# Tian's stabilization problem for toric Fanos

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25 March 2024

*In memory of Eugenio Calabi*

**Abstract**

In 1988, Tian posed the stabilization problem for equivariant global log canonical thresholds. We solve it in the case of toric Fano manifolds. This is the first general result on Tian's problem. A key new estimate involves expressing complex singularity exponents associated to orbits of a group action in terms of support and gauge functions from convex geometry. These techniques also yield a resolution of another conjecture of Tian from 2012 on more general thresholds associated to Grassmannians of plurianticanonical series.

# Contents

# 1 Introduction

This article is the first in a series in which we study asymptotics of invariants related to existence of canonical metrics on Kähler manifolds.

In the present article we focus on Tian's $\alpha_{k,G}$- and $\alpha_{k,m,G}$-invariants that were defined in 1988 and 1991, and to a large extent are still rather mysterious. The presence of the compact symmetry group $G$ is a major source of difficulty and new ideas are needed here as these invariants have not been previously systematically studied or computed. In particular, we provide a formula for such invariants, valid for all toric Fano manifolds, leading to a resolution of Tian's stabilization conjecture in this setting, which is also the first general result on Tian's conjecture.

A sequel to this article deals with the $\delta_k$-invariants of Fujita–Odaka (also called $k$-th stability thresholds) on toric Fano manifolds. It turns out that for these invariants there is a quantitative dichotomy regarding stabilization, and when stabilization fails we derive their complete asymptotic expansion.

## 1.1 Tian's stabilization problem

Let $(X, L, \omega)$ be a polarized Kähler manifold of dimension $n$ with $L$ a very ample line bundle over $X$, and $\omega$ a Kähler form representing $c_1(L)$. The space

$$\mathcal{H}_L := \{\varphi \ : \ \omega_\varphi := \omega + \sqrt{-1}\partial\overline{\partial}\varphi > 0\} \subset C^\infty(X) \tag{1}$$

of Kähler potentials of metrics cohomologous to $\omega$ was introduced by Calabi in a short visionary talk in the Joint AMS–MAA Annual Meetings held at Johns Hopkins University in December, 1953 [7]. In a groundbreaking article some 35 years later, Tian proved (motivated by a question of Yau [34, p. 139]) that $\mathcal{H}_L$ is approximated (or "quantized") in the $C^2$ sense by the finite-dimensional spaces $\mathcal{H}_k$ consisting of pull-backs of Fubini–Study metrics on $\mathbb{P}(H^0(X, L^k)^*)$ under all possible Kodaira embeddings induced by $H^0(X, L^k)$ [31]. A decade later this was improved to a complete asymptotic expansion [9, 35] and so the $\mathcal{H}_k$ can be considered as the Taylor (or Fourier, depending on the point of view) expansion of $\mathcal{H}_L$. The theme that holomorphic and other invariants associated to $X$ and $\mathcal{H}$ may be quantized using the spaces $\mathcal{H}_k$ has dominated Kähler geometry for the last 35 years.

In the 1980's, Futaki's invariant was a new obstruction for the existence of Kähler–Einstein metrics, but there were no invariants that guaranteed existence. At best, there were constructions that utilized symmetry to reduce the Kähler–Einstein equation to a simpler equation that could be solved and lead to specific examples (another theme pioneered by Calabi). Given a maximal compact subgroup $G$ of the automorphism group $\operatorname{Aut} X$, Tian introduced the invariant

$$\alpha_G := \sup\left\{ c > 0 \ : \ \sup_{\varphi\in\mathcal{H}^G} \int_X e^{-c(\varphi - \sup\varphi)}\omega^n < \infty \right\}$$

