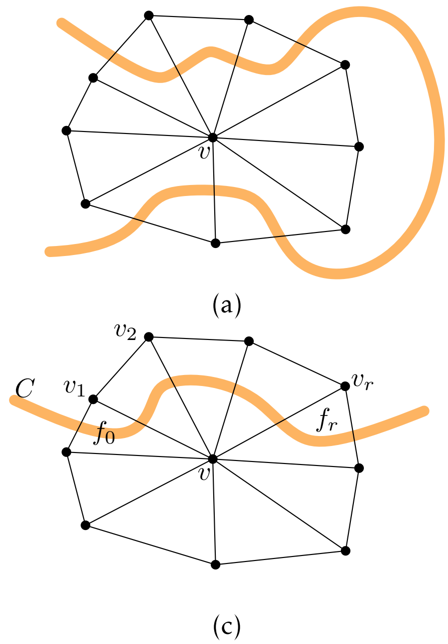


Figure 9: Attempting to reroute a dual cycle (a proper-good curve C) into a new proper-good curve C' so that it contains v.

contains at least $k/4$ faces of G^* that are caressed by C. These faces correspond to an independent set S of vertices of G that are caressed by C. The lemma now follows from Lemma 6. □

Unfortunately, there exists cubic triconnected planar graphs G^* with no large faces that have a Hamiltonian cycle that caresses only four faces of G^*. One such graph is shown in Fig. 10. The main technical result of Dujmović and Morin [28] is to show that such cycles can be modified to produce cycles that caress many faces:

Lemma 8 ([28])**.** *If G^* is an n-vertex cubic triconnected planar graph with no faces of size greater than Δ, then there exists a cycle C in G^* that caresses at least $\Omega(c(G^*)/\Delta^4)$ faces of G^*.*

Proof Sketch. Refer to Fig. 11. The proof of Lemma 8 is far too long to include here in any detail, so we give a high level sketch. For the sake of simplicity, let Δ be a fixed constant. Let C be a cycle in

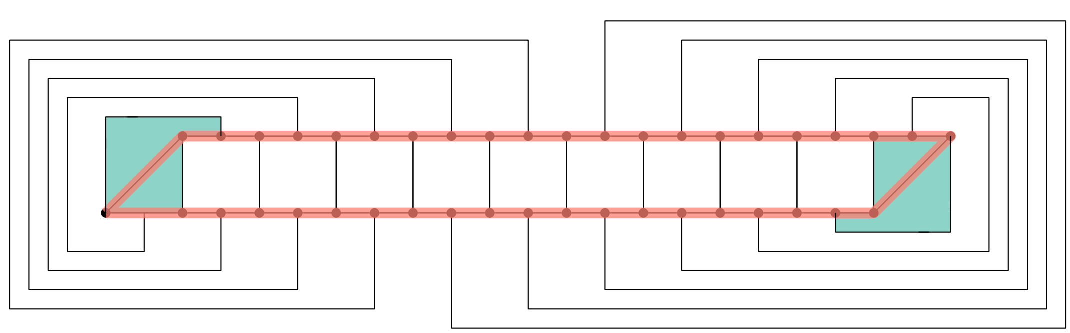


Figure 10: A Hamiltonian cycle in a cubic triconnected planar graph that caresses only four faces.

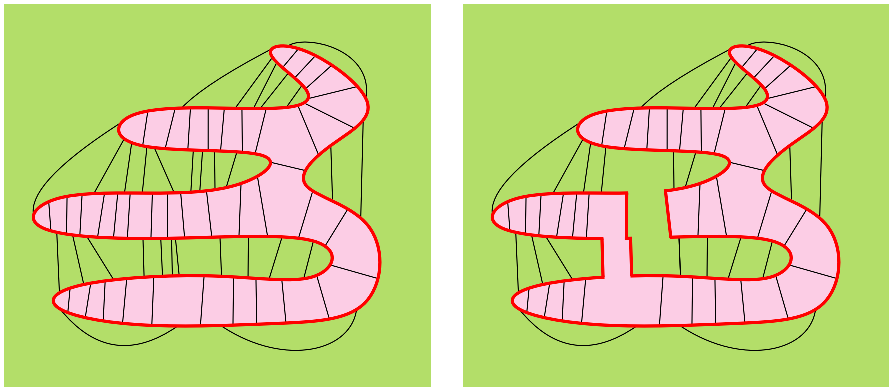


Figure 11: Performing surgery on the trees T_0 (pink) and T_1 (green) in order to increase the number of leaves.

