The $c - d$ conjecture

José I. Latorre[1,2] and Germán Sierra[3]

[1] Centre for Quantum Technologies, National University of Singapore, Singapore.
[2] Quantum Research Centre, Technology Innovation Institute, Abu Dhabi, United Arab Emirates.
[3] Instituto de Física Teórica UAM-CSIC, Universidad Autónoma de Madrid, Spain.

We conjecture a relation between the local dimension d of a local nearest-neighbor critical Hamiltonian in one spatial dimension and the maximum central charge, $c_{\max}$, that it can yield. Specifically, we propose that $c_{\max} \leq d - 1$, establishing a link between the short-distance lattice realization of a model and its emerging long-distance entanglement properties. This inequality can be viewed as a general form of a c-theorem establishing the reduction of effective degrees of freedom between the UV lattice and the IR conformal field theory. We support this conjecture with numerous examples.

Introduction

Quantum systems defined on a lattice can develop long-distance correlations that decay as a power law at zero temperature, depending on the parameters of the Hamiltonian of the system. This phenomenon corresponds to a quantum phase transition which is controlled by the properties of the conformal group.

The emergence of quantum correlations that decay algebraically from the very detailed structure of local interactions remains a subtle issue, where few solid theorems are available. Indeed, it is not fully understood how local interactions keep extending entanglement in a structured way so that correlators decay smoothly and the entanglement entropy obeys scaling laws.

Here, we shall present a conjecture that relates the local dimension of a one-dimensional lattice Hamiltonian to the central charge of the conformal theory that emerges at long distances. It is, thus, a relation of ultraviolet (UV) physics with its infrared (IR) realization. To be precise, we conjecture that a nearest-neighbor critical Hamiltonian, that is translational invariant, where the local degrees of freedom span d dimensions, can only give rise to a conformal theory central charge limited by

$$c_{\max} \leq d - 1. \tag{1}$$

That is, no arbitrarily large central charge can be obtained from a nearest-neighbor Hamiltonian of finite local dimension. We shall named this proposal as the $c - d$ conjecture.

Although the $c - d$ conjecture seems physically plausible, its validity isn't immediately evident. The dimension d is related to the dimensionality of the local Hilbert space and shows up in the original lattice Hamiltonian, which is a dimensionful operator. On the other side, the central charge c can be determined from various sources. These include extracting it from the ground state via the von Neumann entropy of the reduced density matrix [1–5], identifying it through finite size corrections of the ground state energy or free energy [6, 7], or through correlators associated with the energy momentum tensor [8].

These options require having solved the system. They appear as an emergent property of the theory. Alternatively the central charge is a key element in the algebra of operators dictated by the conformal group. The relation between operators in the lattice and conformal operators in the long-distance regime is non-trivial [9, 10]. This avenue is definitely relevant to have a complete proof of the $c - d$ conjecture, if true. Appendix B outlines some components of this effort.

A different way to see the non-trivial nature of the $c - d$ conjecture arises from the existence of local, translational-invariant Hamiltonians that deliver ground states that are maximally entangled, representing superpositions of Motzkin paths [11, 12]. The ground state of the model displays volume law entanglement, showing that this property can be achieved through local Hamiltonians of dimension $d \geq 3$. Yet, the model is no longer conformal and escapes the conjecture. The lesson to be taken from this example is that the dimension d has the potential to distribute significantly larger entanglement than that found in a conformal field theory. Any proof of the $c - d$ conjecture should then make explicit use of the properties of conformal field theories.

The $c - d$ conjecture

There is a simple heuristic argument to understand that no arbitrary large central charge can emerge from finite local dimensions. We may argue this fact using the scaling of entanglement entropy. It is known that the ground state of a critical Hamiltonian carries a von Neumann entropy for a reduced density matrix to L sites that scales as $S \sim \frac{c}{3} \log L$ [3, 4]. On the other hand, if the system is made of a chain of degrees of freedom of dimension d, the maximum entropy of a set of L sites is bounded by $S \leq L \log d$. Necessarily $S \sim \frac{c}{3} \log L < L \log d$. It is then clear that for any large size, it is impossible to accommodate an arbitrary large central charge with a finite d. This argument is yet too coarse to bring further insight into the way that a finite value of d limits the

amount of long distance conformal correlations that can be achieved.