(where $\mathcal{H}^G$ denotes the $G$-invariant elements of $\mathcal{H}_L$), and its quantized version (Definition 4.1)

$$\alpha_{k,G},$$

computed over the $G$-invariant elements of $\mathcal{H}_k$, and obtained the sufficient condition

$$\alpha_G > \frac{n}{n+1} \tag{2}$$

for the existence of a Kähler–Einstein metric when $L = -K_X$ (which we will henceforth assume unless otherwise stated) [29, Theorem 4.1]. Initially, the main interest in the invariants $\alpha_G$ was as the first systematic tool for constructing Kähler–Einstein metrics on Fano manifolds, but later it was also conjectured by Cheltsov and established by Demailly that $\alpha_G$ actually coincides with the $G$-equivariant global log canonical threshold from algebraic geometry [12]. Since $\mathcal{H}_k \subset \mathcal{H}_L$ it follows that $\alpha_G \le \inf_k \alpha_{k,G}$ [31, p. 128], yet this does not help obtain (2). Instead, Tian posed the following difficult question that would reduce the computation of the invariant $\alpha_G$ from the infinite-dimensional space $\mathcal{H}_L$ to a finite-dimensional one $\mathcal{H}_k$, and establish a highly non-trivial relation between the different $\mathcal{H}_k$'s [31, Question 1],[30]:

**Problem 1.1.** *Let $X$ be Fano and $L = -K_X$, and let $G$ be a maximal compact subgroup $G \subset \operatorname{Aut} X$. Is $\alpha_{k,G} = \alpha_G$ for all sufficiently large $k \in \mathbb{N}$?*

It is interesting to note that a slight variation of Tian's $\alpha$- and $\alpha_k$-invariants turned out to lead about three decades later [36, 26] to the very closely related $\delta$- and $\delta_k$-invariants of Fujita–Odaka [18] that are in turn a slight (and ingenius) variation on global log canonical thresholds, and turn out to essentially characterize the existence of Kähler–Einstein metrics. We return to these invariants in a sequel [22].

### 1.2 A Demailly type identity in the presence of symmetry

Another (easier) problem motivating this article concerns the by-now-classical relation between Tian's (holomorphic) invariants and the (algebraic) global log canonical thresholds. The relationship was first conjectured by Cheltsov and proved by Demailly and Shi [12, 27]. However, so far, this relationship has only been shown for the $\alpha$- and $\alpha_k$-invariants, or for the $\alpha_G$-invariant (see [12, Theorem A.3], [27, Proposition 2.1], [12, (A.1)], respectively), and not for the more subtle invariants $\alpha_{k,G}$. Inspired by Demailly, we introduce (Definition 4.4) the $k$-th $G$-equivariant global log canonical threshold

$$\operatorname{glct}_{k,G}$$

as an algebraic counterpart of Tian's $\alpha_{k,G}$ (Definition 4.1). A natural question is:

**Problem 1.2.** *Let $X$ be Fano and $L = -K_X$, and let $G$ be a compact subgroup $G \subset \operatorname{Aut} X$. Is $\operatorname{glct}_{k,G} = \alpha_{k,G}$?*

### 1.3 Results

In this article, we resolve both Problems 1.1 and 1.2 in the toric setting. It is perhaps not well-known, but Calabi was interested in toric geometry and computed certain geodesics in $\mathcal{H}_L$ in the toric setting, although he never published the result [8].

As standard, we allow the slightly more flexible situation of any (and not just a maximal) compact subgroup of the normalizer

$$N((\mathbb{C}^*)^n)$$

of the complex torus $(\mathbb{C}^*)^n$ in $\operatorname{Aut} X$. We refer the reader to §2 where these and other toric notation is set-up carefully. Denote by

$$\operatorname{Aut} P \subseteq GL(M) \cong GL(n, \mathbb{Z}) \tag{3}$$

the subgroup of the automorphism group of the lattice $M$ that leaves the polytope $P$ (26) invariant. It is necessarily a finite group (see §2). In fact, $\operatorname{Aut} P$ is the quotient of the normalizer $N((\mathbb{C}^*)^n)$ of the complex torus $(\mathbb{C}^*)^n$ in $\operatorname{Aut} X$ by $(\mathbb{C}^*)^n$, so that $N((\mathbb{C}^*)^n)$ consists of finitely many components each isomorphic to a complex torus [2, Proposition 3.1]. For $H \subset \operatorname{Aut} P$, let

$$G(H) := H \ltimes (S^1)^n \subset N((\mathbb{C}^*)^n) \subset \operatorname{Aut} X \tag{4}$$

denote the compact group generated by $H$ and $(S^1)^n$ (the latter is the maximal compact subgroup of the complex torus $(\mathbb{C}^*)^n$).

Our first result resolves Problem 1.2 in this generality.