G^*. Say that C *touches* a face of G^* if C and f share at least one edge. Since each face of G^* has at most Δ edges and each edge of C touches two faces of G^* (one inside C and one outside of C), the number of faces of G^* that are touched by C is at least $2|C|/\Delta$. At least $|C|/\Delta$ of these faces are in the interior of C and at least $|C|/\Delta$ of these are in the exterior of C.

The authors define a subgraph H of G^* that includes all the edges of C and such that any cycle in the dual graph H^* of H contains faces inside and outside of C. Removing the edges from H^* that correspond to edges of C produces two trees T_0 and T_1, where T_0 contains faces of H in the interior of C and T_1 contains faces of H in the exterior of C. For each $b \in \{0,1\}$, the tree T_b has two important properties:

1. Each leaf of T_b corresponds to a face f of H that contains at least one face of G^* that is caressed by C.
2. Let $f \in V(T_b)$ be a face of H that has degree δ in T_b and that contains τ faces of G^* touched by C, κ of which are caressed by C. Then $3\kappa + 2\delta \geq \tau$.

The second of these properties says that any node of T_b that contains many faces touched by C either caresses many of these faces or has high degree in T_b. The number of leaves in T_b is $2 + \sum_x(\deg_{T_b}(x) - 2)$, where the sum is over all non-leaf nodes x of T_b. Combining this with the first of these properties implies that C caresses $\Omega(|C|)$ faces of G^* or that T_b has $\Omega(|C|)$ nodes but at most $\epsilon|C|$ leaves, for some small $\epsilon > 0$. In the former case we are done. In the latter case, we conclude that the vast majority of nodes in T_b have degree two.

Since T_0 and T_1 share the boundary C, this can be used to show that T_0 contains a path P_0 of degree-2 nodes and T_1 contains a path P_1 of degree-2 nodes, each of size bounded by a constant (depending on Δ) such that moving the nodes in P_0 from T_0 to T_1 and moving the nodes in P_1 from T_1 to T_0 results in two new subgraphs T_0'' and T_1'' of H^* that contain at least one more leaf than T_0 and T_1. The graphs T_0'' and T_1'' are not necessarily trees, but the boundary they share defines a new cycle C' from which we can define two trees T_0' and T_1' as before. These two new trees differ from T_0 and T_1 only in a small neighbourhood of the paths P_0 and P_1, which ensures that T_0' and T_1' also have at least one more leaf than T_0 and T_1. At this point, the procedure is repeated on C'. Each application of this procedure may decrease the length of the cycle of C by at most a constant (for a fixed Δ), so this procedure can be repeated $\Omega(|C|)$ times, which produces a cycle $\tilde{C}$ that defines two trees $\tilde{T}_0$ and $\tilde{T}_1$ having a total of $\Omega(|C|)$ leaves, each of which contains a face of G^* caressed by C. $\square$

If G^* is the dual of a triangulation G then the size of each face of G^* is the degree of the corresponding vertex in G. This gives the following consequence of Lemma 8.

Corollary 2. *Let G be an n-vertex triangulation of maximum-degree Δ and let G^* be the dual of G. Then G has a free set of size $\Omega(c(G^*)/\Delta^4)$*

A result of Kant and Bodlaender [43] shows that one can add edges to any planar graph G_0 of maximum degree Δ to obtain a triangulation G of maximum degree at most $\lceil 3\Delta/2\rceil + 11$. Combined with Corollary 4, this establishes

Corollary 3. *Let G be an n-vertex planar graph of maximum-degree Δ and let G^* be the dual of G. Then G has a free set of size $\Omega(c(G^*)/\Delta^4)$.*

Corollary 2 shows that, for the class of planar graphs of maximum-degree $\Delta = O(1)$, the dual circumference and the size of the largest free set differ by only a constant factor. Combined with the best known lower bounds on the circumference of cubic triconnected graphs [44], Corollary 2 gives:

Corollary 4. *Let G be an n-vertex triangulation of maximum-degree Δ. Then G has a free set of size $\Omega(n^{0.8}/\Delta^4)$.*

4 Applications

In this section we discuss applications of free sets. For each of these (s) applications, and each of the (t) classes of planar graphs known to have large free sets, the use of free sets gives a result. Rather than provide an exhaustive list of $s \times t$ corollaries, we state the relationship between free sets and the application and give the most general corollary; the one about planar graphs obtained from Theorem 5.

