Symplectic quantization and the Feynman propagator: a new real-time numerical approach to lattice field theory

Martina Giachello[1,2] and Giacomo Gradenigo[1,2]

[1] *Gran Sasso Science Institute, Viale F. Crispi 7, 67100 L'Aquila, Italy*
[2] *INFN-Laboratori Nazionali del Gran Sasso, Via G. Acitelli 22, 67100 Assergi (AQ), Italy*[*]

We present here the first numerical test of symplectic quantization, a new approach to quantum field theory introduced in [1, 2]. Symplectic quantization is characterized by the possibility to sample quantum fluctuations of relativistic fields by means of a deterministic dynamics generated by Hamilton-like equations, the latter evolving with respect to an additional time parameter τ. In the present work we study numerically the symplectic quantization dynamics for a real scalar field in 1+1 space-time dimensions and with $\lambda\phi^4$ non-linear interaction. We find that for $\lambda \ll 1$ the Fourier spectrum of the two-point correlation function obtained numerically reproduces qualitatively well the shape of the Feynman propagator. Within symplectic quantization the expectation over quantum fluctuations is computed as a dynamical average along the trajectories parametrized by the intrinsic time τ. As a numerical strategy to study quantum fluctuations of fields directly in Lorentzian space-time, we believe that symplectic quantization will be of key importance for the study of non-equilibrium relaxational dynamics in quantum field theory and the phenomenology of metastable bound states with very short life-time, something usually not accessible by numerical methods based on Euclidean field theory.

CONTENTS

I. INTRODUCTION

Since its invention by Kenneth Wilson [3], lattice field theory had an enormous development [4, 5] as a method to handle non-perturbative problems in quantum field theory, in particular concerning the theory of strong interactions with problems such as the estimate of hadronic masses [6] or heavy ions collisions [7]. Nevertheless, despite its great achievements, any numerical approach to quantum field theory on the lattice retains a major limitation: any importance sampling protocol is well defined only for Euclidean field theory, which in turn is obtained by Wick-rotating real time into imaginary time. This standard procedure allows to transform the Feynman path integral, characterized by the oscillating factor $\exp(iS[\phi]/\hbar)$, $S[\phi]$ being the action of the field theory considered and ϕ a generic quantum field, into a normalizable probability density $\exp(-S_E[\phi]/\hbar)$, with $\hbar$ playing the same role of temperature in the Boltzmann weight of statistical mechanics. The mapping to imaginary time is therefore necessary to set up any importance sampling numerical protocol to study the quantum fluctuations of fields. But while on the one hand Wick rotation from real to imaginary time is the main trick to allow a numerical approach, on the other hand it also represents the main *limitation* of all numerical approaches to quantum field theory. In particular, the use of Euclidean field theory prevents to represent on the lattice any process or phenomenon intrinsically related to the causal structure of space-time, in particular all processes on the light cone, which is not even defined in Euclidean field theory. It turns out that the probability density $\exp(-S_E[\phi]/\hbar)$ works as an "equilibrium" measure for quantum fluctuations: for instance it allows to reproduce with extreme precision the physics of *stable/equilibrium* bound states of strong interactions [6] while it does not allow to study the metastable resonances with short lifetimes, like for instance tetraquark or pentaquark states [8, 9], or the dynamics of scattering processes with a strong relativistic character, namely processes with a different number of degrees of freedom in $\langle \mathrm{IN}|$ and $|\mathrm{OUT}\rangle$ states. It is for this reason that we believe it is of crucial interest the possibility to test numerically any new proposal for a quantum field theory formulation which allows first to define and then to study the dynamics of quantum fields fluctuations directly in Minkowski space-time.

* martina.giachello@gssi.it

An interesting idea in this direction, namely the proposal of a functional approach to field theory which is well defined from the probabilistic point of view in Lorentzian space-time, has been recently put forward by one of us and goes under the name of *"symplectic quantization"* [1, 2, 10]. According to this approach, for a given quantum field $\phi(x)$, with $x = (ct, \mathbf{x})$ a point in four-dimensional space-time, one assumes a dependence on an additional time parameter τ:

$$\phi(x) \ \rightarrow \ \phi(x, \tau), \tag{1}$$

which controls the continuos sequence of quantum fluctuations in each point of space-time. Theories with such an additional time parameter are not a novelty, the whole Parisi-Wu stochastic quantization approach being based on this idea [11, 12]. The very simple idea of stochastic quantization is to introduce a *fictitious* dynamics for the quantum fluctuations of a Euclidean field theory. This dynamics, following the analogy with statistical mechanics, is the one of a Langevin equation for the field $\phi(x, \tau)$:

$$\frac{d\phi}{d\tau} = -\frac{\delta S_E[\phi]}{\delta\phi(x, \tau)} + \eta(x, \tau), \tag{2}$$

where $S_E[\phi]$ is the Euclidean action and $\eta(x, \tau)$ is a *"white noise"*, that is a stochastic process characterized by its first moments

$$\begin{aligned} &\langle \eta(x, \tau) \rangle = 0 \\ &\langle \eta(x, \tau)\eta(y, \tau') \rangle = 2\ \hbar\ \delta(\tau - \tau')\delta^4(x - y). \end{aligned} \tag{3}$$

The Euclidean weight of field fluctuations comes along to Langevin equation in the following manner. To a *non-linear* stochastic equation for the field such as Eq. (2) it is usually associated a *linear* equation for the probability distribution of the field, the Fokker-Planck equation:

$$\begin{aligned} \frac{d}{d\tau} P_\tau(\phi) &= \left[\frac{\delta^2}{\delta\phi(x)^2} + \frac{\delta}{\delta\phi(x)} \cdot S_E[\phi] \right] P_\tau(\phi) \\ &= L_{\mathrm{FP}}[\phi] \circ P_\tau(\phi). \end{aligned} \tag{4}$$

The link with Euclidean field theory comes with the fact that Eq. (4) admits a stationary solution, i.e., a distribution $P_{\mathrm{eq}}(\phi)$ such that $L_{\mathrm{FP}} \circ P_{\mathrm{eq}}(\phi) = 0$, which reads as:

$$P_{\mathrm{eq}}(\phi) = \frac{1}{\mathcal{Z}} \exp\left(-\frac{1}{\hbar} S_E[\phi] \right) \tag{5}$$

From the point of view of the Langevin dynamics the probability density $P_{\mathrm{eq}}(\phi)$ is the one sampled at stationarity for large values of τ. Within the stochastic quantization approach, Euclidean multipoint correlation functions are therefore obtained as the infinite-time limit of equal-time multipoint correlations computed along the stochastic dynamics, in such a way that in practice both the specific "trajectory" of the noise $\eta(x, \tau)$ and the additional time τ play really the role of dummy auxiliary variables which disappear from the final result. The presence of the additional time τ therefore did not stimulate any conceptual discussion on its interpretation, being it regarded just a sort of "computational trick". Therefore, while on the one hand stochastic quantization had the merit of introducing the key idea of an additional time controlling the sequence of quantum fluctuations, often referred to in the literature as *"fictitious time"* [11], on the other hand it did not trigger a substantial conceptual advance on the foundations of quantum field theory, being such time just an auxiliary variable and being all transients in the dynamics regarded just as *unphysical*.

Before introducing explicitly the symplectic quantization formalism let us mention, among the previous developments of stochastic quantization, the one which moved towards a similar direction, namely the use of a deterministic dynamics in auxiliary time to sample quantum fluctuations [13]. The crucial novelty of this generalized Hamiltonian dynamics, introduced to overcome the numerical difficulties in the implementation of lattice gauge theory with fermions [13], was the presence of *conjugated momenta* proportional to the rate of variation of fields with respect to τ. On the basis of the equivalence between the Euclidean measure of Eq. (5) and a corresponding *microcanonical* one, it was noted that a deterministic dynamics of the kind

$$\frac{d^2\phi}{d\tau^2} = -\frac{\delta S_E[\phi]}{\delta\phi(x, \tau)} \tag{6}$$

was leading to the same asymptotic probability distribution of fields of the stochastic dynamics in Eq. (2), with the major computational advantage of being more suited to parallel updates of the variables, something particularly useful in the case of the non-local bosonic actions obtained from the integration of fermionic variables [5]. The introduction of canonical conjugated momenta of the quantum fields with respect to the intrinsic time was throughly discussed in [14], also in this case within the Euclidean field theory framework.