The $c - d$ conjecture is, in its present form, limited to nearest-neighbor interactions. It is easy to see that non nearest-neighbor interactions can be handled as nearest-neighbor ones on a larger local Hilbert space.

It is important to again emphasize the requirement of conformal symmetry. As mentioned before, local interactions can deliver volume law entanglement, which is far stronger than the area law observed in conformal field theory. In particular, it is known that for spin $j \geq 3/2$ there are local interactions whose ground state is described by large superpositions of Motskin paths [11, 12]. These hamiltonians are not conformal theories and have ground states that display volume law entanglement. The role of conformal invariance is then seen as a complex combination of constraints that have to be provided by d locals degrees of freedom, hence the limitation that $c_{\max} \leq d - 1$.

Let us now establish the $c - d$ conjecture with some more detail. Let us consider a one-dimensional quantum lattice system, with a local Hilbert space of dimension d, that is $\mathcal{H}_1 = \mathbb{C}^{\otimes d}$, and $L > 2$ sites. The total Hilbert space is $\mathcal{H}_L = \mathcal{H}_1^{\otimes L}$. We define a Hamiltonian acting on $\mathcal{H}_L$

$$H_L = \sum_{i=1}^{L} h_{i,i+1}, \qquad h_{L,L+1} = h_{L,1}, \qquad (2)$$

$$h_{i,i+1} = \mathbf{1} \otimes \overset{i-1}{\cdots} \otimes h \otimes \overset{L-i-1}{\cdots} \otimes \mathbf{1}, \quad i = 1, \dots, L-1,$$

where h is an hermitean $d^2 \times d^2$ matrix. We have assumed periodic boundary conditions, expressed as $h_{L,L+1} = h_{L,1}$, ensuring that H_L remains unchanged under the translation of the lattice by one site.

Conjecture: If the low energy spectrum of the Hamiltonian (2) is described by a collection of unitary CFTs with central charges c_a, then

$$c_T \leq d - 1, \qquad \text{where} \quad c_T = \sum_a c_a . \qquad (3)$$

The case $d = 1$ is evidently fulfilled since there is only a single state in $\mathcal{H}_L$ in a unitary CFT with $c = 0$ that corresponds to the vacuum. Another consistency check of Eq.(3) is to consider the tensor product of two quantum models with local dimensions d_1 and d_2, each of them satisfying Eq.(3), that is $\sum_a c_{1,a} \leq d_1 - 1$ and $\sum_b c_{2,b} \leq d_2 - 1$. Compounding the two theories as a direct product gives

$$\sum_a c_{1,a} + \sum_b c_{2,b} \leq d_1 + d_2 - 2 < d_1 d_2 - 1, \qquad (4)$$

since the direct product Hilbert space has dimension $d_1 d_2$.

Let us note here the apparent similarity of the $c - d$ conjecture in Eq.(3) with the c-theorem [13]. The statement of irreversibility of renormalization group flows [13] states that in two space-time dimensions the UV fixed point carries a central charge c_{UV} which is an upper bound for the one in the IR c_{IR}, that is $c_{IR} \leq c_{UV}$. The c-theorem is formulated for relativistic QFTs, so it cannot be straightforwardly applied to our problem. We may however interpret Eq.(3) as an expanded version of the c-theorem where c_{UV} is replaced by $d - 1$. An additional difficulty to use QFT techniques in trying to prove Eq.(3) is that the low energy physics of the Hamiltonian (2) may correspond to CFTs with different speeds of propagation, say v_a, as in the case of the Hubbard model, away from half filling, where the spin wave velocity v_s and charge velocity v_c can be different [14].

Let us make the observation that the $c - d$ conjecture is limited by some bounds. This can be argued by analyzing how the combination of two theories works. Let us accept that $c_{max} = f(d)$. Two independent theories can be compounded onto a single theory of larger dimension. That is, if $c_{max,1} = f(d_1)$ and $c_{max,2} = f(d_2)$, it follows that

$$f(d_1) + f(d_2) \leq f(d_1 \times d_2) , \qquad (5)$$

as the dimension of the total Hilbert space is the product of the dimensions of the two separate ones. Let us further assume that the asymptotic scaling of $f(d)$ is a power law with exponent α. Form Eq.(5), it follows that

$$f(d) \sim d^\alpha \longrightarrow \alpha \geq \log 2 / \log d. \qquad (6)$$

For $d = 2$, the case of spin $1/2$ chains, the bound becomes $\alpha \geq 1$. This bound becomes weaker as the dimension d grows. As a matter of fact, in the limit of very large d, the functionality $f(d) = \log d$ saturates Eq.(6). We shall show later on that some families of models do follow this log relation.