4.1 Untangling

As discussed in the introduction, the original application of free sets was untangling. By Lemma 1, each lower bound (on the size of the largest free set in a graph family) obtained in Sections 3.1 to 3.3, readily turns into the square root of the same bound for the untangling problem on that family. For example, Theorem 5 and Lemma 1 yield:

Corollary 5. *Let $\mathcal{G}$ be the class of all planar graphs. For every n-vertex graph $G \in \mathcal{G}$, $\mathrm{fix}(G) \in \Omega(n^{1/4})$. In other words, $\mathrm{fix}_{\mathcal{G}}(n) \geq (n/2)^{1/4}$.*

We do not know if this bound is asymptotically optimal. Cano et al. [20] proved that for the class of planar graphs $\mathcal{G}$, $\mathrm{fix}_{\mathcal{G}}(n) \in O(n^{0.4965})$.

On the contrary, applying the known bounds on the largest free sets coupled with Lemma 1 gives tight bounds for some graph families. Specifically, the linear bounds on free sets in Section 3.1 and Lemma 1 imply $\Omega(n^{1/2})$ untangling bound for these families. Bose et al. [17] showed that there is a class of trees, in fact forest of stars, $\mathcal{T}$, such that $\mathrm{fix}_{\mathcal{T}} \in O(n^{1/2})$. Thus the $\Omega(n^{1/2})$ untangling bound is tight for all families in Section 3.1, except for the class of cubic triconnected planar graphs. The stars in $\mathcal{T}$ have unbounded degree. Thus a tight bound on untangling cubic triconnected planar graph is unknown.

4.2 Universal point subsets

A set of points P is *universal* for a set of planar graphs if every graph from the set has a straight-line crossing-free drawing where all of its vertices map to distinct points in P. Arguably, the most famous open problem in graph drawing, attributed to Bojan Mohar (1988), asks if there exist a constant c such that, for every n there exists a pointset of size $c \cdot n$ that is universal for the class of all n-vertex planar graphs. It is known that for all, large enough n, no universal pointset of size n exists for the class of n-vertex planar graphs – as first proved by de Fraysseix et al. [34]. The same authors also proved that the $O(n) \times O(n)$ integer grid is universal for all n-vertex planar graphs and thus universal pointsets of size $O(n^2)$ exists. Currently the best known lower bound on the size of a smallest universal pointset for n-vertex planar graphs is $1.239n - o(n)$ [50] and the best known upper bound is $n^2/4 - O(n)$ [6]. Closing the gap between $\Omega(n)$ and $O(n^2)$ is a major, and likely very difficult, graph drawing problem, open since 1988 [33, 34].

Various related notions have been introduced and studied in the literature with the aim of better understanding the universal pointset problem. A set P of k points in the plane is a *universal point subset* for a family of planar graphs, if for every graph G in the family, there exists a straight-line crossing-free drawing in which k vertices of G are placed at the k points in P. Universal point subsets were introduced by Angelini, Binucci, Evans, Hurtado, Liotta, Mchedlidze, Meijer, and Okamoto [2]. They proved that (a very flat convex chain of) $\lceil\sqrt{n}\rceil$ points is a universal point subset for the class of all n-vertex planar graphs. Giacomo, Liotta, and Mchedlidze [37] then proved that every set of $\Omega(\log n)$ points, as well as, every set of $\lceil n^{1/3}\rceil$ points in convex position is a universal point subset for the class of n-vertex planar graphs.

The following lemma and its consequences, first observed by Dujmović [27], is immediate from the definition of free sets.

Lemma 9. *Let k be a positive integer and $\mathcal{F}$ a family of planar graphs such that every graph in the family has a free set of size at least k. Then every set of k points in the plane is a universal point subset for $\mathcal{F}$.*

Note that this lemma does not just prove that there exists a set of k points that is a universal point subset for $\mathcal{G}$, it proves that *every* set of k points is a universal point subset for $\mathcal{G}$. This distinction turns out to be useful in other applications. Combining Lemma 9 and Theorem 5 gives:

Corollary 6. *Every set of $\lceil\sqrt{n/2}\rceil$ points is a universal point subset for the class of n-vertex planar graphs.*

It has been known for a long time that every set of n points in general position is a universal point (sub)set for the family of n-vertex outerplanar graphs [15, 22, 39]. Lemma 9 implies that all the families studied in Section 3.1, also admit universal point subsets of linear size. This includes, for example, planar graphs of treewidth at most 3 by Theorem 2, a strict superclass of the class of outerplanar graphs.