The innovative proposal of symplectic quantization as reported in [1, 2] is that the correct sampling of quantum fluctuation in a Lorentzian space-time can be *only* achieved in a sort of generalized microcanonical ensemble, built by adding the above mentioned conjugated momenta with respect to the intrinsic time τ to the relativistic action with Minkowski signature. This intuition has been strongly inspired from the evidence that in statistical mechanics there are physically important situations where only the microcanonical ensemble is well defined [15, 16], namely physical phenomena which cannot be described within the canonical ensemble.

In the present work we show how to sample the quantum fluctuations of a relativistically invariant scalar field theory in 1+1 coordinate space-time dimensions by studying a generalized Hamiltonian dynamics in the intrinsic time τ, showing that a stationary regime can be reached where the correct measure of the Feynman propagator is obtained: this is an unprecedented result in the whole literature of lattice field theory.

II. SYMPLECTIC QUANTIZATION: FROM DYNAMICS TO ENSEMBLE AVERAGES

Let us summarize here the main steps for the derivation of the symplectic quantization dynamics. First of all, inspired by the stochastic quantization approach [11, 12], we assume that quantum fields $\phi(x,\tau)$ depend on an additional time variable τ which parametrizes the dynamics of quantum fluctuations in a given point of Minkowski space-time. Since for a relativistic quantum field theory the ambient space includes observer's time, necessarily the intrinsic time τ must be a different variable, as thoroughly discussed in [1, 2]. The symplectic quantization approach to field theory assumes, consistently with the existence of an intrinsic time τ, the existence of conjugated momenta of the kind

$$\pi(x,\tau) \propto \dot{\phi}(x,\tau), \tag{7}$$

which can be obtained as follows. First, we introduce a generalized Lagrangian of the kind

$$\mathbb{L}(\phi,\dot{\phi}) = \int d^d x \; \left[\frac{1}{2c_s^2}\dot{\phi}^2(x) + S[\phi]\right], \tag{8}$$

where c_s is in natural units a dimensionless parameter, and $S[\phi]$ is the standard action for a quantum field, e.g.,

$$\begin{aligned} S[\phi] &= \int d^d x \left(\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - V[\phi(x)]\right) \\ &= \int d^d x \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial x^0}\right)^2 - \frac{1}{2}\sum_{i=1}^{d}\left(\frac{\partial\phi}{\partial x^i}\right)^2 - V[\phi(x)]\right] \end{aligned} \tag{9}$$

where the potential is, for instance

$$V[\phi] = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4. \tag{10}$$

By means of a Legendre transform one then passes to the Hamiltonian:

$$\begin{aligned} \mathbb{H}[\phi,\pi] &= \frac{1}{2}\int d^d x \; c_s^2 \; \pi^2(x) - S[\phi] \\ &= \int d^d x \left[\frac{c_s^2}{2}\pi^2(x) - \frac{1}{2}\left(\frac{\partial\phi}{\partial x^0}\right)^2 + \frac{1}{2}\sum_{i=1}^{d}\left(\frac{\partial\phi}{\partial x^i}\right)^2 + V[\phi]\right] \\ &= \int d^d x \; \left[\frac{c_s^2}{2}\pi^2(x) + \frac{1}{2}\phi\;\partial_0^2\phi - \sum_{i=1}^{d}\phi\;\partial_i^2\phi + V[\phi]\right] \end{aligned} \tag{11}$$

For simplicity we will assume $c_s = 1$ from here on. From Eq. (11) we have that, within the symplectic quantization approach, the dynamics of quantum fluctuations is the one governed by the following Hamilton equations:

$$\begin{aligned} \dot{\phi}(x) &= \frac{\delta\mathbb{H}[\phi,\pi]}{\delta\pi(x)} \\ \dot{\pi}(x) &= -\frac{\delta\mathbb{H}[\phi,\pi]}{\delta\phi(x)}, \end{aligned} \tag{12}$$

from which one gets

$$\ddot{\phi}(x,\tau) = -\partial_0^2\phi(x,\tau) + \sum_{i=1}^{d}\partial_i^2\phi(x,\tau) - \frac{\delta V[\phi]}{\delta\phi(x,\tau)}. \tag{13}$$

At this stage one can legitimately wonder how a classical deterministic theory can account for quantum fluctuations. Let us notice that in the expression of the generalized Hamiltonian $\mathbb{H}[\phi,\pi]$ we can recognize a *"generalized potential energy"* $\mathbb{V}[\phi]$, corresponding to the original relavistic action, and a *"generalized kinetic energy"* $\mathbb{K}[\pi]$, namely the quadratic part related to the new conjugated momenta:

$$\begin{aligned} \mathbb{H}[\phi,\pi] &= \mathbb{K}[\pi] + \mathbb{V}[\phi] \\ \mathbb{V}[\phi] &= -S[\phi] \end{aligned} \tag{14}$$

$$\mathbb{K}[\pi] = \frac{1}{2}\int d^d x \; \pi(x). \tag{15}$$

The "classical" (in the traditional sense) solution for the field correspond to minima of the new generalized potential $\mathbb{V}[\phi]$, whereas the quantum fluctuations are naturaly sampled along the generalized Hamiltonian dynamics, where the initial value of $\mathbb{H}[\phi,\pi]$ is constant but the potential energy, namely $S[\phi]$, fluctuates.

Having defined the above deterministic dynamics, Eq. (13), for the quantum fluctuations of the field $\phi(x,\tau)$, one can then legitimately wonder how this functional formalisms connects to the standard one, for instance to the standard Feynman path-integral formulation of quantum field theory. The connection between the dynamic approach of symplectic quantization and the Feynman path integral comes by means of an ergodic hypothesis on the Hamiltonian dynamics in Eq. (12): if we assume that this dynamics samples at long time τ the constant generalized energy hypersurface with uniform probability [1, 2], then we can associate to the dynamics of Eq. (12) the following measure:

$$\rho_{\text{micro}}[\phi(x)] = \frac{1}{\Omega(\mathcal{A})}\;\delta\left(\mathcal{A} - \mathbb{H}[\phi,\pi]\right), \tag{16}$$

where $\Omega(\mathcal{A})$ is a sort of *microcanonical* partition function

$$\Omega(\mathcal{A}) = \int \mathcal{D}\phi\mathcal{D}\pi\;\delta\left(\mathcal{A} - \mathbb{H}[\phi,\pi]\right), \tag{17}$$

with $\mathcal{D}\phi = \prod_x d\phi(x)$ and $\mathcal{D}\pi = \prod_x d\pi(x)$ the standard notation for functional integration. From the above partition function we can define the microcanonical adimensional entropy of symplectic quantization:

$$\Sigma_{\text{sym}}(\mathcal{A}) = \ln \Omega(\mathcal{A}) \tag{18}$$

The *ergodicity assumption* for the symplectic quantization dynamics amounts to say that, considering $\mathcal{O}[\phi(x)]$ a generic observable of the quantum fields, symplectic quantization can be related to the standard path-integral formulation of field theory by claiming that for generic initial conditions the following equivalence between averages holds:

$$\lim_{\Delta\tau\to\infty} \frac{1}{\Delta\tau} \int_{\tau_0}^{\Delta\tau} d\tau \ \mathcal{O}[\phi(x,\tau)] = \int \mathcal{D}\phi\mathcal{D}\pi \ \rho_{\text{micro}}[\phi(x)] \ \mathcal{O}[\phi(x)], \tag{19}$$

where τ_0 is a large enough time for the system to have reached stationarity and "lost memory" of initial conditions. How to relate then the microcanonical partition function in Eq. (17) to the path integral? It is quite intuitive to understand that the two expression must be related by some sort of statistical ensemble change. The crucial point of this change of ensemble, how stressed already in [1], is that the microcanonical partition function $\Omega(\mathcal{A})$ is built on the conservation of a non-positive quantity, the generalized Hamiltonian $\mathbb{H}[\phi,\pi]$. The latter from the point of view of physical dimensions is an action and therefore takes both arbitrarily large positive and negative values due to the negative sign in front of the coordinate-time derivative term in the second line of Eq. (11). The absence of positive definiteness for the generalized Hamiltonian $\mathbb{H}[\phi,\pi]$, which is the true relativistic signature of the theory, is what forbids a standard change of ensemble with a Laplace transform, that is customary in statistical mechanics when passing from microcanonical to canonical ensemble. The only integral transform which allows us to map *formally* the ensemble where $\mathbb{H}[\phi,\pi]$ is constrained to the one where it is free to fluctuate is the Fourier transform. It is by Fourier transforming the microcanonical partition function $\Omega(A)$ that one obtains straightforwardly the Feynman path integral:

$$\begin{aligned} \mathcal{Z}(z) &= \int_{-\infty}^{\infty} dA \ e^{-iz\mathcal{A}} \ \Omega(\mathcal{A}) \\ &= \int \mathcal{D}\phi \ \mathcal{D}\pi \ e^{-\frac{i}{2}z\int d^dx \ \pi^2(x) + izS[\phi]} \\ &= \mathcal{N}(z) \int \mathcal{D}\phi \ e^{izS[\phi]}, \end{aligned} \tag{20}$$

where z is a variable conjugated to the action and in the second line of Eq. (20) we have integrated out momenta thanks to the quadratic dependence on them, contributing the infinite normalization constant $\mathcal{N}(z)$, which is typical of path integrals. Finally, if we fix $z = \hbar^{-1}$ into the last line of Eq. (20) we have the Feynman path integral:

$$\mathcal{Z}(\hbar) = \int_{-\infty}^{\infty} d\mathcal{A} \ e^{-iA/\hbar} \ \Omega(\mathcal{A}) \propto \int \mathcal{D}\phi \ e^{\frac{i}{\hbar}S[\phi]}. \tag{21}$$

The one above is to our knowledge the first derivation from first principles of the Feynman path-integral formula in the context of a more extended framework. We could say that this larger framework is the statistical mechanics of action-preserving systems, opposed to the statistical mechanics of energy-preserving systems, which is the standard one. And it is precisely the fact that statistical ensembles are built on the conservation of a non positive-defined quantity, which is the true landmark of the relativistic nature of the theory, that determines the fact that *locally* we can only access complex probability amplitudes and not real probabilities. From the perspective of symplectic quantization the replacement at the local level of probabilities with probability amplitudes is therefore a direct consequence of special relativity and a wise use of statistical ensembles. To better understand this statement let us consider an unrealistic situation where the symplectic action (generalized Hamiltonian) $\mathbb{H}[\phi,\pi]$ was positive definite. In this case one could change ensemble with Laplace rather than Fourier transform,

$$\mathcal{Z}(\mu) = \int_0^{\infty} d\mathcal{A} \ e^{-\mu\mathcal{A}} \ \Omega(\mathcal{A}) \propto \int \mathcal{D}\phi \ e^{\mu S[\phi]}, \tag{22}$$

leading to a theory which is perfectly equivalent to standard statistical mechanics in the canonical ensemble: locally there is a probability density for the field configuration, $\rho(\phi) \propto e^{\mu S[\phi]}$. Let us notice that the factor $\rho(\phi)$ is intuitively well defined as a local probability density because for typical configuration of the field, far from those corresponding to ultrarelavistic particles, the relativistic action is usually negative $S[\phi] < 0$.

We have just shown how the standard path-integral formulation can be recovered, on the basis of an ergodicity assumption, from the symplectic quantization dynamics approach and which is the role played by $\hbar$ within this, let us say, *change of ensemble*. At the same time it is not only legitimate but also necessary to wonder if and how there is a *quantization constraint* involving $\hbar$ which can be imposed directly on the microcanonical ensemble of symplectic quantization. The indication coming from the stochastic quantizationn framework is that $\hbar$ must play a role analogous to that of temperature. Therefore, as suggested in [10], we believe that the most natural assumption for the role of $\hbar$ in the symplectic quantization formalism is to be analogous to the microcanonical temperature:

$$\frac{1}{\hbar} = \frac{d\Sigma_{\text{sym}}(\mathcal{A})}{d\mathcal{A}} \tag{23}$$

Although satisfactory conceptually and formally consistent, a definition of $\hbar$ as in Eq. (23) is very difficult to implement in practice. For this reason we will resort in this paper to another more trivial but effective way to impose the quantization constraint in the symplectic quantization dynamics, the one analogous to the way which is customarly used to assign the temperature in the contex a microcanonical molecular dynamics. Usually, if we have N degrees of freedom and we wish the system to be on the fixed energy hypersurface such that $T^{-1} = \partial S(E)/\partial E$, we simply assign initial conditions such that the total energy is $E = Nk_BT$: here we follow the same strategy. In particular, counting as "degrees of freedom" the number of components in reciprocal space of the Fourier transform of the fields, i.e., $\pi(k)$ and $\phi(k)$, in order to set at $\hbar$ the typical scale of generalized energy for each mode we can choose initial conditions in the ensemble characterized at stationarity by the following condition:

$$\langle \pi^*(x)\pi(y)\rangle = \frac{\hbar}{2}\ \delta^{(4)}(x-y), \tag{24}$$

where the angular brackes indicates intrinsic time average along the symplectic quantization dynamics:

$$\langle \pi^*(x)\pi(y)\rangle = \lim_{\Delta\tau\to\infty} \frac{1}{\Delta\tau}\int_{\tau_0}^{\Delta\tau} d\tau\ \pi(x,\tau)\pi(y,\tau) \tag{25}$$

The last equation can be rewritten for a discretized d-dimensional space-time lattice with lattice spacing a, as for instance is the case for the numerical simulations which we are going to discuss here:

$$\langle \pi^*(x_i)\pi(x_j)\rangle = \frac{\hbar}{2}\ \frac{\delta_{ij}}{a^d}, \tag{26}$$

where δ_{ij} is the Kronecker delta. By Fourier transforming Eq. (24) it is then straightforward to get

$$\langle \pi^*(k)\pi(k)\rangle = \frac{\hbar}{2}, \tag{27}$$

so that in Fourier space the "kinetic" contribution coming from each degree of freedom to the total action amounts to $\hbar/2$. The relation in Eq. (27) can be also applied to the discretized momenta usually considered for a numerical simulation on the lattice:

$$\langle \pi^*(k_i)\pi(k_i)\rangle = \frac{\hbar}{2} \qquad \forall\ i. \tag{28}$$

This will be the sort of quantization constraint which will be applied to all our numerical simulations, choosing initial conditions which are compatible with that. Since we have chosen to work with natural units we will replace $\hbar = 1$ everywhere in the above formulas. Suitable initial conditions to expect something such as Eq. (28) at stationarity is for instance the following:

$$|\pi^*(k_i;\tau=0)|^2 = \hbar \qquad \forall\ i, \tag{29}$$

which will be used for all simulations presented in this work.

III. SIMULATION DETAILS

The deterministic dynamics of symplectic quantization can be defined for both Euclidean and Minkowski metric: to validate the new approach we have tested both scenarios. In order to do that we have discretized the Hamiltonian equations of motion, writing them in a general form where the nature of the metric is specified by the variable $s = \{0,1\}$. All equations are written in natural units $\hbar = c = 1$.

In the present work we have considered a 1 + 1 lattice with either Euclidean or Minkowski metric, which we denote as Γ:

$$\Gamma : \left\{ x : x_\mu = an_\mu\,, \quad n_\mu = -\frac{N}{2},...,\frac{N}{2} \quad \mu = 0,1 \right\}. \tag{30}$$

Due to the finite size of the simulation grid momenta are also discretized:

$$p_\mu = \frac{2\pi}{a}\frac{k_\mu}{N} \qquad |p_\mu| < \frac{\pi}{a}, \tag{31}$$

where $\mu = 0,1$, $k_\mu \in [-N/2, N/2]$, $L = Na$ is the lattice side and a is the lattice spacing. The discretized Hamiltonian of symplectic quantization reads then as

$$\begin{aligned}\mathbb{H}[\phi,\pi] = \frac{1}{2}\sum_{x\in\Gamma}\Bigg[&\pi(x)^2 - (-1)^s\frac{1}{a^2}\phi(x)\Delta^{(0)}\phi(x) \\ &- \frac{1}{a^2}\phi(x)\Delta^{(1)}\phi(x) + m^2\phi^2(x) + \frac{\lambda}{4}\phi^4(x)\Bigg],\end{aligned} \tag{32}$$

where the symbol $\Delta^{(\mu)}\phi(x)$ denotes the discrete one-dimensional Laplacian along the μ-th coordinate axis:

$$\Delta^{(\mu)}\phi(x) = \phi(x+a^\mu) + \phi(x-a^\mu) - 2\phi(x). \tag{33}$$

We have used a general expression in Eq. (32), which, depending on the value chosen for the integer index $s = \{0,1\}$, describes a theory with Euclidean, $s = 0$, or Minkowskian, $s = 1$, metric. From the expression of the Hamiltonian in Eq. (32) we have that the force acting on the field on a two-dimensional lattice is:

$$F[\phi(x)] = -\frac{\delta \mathbb{H}[\phi,\pi]}{\delta\phi(x)} = \\ = \frac{(-1)^s}{a^2}\Delta^{(0)}\phi(x) + \frac{1}{a^2}\Delta^{(1)}\phi(x) - m^2\phi(x) - \lambda\phi^3(x), \tag{34}$$

so that the equation of motion for the field itself is:

$$\frac{d\phi(x,\tau)}{d\tau^2} = F[\phi(x,\tau)]. \tag{35}$$

Equations (34),(35) define the Hamiltonian dynamics which we have studied numerically using the leap-frog algorithm, a symplectic algorithm described in Appendix A, which guarantees the conservation of (generalized) energy at the order $\mathcal{O}(\tau^2)$.