**Proposition 1.3.** *Let $X$ be toric Fano and $L = -K_X$. Let $P \subset M_{\mathbb{R}}$ (see (17), (26)) be the polytope associated to $(X, -K_X)$, let $H \subset \operatorname{Aut} P$, and let $G(H)$ be as in* (4)*. Then* $\operatorname{glct}_{k,G(H)} = \alpha_{k,G(H)}$.

Using this result, and several new estimates, we can resolve Tian's Problem 1.1 in the toric setting in a surprisingly strong sense, showing that equality holds for *all* $k \in \mathbb{N}$. We also allow for all groups $G(H)$ (and not just the maximal toric one $G(\operatorname{Aut} P)$).

To state the precise result we introduce some more notation. For $H \subset \operatorname{Aut} P$, denote by

$$P^H := \big\{ y \in P \,:\, h.y = y, \quad \forall h \in H \big\} \subset P \subset M_{\mathbb{R}} \tag{5}$$

the fixed-point set of $H$ in $P$, and let

$$\pi_H := \frac{1}{|H|} \sum_{\eta \in H} \eta \in \operatorname{End}(M_{\mathbb{Q}}) \tag{6}$$

be the map that takes a point in $M_{\mathbb{R}}$ to the average of its $H$-orbit. Note that $\pi_H$ is a projection map (see §6.3 for details).

**Theorem 1.4.** *Let $X$ be toric Fano associated to a fan $\Delta$ whose rays are generated by primitive elements $v_i$ in the lattice $N$ dual to $M$. Let $P \subset M_{\mathbb{R}}$ (see (17), (26)) be the polytope associated to $(X, -K_X)$, let $H \subset \operatorname{Aut} P$, and let $G(H)$ be as in* (4)*. Then for any $k \in \mathbb{N}$,*

$$\begin{aligned} \alpha_{k,G(H)} &= \sup \left\{ c \in (0,1) \,:\, -\frac{c}{1-c} P^H \subset P \right\} \\ &= \min_{u \in \operatorname{Ver} P^H} \frac{1}{\max_i \langle u, v_i \rangle + 1} \\ &= \min_{u \in \pi_H(\operatorname{Ver} P)} \frac{1}{\max_i \langle u, v_i \rangle + 1}, \end{aligned} \tag{7}$$

*where $P^H$ and $\pi_H$ are defined in* (5)–(6) *and* $\operatorname{Ver}(\,\cdot\,)$ *denotes the vertex set of a polytope. In particular, $\alpha_{k,G(H)}$ is independent of $k \in \mathbb{N}$ and is equal to $\alpha_{G(H)}$.*

There are a few new ingredients in the proof of Theorem 1.4. The first is a useful formula for the spaces $\mathcal{H}_k^{G(H)}$ in terms of the $H$-orbits of the finite group action (Lemma 4.5). This together with a trick that amounts to estimating the singularities associated to a basis of sections in terms of the finite group action orbit of a section yields a useful formula for $\alpha_{k,G(H)}$ (Proposition 4.6) as well as the equality $\alpha_{k,G(H)} = \operatorname{glct}_{k,G(H)}$, i.e., a solution to Problem 1.2 (Proposition 1.3). These then yield a corresponding useful formula for $\alpha_{G(H)}$ (Corollary 4.7). One may prove using the results of §3–§4 that $\alpha_{k,G(H)} \geq \alpha_{k\ell,G(H)}$ for any fixed $k$ and all $\ell \in \mathbb{N}$ (Proposition 5.1). In Proposition 5.3 it is shown that there is a special $k_0$ for which $\alpha_{k_0\ell,G(H)} = \alpha_{G(H)}$ for all $\ell \in \mathbb{N}$, as well as observed that this does not seem to imply Tian's conjecture (Remark 5.6). Finally, key new estimates occur in §6. First, we show that rather general complex singularity exponents associated to collections of toric monomials are independent of $k$ (Proposition 6.2). The proof of this uses a new observation about the relation between the support functions of collections of lattice points associated to the toric monomials and complex singularity exponents. We then apply this to our $G(H)$-invariant setting, using the aforementioned expression of $\mathcal{H}_k^{G(H)}$ and a reduction lemma to the $H$-invariant subspace (Lemma 6.10), to conclude the proof of Theorem 1.4.

