

Belief Samples Are All You Need For Social Learning

Mahyar JafariNodeh, Amir Ajorlou, and Ali Jadbabaie

Abstract—In this paper, we consider the problem of social learning, where a group of agents embedded in a social network are interested in learning an underlying state of the world. Agents have incomplete, noisy, and heterogeneous sources of information, providing them with recurring private observations of the underlying state of the world. Agents can share their learning experience with their peers by taking actions observable to them, with values from a finite feasible set of states. Actions can be interpreted as samples from the beliefs which agents may form and update on what the true state of the world is. Sharing samples, in place of full beliefs, is motivated by the limited communication, cognitive, and information-processing resources available to agents especially in large populations. Previous work ([1]) poses the question as to whether learning with probability one is still achievable if agents are only allowed to communicate samples from their beliefs. We provide a definite positive answer to this question, assuming a strongly connected network and a “collective distinguishability” assumption, which are both required for learning even in full-belief-sharing settings. In our proposed belief update mechanism, each agent’s belief is a normalized weighted geometric interpolation between a fully Bayesian private belief — aggregating information from the private source — and an ensemble of empirical distributions of the samples shared by her neighbors over time. By carefully constructing asymptotic almost-sure lower/upper bounds on the frequency of shared samples matching the true state/or not, we rigorously prove the convergence of all the beliefs to the true state, with probability one.

I. INTRODUCTION AND RELATED WORK

In recent years, there has been a surge in research exploring mechanisms of belief formation and evolution in large populations, where individual agents have information of varying quality and precision, information exchange is limited and localized, and the sources, reliability, and trustworthiness of information is unclear. The body of literature on social learning, particularly within the realm of non-Bayesian models, reveals a nuanced landscape where individual cognitive capabilities, network structures, and the flow of information converge to shape collective outcomes.

The DeGroot model presented in [2] is a simple model of consensus formation, where individuals update their beliefs by taking weighted averages of their neighbors’ beliefs. This model provided a mathematical framework for analyzing the convergence of beliefs in a network setting. Authors in [3] have examined how the structure of social networks influences the accuracy of collective belief formation, highlighting the importance of network centrality and the distribution of initial opinions. Conditions under which communities can learn the true state of the world—despite the presence of biased agents— have been investigated in [4], contributing to our

understanding of the robustness of social learning processes to misinformation and bias. [5] explored the implications of limited information processing capabilities on social learning outcomes, demonstrating how cognitive constraints can lead to the persistence of incorrect beliefs within networks. The work in [6] focused on computational rationality, providing valuable insights into how individuals make decisions under uncertainty by approximating Bayesian inference, relevant for understanding the cognitive underpinnings of social learning.

More recently, authors in [7] offered a comprehensive analysis of non-Bayesian social learning, identifying the fundamental forces that drive learning, non-learning, and mis-learning in social networks. Another closely related work is [8], where agents make recurring private noisy observations of an underlying state of the world and repeatedly engage in communicating their beliefs on the state with their peers. Agents use Bayes rule to update their beliefs upon making new observations. Subsequently and after receiving her peers’ beliefs in each round, each agent then updates her belief to a convex combination of her own belief and those of her peers. It is then shown that under the so called “collective distinguishability assumption” and provided a strongly connected communication network, all agents learn the true state with probability one.

A key behavioral assumption in many approaches to non-Bayesian social learning (including [7], [8]) is that agents are capable of repeatedly communicating their full belief distributions with their peers. As pointed out in [1], decision-makers in large populations are likely not to satisfy such a cognitive demand, given the limited/costly communication and information processing resources. Motivated by such limitations, authors in [1] pose the question as to whether almost sure learning is achievable if agents are only allowed to communicate samples from their beliefs. They analyze the learning process under a sample-based variation of the model in [8], and show that collective distinguishability is not sufficient for learning anymore.¹

In this paper, we contribute to this line of work by proposing a framework where agents only communicate samples from their beliefs, and yet learning is achievable with probability one. Each agent’s belief in our model is a geometric interpolation between a fully Bayesian private belief — aggregating information from a private source — and an ensemble of empirical distributions of the actions shared by her neighbors (normalized to add up to 1). By carefully constructing asymptotic almost sure lower/upper bounds on the frequency of the shared actions communicating the true/wrong state, we prove the convergence of all the beliefs to the true state with

¹The potential for mislearning when relaying actions instead of information is also underscored in [9], [10].

probability one.

II. MATHEMATICAL MODEL

We consider a set of n agents denoted by $[n] = \{1, \dots, n\}$, who aim to learn an underlying state of the world θ . This state is a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and takes values in a finite set Θ , and take its size to be m (i.e. $|\Theta| = m$).

We adopt the same information structure as in [1]: At each time period $t = 1, 2, \dots$ and conditional on the state θ , each agent i observes a private signal $\omega_{it} \in S_i$ generated by the likelihood function $l_i(\cdot|\theta) \in \Delta_{S_i}$. Here, the finite set S_i denotes agent i 's signal space and Δ_{S_i} the set of probability measures on S_i . We denote the profile of each agent's signals by $\omega_i^t := (\omega_{i1}, \dots, \omega_{it})$. We assume that the observation profiles $\{\omega_{it}\}_{i=1}^n$ are independent over time, and that $l_i(\omega_i|\theta) > 0$ for all $i \in [n]$ and $(\omega_i, \theta) \in S_i \times \Theta$.