An important point for the study of this paper is the definition of boundary conditions. We used two different kind of boundary conditions for the simulations. For all results on Euclidean lattice and for the study of dynamics stability with or without non-linear interaction on Minkowski lattice, discussed respectively in Sec. IV and in Sec. V, we have used standard periodic boundary conditions on the lattice. Differently, in Sec. VI, aimed at studying the free propagation of physical signals across the lattice we used *fringe* boundary conditions [17], introduced with the purpose of mimiking the existence of an infinite lattice outside the simulation grid. Fringe boundary conditions are realized considering a larger lattice, which we denote as Γ_f, where the subscript *"f"* is for fringe, which is composed by the original lattice Γ plus several additional layer of points which we denote as Γ_{ext}, in such a way that the *fringe* lattice is $\Gamma_f = \Gamma + \Gamma_{\text{ext}}$. For the fringe lattice one also considers periodic boundary conditions, but the generalized Hamiltonian for points belonging to Γ and to Γ_{ext} is different. Namely, the fringe lattice is characterized by the Hamiltonian:

$$\mathbb{H}_f[\pi,\phi] = \mathbb{H}_{\text{ext}}[\pi,\phi] + \mathbb{H}[\pi,\phi], \tag{36}$$

where $\mathbb{H}[\pi,\phi]$ is the original discretized Hamiltonian of the system, see Eq. 32, while $\mathbb{H}_{\text{ext}}[\pi,\phi]$ reads as

$$\mathbb{H}_{\text{ext}}[\pi,\phi] = \frac{1}{2}\sum_{x\in\Gamma}\Bigg[\pi(x)^2 + m^2\phi^2(x) + \frac{\lambda}{4}\phi^4(x) \\ + \alpha\left(\frac{1}{a^2}\phi(x)\Delta^{(0)}\phi(x) - \frac{1}{a^2}\phi(x)\Delta^{(1)}\phi(x)\right)\Bigg],, \tag{37}$$

where the coefficient α is very small, $\alpha \ll 1$. This choice of boundary conditions allows us to have a free propagation of signals across the boundary layer of Γ, our true simulation lattice, but the signal is then stronlgy damped when going across Γ_{ext}, the "external" boundary layer before making sort of interference at the periodic boundaries at the border of Γ_{ext}. This choice of boundary conditions allows us not only to deal with an overall system which is still Hamiltonian (apart from small corrections scaling as $1/L$), but also to have quite satisfactory results for the study of the Feynman propagator, as shown in Sec. VI.

We have done all simulations for a lattice with side $L = 128$, lattice spacing $a = 1.0$ and using an integration time-step $\delta\tau = 0.001$. According to the discussion in the previous section, we have fixed the energy scale by choosing initial conditions such that each degree of freedom in Fourier space carries a *"quantum"* of energy $\hbar = 1$. We have therefore assigned an initial total energy equal to $L \cdot L = 16384$ for all simulations. Since we have studied both linear and non-linear interactions, in order to set precisely the initial value of the energy, we started all simulations with:

$$|\phi(k;0)|^2 = 0 \qquad \forall\ k \\ |\pi(k;0)|^2 = 1 \qquad \forall\ k. \tag{38}$$

IV. EUCLIDEAN PROPAGATOR

Our first test of the symplectic quantization approach consists in the study of its deterministic dynamics in the case of a two-dimensional Euclidean lattice, showing that it provides the correct two-point correlation function, also consistently with the results of stochastic quantization. For the simulation on the Euclidean lattice we have used simple periodic boundary conditions, since all correlation functions decay exponentially with the distance and there should be no signals propagating underdamped across the system.

Let us then recall here how the expectation values over quantum fluctuations of fields are computed within the symplectic quantization approach dynamics. If we indicate with $\phi_{\mathbb{H}}(x,\tau)$ the solutions of the Hamiltonian equations of motion written in Eq. (35), we have that the expectation value of a generic n-point correlation function can be computed as follows:

$$\langle\phi(x_1),\ldots,\phi(x_n)\rangle = \\ = \lim_{\Delta\tau\to\infty}\frac{1}{\Delta\tau}\int_{\tau_0}^{\tau_0+\Delta\tau} d\tau\ \phi_{\mathbb{H}}(x_1,\tau)\ldots\phi_{\mathbb{H}}(x_n,\tau), \tag{39}$$

where τ_0 is a large enough time, for which the system has reached equilibrium and forgot any detail on the initial conditions of the dynamics. For a free field theory the propagator on a two-dimensional lattice take the simple form:

$$\tilde{G}(p;a) = \left[\frac{4}{a^2}\sin^2\left(\frac{ak_0}{2}\right) + \frac{4}{a^2}\sin^2\left(\frac{ak_1}{2}\right) + m^2\right]^{-1} \tag{40}$$

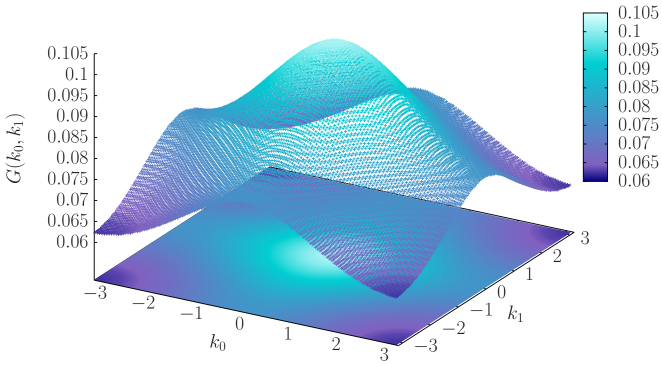

Figure 1. *Real part of the two-point correlation function Fourier spectrum (Euclidean propagator) for a $\lambda\phi^4$ theory in $d = 2$ euclidean dimensions. Numerical value from the interacting theory with nonlinearity $\lambda = 0.001$, lattice spacing $a = 1.0$, lattice side $L = 128$, mass $m = 3.0$.*

If we define the Fourier component of the field as

$$\hat{\phi}(k,\tau) = \frac{a^2}{2\pi}\sum_{x\in\Gamma} e^{-i(k_0x_0+k_1x_1)}\ \phi(x,\tau), \tag{41}$$

we can then write the Fourier spectrum of the two-point correlation function, according to the discretized-time version of Eq. (39), as the following dynamical average:

$$G(k) = \langle\hat{\phi}^*(k)\hat{\phi}(k)\rangle = \frac{1}{\Delta\tau}\sum_{i=0}^{M}\phi^*(k,\tau_0+\delta\tau_i)\phi(k,\tau_0+\delta\tau_i), \tag{42}$$

where $\delta\tau_i = i\cdot\delta\tau$. For the Euclidean lattice we have studied the lattice dynamics with the parameters and initial conditions given at the end of Sec. III, considering, in addition, value of mass $m = 3.0$ and nonlinearity coefficient $\lambda = 0.001$: the numerical value of the propagator in Fourier space perfectly reproduces the expected dumbell shape, as shown in Fig. [1]. We have also checked that the two-point correlation function exhibits in real space the typical exponential decay $C(x) \sim e^{-mx}$.

V. MINKOWSKI LATTICE: LINEAR AND NON-LINEAR THEORY

The numerical and analytical study of the free field theory in 1+1 Minkowski spacetime presents a new problem with respect to the Euclidean space: the dynamics of quantum fluctuations for the *linear* non-interacting theory in the symplectic quantization approach turns out to be *unstable*. This can be recognized immediately from the free field equations in the continuum.