It is perhaps of some interest to include here a rather immediate application of this circle of ideas to a slightly more technical set of invariants, also introduced by Tian, that we call Tian's Grassmannian $\alpha$-invariants. These invariants are defined a little differently, algebraically, and are denoted $\alpha_{k,m}$ or $\alpha_{k,m,G}$ (Definition 7.1). At least in the non-equivariant setting (as well as in the torus-equivariant setting, see Remark 7.2) these can be considered as generalizations of the $\alpha_k$ as $\alpha_k = \alpha_{k,1}$ [27, 12]. The invariants $\alpha_{k,2}$ were used by Tian implicitly in his proof of Calabi's conjecture for del Pezzo surfaces [31, Appendix A] (cf. [32, Theorem 6.1]) to overcome the most difficult case (of a cubic surface with an Eckardt point) where equality holds in (2), and this was improved by Shi to $\alpha_{k,2} > 2/3 = \alpha_{k,1}$ in that case [27, Theorem 1.3],[10].

Tian also posed a stabilization conjecture for these invariants in 2012 [33, Conjecture 5.3]:

**Conjecture 1.5.** *Let $X$ be Fano. Fix $m \in \mathbb{N}$. For sufficiently large $k$, $\alpha_{k,m,G}$ is constant.*

We completely resolve Conjecture 1.5 in the toric setting. Theorem 1.4 resolved Problem 1.1 in the affirmative (corresponding to the case $m = 1$ of Conjecture 1.5). For $m \geq 2$, Conjecture 1.5 turn out to be only partially true as determined by a novel convex geometric obstruction we introduce:

$$\| \cdot \|_{-P}\Big|_{P\setminus \operatorname{Ver} P} < \max_P \| \cdot \|_{-P}. \tag{$*_P$}$$

Note that this condition depends only on $P$ (and not on $m, k$). The condition $(*_P)$ means that the function $\|\cdot\|_{-P}$ on $P$ achieves its maximum only at the vertices of $P$, i.e.,

$$\operatorname{argmax}_P \| \cdot \|_{-P} \subset \operatorname{Ver} P. \tag{$*_P$}$$

When $(*_P)$ fails the maximum is achieved also at some point that is not a vertex of $P$.

**Theorem 1.6.** *Let $X$ be toric Fano with associated polytope $P$ (27). Conjecture 1.5 holds if and only if $(*_P)$ fails. More precisely, if $(*_P)$ holds,*

$$\alpha_{k,m,(S^1)^n} > \alpha, \quad \text{for } k \in \mathbb{N} \text{ and } m \in \mathbb{N} \setminus \{1\}, \tag{8}$$

*otherwise*

$$\alpha_{k,m,(S^1)^n} = \alpha, \quad \text{for } m \in \mathbb{N} \text{ and for sufficiently large } k \in \mathbb{N}. \tag{9}$$

Theorem 1.6 is proven in §7.1 where we also explain the intuition behind it (see also Examples 8.6 and 8.9). For now, let us elucidate the condition $(*_P)$ a bit. The level set $\{\| \cdot \|_{-P} = \lambda\}$ is the dilation $\lambda\partial(-P)$, and $\{\| \cdot \|_{-P} = \max_P \|\cdot\|_{-P}\}$ is the largest dilation that intersects $P$ (by Lemma 7.4). Thus, condition $(*_P)$ states that $P$ intersects $\max_P \| \cdot \|_{-P}\partial(-P)$ only at vertices. When $(*_P)$ fails, convexity arguments show the intersection will contain a positive-dimensional face of $P$. See Figure 1 for two examples.

**Relation to earlier works.** Theorem 1.4 strengthens and clarifies work of Song [28] and Li–Zhu [24]. Song obtained a formula for $\alpha_G$ but not for $\alpha_{k,G}$. In particular, there seems to be a gap in the proof of [28, Theorem 1.2] that claims that $\alpha_G = \alpha_{k,G}$ for all sufficiently large $k$. This claim relies on proving that for some $k_0 \in \mathbb{N}$ and all $\ell \in \mathbb{N}$, $\alpha_G = \alpha_{k_0\ell,G}$ and then invoking that $\alpha_{k,G}$ is eventually monotone in $k$, and hence must be independent of $k$ for sufficiently large $k$.