It is to be noted that agents, in general, may not be able to identify the true state solely relying on their private observations. This is the case when two states are observationally equivalent to an agent: Two states $\theta \neq \theta'$ are observationally equivalent to agent i if $l_i(\cdot|\theta) = l_i(\cdot|\theta')$. As a remedy, agents engage in repeated communication with each other on a social network where they can make state-related observations from their neighbors. The network is a weighted directed graph parameterized with $(\mathcal{V}, \mathcal{E})$ with adjacency matrix $A = \{a_{ij}\}_{i,j \in [n]^2}$, where the weights a_{ij} are non-negative and $\sum_{j=1}^n a_{ij} = 1$. A positive weight $a_{ij} > 0$ implies that agent j is a neighbor of agent i , and in particular, agent i can observe the action of agent j . We show the set of neighbors of agent i with \mathcal{N}_i . We assume agents have positive self-confidences, that is, the diagonal entries of A are all positive.

In our framework, each agent constructs an empirical distribution of their neighbors' actions. We denote agent j 's action at time t by $c_{jt} \in \Theta$ and the profile of her actions by $c_j^t := (c_{j1}, \dots, c_{jt})$; The indicator function $\mathbf{1}_{c_{jt}}(\theta)$ is then equal to 1 if agent j has declared θ at time t as her opinion. Agents can use their actions as a means to broadcast their opinion on which state they find more likely to be the true state of the world to their neighbors. We elaborate on our proposed strategy for taking actions later in this section. Neighbors of agent j construct an empirical distribution $\hat{\mu}_{jt} \in \Delta_{\Theta}$ of her actions by taking counts of the times she declares θ as her opinion/action for each $\theta \in \Theta$.

For each $\theta \in \Theta$, let $n_{jt}(\theta) := 1 + \sum_{\tau=1}^t \mathbf{1}_{c_{j\tau}}(\theta)$ count how many times agent j takes action θ up to time t . We initialize all counters by 1. We then normalize the counts to construct what we refer to (with a bit misuse of notation) as the empirical distribution of declared actions for agent j :

$$\hat{\mu}_{jt}(\theta) := \frac{n_{jt}(\theta)}{\sum_{\theta' \in \Theta} n_{jt}(\theta')} = \frac{n_{jt}(\theta)}{t + m}. \quad (1)$$

Each agent also holds a private belief $\mu_{it}^P \in \Delta_{\Theta}$ aggregating information from its private source following Bayes update rule:

$$\mu_{it}^P(\theta|\omega_i^t) = \frac{l_i(\omega_{it} | \theta) \cdot \mu_{it-1}^P(\theta|\omega_i^{t-1})}{m_{it}(\omega_{it})}, \quad (2)$$

$$m_{it}(\omega_{it}) = \sum_{\theta \in \Theta} l_i(\omega_{it}|\theta) \mu_{it-1}^P(\theta|\omega_i^{t-1}).$$

We initialize the private beliefs to be uniform, i.e., $\mu_{i0}^P = (\frac{1}{m}, \dots, \frac{1}{m})$. Each agent i then incorporates the empirical distribution of declared opinions of her neighbors $\hat{\mu}_{jt}$ for all $j \in \mathcal{N}_i$ into their private belief μ_{it}^P to form her belief $\mu_{it} \in \Delta_{\Theta}$ on what true state of the word is. They do so by taking the weighted geometric mean of their private beliefs and the empirical distribution of their neighbors' actions, and normalizing it to add up to 1:

$$\mu_{it}(\theta) \propto \mu_{it}^P(\theta)^{a_{ii}} \times \prod_{j \in \mathcal{N}_i} \hat{\mu}_{jt}(\theta)^{a_{ij}}. \quad (3)$$

Notice that the weights a_{ii} and a_{ij} 's capture the trust of agent i in her private source of information and her neighbors' declared opinions, respectively. Each agent i then takes action c_{it} by drawing a sample from her belief μ_{it} (i.e. $c_{it} \sim \mu_{it}$), which is subsequently observed by those who are neighboring her.

III. MODEL DISCUSSION AND PRELIMINARIES

The key contribution of this work is to show that as long as the agents can collectively distinguish the states and the graph is strongly connected, learning occurs with probability one under our proposed framework. Collective distinguishability means that for every two different states θ and θ' , there exists an agent i such that $l_i(\cdot|\theta) \neq l_i(\cdot|\theta')$. We formally define learning below.

Definition 1 ([1]). *Agent $i \in [n]$ learns the true state θ^* along the sample path $w \in \Omega$, if $\lim_{t \rightarrow \infty} \mu_{it}(\theta^*) = 1$ at w .*

It proves insightful to elaborate on connections/distinctions of our work with [8], [1] which study models similar to ours. Instead of sharing samples from beliefs, authors in [8] assume that agents are capable of sharing their full beliefs with their neighbors in each round. Their belief update rule is of the form

$$\mu_{it+1}(\theta) = a_{ii} \frac{l_i(\omega_{it+1}|\theta)}{m_{it}(\omega_{it+1})} \mu_{it}(\theta) + \sum_{j \in \mathcal{N}_i} a_{ij} \mu_{jt}(\theta), \quad (4)$$

where

$$m_{it}(\omega) = \sum_{\theta \in \Theta} l_i(\omega|\theta) \mu_{it}(\theta).$$

They show that under the Collective distinguishability assumption and strongly connected graph, learning occurs with probability one.