In the case of a purely quadratic potential $V[\phi] = \frac{1}{2}m^2\phi^2$, the explicit solution of Eq. (13) can be obtained by exploiting the translational symmetry of space-time, which allow to Fourier transform the equations:

$$\ddot{\phi}(k,\tau) + \omega_k^2\ \phi(k,\tau) = 0, \tag{43}$$

with

$$\omega_k^2 = |\mathbf{k}|^2 + m^2 - k_0^2. \tag{44}$$

The general solution of Eq. (43) can be then written in terms of the initial conditions as

$$\begin{aligned}\phi(k,\tau) &= \phi(k,0)\cos(\omega_k\tau) + \frac{\dot{\phi}(k,0)}{\omega_k}\sin(\omega_k\tau) \quad \forall\ \ \omega_k^2 > 0\\ \phi(k,\tau) &= \phi(k,0)\cosh(z_k\tau) + \frac{\dot{\phi}(k,0)}{z_k}\sinh(z_k\tau) \ \ \forall\ \ \omega_k^2 < 0,\end{aligned} \tag{45}$$

where

$$iz_k = \sqrt{\omega_k^2}. \tag{46}$$

Without any loss of generality and consistently with what we have done numerically on the lattice, one can consider the following initial conditons:

$$\begin{aligned}\phi(k,0) &= 0\\ \dot{\phi}(k,0) &= 1,\end{aligned} \tag{47}$$

so that the general time-dependent solution reads as

$$\begin{aligned}\omega_k^2 > 0 \quad &\Longrightarrow \quad \phi(k,\tau) = \frac{\sin(\omega_k\tau)}{\omega_k}\\ \omega_k^2 < 0 \quad &\Longrightarrow \quad \phi(k,\tau) = \frac{\sinh(z_k\tau)}{z_k}.\end{aligned} \tag{48}$$

Rewriting the generalized Hamiltonian in Fourier space we have

$$\mathbb{H}[\phi,\pi] = \frac{1}{2}\int d^dk\ \ \left(|\pi(k)|^2 + \omega_k^2\ |\phi(k)|^2\right), \tag{49}$$

so that, by plugging into it the time-dependent solutions we have:

$$\begin{aligned}&\omega_k^2 > 0 \quad \Longrightarrow\\ &\mathbb{H}[\phi(\tau),\pi(\tau)] = \frac{1}{2}\int d^dk\ \ \left[\cos^2(\omega_k\tau) + \sin^2(\omega_k\tau)\right]\\ &\omega_k^2 < 0 \quad \Longrightarrow\\ &\mathbb{H}[\phi(\tau),\pi(\tau)] = \frac{1}{2}\int d^dk\ \ \left[\cosh^2(z_k\tau) - \sinh^2(z_k\tau)\right].\end{aligned} \tag{50}$$

Considering the expressions in Eq. (50) we realize that, despite the conservation of the symplectic quantization Hamiltonian, it exists an infinite set of momenta, namely all k's with $\omega_k^2 < 0$, such that the *"potential"* and *"kinetic"* part of the generalized energy in Eq. (49), namely $\mathbb{K}[\phi,\pi]$ and $\mathbb{V}[\phi,\pi]$, both diverge exponentially with τ. This fact presents two problems, one conceptual and the second numerical. The conceptual problem is represented by the fact that, irrespectively to the behaviour of moments $\pi(k,\tau)$, which *might* also be regarded as unphysical auxiliary variables, we have that

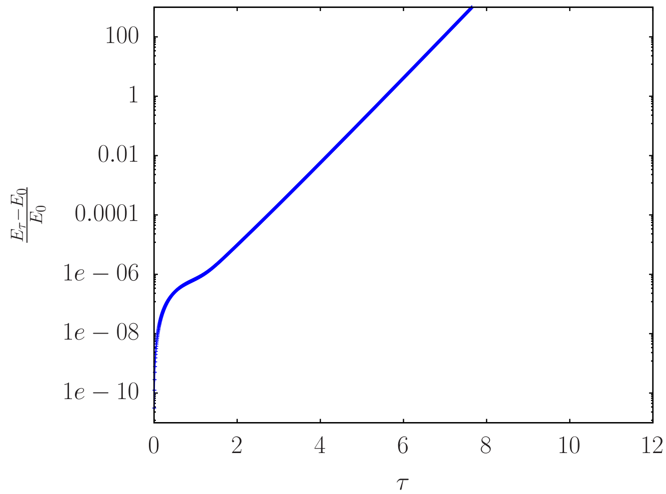

Figure 2. *Behaviour of the normalized energy $(E(\tau)-E_0)/E_0$ vs τ for a scalar free theory ($\lambda = 0$) with $m = 1.0$, $a = 1.0$, $L = 128$, with initial conditions $\pi(k;0) = 1$ and $\phi(k;0) = 0$ for all k's. Notice the exponential growth with τ which sets in after a short transient.*

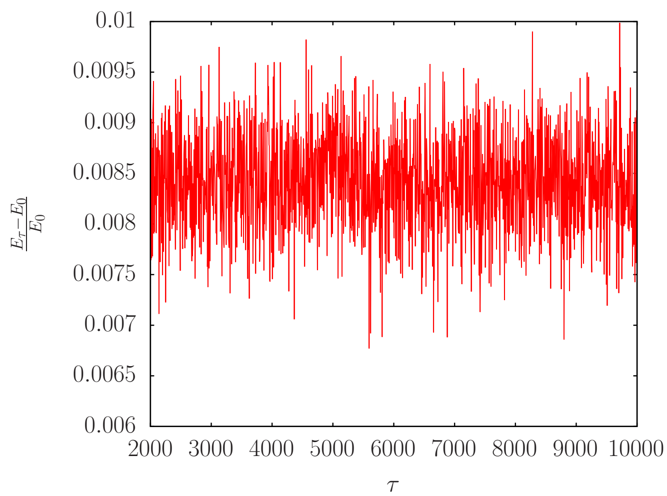

Figure 3. *Behaviour of the normalized energy $(E(\tau)-E_0)/E_0$ vs τ for a scalar theory with a small self-interaction term, $\lambda = 0.001$, and with $m = 1.0$, $a = 1.0$, $L = 128$, with initial conditions $\pi(k;0) = 1$ and $\phi(k;0) = 0$ for all k's. Oscillations are of order δt.*

also the contribution to generalized potential energy $\mathbb{V}[\phi,\pi]$ (corresponding in practice to the relativistic action) of an infinite amount of field modes $\phi(k,\tau)$ diverges exponentially with τ. This divergence of the field amplitude is clearly unphysical: interpreting in fact as *"particles"* the modes of the free field, this would correspond to infinite growth of the action of an isolated particle, which is clearly not observed in the real world.

At the same time, attempting to study numerically the free-field dynamics on Minkowski lattice, the leap-frog algorithm, which proceedes alternating the update of kinetic and potential energy, cannot handle the situation where the overall energy is conserved but the two contributions diverge. Eventually, due to the accumulation of numerical errors, total energy starts to diverge exponentially as well with elapsing time, see Fig. 2 and the discussion below.

Since in the present section we are just interested in the stability of the theory, irrespectively of a realistic study of signals propagation across the lattice, we have considered for symplicity periodic boundary conditions. Let us remark that these conditions would not be appropriate for a more realistic study of two-point correlation functions with Minkowski metric, since in this case we would like to probe the causal structure of space-time: a signal excaping from the lattice at $+ct$ cannot appear back at $-ct$. For a similar reason even fixed boundary conditions would not be appropriate.

