A FROBENIUS INTEGRABILITY THEOREM FOR PLANE FIELDS GENERATED BY QUASICONFORMAL DEFORMATIONS

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ABSTRACT. We generalize the classical Frobenius integrability theorem to plane fields of class C^Q, a regularity class introduced by Reimann [Rei76] for vector fields in Euclidean spaces. A C^Q vector field is uniquely integrable and its flow is a quasiconformal deformation. We show that an a.e. involutive C^Q plane field (defined in a suitable way) in $\mathbb{R}^n$ is integrable, with integral manifolds of class C^1.

1. Introduction

Frobenius's integrability theorem is a fundamental result in differential topology and gives a necessary and sufficient condition for a smooth plane field (i.e., a distribution) to be tangent to a foliation. The result has been generalized to Lipschitz plane fields in [Sim96] and [Ram07]. The goal of this paper is to further extend Frobenius's theorem to a regularity class weaker than Lipschitz.

In [Rei76], Reimann introduced this new regularity class for vector fields and showed that it is situated between Lipschitz and Zygmund. We will call the vector fields in this class Q-vector fields, or of class C^Q (see the definition below). Reimann proved that a Q-vector field is uniquely integrable and that each time t map of its flow is quasiconformal. That is, the flow is a quasiconformal deformation.

1. **Definition.** *A continuous vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is called a Q-vector field or of class C^Q if*

$$\|f\|_Q = \sup \left| \frac{\langle a, f(x+a) - f(x) \rangle}{|a|^2} - \frac{\langle b, f(x+b) - f(x) \rangle}{|b|^2} \right| < \infty,$$

where the supremum is taken over all $x \in \mathbb{R}^n$ and $|a| = |b| \neq 0$.

Let

$$\|f\|_Z = \sup_{x,y \in \mathbb{R}^n, y \neq 0} \frac{|f(x+y) + f(x-y) - 2f(x)|}{|y|}$$

and

$$\|f\|_L = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|}$$

denote the Zygmund and Lipschitz seminorms of $f : \mathbb{R}^n \to \mathbb{R}^n$.

The following properties were shown in [Rei76]:

(1) $\|f\|_Z \leq 4 \|f\|_Q \leq 8 \|f\|_L$. Thus every Lipschitz vector field is of class C^Q. In dimension one, we have $\|f\|_Z = \|f\|_Q$.
(2) If f is a Q-vector field and $n \geq 2$, then f is Frechét differentiable a.e., its classical partial derivatives coincide with its weak derivatives and are locally integrable.

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(3) If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Q-vector field and $n \geq 2$, then the differential equation $\dot{x} = f(x)$ is uniquely integrable, and for every t, the time t map of its flow is $e^{c|t|}$-quasiconformal, for some $c > 0$.

Recall also:

2. **Definition.** *A homeomorphism $h : U \to V$ between open sets in $\mathbb{R}^n$ is said to be K-quasiconformal (K-qc) if the following conditions are satisfied:*

(a) *h is absolutely continuous on almost every line segment in U parallel to the coordinate axes;*
(b) *h is differentiable a.e.;*
(c) $\frac{1}{K} \|Dh(x)\|^n \leq |\det Dh(x)| \leq K\, m(Dh(x))^n$, *for a.e.* $x \in U$,

where, for a linear map A, $m(A)$ denotes the "minimum norm" of A:

$$m(A) = \min\{|Av| : |v| = 1\}.$$

The following characterization of Q-vector fields was proved in [Rei76] (cf., Theorem 3):

3. **Theorem.** *A continuous vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ ($n \geq 2$) is of class C^Q if and only if:*

(a) *f has distributional derivatives which are locally integrable.*
(b) $|f(x)| = O(|x| \log |x|)$, *as* $|x| \to \infty$.
(c) *The anticonformal part Sf of the derivative of f is essentially bounded, where*

$$Sf = \frac{1}{2}[Df + (Df)^T] - \frac{1}{n}\mathrm{Trace}(Df)I.$$

Let X and Y be vector fields of class C^Q. Since both are a.e. differentiable, we can define their Lie bracket in the usual way:

$$[X, Y] = DX(Y) - DY(X).$$

Alternatively,

$$[X, Y]u = X(Yu) - Y(Xu),$$

for every C^∞ function $u : \mathbb{R}^n \to \mathbb{R}$.

For a discussion of properties of the Lie bracket in the setting of rough vector fields (i.e., Lipschitz and below), see [CT21].

Plane fields of class C^Q. We come to the question of how to define a plane field of class C^Q. One possibility is to use the usual route and say that E is class C^Q if it is locally spanned by C^Q vector fields. This leads to some technical difficulties so instead we opt for a slightly stronger definition.