Motivated by the limited communication and cognitive resources available to agents especially in large populations, authors in [1] pose the question as to whether learning with probability one is still achievable if agents are only allowed to communicate samples from their beliefs. They then analyze the learning process under a sample-based variation of (4):

$$\mu_{it+1}(\theta) = a_{ii} \frac{l_i(\omega_{it+1}|\theta)}{m_{it}(\omega_{it+1})} \mu_{it}(\theta) + \sum_{j \in \mathcal{N}_i} a_{ij} \mathbf{1}_{c_{jt}}(\theta).$$

As their main result, they prove that collective distinguishability is not sufficient for learning in this case. Our work complements this chain by proposing a framework where agents only communicate samples from their beliefs, and yet learning occurs with probability one. Each agent's belief is a geometric interpolation between a fully Bayesian private belief–aggregating information from a private source– and an ensemble of empirical distributions of her neighbors' actions, as governed by (1)-(3).²

IV. MAIN RESULTS

In this section, we rigorously analyze the belief dynamics governed by (1)-(3) to establish that learning occurs with probability one, under our proposed framework.

Definition 2. Denoting the true state of the world by θ^* , we say that a state $\theta \in \Theta$ is θ^* -identifiable for agent i if:

$$\theta \neq \theta^* \rightarrow l_i(\cdot|\theta) \neq l_i(\cdot|\theta^*).$$

We also denote $\mathbb{P}_{\theta^*}(\cdot) := \mathbb{P}(\cdot|\theta^*)$, and $\mathbb{E}_{\theta^*}[\cdot] := \mathbb{E}(\cdot|\theta^*)$.

A. Exponentially Fast Decay of the Belief over Identifiable States

We start by showing the exponential decay of private beliefs on the states identifiable from the true state. We first introduce the notion of Rényi divergence.

Definition 3. (α -Rényi divergence) The α -Rényi divergence between two discrete distributions P and Q is defined as,

$$D_\alpha(P\|Q) := \frac{1}{\alpha - 1} \log \left(\sum_{i=1}^k p_i^\alpha q_i^{1-\alpha} \right), \quad \alpha \geq 0.$$

Note that $D_1(P\|Q)$, that is the Rényi divergence for $\alpha = 1$, recovers KL-divergence.

Lemma 1. Let θ be a θ^* -identifiable state for agent $i \in [n]$. Then, for any $\beta_{i\theta}$ with $0 < \beta_{i\theta} < D_1(l_i(\cdot|\theta^*)\|l_i(\cdot|\theta))$, there exists $\gamma_{i\theta} := \gamma(\beta_{i\theta}) > 0$ such that for each $t \in \mathbb{N}$ we have:

$$\mathbb{P}_{\theta^*} \left(\frac{\mu_{it}^P(\theta|\omega_i^t)}{\mu_{it}^P(\theta^*|\omega_i^t)} > e^{-\beta_{i\theta}t} \right) \leq e^{-\gamma_{i\theta}t}.$$

Proof. Using Equation (3) one can write

$$\frac{\mu_{it}^P(\theta|\omega_i^t)}{\mu_{it}^P(\theta^*|\omega_i^t)} = \left(\prod_{\tau=1}^t \frac{l_i(\omega_{i\tau}|\theta)}{l_i(\omega_{i\tau}|\theta^*)} \right) \cdot \frac{\mu_{i0}^P(\theta)}{\mu_{i0}^P(\theta^*)}.$$

Now by noting the independence of $\{\omega_{i\tau}\}_{\tau=1}^t$ due to i.i.d. samples, by Markov inequality, we have

$$\begin{aligned} \mathbb{P}_{\theta^*} \left(\frac{\mu_{it}^P(\theta|\omega_i^t)}{\mu_{it}^P(\theta^*|\omega_i^t)} > e^{-\beta_{i\theta}t} \right) &\leq \frac{\mathbb{E}_{\theta^*} \left[\left(\frac{l_i(\omega_i|\theta)}{l_i(\omega_i|\theta^*)} \right)^{1-\alpha} \right]^t}{e^{-(1-\alpha)\beta_{i\theta}t}} \\ &= \exp(-t(1-\alpha)(D_\alpha(l_i(\cdot|\theta^*)\|l_i(\cdot|\theta)) - \beta_{i\theta})), \end{aligned}$$

where $\alpha \in (0, 1)$, and $D_\alpha(P\|Q)$ is α -Rényi Divergence between distributions P, Q , and $\gamma(\beta) = \max_{\alpha^*} (1 -$

$\alpha)(D_\alpha(l_i(\cdot|\theta^*)\|l_i(\cdot|\theta)) - \beta)$ where α^* is the set of $\alpha \in (0, 1)$ for which $D_\alpha(l_i(\cdot|\theta^*)\|l_i(\cdot|\theta)) > \beta_{i\theta}$. \square

Lemma 1 suggests that as long as a state θ is identifiable for an agent, her belief on θ decays exponentially fast in time, no matter how misinforming her neighbors are, which is formally stated below.

Lemma 2. Let θ be a θ^* -identifiable state for agent $i \in [n]$. Then, for any $0 < \beta_{i\theta} < D_1(l_i(\cdot|\theta^*)\|l_i(\cdot|\theta))$ there exists $\gamma_{i\theta} > 0$ such that for sufficiently large t , we have

$$\mathbb{P}_{\theta^*}(\mu_{it}(\theta) \geq e^{-a_{ii}\beta_{i\theta}t}) \leq e^{-\gamma_{i\theta}t}, \quad (5)$$