Evidence supporting the $c - d$ conjecture

The $c - d$ conjecture can be tested on a large series of known models. Some non-trivial examples of Eq.(3) are shown in the following table, which strongly suggests that it must hold in general.

d	c	Model	CFT	Sym
2	$\frac{1}{2}$	Ising	$\mathcal{M}_{3,4}$	$\mathbb{Z}_2$
3	$\frac{4}{5}$	3 state Potts	$\mathcal{M}_{5,6}$	Z_3
4	1	4 state Potts	$\mathcal{M}_{\infty,\infty}$	Z_4
2	1	$XXZ_{1/2}$	$U(1)$	$U(1)$
4	1	$Ashkin-Teller$	$U(1)$ orbifold	$\mathbb{Z}_2 \otimes \mathbb{Z}_2$
Q	$\frac{2(Q-1)}{Q+2}$	Parafermion	Parafermion	$\mathbb{Z}_Q$
3	$\frac{3}{2}, 1$	FZ	$U(1)$	$U(1)$
$2s+1$	$\frac{3s}{s+1}$	XXX_s	$SU(2)_{2s}$	$SU(2)$
3	$\frac{1}{2}, \frac{7}{10}$	Blume$-$Capel	$\mathcal{M}_{3,4}, \mathcal{M}_{4,5}$	$\mathbb{Z}_2$
3	$3/2$ 2	spin 1 bilinear $-$biquadratic	$SU(2)_2$ $SU(3)_1$	$SO(3)$
N	$N-1$	$SU(N)$	$SU(N)_1$	$SU(N)$

Table: Quantum lattice models with nearest-neighbour interations with local dimension d, central charge c, associated CFT and symmetries of the Hamiltonian. $\mathcal{M}_{m,m+1}$ denotes the minimal model with central charge $c = 1 - \frac{6}{m(m+1)}$, and G_k a Wess-Zumino-Witten model associated to a Lie algebra G and level k.

The details of each model, including Hamiltonian, central charges and related symmetries are presented in the Supplementary Material section.

It is interesting to notice that the $c-d$ conjecture is saturated in some cases. For instance, for $d = 2$, the spin 1/2 antiferromagnetic Heisenberg model has central charge $c = 1$, thus obeying $c = d - 1$. According to the conjecture there is no other model made with spins that can provide a larger central charge.

The saturation of the $c - d$ conjecture bound is also reached by the critical $SU(N)$ spin chain where $d = N$ and $c = N - 1$. When $N = 2$ it coincides with the result previously discussed.

Discussion

The $c-d$ conjecture provides a relation between the UV and IR realizations of a theory. The building blocks of a model may display some symmetry among UV degrees of freedom. At long distances, the effective IR realization of the theory may describe some extra symmetry such as conformal invariance. The $c - d$ conjecture proposes a strict bound on the amount of entanglement which is present in the IR conformal realization of a theory.

The $c-d$ conjecture bares some relation to other efforts to relate UV with IR realizations of a theory. Let us mention here the ideas around anomaly matching, used for exploration of model building, as well as the c-theorem.

Some theories display symmetries that may be broken by an anomaly in the algebra of its operators, yet these anomalous terms are protected against renormalization. In this situation, the UV and IR realizations of the anomaly are strictly related, the anomaly central extension is identical. This feature serves the purpose of exploring possible UV completions of theories from their known IR realization. This scenario is different in spirit to the $c - d$ conjecture, as the latter is not related to the possible presence of some symmetry algebra.

The c-theorem states the irreversibility of renormalization group flows. In one dimension, it was originally proven that the central charge of an ultraviolet fixed point c_{UV} is larger than the corresponding infrared one c_{IR}, that is $c_{IR} \leq c_{UV}$. This theorem is rooted in unitarity [13, 15]. The extension of this theorem to higher dimensions is based on the conformal anomaly terms related to the stress tensor [16–18]. As mentioned before, the $c - d$ conjecture acts as a sort of c-theorem acting from a lattice representation to a conformal field theory one.