Using periodic boundary conditions we have checked numerically that the simplectic quantization dynamics of a free scalar field suffers from the pathology which can be conjectured already from the exact solution: after a certain time the whole energy starts to grow exponentially with τ. In Fig. [2] we present the results of simulations of the free-field with Minkowski metric, all the parameters declared at the end of Sec. III and $m = 1.0$, showing a clear evidence of the exponential divergence with τ. What seemed a good solution to both the conceptual and numerical shortcomings of the free theory has been to consider that the physically relevant theory is only the interacting one: physical fields are always in interactions and the "free-field theory" is just an approximation, with some internal inconsistencies which are revealed by the symplectic quantization approach. Let us for instance consider a potential of the kind

$$V[\phi] = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4, \tag{51}$$

for which the equations of motions in the continuum read as

$$\ddot{\phi}(x,\tau) = \\ -\partial_0^2\phi(x,\tau) + \sum_{i=1}^{d}\partial_i^2\phi(x,\tau) - m\phi(x,\tau) - \lambda\phi^3(x,\tau). \tag{52}$$

Clearly, due to the non-linear term in Eq. (52), it is not possibile anymore to diagonalize the equations in Fourier space, so that both the sin/cos and the sinh/cosh solutions cannot be taken into account as a reference. Yet to be proven mathematically, the stability of Eq. (52) is a quite delicate problem, since in general for many Fourier components the equations are linearly unstable. The intuition suggests that for each point of space-time x the cubic force acts as a restoring term which prevents the amplitude $\phi(x,\tau)$ to grow without

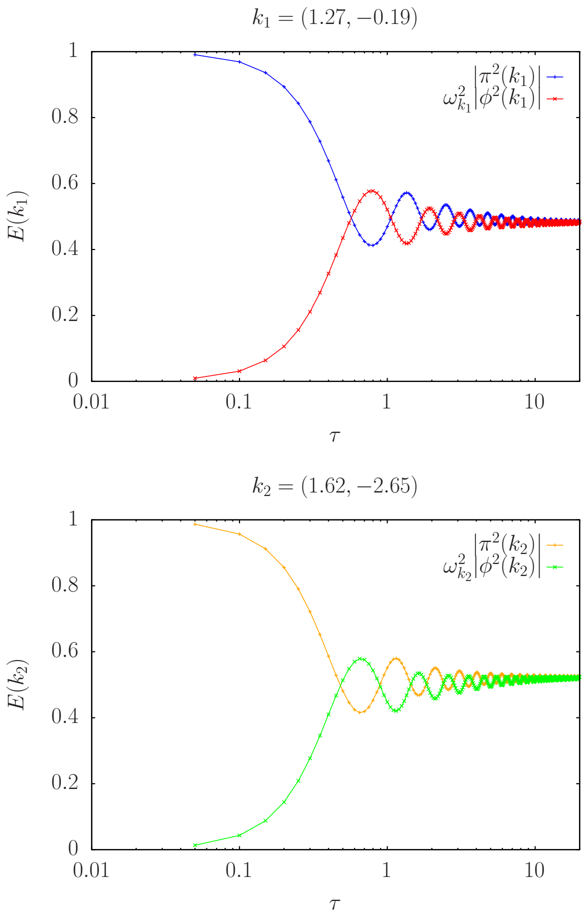

Figure 4. Behaviour of the time averaged harmonic $\overline{E}_{\rm harm}(k,\tau)$ and kinetic $\overline{E}_{\rm kin}(k,\tau)$ energies for two different choices of k, corresponding respectively to small (top panel) and large (bottom panel) scales. Non-linearity coefficient is $\lambda = 0.001$ and lattice parameters are with $m = 3.0$, $a = 1.0$, $L = 128$, with initial conditions $\pi(k;0) = 1$ and $\phi(k;0) = 0$ for all k's. For this choice of parameters there are no unstable modes, i.e. for all k's we have $\omega_k^2 > 0$.

bounds. This intuition has been confirmed, up to the accuracy of our analysis, from our numerical results. By using periodic boundary conditions, the parameters and initial conditions declared at the end of Sec. III, setting the non-linearity coefficient $\lambda = 0.001$ we find that the energy is no more divergent. The system relaxes to a stationary state with oscillations of order $|E(t) - E_0|/E_0 = \mathcal{O}(\delta\tau)$, as is shown in Fig. 5.

Having assessed the stability of the symplectic quantization dynamics in the presence of non-linear interactions and periodic boundary conditions, it is now time to consider, keeping the non-linearity switched on, the more physical case of fringe boundary conditions [17]. This procedure will allow us to sample numerically the Feynman propagator for small non-linearity, as will be discussed in the next section.

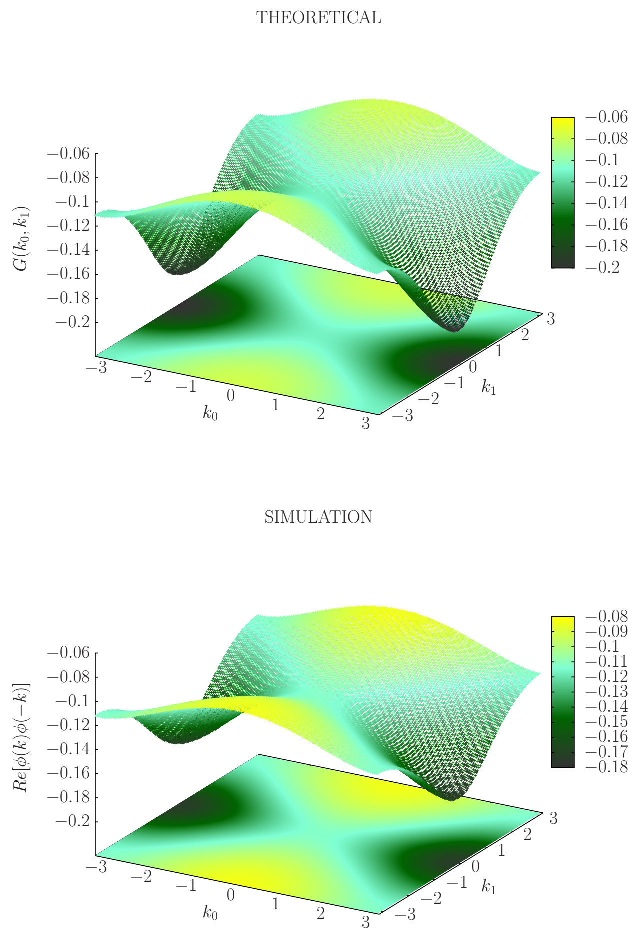

Figure 5. *Real part of the two-point correlation function Fourier spectrum* $G(k_0, k_1) = \langle \phi^*(k_0, k_1)\phi(k_0, k_1)\rangle$ *(Feynman propagator) for a* $\lambda\phi^4$ *theory in* $1+1$ *space-time dimensions. Top): theoretical value of the free propagator with lattice spacing* $a = 1.0$*, lattice side* $L = 128$*, mass* $m = 3.0$*; Down): numerical value from the interacting theory with the same parameters and nonlinearity* $\lambda = 0.001$*. Initial conditions are set to* $\phi(k;0) = 0$ *and* $\pi(k;0) = 1$ *for all* k*'s. For this choice of parameters there are no unstable modes, i.e. for all* k*'s we have* $\omega_k^2 > 0$.

VI. FEYNMAN PROPAGATOR: NUMERICAL RESULTS

In the previous section we have shown how the presence of non-linear interactions solves the instability problem of the linear theory, still keeping periodic boundary conditions. But periodic boundary conditions are clearly unphysical, because one of the directions of our lattice corresponds to ct, so that periodicity of the boundaries is clearly meaningless. We need to devise a strategy to mimick the free propagation of any kind of signal across the boundaries as if outside there was an infinitely large lattice. This strategy is provided by the use of fringe boundary conditions, introduced in Sec. III.

In this part of the paper we will therefore provide the numerical evidence that for perturbative values of the non-linearity coefficient λ we recover qualitatively the correct shape of the free Feynman propagator.

The strategy is very simple: having set the coefficient of the non-linear interaction λ to a small but finite value, $\lambda = 0.001$, we have runned the symplectic dynamics with fringe boundary conditions until stationarity is reached at a certain time, which we call $\tau_{\rm eq}$. According to the premises of Sec. II, where we assumed that at long enough times the symplectic quantization dynamics allows us to sample an equilibrium ensemble, we have checked that equipartition between positional and kinetic degrees of freedom is in fact reached. In Fig. 4 is shown how, for two given choices of $k = \{k_0, k_1\}$ (corresponding respectively to small and large scales), we have that $\overline{E}_{\rm harm}(k,\tau)$ and $\overline{E}_{\rm kin}(k,\tau)$ reach asymptotically a value close to 1/2, starting respectively from $E_{\rm harm}(k,0) = 0$ and $E_{\rm kin}(k,0) = 1$, where the two energies are defined respectively as

$$\begin{aligned} \overline{E}_{\rm harm}(k,\tau) &= \frac{1}{\tau}\int_0^\tau ds\ \frac{1}{2}\omega_k^2|\phi(k,s)|^2 \\ \overline{E}_{\rm kin}(k,\tau) &= \frac{1}{\tau}\int_0^\tau ds\ \frac{1}{2}|\pi(k,s)|^2. \end{aligned} \quad (53)$$

We have found that this standard equipartition condition is fullfilled well when all k's in the lattice are such that $\omega_k^2 > 0$, while the stationary state reached when a finite fraction of the modes is such that $\omega_k^2 < 0$ has less trivial properties, which will be analised in further details elsewhere.