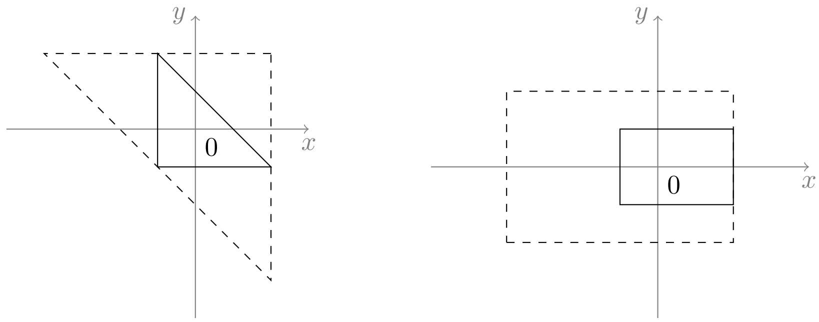


Figure 1: The polytope $P$ (solid line) and the level set $\{\|\cdot\|_{-P} = \max_P \|\cdot\|_{-P}\}$ (dashed line). For $P = \mathrm{co}\{(-1,-1),(2,-1),(-1,2)\}$, the maximum is only attained at the vertices of $P$. In particular, $(*_P)$ holds. For $P = [-1,2] \times [-1,-1]$, the maximum is attained on the line segment $\{2\} \times [-1,-1]$. In particular, $(*_P)$ does not hold.

Unfortunately, the proof of monotonicity is omitted from [28, p. 1257, line 7], and it appears to be difficult to reproduce. It seems that such monotonicity is not currently known (cf. Remark 5.6). Indeed, there is no obvious relationship between the various $\mathcal{H}_k^G$ coming from different Kodaira embeddings. As noted above, Song showed (for $H = \mathrm{Aut}\, P$) that $\alpha_{G(H)} = \alpha_{k_0\ell, G(H)}$ for some $k_0 \in \mathbb{N}$ and all $\ell \in \mathbb{N}$. Li–Zhu showed the same identity (essentially for $H = \{\mathrm{id}\}$) for a group compactification of a reductive complex Lie group and also claimed, similarly to Song, that this implies eventual constancy in $k$ in that setting [24, Theorem 1.3, p. 233]. Unfortunately, also they do not provide a proof of the needed eventual monotonicity or constancy. In the non-equivariant setting, and for rather general Fano varieties for which $\alpha \leq 1$, Birkar showed the deep result that $\alpha = \alpha_{k_0\ell}$ for some $k_0 \in \mathbb{N}$ and all $\ell \in \mathbb{N}$ [5, Theorem 1.7]. However, also this result does not imply Tian stabilization due to the aforementioned unknown monotonicity. Thus, Theorem 1.4 seems to be the first general result on Tian's stabilization Problem 1.1.

Similarly, Theorem 1.6 seems to be the first general result on Conjecture 1.5. Indeed, Li–Zhu showed the same type of result Song obtained in the $m = 1$ setting, i.e., that $\alpha_{k_0\ell, m, (S^1)^n} = \alpha_{(S^1)^n}$ under a condition depending on $k_0$ and $m$, and hence different from our $(*_P)$ (with the minor caveat that their statement as written [24, Theorem 1.4] is incorrect, though can be easily fixed by replacing "facet" by "face", see §7.1). However, again, due to the lack of monotonicity, they do not obtain a resolution of Conjecture 1.5 though they do obtain the first counterexamples to it when $m \geq 2$.

Combining Theorem 1.4 and Demailly's theorem [12, (A.1)] also recovers Song's formula for the $\alpha_{G(\mathrm{Aut}\, P)}$ (that itself generalized Batyrev–Selivanova's formula that $\alpha_{G(\mathrm{Aut}\, P)} = 1$ whenever $P^{\mathrm{Aut}\, P} = \{0\}$ (recall (5)) [2, Theorem 1.1, p. 233]). Cheltsov–Shramov claimed a more general formula for $\alpha_H$ (i.e., without the real torus symmetry included in $G(H)$, recall (4)) however (as kindly pointed out to us by I. Cheltsov) there is an error in the proof of [12, Lemma 5.1] as the toric degeneration used there need not respect the $H$-invariance. Further generalizations of Song's formula for $\alpha_G$ to general polarizations and group compactifications are due to Delcroix [13, 14] and Li–Shi–Yao [23] and our methods should generalize to those settings as well as to the setting of log toric Fano pairs and edge singularities [11, §6–7].