Proof. Using Equation (3) we have

$$\begin{aligned} \frac{\mu_{it}(\theta)}{\mu_{it}(\theta^*)} &= \left(\frac{\mu_{it}^P(\theta)}{\mu_{it}^P(\theta^*)} \right)^{a_{ii}} \times \underbrace{\prod_{j \in \mathcal{N}_i} \left(\frac{\hat{\mu}_{jt}(\theta)}{\hat{\mu}_{jt}(\theta^*)} \right)^{a_{ij}}}_{\text{KOO(Knowledge of Others)}} \quad (6) \\ &\leq \left(\frac{\mu_{it}^P(\theta)}{\mu_{it}^P(\theta^*)} \right)^{a_{ii}} \times (t+1)^{1-a_{ii}}, \quad (7) \end{aligned}$$

where the inequality follows by considering the worst case where the true state is never chosen by the neighbors while θ has been constantly chosen since the beginning. Now by invoking Lemma 1, with probability at least $1 - e^{-\gamma_{i\theta}t}$

$$\frac{\mu_{it}(\theta)}{\mu_{it}(\theta^*)} \leq e^{-a_{ii}\beta_{i\theta}t} \times (t+1)^{1-a_{ii}},$$

\square

However, what about the states that agents cannot distinguish from θ^* ? The answer lies in the knowledge of other users in the network who possess this capability which is characterized by KOO term in Equation (6). In order to utilize this knowledge effectively, we must understand the properties of these users and how their expertise can benefit others. To accomplish this, we need to analyze the empirical distribution of the opinions declared by neighbors, taking into account the frequency of declaring each of θ and θ^* as their action, encapsulated in the parameters $n_{it}(\theta)$ and $n_{it}(\theta^*)$ which denote the number of times each of θ and θ^* are chosen by agent i up to time t .

B. The frequency of declaring true state

In this section we investigate the $\hat{\mu}_{jt}(\theta^*)$ component of KOO term which is capturing the frequency of neighbors declaring true state θ^* as their opinion.

Lemma 2 was proved using the worst case lower bound $n_{it}(\theta^*) \geq 1$ (i.e. users don't take θ^* as their action up to time t), which is clearly an underestimation. To refine this, we must derive a non-trivial lower bound on the number of instances in which users select the true state θ^* as their action. We first derive a lower bound on the belief of agents on θ^* , and will subsequently use it to approximate the number of declared opinions matching the true state using some concentration inequalities.

²This is subsequently normalized to add up to 1.

Lemma 3. For any agent $i \in [n]$, there exists $\gamma_i > 0$ such that for all $t \in \mathbb{N}$ we have:

$$\mathbb{P}_{\theta^*} \left(\mu_{it}(\theta^* | \omega_i^t) \leq \frac{1}{m(t+1)^{1-a_{ii}}} \right) \leq e^{-\gamma_i t}.$$

Proof. Reversing the inequality in Equation (7) we have:

$$\frac{\mu_{it}(\theta^*)}{\mu_{it}(\theta)} \geq \left(\frac{\mu_{it}^P(\theta^*)}{\mu_{it}^P(\theta)} \right)^{a_{ii}} \times 1/(t+1)^{1-a_{ii}}.$$

For each non- θ^* -identifiable θ in the above inequality, we have $\left(\frac{\mu_{it}^P(\theta^*)}{\mu_{it}^P(\theta)} \right)^{a_{ii}} = 1$, so $\frac{\mu_{it}(\theta^*)}{\mu_{it}(\theta)} > 1/(t+1)^{1-a_{ii}}$. For the rest, by Lemma 1 we know that with probability at least $1 - e^{-\gamma_i \theta t}$, the event $\frac{\mu_{it}^P(\theta^*)}{\mu_{it}^P(\theta)} > e^{a_{ii} \beta_i \theta t} > 1$ happens which implies $\frac{\mu_{it}(\theta^*)}{\mu_{it}(\theta)} > 1/(t+1)^{1-a_{ii}}$. By union bound we get

$$\begin{aligned} \mathbb{P}_{\theta^*} \left(\frac{\mu_{it}(\theta^*)}{\mu_{it}(\theta)} > 1/(t+1)^{1-a_{ii}}, \text{ For all } \theta \in \Theta \right) \\ \geq 1 - m e^{-\gamma_i t}. \end{aligned}$$

Hence, we have

$$\mathbb{P}_{\theta^*} (\mu_{it}(\theta^* | \omega_i^t) > \frac{1}{m(t+1)^{1-a_{ii}}}) \geq 1 - m e^{-\gamma_i t}.$$

where $\gamma_i := \gamma(\min_{\theta \neq \theta^*} \beta_i \theta)$ \square

Since the agents are constructing the empirical distributions on a counting manner, we aim to derive at least how many times θ^* is chosen for large t . From this point on, we will consider t to be sufficiently large, and the inequalities that will be used would hold for large enough values of t . To further formalize this, we have the following Theorem.

Lemma 4. For each agent $i \in [n]$ there exists an $\alpha > 0$, and $T_\alpha \in \mathbb{N}$ such that for all $t \geq T_\alpha$ we have:

$$n_{it}(\theta^*) > (t+1)^{1-\alpha}, \quad (8)$$

with probability at least $1 - e^{-(t+1)^{1-\alpha}}$.

Proof. Let $n_{it_0:t}(\theta^*) := X_{it_0}^{\theta^*} + X_{it_0+1}^{\theta^*} + \dots + X_{it}^{\theta^*}$ be the number of times that θ^* is chosen by her, where $\{X_\tau\}_{\tau=t_0}^t$ are i.i.d. Bernoulli random variables defined as $X_{i\tau}^{\theta^*} := \mathbf{1}_{c_{i\tau}}(\theta^*)$. By Lemma 3 and law of conditional expectation, for each time step t we have:

$$\begin{aligned} \mathbb{E}_{\theta^*} [X_t] \\ = \mu_{it}(\theta^* | \mu_{it} > \frac{1}{m(t+1)^{1-a_{ii}}}) \cdot \mathbb{P}_{\theta^*} (\mu_{it} > \frac{1}{m(t+1)^{1-a_{ii}}}) \\ + \mu_{it}(\theta^* | \mu_{it} \leq \frac{1}{m(t+1)^{1-a_{ii}}}) \cdot \mathbb{P}_{\theta^*} (\mu_{it} \leq \frac{1}{m(t+1)^{1-a_{ii}}}) \\ \geq \frac{1 - m \cdot e^{-\gamma_i t}}{m(t+1)^{1-a_{ii}}}. \end{aligned}$$