Let us elaborate on a particular case of a renormalization group flow. Consider a local interaction such that some local high energy levels are not populated, that is, some local degrees of freedom decouple and play no role in the long-distance description of correlations. Then, it is possible to integrate these modes out, by just dropping them, and produce an effective description of the model in terms of the rest of modes. This description remains faithful at long distances. As a matter of fact the initial local dimension d_{UV} is larger than the final one d_{IR}. Then along this RG flow

$$d_{UV} \geq d_{IR} \implies c_{\max,UV} \geq c_{\max,IR} . \qquad (7)$$

This idea is aligned with the $c - d$ conjecture.

The conjecture we have formulated has been applied to Hamiltonians with periodic boundary conditions. We anticipate that a comparable assertion remains valid for open boundary conditions. An intriguing question arises regarding the behavior under non-unitary Hamiltonians, where the central charge is substituted with the effective central charge. These topics will be addressed in other work.

[1] C. Callan and F. Wilczek, *Phys. Lett.* B 333, 55 (1994).
[2] C. Holzhey, F. Larsen and F. Wilczek, *Nucl. Phys.* B 424, 443 (1994).
[3] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, *Phys. Rev. Lett.* 90, 227902 (2003).
[4] P. Calabrese and J. Cardy, *J. Stat. Mech.* P06002 (2004).
[5] V. E. Korepin, *Phys. Rev. Lett.* 92, 096402 (2004).
[6] H. W. J. Blöte, J. L. Cardy, and M. P. Nightingale, *Phys. Rev. Lett.* 56, 742 (1986).
[7] I. Affleck, *Phys. Rev. Lett.* 56, 746 (1986).
[8] A. A. Belavin, A. M. Polyakov, A. B. Zamolodchikov, *Nucl. Phys.* B241, 333 (1984).
[9] W. M. Koo and H. Saleur, *Nucl. Phys.* B 426, 459 (1994).
[10] A. Milsted, G. Vidal, *Phys. Rev.* B 96, 245105 (2017).
[11] R. Movassagh and P. W. Shor, *Proc. Natl. Acad. Sci.* 113, 13278 (2016); arXiv:1408.1657.
[12] S. Bravyi, L. Caha, R. Movassagh, D. Nagaj and P. W. Shor, *Phys. Rev. Lett.* 109, 207202 (2012).
[13] A. B.Zamolodchikov, *JETP Lett* 43, 730 (1986).
[14] F. H. L. Essler, H. Frahm, F. Göhmann, A. Klümper, V. E. Korepin, *The one-dimensional Hubbard model*, Cambridge University Press (2010).
[15] A. Cappelli, D. Friedan and J. I. Latorre, *Nucl. Phys.* B352, 616 (1991).
[16] J. L. Cardy, *Phys. Lett.* B215, 749 (1988).
[17] S. Forte and J. I. Latorre, *Nucl. Phys.* B535, 6709 (1998).
[18] Z. Komargodski and A. Schwimmer, *JHEP* 12,1 (2011).
[19] E. Barouch, B. M. McCoy, and M. Dresden, *Phys. Rev.* A 2, 1075 (1970); ibid. 3, 786 (1971); ibid. 3, 2137 (1971).
[20] P. Pfeuty, *Ann. Phys.* 57, 79 (1970).
[21] F. Y. Wu, *Rev. Mod. Phys*, 54, 235 (1982).
[22] R.J. Baxter, *Exactly solved models in statistical mechanics*, Academic Press, New York (1982).
[23] V.S. Dotsenko, *Nucl. Phys.* B 235, 54 (1984).
[24] G. von Gehlen,V. Rittenberg and H. Ruegg, *J. Phys. A, Math Gen* 19, 107 (1986).
[25] L. P. Kadanoff and A. Brown, *Ann. Phys.* 121, 319 (1979).
[26] A. Luther and I. Peschel, *Phys. Rev.* B 12, 3908 (1975).
[27] I. Affleck, *Phys. Rev. Lett.* 56, 2763 (1986).
[28] J. Ashkin and E. Teller, *Phys. Rev.* 64, 178 (1943).
[29] M. Kohmoto, M. den Nijs, and L. P. Kadanoff, *Phys. Rev.* B 24, 5229 (1981).
[30] S. Yang, *Nucl. Phys.* B285, 183(1987).
[31] H. Saleur, *J. Phys. A* 20 L1127 (1987).
[32] M. Baake, G. von Gehlen and V. Rittenberg, *J. Phys. A* 20, L479 (1987).
[33] V. A. Fateev and A. B. Zamolodchikov, *Phys. Lett. A* 92, 37 (1982).
[34] F.C. Alcaraz and A. Lima-Santos, *Nucl. Phys.* B275, 436 (1986).
[35] F. C. Alcaraz, *J. Phys. A* 20, 2511 (1987).
[36] A. B. Zamolodchikov and V. A. Fateev, *Sov. J. Nucl. Phys.* 32, 298 (1980).
[37] F. C. Alcaraz and M. J. Martins, *Phys. Rev. Lett.* 61, 1529 (1988).
[38] P. Di Francesco, H. Saleur, and J. B. Zuber, *Nucl. Phys.* B300, 393 (1988).
[39] F. C. Alcaraz and M. J. Martins, *Phys. Rev. Lett.* 63, 708 (1989).
[40] P. Kulish, N. Yu. Reshetikhin and E. Sklyanin, *Lett. Math. Phys.* 5, 393 (1981); L. Takhtajan, *Phys. Lett.* 87A, 479 (1982); H. Babujan, *Nucl. Phys.* B215 317 (1983).
[41] I. Affleck, D. Gepner, H. Shulz and T. Ziman, *J. Phys. A* 22, 511 (1999).
[42] I. Affleck and F. D. M. Haldane, *Phys. Rev.* B 36 5291 (1987).
[43] C. Itoi and M. H. Kato, *Phys. Rev.* B 55, 8295 (1997).
[44] N. Y. Reshetikhin, *Theor. Math. Phys.* 63, 555 (1985).
[45] H. H. Tu, R. Orus, *Phys. Rev. Lett.* 107, 077202 (2011).
[46] B. Sutherland, *Phys. Rev.* B12, 3795(1975).
[47] I. Affleck, *Nucl. Phys.* B265,409 (1986); *Nucl. Phys.* B305,582 (1988).
[48] P. Bouwknegt and K. Schoutens, *Nucl. Phys.* B547, 501(1999).
[49] R. Assaraf, P. Azaria, M. Caffarel, and P. Lecheminant, *Phys. Rev.* B 60, 2299 (1999).
[50] F. C. Alcaraz, J. R. Drugowich de Felicio, R. Köberle, and J. F. Stilck, *Phys. Rev.* B 32, 7469 (1985).
[51] J. C. Xavier and F. C. Alcaraz, *Phys. Rev.* B 84, 094410 (2011).