Let E be a continuous k-dimensional plane field on $\mathbb{R}^n$ and let $p \in \mathbb{R}^n$ be arbitrary. Let H_p be an $(n-k)$-dimensional coordinate plane in $\mathbb{R}^n$ such that E_p is transverse to $\{p\} \times H_p \subset T_p\mathbb{R}^n$. Continuity of E implies the existence of a neighborhood U of p such that for every $q \in U$, E_q remains transverse to $\{q\} \times H_p \subset T_q\mathbb{R}^n$. Let K_p be the k-dimensional coordinate subspace of $\mathbb{R}^n$ complementary to H_p and denote by π_p the orthogonal projection $\pi_p : \mathbb{R}^n \to K_p$. Observe that for every $q \in U$, the restriction of $D_q\pi_p$ to E_q is injective.

For an arbitrary vector field X on K_p with compact support in $\pi_p(U)$, define a lift $\hat{X}$ of X to E by

$$\hat{X}_q = (D_q\pi_p|_{E_q})^{-1}(X_{\pi(q)}),$$

for $q \in U$, and $\hat{X}_q = 0$, otherwise. This vector field is a section of E.

4. **Definition.** *A continuous k-dimensional plane field E on $\mathbb{R}^n$ is a Q-plane field or of class C^Q, if for every p, U, H_p, and X as above, the following holds: if X is C^∞ and has compact support, then its lift $\hat{X}$ to E (defined as above) is a Q-vector field.*

Note that if E is of class C^r $(r \geq 1)$ or Lipschitz, then lifts to E of C^∞ vector fields are C^r or Lipschitz, respectively.

5. **Definition.** *A plane field of class C^Q is said to be involutive if for every two C^Q sections X, Y of E, their Lie bracket $[X, Y]$ is an a.e. section of E; i.e.,*

$$[X, Y]_p \in E_p,$$

for a.e. p.

Our main result is:

Frobenius Theorem. *Let E be a k-dimensional plane field of class C^Q on $\mathbb{R}^n$. If E is involutive, then E is integrable, in the following sense. For every $p \in \mathbb{R}^n$, there exists a cubic neighborhood $U = (-\varepsilon, \varepsilon)^n$ of $\mathbf{0}$ in $\mathbb{R}^n$, a neighborhood V of p, and an almost everywhere differentiable homeomorphism*

$$\Phi : U \to V$$

such that Φ maps slices $(-\varepsilon, \varepsilon)^k \times \{\text{const}\}$ to integral manifolds of E in V. Writing $\Psi = \Phi^{-1} : p \mapsto x = (x_1, \ldots, x_n)$, we have that the integral manifolds of E in V are the slices

$$x_{k+1} = \text{constant}, \ldots, x_n = \text{constant}.$$

Every integral manifold of E in V lies in one of these slices and is of class C^1.

2. Preliminaries from the DiPerna-Lions-Ambrosio theory

We briefly recall some basic facts from the DiPerna-Lions-Ambrosio theory (cf., [Amb04, dL89]) of regular Lagrangian flows, which generalize the notion of a flow for "rough" vector fields. These will be needed in the proof of the main result.

The basic idea of DiPerna-Lions-Ambrosio is to exploit (via the theory of characteristics) the connection between the ODE $\dot{x} = X(x)$, where X is a vector field, and the associated transport PDE:

$$\frac{\partial u}{\partial t} + X \cdot \nabla_x u = 0,$$

for a function $u = u(t, x)$.

A common definition of this generalized notion of a flow is the following.

6. **Definition** (Regular Lagrangian flows [CT21]). *We say that $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a regular Lagrangian flow for a vector field X on $\mathbb{R}^n$ if:*

(a) *For a.e. (with respect to the 1-dimensional Lebesgue measure) $t \in \mathbb{R}$ and every measure zero Borel set $A \subset \mathbb{R}^n$, the set $\phi_t^{-1}(A)$ has n-dimensional Lebesgue measure zero, where $\phi_t(x) = \phi(t, x)$.*

(b) *We have $\phi_0 = \text{identity}$ and for a.e. $x \in \mathbb{R}^n$, $t \mapsto \phi_t(x)$ is an absolutely continuous integral curve of X, i.e.,*

$$\frac{d}{dt}\phi_t(x) = X(\phi_t(x)).$$

DiPerna, Ambrosio, and Lions showed that if $X \in L^\infty \cap BV$ has essentially bounded divergence, then the regular Lagrangian flow of X exists and is unique. Regular Lagrangian flows are *stable* in the following sense: if (X_k) is a sequence of smooth vector fields such that $X_k \to X$ strongly in L^1_{loc}, and $(\text{div}(X_k))$ is equibounded in L^∞, then the flows ϕ_t^k of X_k converge strongly to ϕ_t in L^1_{loc}, for every $t \in \mathbb{R}$.