Having thus assessed that the system reaches some equilibrium/stationary state within some time $\tau_{\rm eq}$, we have coputed for all times $\tau > \tau_{\rm eq}$ the Fourier spectrum of the two-point correlation function $G(k) = \langle \phi^*(k)\phi(k)\rangle$ by averaging (quantum) fluctuations over intrinsic time. That is, we have defined an interval $\Delta\tau$ large enough and we have computed

$$\langle \phi^*(k)\phi(k)\rangle = \frac{1}{\Delta\tau}\sum_{i=0}^{M}\phi^*(k,\tau_{\rm eq}+\tau_i)\phi(k,\tau_{\rm eq}+\tau_i), \quad (54)$$

where $\tau_i = i\cdot\delta\tau$ and $\Delta\tau = M\delta\tau$.

In Fig. 5 we show (bottom panel) the result for the Fourier spectrum of the two-point correlation function obtained by setting all the parameters of the simulation and the initial conditions as declared at the end of Sec. III, apart from the value of the mass that is set here at $m = 3.0$ in order to better appreciate the shape of the propagator, and taking the value $\lambda = 0.001$ for the non-linearity parameter. In order to compare our numerical data at small non-linearity with the theory, we have also reported in the top panel of Fig.5 the theoretical shape of the free Feynman propagator $G_{\rm th}(k_0, k_1)$ on a discretized space-time grid in $1+1$ dimensions, using for the lattice the same parameters of the simulation, i.e., $a = 1.0$, $m = 1.0$, and $L = 128$, where $G_{\rm th}(k_0, k_1)$ reads as

$$G_{\rm th}(k_0,k_1) = \left[\frac{4}{a^2}\sin^2\left(\frac{ak_0}{2}\right) - \frac{4}{a^2}\sin^2\left(\frac{ak_1}{2}\right) - m^2\right]^{-1}. \quad (55)$$

Let us stress the beautiful qualitative agreement between the theoretical prediction of the free propagator and the numerical results: at variance with the Euclidean propagator, which is a function decreasing monotonically in all directions moving away from the origin (see Fig. 1 above), we find that the Feynman propagator sampled numerically here has the characteristic shape of a saddle, denoting a different behaviour between *time-like* direc-

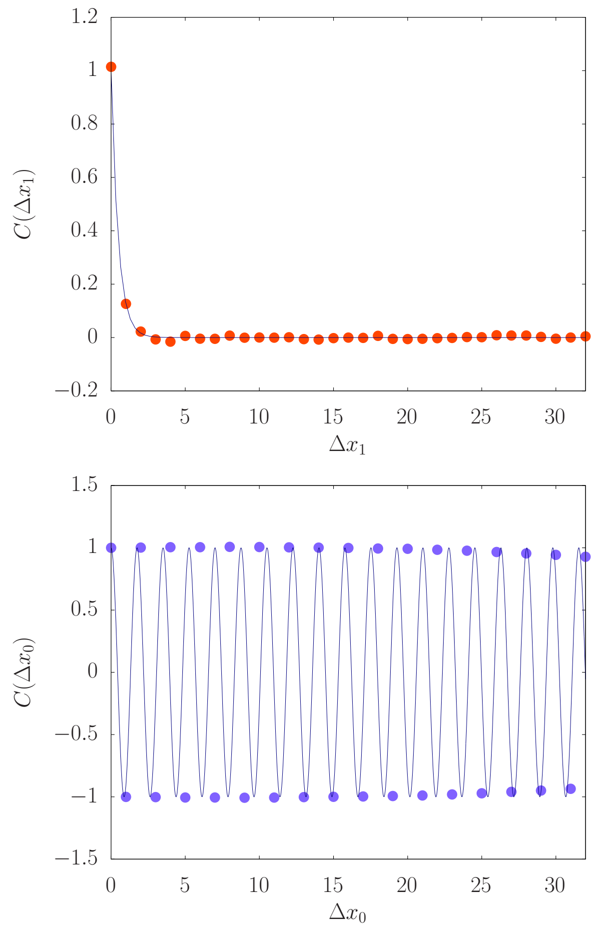

Figure 6. *Real space two-point correlation function for a $\lambda\phi^4$ theory in $1+1$ space-time dimensions with fringe boundary conditions, lattice spacing $a = 1.0$, lattice side $L = 128$, mass $m = 1.0$ and nonlinearity $\lambda = 0.001$. Top): exponential decay along the direction parallel to the x_1 axis; Bottom): oscillations along the direction parallel to the $x_0 = ct$ axis.*

tions and *space-like* directions. This is the first and incontrovertible strong evidence that the symplectic quantization approach opens up new possibilities so far out of reach within the Euclidean formulation of lattice field theory. Even more clear is the signature of the causal structure of space-time probed by means of the new approach if we look at the two-point correlation function in real space. According to the theoretical predictions for the free theory in the continuum one would expect undamped oscillations along the purely *time-like* directions and an exponential decay along the purely *space-like* directions for the Feynman propagator $\Delta_F(x-y)$:

$$\Delta_F(x-y) = \frac{1}{(2\pi)^2}\int d^2k\ \frac{e^{ik(x-y)}}{k^2-m^2}, \tag{56}$$

with

$$\begin{aligned} \Delta_F(x-y) &\sim e^{im|x-y|} \quad \text{for} \quad x-y \,||\, x_0 \\ \Delta_F(x-y) &\sim e^{-m|x-y|} \quad \text{for} \quad x-y \,||\, x_1. \end{aligned} \tag{57}$$

Cleary, when the same correlation function is sampled on a finite and discrete grid there will be finite-size effects at play so that, for instance, also the oscillations along the time-like direction will be slightly modulated by a tiny exponential decay: this is precisely what we find in numerical simulations. In Fig. [6] are shown, respectively in top and bottom panels, the exponential decay along the purely space-like direction and the oscillations along the purely time-like direction, obtained for the following choice of parameters: $a = 1.0$, $L = 128$, mass $m = 1.0$ and nonlinearity $\lambda = 0.001$. Let us notice that the value of the mass which can be obtained from either the fit of the exponential decay as $C(\Delta x_1) \sim e^{-m\Delta x_1}$ or the oscillating part as $C(\Delta x_0) \sim e^{im\Delta x_0}$ is $m \sim 2.06 \pm 0.04$, i.e., quite different from the value $m = 1$ put in the Lagrangian. This effect, which we do not find for the deterministic dynamics in Euclidean space-time, is most probably a finite-size effect related to nature of the new fringe boundary conditions adopted. We will devote our next effort in the numerical investigation of symplectic quantization to carefully study the finite-size effects induced by fringe boundaries

VII. CONCLUSIONS AND PERSPECTIVES

In this work we have presented the first numerical test of symplectic quantization, a new functional approach to quantum field theory [1, 2] which allows for an importance sampling procedure directly in Minkowski space-time. The whole idea, which parallels the one of stochastic quantization, is based on the assumption that fields has a dependence of an additional time parameter, the intrinsic time τ, with respect to which conjugated momenta $\pi(x)$ are defined. Quantum fluctuations of the fields are sampled by means of a deterministic dynamics flowing along the new time τ, which controls the internal dynamics of the system and is distinguished from the coordinate time of observers and clocks. Such a dynamics is generated by a generalized Hamiltonian where the original relativistic action plays the role of a potential energy part and therefore fluctuates naturally along the flow of τ. This whole construction does not need any sort of rotation from real to immaginary time to be consistent and to efficiently allow the numerical sampling of field fluctuations. Furthermore, under the hypothesis of ergodicity, symplectic quantization allows to define a generalized microcanonical ensemble which represents a probabilistically well defined functional approach to quantum field theory and yields by means of a simple Fourier transform the standard Feynman path integral, as discusse thoroughly in [1, 10]. The main result of this paper has been the evidence that the shape of the free Feynman propagator can be efficiently sampled for a $\lambda\phi^4$ real theory for a small value of λ. The next step along this work programm will be a careful study of finite-size effects induced by fringe boundary conditions and the test of the symplectic quantization approach in the presence of non-perturbative values of the nonlinearity λ.