SUPPLEMENTARY MATERIAL

Evidence for the $c - d$ conjecture

We present below a collection of Hamiltonians of the form (2) that are critical and described by a unitary CFT. The values of the local dimension of the Hilbert space, d, the central charge c, the corresponding CFT and symmetry are collected in Table 1 in the main text.

- Ising Model [19, 20]

$$H_{\text{Ising}} = - \sum_{i=1}^{L} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^z) \,, \qquad (8)$$

where $\sigma_i^{x,z}$ are Pauli matrices. The associated CFT is the minimal model $\mathcal{M}_{3,4}$ [8] with central charge $c = \frac{1}{2}$.

- Q-state Potts model [21, 22]

$$H_Q = - \sum_{i=1}^{L} \sum_{n=1}^{Q-1} \left((X_i X_{i+1}^\dagger)^n + Z_i^n \right) \,, \qquad (9)$$

where X and Z are $Q \times Q$ matrices satisfying the relations $X^Q = Z^Q = \mathbf{1}$, $XZ = e^{2\pi i/Q} ZX$. The case $Q = 2$ coincides with the Ising model. This model is critical for $Q = 2, 3, 4$ with central charge $c_m = 1 - \frac{6}{m(m+1)}$, where $\sqrt{Q} = 2 \cos(\frac{\pi}{m+1})$. The cases $Q = 3$ and $Q = 4$ correspond to $c_5 = \frac{4}{5}$ [23] and $c_\infty = 1$ [24].