4.3 Simultaneous Geometric Embeddings

Simultaneous geometric embeddings were introduced by Braß, Cenek, Duncan, Efrat, Erten, Ismailescu, Kobourov, Lubiw, and Mitchell [19]. Since then there has been a plethora of work on the subject on many variants of the problem– see, for example a survey by Bläsius, Kobourov, and Rutter [13]. Common variants of the problem include those in which the mapping between the vertices of the two graphs is given and those in which the mapping is not given.

4.3.1 Without Mapping

A sequence of graphs $G_1, G_2, \dots, G_r$ where $|V(G_1)| \geq |V(G_2)| \geq \dots |V(G_r)|$ are said to have a *simultaneous geometric embedding without mapping (SGE-nomap)* if there exists a pointset P of size $|V(G_1)|$ such that each of $G_1, \dots, G_r$ has a straight-line crossing-free drawing where all of its vertices are mapped to distinct points in P. The following well known and still wildly open problem was asked by Braß et al. [19] in 2003, and is also listed among selected list of graph drawing problems in [18] (see also Problem 12): Does every pair of n-vertex planar graphs has SGE-nomap (for every positive integer n)? The statement is known *not* to be true when "pair" (that is, $r = 2$) is replaced by a bigger constant [21, 51], currently being $r = 30$ [54]. More generally, Steiner [54] showed that for every large enough n there exists a sequence of at most $(3+o(1))\log_2(n)$ n-vertex planar graphs that do not have a SGE-nomap.

While the problem of Braß et al. [19] still seems to be out of reach, in a different direction Angelini, Evans, Frati, and Gudmundsson [3] write: "What is the largest $k \le n$ such that every n-vertex planar graph and every k-vertex planar graph admit a geometric simultaneous embedding with no mapping? Surprisingly, we are not aware of any super-constant lower bound for the value of k?" The corollary below, noticed first by Dujmović [27], answered their question with the help of free sets, as detailed in the next lemma.

Lemma 10. *Let G_1 be a planar graph with free set of size at least k. Let G_2 be a planar graph on at most k vertices. Then G_1 and G_2 admit SGE-nomap.*

Proof. By Fáry's theorem, G_2 has a straight-line crossing-free drawing on some set, P_2, of $|V(G_2)| \le k$ points. Since G_1 has a free set S of size at least k, G_1 has a straight-line crossing-free drawing where $|P_2|$ vertices of G_1 are mapped to distinct points in P_2. Consider now the set of points, P, defined by the vertices in the drawing of G_1. This set is our desired pointset as it is a set of $|V(G_1)|$ points such that each of G_1 and G_2 has a straight-line crossing-free drawing where all of its vertices are mapped to distinct points of P. □

Lemma 10 and Theorem 5 immediately imply the following, aforementioned result.

Corollary 7. *For every n and every $k \le \sqrt{n/2}$, every n-vertex planar graph and every k-vertex planar graph admit a SGE-nomap.*

One can combine the many lower bounds (on the size of the largest free set in a graph family) obtained in Sections 3.1 to 3.3 to obtain results of similar flavour to Corollary Corollary 7. One example is the following, for every n and every $k \le n/4$, every n-vertex planar graph and every k-vertex triconnected cubic planar graph admit a SGE-nomap.

Note that universal point subsets of size k do not imply results on SGE-nomap, akin to 7. In particular, imagine that, as in the proof of Lemma 10, one starts with a straight-line crossing-free drawing of the smaller graph G_2. Even if the larger graph G_1 is in a class of graphs that have universal point subsets of size k, there is no guarantee that G_1 can be drawn on the specific pointset of size k used by the drawing of G_2. To be able to use universal point subsets for SGE-nomap problems, one needs the stronger variant of "all pointsets of size k" to be subset universal—something that is guaranteed by free sets of size k.