So by using the Chernoff bound for $\delta \in (0, 1)$, we have

$$\begin{aligned} \mathbb{P}_{\theta^*} \left(n_{it_0:t}(\theta^*) < (1-\delta) \cdot \sum_{\tau=t_0}^t \frac{1 - m \cdot e^{-\gamma_i \tau}}{m(\tau+1)^{1-a_{ii}}} \right) \\ \leq \mathbb{P}_{\theta^*} (n_{it_0:t}(\theta^*) < (1-\delta) \cdot \mathbb{E}_{\theta^*} [n_{it_0:t}(\theta^*)]) \\ \leq \exp\left(-\frac{\delta^2 \cdot \mathbb{E}_{\theta^*} [n_{it_0:t}(\theta^*)]}{2}\right), \end{aligned}$$

which implies with probability at least $1 - e^{-\delta^2 \cdot \mathbb{E}_{\theta^*} [n_{it_0:t}(\theta^*)]/2}$ we have:

$$\begin{aligned} \frac{n_{it_0:t}(\theta^*)}{1-\delta} &\geq \sum_{\tau=t_0}^t \frac{1 - m \cdot e^{-\gamma_i \tau}}{m(\tau+1)^{1-a_{ii}}} \\ &\geq \frac{(1 - m \cdot e^{-\gamma_i t_0})}{m} \cdot \underbrace{\sum_{\tau=t_0}^t \frac{1}{(\tau+1)^{1-a_{ii}}}}_{:=h_{1-a_{ii}}(t_0, t)} \\ &= \frac{(1 - m \cdot e^{-\gamma_i t_0})}{m} \cdot h_{1-a_{ii}}(t_0, t) \\ &\stackrel{(a)}{\geq} \frac{(1 - m \cdot e^{-\gamma_i t_0})}{m} \int_{t_0+1}^{t+2} \frac{1}{\tau^{\alpha_i}} d\tau \\ &\geq \frac{(1 - m \cdot e^{-\gamma_i t_0})}{m} \cdot \frac{(t+2)^{1-\alpha_i} - (t_0+1)^{1-\alpha_i}}{1-\alpha_i} \\ &\gtrsim c_i (t+1)^{1-\alpha_i}, \end{aligned}$$

where $\alpha_i := 1 - a_{ii}$, and $0 < c_i < \frac{(1 - m \cdot e^{-\gamma_i t_0})}{m(1-\alpha_i)}$. (a) follows because of the inequality $\int_1^{N+1} \frac{1}{t^\alpha} dt \leq \sum_{i=1}^N \frac{1}{i^\alpha}$. It could also be observed that, by choosing $t_0 \in o(t)$, we have $h(t_0, t) \in \mathcal{O}(h(1, t))$, and $h(1, t_0) \in o(h(t_0, t))$, while we also need to take $t_0 > \log(m)/\gamma_i$.

Hence we can write

$$\begin{aligned} \frac{n_{it}(\theta^*)}{1-\delta} &\geq \frac{n_{it_0:t}(\theta^*)}{1-\delta} \\ &\gtrsim c_i \cdot (t+1)^{1-\alpha_i}. \end{aligned} \quad (9) \quad (10)$$

By utilizing this, we can improve the inequality in Equation (7); Since at the first place we bounded it at worst case considering that $n_{jt}(\theta^*) = 1$. Rewriting it we will have:

$$\begin{aligned} \frac{\mu_{it}(\theta)}{\mu_{it}(\theta^*)} &\lesssim \left(\frac{\mu_{it}^P(\theta)}{\mu_{it}^P(\theta^*)} \right)^{a_{ii}} \times \prod_{j \in \mathcal{N}_i} \left(\frac{t+1}{c_j \cdot (t+1)^{1-\alpha_j}} \right)^{a_{ij}} \\ &\leq \tilde{c}_i \left(\frac{\mu_{it}^P(\theta)}{\mu_{it}^P(\theta^*)} \right)^{a_{ii}} \times (t+1)^{\sum_{j \in \mathcal{N}_i} a_{ij} \alpha_j}, \end{aligned} \quad (11)$$

where $\tilde{c}_i = \prod_{j \in \mathcal{N}_i} (c_j \cdot (1-\delta))^{a_{ij}} > 1$. Note that since we are using (10) for all the neighbors, by using Union Bound, 11 will hold with probability at least

$$1 - \sum_{j \in \mathcal{N}_i} e^{-c_i \delta^2 (t+1)^{1-\alpha_i} / 2}$$

It appers the power of $(t+1)$ on the R.H.S of 11 is exhibiting an iterative pattern of $(t+1)^{\alpha_i(m)}$ for $(\alpha_i(m) : m \in [0, 1, \dots])$. where

$$\begin{cases} \alpha_i(0) = 1 - a_{ii} \\ \alpha_i(k+1) = \sum_{j \neq i} a_{ij} \alpha_j(k) \quad \forall k \in [0, 1, \dots] \end{cases}$$

writing this down for all users in matrix we get the following matrix form:

$$\boldsymbol{\alpha}(m+1) = A' \boldsymbol{\alpha}(m),$$

where $\boldsymbol{\alpha}(m) = [\alpha_1(m), \dots, \alpha_n(m)]^\top$, and A' is the adjacency matrix with its diagonals equal to zero. The matrix forms now can be exploited to see