If X is of class C^Q, each time t-map ϕ_t of its flow (in the usual sense) is K-quasiconformal for some K, hence preserves sets of Lebesgue measure zero (see [Kos09]). Thus $\{\phi_t\}$ is the regular Lagrangian flow of X.

7. **Lemma.** *Let X is a C^Q vector field with divergence in L^∞ and denote by $\{\phi_t\}$ its flow. Then $\|D\phi_t\|_{L^\infty} < \infty$, for each t, i.e., ϕ_t is Lipschitz.*

Proof. We follow [CT21]. Assume for a moment that X is smooth. Then by Liouville's Theorem,

$$\frac{d}{dt} \det D\phi_t(x) = \text{div}(X)(\phi_t(x)) \cdot \det D\phi_t(x),$$

which implies, via Gronwall's inequality that

$$\exp(-T \|\text{div}(X)\|_{L^\infty}) \leq \det D\phi_t(x) \leq \exp(T \|\text{div}(X)\|_{L^\infty}), \tag{1}$$

for all $T > 0$ and $-T \leq t \leq T$. Thus $\|\det D\phi_t\|_{L^\infty} < \infty$, for each t.

To make this work for a C^Q vector field X, consider $\det D\phi_t$ as a density, and take the mollifications X^ε of X (see, e.g., [Eva98]); let ϕ_t^ε be the flow of X^ε. By the stability of regular Lagrangian flows, we have

$$\det D\phi_t^\varepsilon \overset{*}{\rightharpoonup} \det D\phi_t$$

as $\varepsilon \to 0$ in L^∞, which again yields (1). Since ϕ_t is also K-quasiconformal, it follows (see part (c) in Def. 2) that

$$\|D\phi_t\|_{L^\infty} \leq K^{1/n} \|\det D\phi_t\|_{L^\infty}^{1/n} < \infty,$$

for $-T \leq t \leq T$, as desired. In particular, for any essentially bounded vector field Y, $\|D\phi_t(Y)\|_{L^\infty} < \infty$, for $-T \leq t \leq T$. □

The following corollary is a consequence of Theorem 1.1 in [CT21] and Lemma 7.

8. **Corollary.** *Let X, Y be bounded Q-vector fields with essentially bounded divergence, and let $\Phi = \{\phi_t\}$, $\Psi = \{\psi_t\}$ be their flows. Then the following statements are equivalent:*

(a) *Φ and Ψ commute as flows, i.e.,*

$$\phi_t \circ \psi_s(x) = \psi_s \circ \phi_t(x),$$

for a.e. $x \in \mathbb{R}^n$ and all $s, t \in \mathbb{R}$.

(b) *$[X, Y] = \mathbf{0}$, almost everywhere.*

3. Proof of the main result

The proof is a generalization of the standard proof for the smooth case, which can be found in, say, Lee [Lee13]. We will show that locally E admits a frame consisting of commuting C^Q vector fields, the composition of whose flows then defines the desired coordinate system.

Assume that E is an involutive k-dimensional plane field of class C^Q and let $p = (p_1, \dots, p_n) \in \mathbb{R}^n$ be an arbitrary point. Without loss we can assume that E_p is transverse to the subspace of $T_p\mathbb{R}^n$ spanned by

$$\left.\frac{\partial}{\partial x_{k+1}}\right|_p, \dots, \left.\frac{\partial}{\partial x_n}\right|_p.$$

Let $\pi : \mathbb{R}^n \to \mathbb{R}^k \times \{\mathbf{0}\}$ be the projection $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_k, \overbrace{0, \dots, 0}^{n-k})$. Then there exists a neighborhood W of p such that for every $q \in W$, $D_q\pi$ is injective when restricted to E_q. Let U be a neighborhood of p such that $\overline{U} \subset W$.

Let $\beta : \mathbb{R}^k \to \mathbb{R}$ be a C^∞ bump function such that $0 \le \beta \le 1$, $\beta = 1$ on $\pi(U)$, and $\beta = 0$ on the complement of $\pi(W)$. For $1 \le i \le k$, set

$$V_i = \beta \frac{\partial}{\partial x_i}.$$

Then V_i is a C^∞ vector field on $\mathbb{R}^n$ with compact support in $\pi(U)$. Denote its lift to E via π by X_i; i.e.,

$$X_i(q) = (D_q\pi|_{E_q})^{-1}(V_i(\pi(q))).$$

Since E is of class C^Q, X_i is also C^Q. Moreover, on U, we have

$$\pi_*([X_i, X_j]) = [\pi_*(X_i), \pi_*(X_j)] = [V_i, V_j] = \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0.$$

By involutivity of E, $[X_i, X_j]$ is a section of E a.e.. Since the restriction of $D\pi$ to E is injective, it follows that $[X_i, X_j] = 0$, a.e. on U.