Finally, it is also worth mentioning that Tian also posed more general conjectures [33, Conjecture 5.4] for general polarizations (i.e., $L$ not being $-K_X$) for which there are already some counterexamples [1].

**Organization.** Section 2 sets up the necessary notation concerning toric varieties and convex analysis. Section 3 constructs natural equivariant reference Hermitian metrics and volume forms. Proposition 1.3 is proved in §4.4. Section 5 explains a trick that allows to deal with divisible $k \in \mathbb{N}$ but also highlights the difficulties in dealing with general $k$. Theorem 1.4 is proved in §6. Theorem 1.6 is proved in §7. We conclude with examples in §8.

**Acknowledgments.** Thanks to C. Birkar, H. Blum, I. Cheltsov, and K. Zhang for helpful references. Research supported in part by NSF grants DMS-1906370,2204347, BSF grant 2020329, and an Ann G. Wylie Dissertation Fellowship.

# 2 Toric and convex analysis set-up

## 2.1 Notions from convexity

Consider an $n$-dimensional real vector space $V \cong \mathbb{R}^n$ and let $V^* \cong \mathbb{R}^n$ denote its dual with the pairing denoted by $\langle \cdot, \cdot \rangle$. Given a set $A \subset V$, denote by

$$A^\circ = \{y \in V^* \,:\, \langle x, y \rangle \leq 1, \ \forall x \in A\}$$

the polar of $A$ [25, p. 125], and by

$$\operatorname{co} A \tag{10}$$

the convex hull of $A$ [25, p. 12]. For a finite set [25, Theorem 2.3],

$$\operatorname{co}\{p_1, \ldots, p_\ell\} = \Big\{ \textstyle\sum_{i=1}^{\ell} \lambda_i p_i \,:\, \sum_{i=1}^{\ell} \lambda_i = 1, \ \lambda \in [0,1]^\ell \Big\}. \tag{11}$$

Also, set

$$-K := \{-x \,:\, x \in K\}.$$

Note $(-K)^\circ = -K^\circ$. Also, $K^\circ = (\operatorname{co} K)^\circ \subset V^*$ whenever $K \subset V$. The polar can also be described via the support function $h_K : V^* \to \mathbb{R}$,

$$h_K(y) := \sup_{x \in K} \langle x, y \rangle, \quad y \in V^* \cong \mathbb{R}^n, \tag{12}$$

by $K^\circ = \{h_K \leq 1\}$.

A dual notion to the support function is the near-norm function

$$\|x\|_K := \inf \{t \geq 0 \,:\, x \in tK\}, \tag{13}$$

associated to any compact convex set $K$ with $0 \in \operatorname{int} K$. Note that [25, Corollary 14.5]

$$\| \cdot \|_K = h_{K^\circ}. \tag{14}$$

Note that $\| \cdot \|_K$ is a norm when $K$ is centrally symmetric (i.e., $K = -K$), otherwise it is only a near-norm in the sense that it satisfies all the properties of a norm but is only $\mathbb{R}_+$-homogeneous: $\|\lambda x\|_K = \|x\|_K$ for $\lambda \in \mathbb{R}_+$ (and not fully $\mathbb{R}$-homogeneous). The infimum in (13) is achieved: for a minimizing sequence $\{t_i\}$, $\frac{x}{t_i} \in K$ for any $i$, so

$$x \in \|x\|_K K \tag{15}$$

since $K$ is closed.