ACKNOWLEDGMENTS

We thank M. Bonvini, R. Livi, F. Mercati, M. Papinutto, A. Polosa, A. Ponno, L. Salasnich, N. Tantalo and F. Viola for valuable discussions on the subject of this paper. In particular we want to thank M. Bonvini, F. Mercati and L. Livi for several interactions at the beginning of this project. G.G. thanks the Physics department of Sapienza for its kind hospitality during several periods during the preparation of this work. G.G. acknowledge partial support from the project MIUR-PRIN2022, *"Emergent Dynamical Patterns of Disordered Systems with Applications to Natural Communities"*, code 2022WPHMXK.

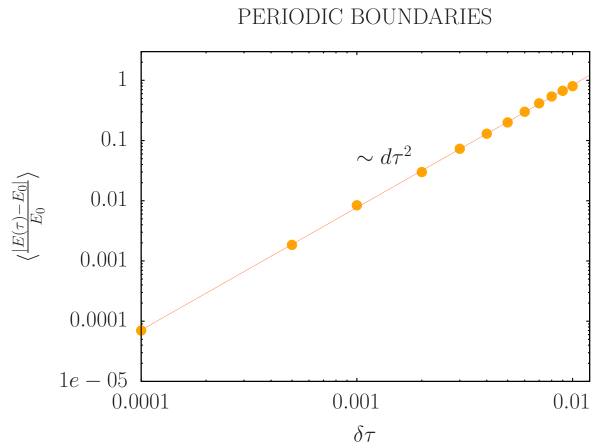

Figure 7. Energy fluctuations $\delta E(\delta\tau)$ as a function of the timestep $\delta\tau$ of the numerical algorithm in the case of Minkowski metric and *periodic* boundary conditions. Energy conservation at the algorithmic precision, i.e. $\delta E(\delta\tau) \sim \delta\tau^2$, is fulfilled.

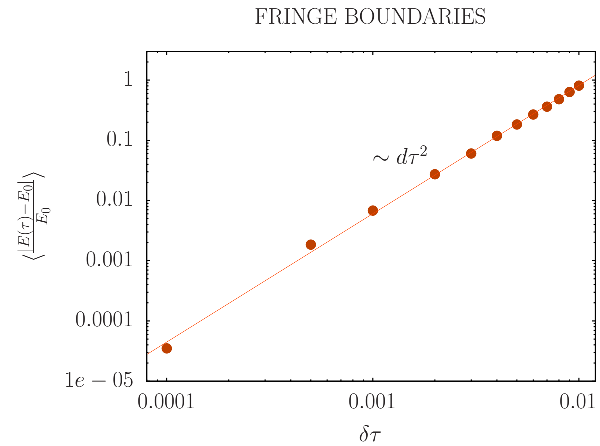

Figure 8. Energy fluctuations $\delta E(\delta\tau)$ as a function of the timestep $\delta\tau$ of the numerical algorithm in the case of Minkowski metric and *fringe* boundary conditions. Energy conservation at the algorithmic precision, i.e. $\delta E(\delta\tau) \sim \delta\tau^2$, is fulfilled.

Appendix A: Numerical Algorithm

All numerical cacalculations in this paper have been performed using a siplitting algorithm of second order, which takes advance of the generalized Hamiltonian separability. Using the notation of Sec. II, the algorithm can be characterized as a map

$$\Psi_{\delta\tau} : \phi(x,\tau), \pi(x,\tau) \quad \longrightarrow \quad \phi(x,\tau+\delta\tau), \pi(x,\tau+\delta\tau), \tag{A1}$$

with the following structure

$$\Psi_{\delta\tau} = \Phi_{\mathbb{K}}^{\delta\tau/2} \circ \Phi_{\mathbb{V}}^{\delta\tau} \circ \Phi_{\mathbb{K}}^{\delta\tau/2}, \tag{A2}$$

where $\Phi_{\mathbb{K}}^{\delta\tau/2}$ denotes the Hamiltonian flow of $\mathbb{K}[\pi]$, i.e., the flow of generalized momenta, while $\Phi_{\mathbb{V}}^{\delta\tau}$ denotes the Hamiltonian flow of $\mathbb{V}[\phi]$, i.e., the flow of generalized coordinates (in this case, the field). In formulae, each time step of the algorithm is represented by the following sequence of operations, to be realized for each point of x of the lattice:

$$\begin{aligned} \pi(x,\tau+\delta\tau/2) &= \pi(x,\tau) + \frac{\delta\tau}{2}\cdot F[\phi(x,\tau)] \qquad \forall\ x \\ \phi(x,\tau+\delta\tau) &= \phi(x,\tau) + \delta\tau\cdot\pi(x,\tau+\delta\tau/2) \quad \forall\ x \\ \pi(x,\tau+\delta\tau) &= \pi(x,\tau+\delta\tau/2) + \frac{\delta\tau}{2}\cdot F[\phi(x,\tau+\delta\tau)] \quad \forall\ x \end{aligned} \tag{A3}$$

The splitting algorithm which we have just described is usually known as the leapfrog algorithm, the name coming from the fact the updated of generalized positions and velocities takes place at interleaved time points. Given $E_0 = \mathbb{H}[\phi(x,0), \pi(x,0)]$ and $E(\tau) = \mathbb{H}[\phi(x,\tau), \pi(x,\tau)]$, where $\phi(x,\tau)$ and $\pi(x,\tau)$ are the numerical solutions computed at τ, the leapfrog dynamics has the following algorithmic bound on energy fluctuations

$$\delta E(\delta\tau) = \langle |E(\tau)/E_0 - 1| \rangle \ \propto \ \delta\tau^2 \tag{A4}$$

We have verified that the bound in Eq. (A4) is fulfilled by the fluctuations of both the Hamiltonian $E(\tau) = \mathbb{H}[\phi(x,\tau), \pi(x,\tau)]$ in the case of Minkowski metric with periodic boundary conditions and the total Hamiltonian (system + boundary layers) $\mathbb{H}_f[\phi(x,\tau), \pi(x,\tau)]$ in the case of *fringe* boundary conditions (See Eq. (36) and the following discussion for the definition of $\mathbb{H}_f[\phi,\pi]$). In Fig.7 and Fig.8 is shown the behavior of $\delta E(\delta\tau)$ as a function of $\delta\tau$ respectively for the case of periodic and fringe boundary conditions.

[1] G. Gradenigo and R. Livi, Foundations of Physics **51**, 10.1007/s10701-021-00470-9 (2021).
[2] G. Gradenigo, Foundations of Physics **51**, 10.1007/s10701-021-00468-3 (2021).
[3] K. G. Wilson, Phys. Rev. D **10**, 2445 (1974).
[4] M. Creutz, *Quarks, Gluons and Lattices* (Oxford University Press, 1983).
[5] I. Montvay and G. Munster, *Quantum fields on a lattice*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 1997).
[6] Z. Fodor and C. Hoelbling, Reviews of Modern Physics **84**, 449–495 (2012).
[7] C. Ratti, Reports on Progress in Physics **81**, 084301 (2018).
[8] L. Maiani, A. Polosa, and V. Riquer, Physics Letters B **749**, 289–291 (2015).
[9] A. Esposito, A. Pilloni, and A. Polosa, Physics Reports **668**, 1–97 (2017).
[10] G. Gradenigo, R. Livi, and L. Salasnich, Symplectic quantization iii: Non-relativistic limit (2024), arXiv:2401.12355 [cond-mat.stat-mech].
[11] G. Parisi and Y.-s. Wu, Sci. Sin. **24**, 483 (1981).
[12] P. H. Damgaard and H. Huffel, Phys. Rept. **152**, 227 (1987).
[13] D. J. E. Callaway and A. Rahman, Phys. Rev. D **28**, 1506 (1983).
[14] V. de Alfaro, S. Fubini, and G. Furlan, Nuovo Cim. A **74**, 365 (1983).
[15] G. Gradenigo, S. Iubini, R. Livi, and S. N. Majumdar, Journal of Statistical Mechanics: Theory and Experiment **2021**, 023201 (2021).
[16] G. Gradenigo, S. Iubini, R. Livi, and S. N. Majumdar, The European Physical Journal E **44**, 10.1140/epje/s10189-021-00046-5 (2021).
[17] J. Nordström, N. Nordin, and D. Henningson, SIAM Journal on Scientific Computing **20**, 1365 (1999).