4.3.2 With Mapping

Two n-vertex planar graphs G_1 and G_2 on *the same vertex set*, $V := V(G_1) = V(G_2)$, are said to have a *k-partial simultaneous geometric embedding with mapping* (*k-PSGE-withmap*) if there exists a set $V' := \{v_1,\dots,v_k\} \subseteq V$, and a set $P := \{p_1,\dots,p_k\}$ of points such that each of G_1 and G_2 has a straight-line crossing-free drawing in which v_i is mapped to p_i, for each $i \in \{1,\dots,k\}$. PSGE-withmap on the whole vertex set (i.e. n-PSGE-withmap) has been widely studied leading to mostly negative results (thus giving another motivation to introduce this partial version). For example, it is known that, for every large enough n, there are pairs of n-vertex planar graphs that do not have n-PSGE-withmap [19]. In fact the same is true for graphs from very simple families of planar graphs, for example: for an n-vertex tree and an n-vertex path [5], for an n-vertex planar graph and an n-vertex matching [5] and for three n-vertex paths [19].

The k-PSGE-withmap problem was introduced[8] by Evans et al. [31] who proved that any two n-vertex trees have an $11n/17$-PSGE-withmap. Their proof uses their column planarity, which is

[8] In [31], they use the abbreviation k-PSGE for what we call k-PSGE-withmap. Also, the definition of k-PSGE-withmap, as introduced in Evans et al. [31], has one additional requirement, which states that if $v, w \in V$ are mapped to the same

the topic of the next section. Barba, Hoffmann, and Kusters [9] proved that any two n-vertex outerplanar graphs have an $n/4$-PSGE-withmap. Evans et al. [31] also observed that Corollary 5, the untangling result, implies that every pair of n-vertex planar graphs has an $\Omega(n^{1/4})$-PSGE-withmap. Namely, start with a straight-line crossing-free drawing of G_1. Since G_1 and G_2 have the same vertex set, the drawing of G_1 (or rather the positions of its vertices in the plane) defines a straight-line drawing of G_2 (that almost certainly has crossings). Untangling G_2 while keeping $\Omega(n^{1/4})$ of its vertices fixed (which is possible by Corollary 5) gives the result.

Theorem 7 ([31]). *Every pair of n-vertex planar graphs has an $\Omega(n^{1/4})$-PSGE-withmap.*

However, the above untangling argument fails if we try to apply it one more time. The following generalization of the k-PSGE problem (to more than two graphs) illustrates this. Given any set $Q := \{G_1,\dots,G_r\}$ of planar graphs on *the same vertex set*, V, we say that the graphs in Q have a *k-partial simultaneous geometric embedding with mapping* (k-PSGE-withmap) if there exists a set $V' := \{v_1,\dots,v_k\} \subseteq V$, and a set $P := \{p_1,\dots,p_k\}$ of points such that each graph in Q has a straight-line crossing-free drawing in which v_i is mapped to p_i, for each $i \in \{1,\dots,k\}$.

If we try to mimic the earlier untangling argument that proves Theorem 7, it fails for $r = 3$ already since we need to be able to guarantee that when G_3 is untangled the set of its vertices that stays fixed has a large intersection with the set that remained fixed when untangling G_2. It is here that we need the stronger version of Theorem 5, namely Theorem 6.

We start by presenting a lemma about PSGE-withmap for two graphs via free-sets. Its proof uses the common trick of taking advantage of the fact that points in the plane have two degrees of freedom, and thus one ordering can be imposed on the x-coordinates and a different ordering on y-coordinates (see for example [8]).

Lemma 11. *Let G_1 and G_2 be two planar graphs on the same vertex set $V := V(G_1) = V(G_2)$. Let $S \subseteq V$ be an (unordered) free set in G_1 and an (unordered) free set in G_2, then G_1 and G_2 have a $|S|$-PSGE-withmap.*

Proof. Let $S := \{v_1,\dots,v_s\}$, let S_1 be a permutation of S such that S_1 is an ordered free set in G_1, and let S_2 be permutation of S such that S_2 is an ordered free set in G_2. Define the pointset $P := \{p_1,\dots,p_s\}$ in which p_i has x-coordinate equal to the position of v_i in S_1 and y-coordinate equal to the position of v_i in S_2, for each $i \in \{1,\dots,s\}$. By the definition of free set, G_1 has straight-line crossing-free drawing drawing with v_i at p_i, for each $i \in \{1,\dots,s\}$. By the definition of free set—with the roles of x-coordinates and y-coordinates reversed—G_2 has a straight-line crossing-free drawing with v_i at p_i, for each $i \in \{1,\dots,s\}$. Thus G_1 and G_2 have a $|S|$-PSGE-withmap. □