**Lemma 2.1.** *Let $V$ be an $\mathbb{R}$-vector space. Consider the polytope*

$$A = \bigcap_{j=1}^{d} \{x \in V \,:\, \langle x, v_j \rangle \leq 1\},$$

*where $v_j \in V^*$. Then,*

$$\|x\|_A = \max_{1 \leq j \leq d} \langle x, v_j \rangle .$$

*Proof.* Notice that $x \in tA$ if and only if for any $1 \leq j \leq d$, $\langle x, v_j \rangle \leq t$. Thus,

$$\|x\|_A = \inf \left\{ t \geq 0 \,:\, \max_{1 \leq j \leq d} \langle x, v_j \rangle \leq t \right\} = \max_{1 \leq j \leq d} \langle x, v_j \rangle .$$

Alternatively, observe that $A = \{v_1, \ldots, v_d\}^\circ$ and $A^\circ = (\{v_1, \ldots, v_d\}^\circ)^\circ = \mathrm{co}\{v_1, \ldots, v_d\}$ [25, Theorem 14.5] and then use (55) and (14). $\square$

## 2.2 Toric algebra

Consider a lattice of rank $n$ and its dual lattice

$$N, \quad M := N^* := \mathrm{Hom}(N, \mathbb{Z}). \tag{16}$$

Both $N$ and $M$ are isomorphic to $\mathbb{Z}^n$ but we do not specify the isomorphism (see, e.g., the proof of Lemma 2.3 for this point). The notation is useful as it serves to distinguish between objects living in one lattice and its dual (although of course in computations we simply work on $\mathbb{Z}^n$, see §8). Denote the corresponding $\mathbb{R}$-vector space and its dual (both isomorphic to $\mathbb{R}^n$)

$$N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}, \quad M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} = N_{\mathbb{R}}^*. \tag{17}$$

A rational convex polyhedral cone in $N_{\mathbb{R}}$ takes the form

$$\sigma = \sigma(v_1, \ldots, v_d) := \left\{ \textstyle\sum_{i=1}^{d} a_i v_i \,:\, a_i \geq 0, v_i \in N \right\}. \tag{18}$$

The rays $\mathbb{R}_+ v_i, i \in \{1, \ldots, d\}$ are called the generators of the cone [19, p. 9]. They are (1-dimensional) cones themselves, of course. Our convention will be that the

$$v_i, \quad i \in \{1, \ldots, d\}, \quad \text{are primitive elements of the lattice } N, \tag{19}$$

which means there is no $m \in \mathbb{N} \setminus \{1\}$ such that $v_i/m \in N$. A cone is called strongly convex if $\sigma \cap -\sigma = \{0\}$ [19, p. 14]. A face of $\sigma$ is any intersection of $\sigma$ with a supporting hyperplane.

**Definition 2.2.** A *fan* $\Delta = \{\sigma_i\}_{i=1}^{\delta}$ in $N$ is a finite set of rational strongly convex polyhedral cones $\sigma_i$ in $N_{\mathbb{R}}$ such that:

(i) each face of a cone in $\Delta$ is also (a cone) in $\Delta$,

(ii) the intersection of two cones in $\Delta$ is a face of each.

Such a fan gives rise to a toric variety $X(\Delta)$: each cone $\sigma_i$ in $\Delta$ gives rise to an affine toric variety [19, §1.3], that serves as (a Zariski open) chart in $X(\Delta)$ with the transition between the charts constructed by (i) and (ii) above [19, p. 21]. For instance, the zero cone corresponds to the open dense orbit $(\mathbb{C}^*)^n$ [3, p. 64], and more generally there is a bijection between the cones $\{\sigma_i\}_{i=1}^{\delta}$ and the orbits of the complex torus $(\mathbb{C}^*)^n$ in $X(\Delta)$ [3, Proposition 5.6.2], with the non-zero cones corresponding precisely to all the toric subvarieties of $X(\Delta)$ of positive codimension.

When $X(\Delta)$ is a smooth toric Fano variety (as we always assume), the fan $\Delta$ must arise from an integral polytope as follows (but in general, i.e., for singular toric varieties, this need not be the case [19, p. 25]). Let $\Delta$ be a fan such that $X(\Delta)$ is smooth Fano and let $\sigma_1, \ldots, \sigma_d$ be its 1-dimensional cones (i.e., rays) generated by primitive generators

$$\Delta_1 := \{v_1, \ldots, v_d\} \subset N, \tag{20}$$

so $\sigma_i = \mathbb{R}_+ v_i$, and set (recall (10))

$$Q := \operatorname{co} \Delta_1 = \operatorname{co}\{v_1, \ldots, v_d\} = \operatorname{co} \Delta_1 \subset N_{\mathbb{R}}. \tag{21}$$