$$\|\boldsymbol{\alpha}(m+1)\|_\infty < \|A'\|_\infty \cdot \|\boldsymbol{\alpha}(m)\|_\infty < \|A'\|_\infty^m \|\boldsymbol{\alpha}(0)\|_\infty,$$

which implies that α_i s could be made desirably small by increasing m noticing that $\|A'\|_\infty < 1$. This improves (10) at the cost of decreasing c_i s in

$$\begin{aligned} & 1 - \sum_{j \in \mathcal{N}_i} e^{-c_i \delta^2 (t+1)^{1-\alpha_i} / 2} - m e^{-\gamma_i t} \\ & \gtrsim 1 - e^{-(t+1)^{1-\alpha}} \end{aligned}$$

C. The frequency of declaring a state $\theta \neq \theta^*$

Proposition 1. *Each agent $i \in [n]$, chooses each of her θ^* -identifiable states (θ) finitely many times.*

Proof. We know that the event $\{c_{it} = \theta\}$ (which denotes if user i announces θ at time t) occurs with probability $p_{it}(\theta) := (\mu_{it}^S(\theta))$. Using Corollary 2, we can write

$$\begin{aligned} & \sum_{i=1}^{\infty} p_{it}(\theta) \\ & = \sum_{t=1}^{\infty} \left[p_{it}(\theta) | p_{it} > e^{-a_{ii}\beta_i \theta t} \cdot \mathbb{P}_{\theta^*}(p_{it} > e^{-a_{ii}\beta_i \theta t}) \right. \\ & \quad \left. + p_{it}(\theta) | p_{it} \leq e^{-a_{ii}\beta_i \theta t} \cdot \mathbb{P}_{\theta^*}(p_{it} < e^{-a_{ii}\beta_i \theta t}) \right] \\ & \leq \sum_{t=1}^{\infty} [e^{-\gamma_i \theta t} + e^{-a_{ii}\beta_i \theta t}] < \infty, \end{aligned}$$

where c is sufficiently large so that for $(t > c)$ Lemma 2 holds; Thus by using the Borell-Contelli Theorem, we deduce that the event $\{C_{it} = \theta\}$ happens finitely many times. \square

Remark 1. *Note that above Proposition doesn't imply uniform boundedness of $n_{it}(\theta)$, meaning that there is a constant that it is smaller than for all times.*

Fix some $\theta \neq \theta^*$. As the next milestone, we aim to carefully construct upper bounds on $n_{it}(\theta)$ of the form

$$n_{it}(\theta) \leq (1+t)^{\beta_i}, \quad (12)$$

with probability at least $1 - e^{-\gamma_i \sqrt{t}}$ for all $t \geq T_i$, for some $\beta_i, \gamma_i \geq 0$ and $T_i \in \mathbb{N}$. Observing that $n_{it}(\theta) \leq 1+t$ by definition, a trivial choice is $\beta_i = 1$, $\gamma_i = 0$, $T_i = 1$. It turns out we can do much better. It proves convenient to define the notion of expert agents.

Definition 4. *The set of θ -expert agents \mathcal{J}_θ consists of agents in the network who can distinguish θ from θ^* . The distance of an agent from this set is defined as the length of the shortest path connecting her to an agent in this set on the graph associated with A .³ For any $i \in [n]$, we also define σ_i to be the node immediately proceeding i on the shortest path to this set (if there are multiple shortest paths, we choose one at random).*

Let us start by improving the choice of (β_i, γ_i, T_i) for θ -expert agents.

³Note that the length of a path here is the number of edges on the path and not the sum of the weights of the edges on it.

Lemma 5. *For any θ -expert agent $i \in [n]$ and any $\beta_i > \frac{3}{4}$, there exist $\gamma_i > 0$ and $T_i \in \mathbb{N}$ such that*

$$n_{it}(\theta) \leq (t+1)^{\beta_i}$$

with probability at least $1 - e^{-\gamma_i \sqrt{t}}$ for $t \geq T_i$.

Proof. Using conditional expectation, and constructing a model, based on Bernoulli random variables to denote each time action θ is taken, we have $n_{it}(\theta) = X_1 + \dots + X_t$, using the same argument as in the proof of Proposition 1, we get $\mathbb{E}_{\theta^*}[n_{it}(\theta)] \leq c + \sum_{\tau=c+1}^t [e^{-\gamma_i \theta \tau} + e^{-a_{ii}\beta_i \theta \tau}] < \infty$, and by using the Hoeffding inequality [11] for bounded random variables we have we have

$$\begin{aligned} & \mathbb{P}_{\theta^*} \left(n_{it}(\theta) > \left(\sum_{\tau=1}^t [e^{-\gamma_i \theta \tau} + e^{-a_{ii}\beta_i \theta \tau}] \right) + t^{\beta_i} \right) \\ & \leq \mathbb{P}_{\theta^*} \left(n_{it}(\theta) > \mathbb{E}_{\theta^*} [n_{jt}(\theta)] + t^{\beta_i} \right) \\ & \leq \exp(-2t^{2\beta_i-1}) \lesssim \exp(-\gamma_i \sqrt{t}) \end{aligned}$$

for some choice of γ_i , which implies with probability $1 - e^{-\gamma_i \sqrt{t}}$ we have $n_{it}(\theta) \lesssim (t+1)^{\beta_i}$ \square

The following result enables us to come up with improved upper bounds of the form (12) for an agent exploiting the potentially improved bounds of her neighbors.