Theorem 6 and Lemma 11 gives another proof of Theorem 7. By Theorem 6 (applied to G_1 and V), there exists $S_1 \subseteq V$ of size at least $\sqrt{n/2}$ that is a free set in G_1. By Theorem 6 (applied to G_2 and S_1), there exists $S \subseteq S_1$ of size at least $\sqrt{|S_1|/2} \geq n^{1/4}/2^{3/4}$. Thus S is a free set in G_1 and in G_2. By Lemma 11, G_1 and G_2 have a $\lceil n^{1/4}/2^{3/4}\rceil$-PSGE-withmap. In exactly the same way, Lemmata 3 and 11 give a proof that any two n-vertex graphs each of which is one of the following: a tree, an outerplanar graph, Halin graph, or a square graph, have a $(n/4)$-PSGE-withmap.

Lemma 12. *Let $Q := \{G_1,\dots,G_r\}$ be a set of $r \geq 2$ planar graphs on the same vertex set V. Let $S \subseteq V$ be an unordered free set in G_i, for each $i \in \{1,\dots,r\}$. Then the graphs in Q have a $|S|^{1/2^{(r-2)}}$-PSGE-withmap.*

Proof. For each $i \in \{2,\dots,r\}$, let S_i be a permutation of S that is an ordered free set in G_i. Suppose that, for some $i \in \{3,\dots,r\}$, S'_{i-1} is a subsequence of S_2 that defines an (ordered) free set in each of

point in D_i and D_j, then $v = w$. However this additional requirement can always be met by the fact that it is possible to perturb any subset of vertices in a straight-line crossing-free drawing without introducing crossings. (This fact is used for example in the proof that every free-collinear set is free.)

$G_2, \ldots, G_{i-1}$. Since S_i and its reversal are both free sets in G_i, Dilworth's Theorem implies that there is a subsequence S'_i of S'_{i-1} of size at least $|S'_{i-1}|^{1/2}$ that is a free set in G_i and is therefore a free set in each of $G_1, \ldots, G_i$. By starting with $S'_2 := S_2$, it follows that there exists an ordered set $S' := S'_r$ of size $|S|^{1/2^{r-2}}$ that is a free set in each of $G_2, \ldots, G_r$.

Now apply Lemma 11 to the graphs G_1 and G_2 with the set S to obtain straight-line crossing-free drawing of both graphs in which each vertex in S (and therefore, the vertices in S') appear at the same location in both drawings. In these drawings, the y-coordinates of the vertices in S' are increasing. By the definition of free set (exchanging the roles of x- and y-coordinates), each of $G_3, \ldots, G_k$ has a straight-line crossing-free drawing in which each vertex in S' has the same location in each drawing. □

Starting with $X := V$ and using r applications of Theorem 6 yields a set S of size at least $n^{1/2^r}/2$ that is a free set in each of $G_1, \ldots, G_r$. Applying Lemma 12 to S and $G_1, \ldots, G_r$ gives the following corollary:

Corollary 8. *Let $Q := \{G_1, \ldots, G_r\}$ be a set of $r \geq 2$ planar graphs on the same vertex set V. Then the graphs in Q have a $(n^{1/4^{r-1}}/2)$-PSGE-withmap.*

If the graphs in Q each come from one of the classes covered by Corollary 1, then repeated applications of Corollary 1 gives a set S of size at least $n/2^r$ that is free in each of $G_1, \ldots, G_r$, in which case the bound in Corollary 8 becomes $n^{1/2^{r-2}}/4$.

4.4 Column planarity

Given a planar graph G, a set $R \subseteq V(G)$ is *column planar* in G if the vertices of R can be assigned distinct x-coordinates such that for any arbitrary assignment of y-coordinates to the vertices in R (under the condition that the resulting $|R|$ points have no three on a line), there exists a straight-line crossing-free drawing of G in which each vertex $v \in R$ is placed to its assigned coordinates. R is *strongly column planar* if the "no three on a line condition" is removed from the definition. The following lemma is immediate from the definition of free sets and strongly column planar sets.