- $XXZ_{1/2}$ model [22]

$$H_{XXZ} = \sum_{i=1}^{L} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z \right) \,, \qquad (10)$$

where σ_i^a are the Pauli matrices. For any Δ there is a $U(1)$ symmetry $\sigma^z \to \sigma^z$, $\sigma^\pm \to e^{\pm i\theta}\sigma^\pm$ that is enlarged to $SU(2)$ at $\Delta = 1$. This model is conformal invariant for $-1 < \Delta \leq 1$ with $c = 1$. It is described by a compact boson with radius $R = \sqrt{\frac{2}{\pi}\arccos(-\Delta)}$ [25, 26]. For $\Delta = 0$ the Hamiltonian (10) can be mapped, via a Jordan-Wigner transformation, into a hopping Hamiltonian of a spinless fermion. At $\Delta = 1$, the CFT is described by the WZW model $SU(2)_k$ at level $k = 1$ [27].

- Ashkin-Teller model [28]

$$
\begin{aligned}
H_{AT} = & -\sum_{i=1}^{L} (\sigma_i^z + \tau_i^z + \Delta\sigma_i^z\tau_i^z \\
& + \sigma_i^x\sigma_{i+1}^x + \tau_i^x\tau_{i+1}^x + \Delta\sigma_i^x\sigma_{i+1}^x\tau_i^x\tau_{i+1}^x) \,,
\end{aligned} \tag{11}
$$

where σ_i^a and τ_i^a are two sets of commuting Pauli matrices. This model describes the coupling of two critical Ising models via the four spin term. It has $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. The model is critical in the line $-1/\sqrt{2} < \Delta \leq 1$ corresponding to a Z_2-orbifold boson with radius $R_0 = \sqrt{\pi/(2\arccos(-\Delta))}$ [29–32]

- Parafermionic Z_Q [33]

$$
H_Q = -\sum_{i=1}^{L}\sum_{n=1}^{Q-1} \frac{1}{\sin(\pi n/Q)} \left((X_i X_{i+1}^\dagger)^n + Z_i^n \right) \tag{12}
$$

where X and Z denote identical matrices as those used in the Q-state Potts model. This model is conformal invariant for any $Q \geq 2$ with $c_Q = 2(Q-1)/(Q+2)$. For $Q = 2$ and 3 one recovers the Ising and the 3-state Potts models, while for $Q = 4$ one finds the Ashkin-Teller model with $\Delta = 1/\sqrt{2}$ [34, 35].

- Fateev-Zamolodchikov model [36]

$$
\begin{aligned}
H_{FZ} = & \ \varepsilon \sum_{i=1}^{L} \left[\sigma_i - (\sigma_i^z)^2 - 2(\cos\gamma - 1)(\sigma_i^\perp \sigma_i^z + \sigma_i^z \sigma_i^\perp) \right. \\
& - 2\sin^\gamma(\sigma_i^z - (\sigma_i^z)^2 + 2(S_i^z)^2) \Big] \,,
\end{aligned} \tag{13}
$$

where $\vec{S} = (S^x, S^y, S^z)$ are the spin 1 $SU(2)$ matrices, $\sigma_i^z = S_i^z S_{i+1}^z$ and $\sigma_i = \vec{S}_i \cdot \vec{S}_{i+1} = \sigma_i^z + \sigma_i^\perp$. The model is critical for $0 \leq \gamma \leq \frac{\pi}{2}$, with $c = \frac{3}{2}$ for $\varepsilon = +1$ [37, 38] (antiferromagnetic model) and $c = 1$ for $\varepsilon = -1$ [39](ferromagnetic model).

- Higher spin XXX$_s$ model [40]

$$
H_{XXX_s} = \sum_{i=1}^{L}\sum_{k=1}^{2s} a_k P_k(\vec{S}_i + \vec{S}_{i+1}) \,, \tag{14}
$$

where P_k are projectors out spin k and the constants are given by

$$
a_k = \sum_{\ell=1}^{k} \frac{1}{\ell} \,.
$$

The case $s = 1/2$ coincides with the Heisenberg Hamiltonian $\Delta = 1$ (13). The spin 1 and 3/2 Hamiltonians are given, up to an additive constant, by

$$
H_{XXX_1} = \frac{1}{4}\sum_{i=1}^{L}\left[\vec{S}_i \cdot \vec{S}_{i+1} - (\vec{S}_i \cdot \vec{S}_{i+1})^2\right],
$$

$$
\begin{aligned}
H_{XXX_{3/2}} = \sum_{i=1}^{L} & \left[-\frac{1}{16}\vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{54}(\vec{S}_i \cdot \vec{S}_{i+1})^2 \right. \\
& \left. + \frac{1}{27}(\vec{S}_i \cdot \vec{S}_{i+1})^3 \right].
\end{aligned}
$$