Lemma 6. *Consider agent $i \notin \mathcal{J}_\theta$ and assume that her neighbors $j \in \mathcal{N}_i$ satisfy*

$$n_{jt}(\theta) \leq (1+t)^{\beta_j}, \quad (13)$$

with probability at least $1 - e^{-\gamma_j \sqrt{t}}$ for all $t \geq T_j$, for some $\{(\beta_j, \gamma_j, T_j)\}_{j \in \mathcal{N}_i}$. Choose any $\beta_i > a_{ii} + \sum_{j \neq i} a_{ij} \beta_j$, then there exists $\gamma_i > 0$ and $T_i \in \mathbb{N}$ such that

$$n_{it}(\theta) \leq (1+t)^{\beta_i}, \quad (14)$$

with probability at least $1 - e^{-\gamma_i \sqrt{t}}$ for all $t \geq T_i$.

Proof. We have:

$$\begin{aligned} \frac{\mu_{it}(\theta)}{\mu_{it}(\theta^*)} & = \left(\frac{\mu_{it}^P(\theta)}{\mu_{it}^P(\theta^*)} \right)^{a_{ii}} \times \prod_{j \in \mathcal{N}_i} \left(\frac{\hat{\mu}_{jt}(\theta)}{\hat{\mu}_{jt}(\theta^*)} \right)^{a_{ij}} \\ & \lesssim \tilde{c}_i \prod_{j \in \mathcal{N}_i} \left(\frac{(t+1)^{\beta_j}}{(t+1)^{1-\alpha_j}} \right)^{a_{ij}} \\ & \leq \tilde{c}_i \times (t+1)^{(\sum_{j \in \mathcal{N}_i} a_{ij}(\beta_j - (1-\alpha_j^*)))} \end{aligned}$$

Which implies:

$$\mu_{it}(\theta) \lesssim \tilde{c}_i \times (t+1)^{(\sum_{j \in \mathcal{N}_i} a_{ij}(\beta_j - (1-\alpha_j^*)))},$$

with probability at least $p_t := 1 - \sum_{j \in \mathcal{N}_i} (e^{-\gamma_j \sqrt{t}} + e^{-b_j \delta^2 (t+1)^{1-\alpha_j} / 2})$. So by law of conditional expectation we will have:

$$\begin{aligned} & \mathbb{E}_{\theta^*} [n_{it}(\theta)] \leq c \\ & + \tilde{c}_i \times \sum_{\tau=c+1}^t (\tau+1)^{(\sum_{j \in \mathcal{N}_i} a_{ij}(\beta_j - (1-\alpha_j^*)))} \times p_\tau + (1-p_\tau) \\ & \leq \tilde{c} + (t+1)^{(\sum_{j \in \mathcal{N}_i} a_{ij}(\beta_j - (1-\alpha_j^*))) + 1} \lesssim (t+1)^{\beta_i}, \end{aligned}$$

where β_i is chosen to satisfy $\beta_i > a_{ii} + \sum_{j \in \mathcal{N}_i} a_{ij} \beta_j$. Using Hoeffding inequality we get:

$$\begin{aligned} \mathbb{P}_{\theta^*} (n_{it}(\theta) \gtrsim (t+1)^{\beta_i}) &\leq \mathbb{P}_{\theta^*} \left(n_{it}(\theta) > c \right. \\ &\quad \left. + \tilde{c}_i \sum_{\tau=c+1}^t (t+1)^{(\sum_{j \in \mathcal{N}_i} a_{ij}(\beta_j - (1-\alpha_i^*)))} + t^{\beta_i} \right) \\ &\leq \mathbb{P}_{\theta^*} \left(n_{it}(\theta) > \mathbb{E}_{\theta^*} [n_{jt}(\theta)] + t^{\beta_i} \right) \\ &\leq \exp(-2t^{2\beta_i-1}) \lesssim \exp(-\gamma_i \sqrt{t}) \end{aligned}$$

which proves the claim. \square

Let us illustrate how one can use the above lemma to come up with non-trivial bounds of the form (12) for an agent i with $\text{dist}(\mathcal{J}_\theta, i) = 1$. From definition, we have $\text{dist}(\mathcal{J}_\theta, \sigma_i) = 0$ (σ_i is the neighbor of i that is a θ -expert). It thus follows from Lemma 5 that for any choice of $\beta_{\sigma_i} > \frac{3}{4}$ there exists $\gamma_{\sigma_i} \geq 0$ and $T_{\sigma_i} \in \mathbb{N}$ satisfying the bound of the form (12) for agent σ_i . Let us use the trivial triplet $(\beta, \gamma, T) = (1, 0, 1)$ for the rest of the neighbors of agent i . The condition on β_i from the above lemma then becomes:

$$\beta_i > a_{ii} + \sum_{j \neq i, \sigma_i} a_{ij} + a_{i\sigma_i} \beta_{\sigma_i} = 1 - a_{i\sigma_i} (1 - \beta_{\sigma_i})$$

Recalling that any number greater than $\bar{\beta}^0 := \frac{3}{4}$ is a feasible choice for β_{σ_i} , the possible choices for β_i becomes any $\beta_i > 1 - \frac{a_{i\sigma_i}}{4}$. Now, let

$$\bar{\beta}^1 := \max_{i: \text{dist}(\mathcal{J}_\theta, i)=1} 1 - \frac{a_{i\sigma_i}}{4}.$$

Then, for any agent i at distance 1 from \mathcal{J}_θ , any $\beta_i > \bar{\beta}^1$ is a feasible choice for a bound of the form (12). Notice that this is a non-trivial bound since $\bar{\beta}^1 < 1$. Recursively applying the above argument, we can construct non-trivial bounds of the form (12) for agents at any distance from \mathcal{J}_θ , as established in the next lemma.