Lemma 13. *A (sub)set of vertices of a planar graph is strongly column planar set if and only if it is an unordered free set.*

Being strongly column planar implies being column planar, thus all the bounds from Sections 3.1 to 3.3 apply to column planarity. For example, Theorem 5 implies that every n-vertex planar graph has a column planar set of size $\Omega(\sqrt{n})$. Corollary 3 implies that every n-vertex bounded-degree planar graph has a column planar set of size $\Omega(n^{0.8})$; and so on.

Column planar sets were introduced by Evans et al. [31] motivated by applications to partial simultaneous geometric embeddings. Notions similar to column planarity were studied by Estrella-Balderrama, Fowler, and Kobourov [30] and Di Giacomo, Didimo, van Kreveld, Liotta, and Speckmann [25]. Barba et al. [8] proved that n-vertex trees have column planar sets of size $14n/17$. Note that that does not imply that trees have free sets of that size, since being column planar does not imply being strongly column planar. In particular, if G is a 3-cycles, then $V(G)$ is column planar set for any assignment of x-coordinates to $V(G)$. However, no x-coordinate assignment can be turned into a straight-line crossing-free drawing with all the vertices having the same y-coordinate. Thus $V(G)$ is not strongly column planar.

5 A One-Bend Variant

A *k-bend crossing-free drawing* of a planar graph G is a crossing-free drawing of G in which each edge is represented by a polygonal chain consisting of at most k line segments. Thus, a 0-bend

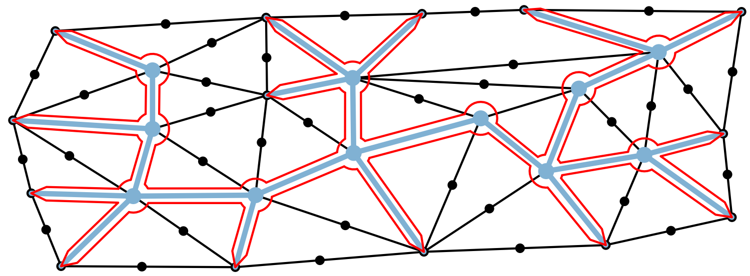


Figure 12: The leaves of a spanning tree of G are a 1-bend collinear set of G.

crossing-free drawing is a straight-line crossing-free drawing. This leads naturally to a definition of a k-bend free set introduced by Bose, Dujmović, Houdrouge, Morin, and Odak [16]. An ordered set $S := (v_1, \dots, v_s)$ of vertices in a planar graph G is a *k-bend free set* if, for any $x_1 < \cdots < x_s$ and any $y_1, \dots, y_s$, there exists a k-bend crossing-free drawing Γ of G such that (x_i, y_i) is the location of v_i in Γ, for each $i \in \{1, \dots, s\}$. As an application of Theorem 9, below, Bose et al. [16] prove the following result:

Theorem 8 ([16]). *Every n-vertex planar graph has a 1-bend free set of size at least $11n/21$.*

They prove Theorem 8 by proving the following result, which the main result in [16]:

Theorem 9 ([16]). *Every n-vertex triangulation has a spanning tree with at least $11n/21$ leaves.*

Theorem 9 improves a longstanding bound of $n/2$ that has at least two different proofs [1, 4]. The proof of Theorem 9 is well outside the scope of this survey. Instead, we prove the following lemma that, along with Theorem 9, immediately implies Theorem 8.

Lemma 14. *Let G be a planar graph and let T be a spanning tree of G. Then the leaves of T are a one-bend collinear set in G.*

Proof. Refer to Fig. 12. Fix some straight-line crossing-free drawing of G, which also fixes a straight-line crossing-free drawing of T. Fatten the drawing of T by some arbitrarily small value $\epsilon > 0$ by taking the Minkowsky sum of T with a disc of radius ϵ. The boundary of this fattened tree is a simple closed curve C_0. Construct a curve C by deforming C_0 in an ϵ-neighbourhood of each leaf v of T so that it contains v. If ϵ is sufficiently small, then C is a simple closed curve that intersects each edge vw of G in at most 2 points: one point within distance ϵ of v and one point within distance ϵ of w. Let $S := (v_1, \dots, v_s)$ be the leaves of T in the order they are encountered while traversing C. Subdivide each edge of G by placing a vertex in the center of the edge and call the resulting graph G'. Then C is a proper-good curve for G' that contains S. Therefore S is a free set in G'. Therefore, for any $x_1 < \cdots < x_s$ and any $y_1, \dots, y_s$, G' has a straight-line crossing-free drawing in which v_i is placed at (x_i, y_i) for each $i \in \{1, \dots, s\}$. Any such drawing of G' gives a 1-bend crossing-free drawing of G in which v_i is placed at (x_i, y_i) for each $i \in \{1, \dots, s\}$. Thus, S is a 1-bend collinear set in G. ∎

The obvious question is whether the bound in Theorem 8 is of the correct form. Perhaps planar graphs have 1-bend collinear sets of size $n - o(n)$. We can rule this possibility out with the following construction.