9. **Lemma.** *X_i has essentially bounded divergence, for $1 \le i \le k$.*

Proof. Fix $1 \le i \le k$. It follows by construction of X_i that on U we have:

$$X_i = \frac{\partial}{\partial x_i} + \sum_{j=k+1}^{n} a_j \frac{\partial}{\partial x_j},$$

for some continuous a.e. differentiable functions a_j on U. Since X_i is C^Q, $S(X_i)$ is essentially bounded. It is easy to check that the (j, j)-component of $S(X_i)$ equals

$$S(X_i)_{jj} = -\frac{1}{n} \sum_{\ell=k+1}^{n} \frac{\partial a_\ell}{\partial x_\ell} = -\frac{1}{n}\operatorname{div}(X_i).$$

Thus the divergence of X_i is essentially bounded. □

Denote the flows of $X_1, \dots, X_k$ by $\phi_t^1, \dots, \phi_t^k$, respectively. By Corollary 8, they commute.

Define a map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Phi(x_1, \dots, x_n) = (\phi_{x_1}^1 \circ \cdots \phi_{x_k}^k)(p_1, \dots, p_k, p_{k+1} + x_{k+1}, \dots, p_n + x_n).$$

Then Φ is continuous and differentiable a.e.. We claim that there exists $\varepsilon > 0$ such that the restriction of Φ to the cube $C_\varepsilon = (-\varepsilon, \varepsilon)^n$ is injective. Observe that were Φ of class C^1, this would follow immediately from the Inverse Function Theorem.

Let $\varepsilon > 0$ be small enough so that the closure of C_ε is contained in $\Phi^{-1}(U)$. We claim that Φ is injective on the closure of C_ε. Assume that

$$\Phi(x_1, \dots, x_n) = \Phi(y_1, \dots, y_n),$$

i.e.,

$$(\phi^1_{x_1}\circ\cdots\phi^k_{x_k})(p_1,\dots,p_k,p_{k+1}+x_{k+1},\dots,p_n+x_n) = (\phi^1_{y_1}\circ\cdots\phi^k_{y_k})(p_1,\dots,p_k,p_{k+1}+y_{k+1},\dots,p_n+y_n),$$

for some $(x_1,\dots,x_n),(y_1,\dots,y_n)\in\overline{C}_\varepsilon$. Denote the flow of $\partial/\partial x_i$ by ψ^i_t. Since $\pi\circ\phi^i_t=\psi^i_t\circ\pi$, by projecting $\Phi(x_1,\dots,x_n)$ and $\Phi(y_1,\dots,y_n)$ via π, we obtain

$$(\psi^1_{x_1}\circ\cdots\psi^k_{x_k})(p_1,\dots,p_k) = (\psi^1_{y_1}\circ\cdots\psi^k_{y_k})(p_1,\dots,p_k),$$

which implies that $x_i=y_i$, for $1\le i\le k$.

Thus

$$(\phi^1_{x_1}\circ\cdots\phi^k_{x_k})(p_1,\dots,p_k,p_{k+1}+x_{k+1},\dots,p_n+x_n) = (\phi^1_{x_1}\circ\cdots\phi^k_{x_k})(p_1,\dots,p_k,p_{k+1}+y_{k+1},\dots,p_n+y_n)$$

which clearly implies $x_i=y_i$, for $k+1\le i\le n$. Therefore, Φ is 1–1 on $\overline{C}_\varepsilon$. By the continuity of Φ and compactness of $\overline{C}_\varepsilon$, it follows that $\Phi: C_\varepsilon\to\Phi(C_\varepsilon)$ is a homeomorphism.

Let S_c be a slice $(-\varepsilon,\varepsilon)^k\times\{c\}\subset C_\varepsilon$ (where $c\in\mathbb{R}^{n-k}$) and let

$$\alpha(t)=(x_1(t),\dots,x_k(t),c)$$

be an arbitrary C^1 path in S_c. Then

$$\gamma(t)=\Phi(\alpha(t))=\phi^1_{x_1(t)}\circ\cdots\circ\phi^k_{x_k(t)}(\text{const}).$$

The chain rule and commutativity of the flows ϕ^i_t implies that $\gamma'(t)$ is a linear combination of $X_1,\dots,X_k$, hence tangent to E. Therefore, $\Phi(S_c)$ is an integral manifold of E.

Since the tangent bundle of each integral manifold N of E is a continuous plane field (namely, E restricted to N), it follows that N is a C^1 manifold. This completes the proof. $\square$

Remark. The following questions would be of interest for further exploration:

(a) Does the main result still hold if a C^Q plane field is defined as locally spanned by C^Q vector fields with compact support?

(b) What can be said about foliations tangent to integrable C^Q plane fields?

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