Lemma 7. *Let $h := \max_{i \in [n]} \text{dist}(\mathcal{J}_\theta, i)$. Consider the sequence $\{\bar{\beta}^l\}_{l=0}^h$ defined by the recursion*

$$\bar{\beta}^{\ell+1} = \max_{i: \text{dist}(\mathcal{J}_\theta, i)=\ell+1} 1 - a_{i\sigma_i} (1 - \bar{\beta}^\ell), \quad (15)$$

with $\bar{\beta}^0 = \frac{3}{4}$. Then,

i) $\bar{\beta}^\ell < 1$ for $l = 0, 1, \dots, h$.

ii) For any $i \in [n]$ and $\beta_i > \bar{\beta}^{\text{dist}(\mathcal{J}_\theta, i)}$, there exists $\gamma_i \geq 0$ and $T_i \in \mathbb{N}$ such that

$$n_{it}(\theta) \leq (1+t)^{\beta_i}, \quad (16)$$

with probability at least $1 - e^{-\gamma_i \sqrt{t}}$ for all $t \geq T_i$.

Proof. Consider $i \in [n]$ with $\text{dist}(\mathcal{J}_\theta, i) = \ell + 1$. The proof is then the same as the argument made above the lemma, using the bound of the form in (12) with $\beta_{\sigma_i} = \bar{\beta}^\ell$, and the trivial bounds (1,0,1) for the rest of the nodes. \square

Theorem 1. *Within a strongly connected network of agents obeying the belief update rules governed by (1)-(3), and*

assuming that for any $\theta \neq \theta^$ there exists at least one θ -expert agent, all the agents learn the true state θ^* with probability one in the sense that for all $i \in [n]$*

$$\mathbb{P}_{\theta^*} \left(\lim_{t \rightarrow \infty} \mu_{it}(\theta^*) = 1 \right) = 1$$

Proof. Equivalently, we may show that for any $\theta \neq \theta^*$:

$$\mathbb{P}_{\theta^*} \left(\lim_{t \rightarrow \infty} \mu_{it}(\theta) = 0 \right) = 1$$

We consider two cases:

- 1) if agent i can distinguish θ from θ^* the proof follows from Lemma 2.
- 2) if $i \notin \mathcal{J}_\theta$, then the proof follows from the observation that for any $\tilde{\beta}_i > a_{ii} + \sum_{j \neq i} a_{ij} \bar{\beta}^{\text{dist}(\mathcal{J}_\theta, j)}$, there exists $\gamma_i > 0$ and $T_i \in \mathbb{N}$ such that with probability at least $1 - e^{-\gamma_i \sqrt{t}}$ we have

$$\mu_{it}(\theta) \leq (t+1)^{\tilde{\beta}_i-1}, \quad (17)$$

for all $t \geq T_i$. The proof immediately follows noticing that $a_{ii} + \sum_{j \neq i} a_{ij} \bar{\beta}^{\text{dist}(\mathcal{J}_\theta, j)} < \sum_{j=1}^n a_{ij} = 1$. \square

REFERENCES

- [1] Rabih Salhab, Amir Ajorlou, and Ali Jadbabaie, "Social learning with sparse belief samples," in *2020 59th IEEE Conference on Decision and Control (CDC)*, 2020, pp. 1792–1797.
- [2] Morris H. DeGroot, "Reaching a consensus," *Journal of the American Statistical Association*, vol. 69, no. 345, pp. 118–121, 1974.
- [3] Benjamin Golub and Matthew O. Jackson, "Naïve learning in social networks and the wisdom of crowds," *American Economic Journal: Microeconomics*, vol. 2, no. 1, pp. 112–49, February 2010.
- [4] Daron Acemoglu, Munther A. Dahleh, Ilan Lobel, and Asuman Ozdaglar, "Bayesian Learning in Social Networks," *The Review of Economic Studies*, vol. 78, no. 4, pp. 1201–1236, 03 2011.
- [5] Arun G. Chandrasekhar, Cynthia Kinnan, and Horacio Larreguy, "Social networks as contract enforcement: Evidence from a lab experiment in the field," *American Economic Journal: Applied Economics*, vol. 10, no. 4, pp. 43–78, 2018.
- [6] Samuel J. Gershman, Eric J. Horvitz, and Joshua B. Tenenbaum, "Computational rationality: A converging paradigm for intelligence in brains, minds, and machines," *Science*, vol. 349, no. 6245, pp. 273–278, 2015.
- [7] Pooya Molavi, Alireza Tahbaz-Salehi, and Ali Jadbabaie, "A theory of non-bayesian social learning," *Econometrica*, vol. 86, no. 2, pp. 445–490, 2018.
- [8] Ali Jadbabaie, Pooya Molavi, Alvaro Sandroni, and Alireza Tahbaz-Salehi, "Non-Bayesian social learning," *Games and Economic Behavior*, vol. 76, no. 1, pp. 210–225, 2012.
- [9] Sushil Bikhchandani, David Hirshleifer, and Ivo Welch, "Learning from the behavior of others: Conformity, fads, and informational cascades," *Journal of Economic Perspectives*, vol. 12, no. 3, pp. 151–170, September 1998.
- [10] Abhijit V. Banerjee, "A simple model of herd behavior," *The Quarterly Journal of Economics*, vol. 107, no. 3, pp. 797–817, 1992.
- [11] Wassily Hoeffding, "Probability inequalities for sums of bounded random variables," *Journal of the American Statistical Association*, vol. 58, no. 301, pp. 13–30, 1963.