Theorem 10. *For infinitely many values of n, there exists an n-vertex planar graph that has no 1-bend collinear set of size greater than $10n/11$.*

Proof. The Goldner–Harary graph, G_0 is an 11-vertex triangulation that is not Hamiltonian. We claim that any 1-bend free set in G_0 has size at most 10. Suppose, for the sake of contradiction, that $S := V(G_0)$ is a 1-bend free set in G_0. Then G has a 1-bend crossing-free drawing with all vertices of S placed on the x-axis. Since this is a 1-bend crossing-free drawing and all vertices of G_0 are on the x-axis, no edge properly crosses the x-axis. Therefore, this is a 2-page book embedding of G_0. This is impossible because any triangulation with a 2-page book embedding is Hamiltonian.

For any positive integer k, let G to be a graph consisting of k disjoint copies of G_0. Then G has $n = 11k$ vertices. Any 1-bend free set in G must exclude one vertex from each copy of G_0 and therefore has size at most $10k = 10n/11$. □

6 Open Problems

We conclude with a list of open problems:

1. What is the largest value α such that every n-vertex planar graph has a free set of size $\Omega(n^\alpha)$? Currently, we know that $1/2 \le \alpha \le 0.9859$. Is α equal to the shortness exponent σ for cubic triconnected planar graphs? In other words, does every triangulation G contain a free set of size $\Omega(c(G^*))$?

2. Does every n-vertex planar graph of maximum degree 3, 4, 5, or 6 contain a free set of size $\Omega(n)$

3. Does every n-vertex planar graph of treewidth 4 contain a free set of size $\Omega(n)$?

4. What is the largest value of β such that every n-vertex planar graph has a 1-bend free set of size $\beta n - o(n)$?

5. Say that a subset S of vertices in a planar graph G is *mostly free* if, for every set P of $|S|$ points *in general position*, there exists a straight-line crossing-free drawing of G in which each vertex in S is drawn at a distinct point in P. Is there a constant $\gamma > 0$ such that every mostly free set S contains a free subset of size at least $\gamma|S|$? The class of outerplanar graphs shows that $\gamma \le 2/3$ since any set of n points in general position is universal for the class of n-vertex outerplanar graphs [15, 22, 39] so the entire vertex set of every outerplanar graph is mostly free, but there exists n-vertex outerplanar graphs with no free set of size greater than $\lceil 2n \rceil$.

6. Theorem 6 and Corollary 1 allow us to choose any set X of vertices in G and find a large subset of X that is a free set of G, and this turns out to be useful in a number of applications. Corollary 3 and Theorem 4, which currently give the strongest lower bounds for planar graphs of bounded degree and triconnected cubic planar graphs, do not have this flexibility.

   (a) Is the following strengthening of Corollary 3 true: For any planar graph G of bounded degree and any subset X of vertices of G, there exists a set $S \subseteq X$ of size $\Omega(|X|^{0.8})$ that is a free set of G?

   (b) Is the following strengthening of Theorem 4 true: For any triconnected cubic planar graph G and any subset X of vertices of G, there exists a set $S \subseteq X$ of size $\Omega(|X|)$ that is a free set of G?

In some of the applications discussed in Section 4, free sets are a convenient tool but may be more powerful than necessary, and better bounds may be possible using more direct methods. Here are two examples:

5. Do universal point subset of size $\Omega(n^{1/2+\epsilon})$ for some $\epsilon > 0$ exist for the class of n-vertex planar graphs. Currently, there is nothing that rules out a bound of $\Omega(n)$.

6. Although free sets and an application of Dilworth's Theorem currently give the best bounds for untangling planar graphs and several other graph classes, better bounds may be possible using more direct techniques. The $\Omega(n^{2/3})$ bound on $\mathrm{fix}_{\mathcal{C}}(G)$ for untangling cycles [23] gives one example in which free sets are not the best approach.

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