The underlying CFT is given by the WZW model $SU(2)_k$ with $k = 2s$ [27, 41, 42]

- Spin 1 bilinear-biquadratic Heisenberg model

$$
H = \sum_{i=1}^{N} \left[\cos\theta\,(\mathbf{S}_i \cdot \mathbf{S}_{i+1}) + \sin\theta\,(\mathbf{S}_i \cdot \mathbf{S}_{i+1})^2 \right] \,, \tag{15}
$$

where $\mathbf{S}_i$ are spin 1 matrices. As a function of the angle θ, this model exhibits several critical phases. At $\theta = -\frac{\pi}{4}$ it coincides with the spin 1 XXX model considered above that has $c = \frac{3}{2}$, while at $\theta = \frac{\pi}{4}$ it is described by the $SU(3)_1$ model with $c = 2$ [43]. The entire region $\lambda \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$ is critical with $c = 2$. Further extensions to $SO(N)$ can be found in [44, 45].

- $SU(N)$ model [46]

$$
H = \sum_{i=1}^{L} P_{i,i+1} \,, \tag{16}
$$

where $P_{i,i+1}$ is the permutation operator acting on N values of spins (colors) that corresponds to the fundamental representation of $SU(N)$. For $N = 2$ this Hamiltonian reduces to Heisenberg spin $1/2$. The low energy physics of this model is described by the WZW model $SU(N)_1$ which has central charge $c = N - 1$ [47–49].

- Blume-Capel model [50, 51]

$$
H_{BC} = -\sum_{i=1}^{L} \left(S_i^z S_{i+1}^z - \delta(S_i^z)^2 - \gamma S_i^x \right) \tag{17}
$$

where S^x, S^z are spin-1 $SU(2)$ matrices. For $\gamma > \gamma_{\mathrm{tr}} \simeq 0.41653$ the model has a critical line $\delta_c(\gamma)$ in the Ising universality class, $c = 1/2$. At $\delta_{\mathrm{tr}} \simeq 0.91024$ there is a tricritical point described by the tricritical Ising model with $c = 7/10$.

A variation of the Koo and Saleur formula

In this appendix we present a potential approach to prove the $c - d$ conjecture. Our method draws upon the Koo and Saleur lattice representation of the Virasoro operators [9]. This representation was previously employed in Ref. [10] to extract the central charge and additional conformal data of critical Hamiltonians through numerical techniques. Koo and Saleur introduced the Fourier modes H_n of the lattice Hamiltonian density h_j,

$$H_n = \frac{N}{2\pi} \sum_{j=1}^{N} e^{ijn\frac{2\pi}{N}} h_j, \qquad H_0 = \frac{N}{2\pi} H. \qquad (18)$$

In the case of a critical Hamiltonian, these modes will have the continuum counterparts

$$\begin{aligned} H_n^{CFT} &= \frac{L}{2\pi} \int_0^L dx e^{inx\frac{2\pi}{L}} h^{CFT}(x) \qquad (19) \\ &= L_n + \bar{L}_n - \frac{c}{12}\delta_{n,0}, \quad n \in \mathbb{Z}, \end{aligned}$$

where L_n and $\bar{L}_n$ are the chiral and antichiral Virasoro operators that satisfy the algebra

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \\ [\bar{L}_n, \bar{L}_m] &= (n-m)\bar{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \\ [L_n, \bar{L}_m] &= 0, \qquad (20) \end{aligned}$$

where c is the central charge.

In general, the Hamiltonian H may not satisfy the normalization standards of Conformal Field Theory. To address this, let's temporarily label H as $\tilde{H}$ and similarly $\tilde{h}_j$ as h_j, then introduce the normalized operators as defined in [10].

$$H = a\tilde{H} + bN, \quad h_j = a\tilde{h}_j + b. \qquad (21)$$

We now introduce the states $|I\rangle$ and $|T\rangle$, defined as follows

$$\tilde{H}|I\rangle = \tilde{E}_I |I\rangle, \quad \tilde{H}|T\rangle = \tilde{E}_T |T\rangle, \qquad (22)$$

where $|I\rangle$ represents the unique ground state of $\tilde{H}$, and $|T\rangle$ denotes the eigenstate with momentum $P_T = 2 \times \frac{2\pi}{N}$, having the maximum overlap with $\tilde{H}_{-2}|I\rangle$. The normalized energies of these states must fulfill $E_T^{CFT} - E_I^{CFT} = \Delta_T \frac{2\pi}{N} = 2 \times \frac{2\pi}{N}$, given that $\Delta_T = 2$. This condition determines the value of the parameter a, where $a = \frac{4\pi}{N}/(\tilde{E}_T - \tilde{E}_I)$, ensuring that the lattice energies satisfy $E_T - E_I = 2 \times \frac{2\pi}{N}$. Finally, by recognizing the relationship in CFT given by

$$\sqrt{\frac{c}{2}}|T\rangle = H_{-2}^{CFT}|I\rangle, \qquad \sqrt{\frac{c}{2}}|\bar{T}\rangle = H_2^{CFT}|I\rangle, \qquad (23)$$
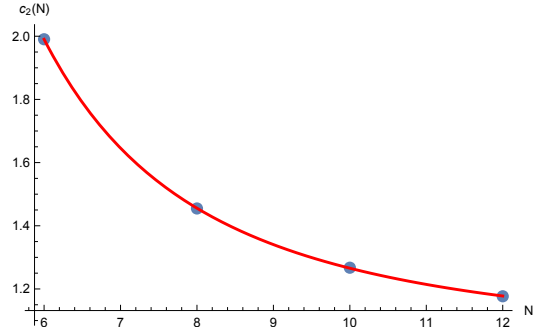

FIG. 1: The bound to the IR central charge of the Heisenberg model (XXX) using Eq. (25) is shown as a function of the number of qubits N. A fit to an expression of the form $c + a/N^2 + b/N^4$ provides an estimate for the central charge $c \sim 1.0175$.

one arrives at the fundamental equation for determining the central charge c [9]:

$$c \sim 2\langle I|H_2 H_{-2}|I\rangle. \qquad (24)$$

This formula, employed in [10] to ascertain the value of c in the regime of large N, relies on solving Eqs. (22) to normalize the Hamiltonian, as discussed earlier. Here, we propose an alternative formula that solely necessitates knowledge of the ground state. It is expressed as:

$$c_2(N) = \frac{(2\langle I|H_2 H_{-2}|I\rangle)^3}{\langle I|H_2[H, H_{-2}]|I\rangle^2}. \qquad (25)$$

To justify this equation, we first note that under an overall scaling, where $H \to \lambda H$ and $H_{\pm 2} \to \lambda H_{\pm 2}$, the quantity $c_2(N)$ remains unchanged. Additionally, if the model approaches a critical theory, we can substitute the state $|I\rangle$ with the CFT vacuum $|0\rangle$, replace the Hamiltonian with its expression in terms of the Virasoro operators $H \to L_0 + \bar{L}_0 - \frac{c}{12}$, and $H_{\pm 2}$ with $L_{\pm 2}$. This leads us to:

$$\lim_{N\to\infty} c_2(N) = \frac{(0|L_2 L_{-2}|0\rangle)^3}{\langle 0|L_2[L_0 + \bar{L}_0, L_{-2}]|0\rangle^2}. \qquad (26)$$

Ultimately, the right-hand side of this equation simplifies to the central charge c of the CFT, utilizing the Virasoro algebra and the conditions $L_n|0\rangle = 0$ ($n = 0, 2$) and $\bar{L}_0|0\rangle = 0$.

The preceding arguments are undoubtedly heuristic, and a more comprehensive investigation is required to substantiate them. An indication supporting the correctness of eq. (25) is provided by Fig. 1, which illustrates how $c_2(N)$ approaches the central charge of the XXX Heisenberg model, $c_{\text{Heisenberg}} = 1$, as N increases. This shows that the techniques related to mapping lattice operators to CFT ones, in the weak sense, are a useful tool to investigate the universality class of a model.