Then $\Delta$ is equal to the collection of cones over each face of $Q$ plus the zero cone [19, p. 26], in other words if $F \subset Q$ is a face, then

$$\sigma_F := \{rx \in N_{\mathbb{R}} \,:\, r \geq 0, x \in F\} \tag{22}$$

is the union of all rays through $F$ and the origin, and

$$\Delta = \{\sigma_F\}_{F \subset Q}.$$

Denote by

$$\operatorname{Ver} A \tag{23}$$

the vertices of a polytope $A$. Note that $\operatorname{Ver} F \subset \operatorname{Ver} Q = \Delta_1$, and by (18)–(19),

$$\sigma_F = \sigma(\operatorname{Ver} F). \tag{24}$$

Smoothness of $X$ means that the generators of $\sigma_F$ form a $\mathbb{Z}$-basis for $N$ [19, p. 29]. By (24) this means

$$\text{the vertices of } F \text{ form a } \mathbb{Z}\text{-basis for } N \text{ (for any facet } F \subset Q). \tag{25}$$

Thus each facet $F$ of $Q$ is an $(n-1)$-simplex whose vertices form a $\mathbb{Z}$-basis of $N$. In Lemma 2.3 we show this means the vertices of the polar polytope belong to the dual lattice $M$.

When $L = -K_X$, there is an $\operatorname{Aut} X$ action on $H^0(X, -kK_X)$ for every $k \in \mathbb{N}$. To get an induced linear action on $M_{\mathbb{Q}}$ we must restrict to the normalizer $N((\mathbb{C}^*)^n)$ of the complex torus $(\mathbb{C}^*)^n$ in $\operatorname{Aut} X$. The representation of $(\mathbb{C}^*)^n$ on $H^0(X, -kK_X)$ splits into 1-dimensional spaces, whose generators are called the monomial basis. There is a one-to-one correspondence between the monomial basis of $H^0(X, -kK_X)$ and points in $kP \cap M$, and the quotient $N((\mathbb{C}^*)^n)/(\mathbb{C}^*)^n$ is a linear group, that can be identified with $\operatorname{Aut} P \subset GL(M) \cong GL(n, \mathbb{Z})$ (3). Since $P$ is defined as the convex hull of vertices in $M$ it follows that $\operatorname{Aut} P$ is finite. Alternatively, this can be seen by observing that $N_{\mathbb{R}}$ is canonically isomorphic to the quotient of $(\mathbb{C}^*)^n$ by its maximal compact subgroup $(S^1)^n$ [2, p. 229] and the induced action on $M_{\mathbb{R}}$ is then defined by transposing via the pairing. Conversely, all compact subgroups of $N((\mathbb{C}^*)^n)$ that contain $(S^1)^n$ are generated by $(S^1)^n$ and a finite subgroup $H$ of $\operatorname{Aut} P$ [2, Proposition 3.1], and we denote such a group by $G(H) \subset \operatorname{Aut} X$ as in (4). We describe in the proof of Lemma 3.1 concretely how the action of $\operatorname{Aut} P$ is expressed in coordinates. For a finite group $H$ or finite set $\mathcal{A}$ we denote by

$$|H|, \text{ respectively } |\mathcal{A}|,$$

its order or cardinality.

Oftentimes we will work with

$$P := -Q^\circ = \{-v_1, \ldots, -v_d\}^\circ = -\{v_1, \ldots, v_d\}^\circ \subset M_{\mathbb{R}}, \tag{26}$$

as it has the nice geometric property of faces of (real) dimension $k$ corresponding to toric subvarieties of dimension $k$, and since the metric properties (e.g., volume) of $P$ correspond to those of $X$. (Moreover, $P$ can also be realized as the Delzant (moment) polytope associated to any $(S^1)^n$-invariant Kähler metric representing the anticanonical class.) In particular,

$$P = \bigcap_{i=1}^{d} \Big\{ y \in M_{\mathbb{R}} \,:\, \langle y, -v_i \rangle \leq 1 \Big\} = \{h_{-\Delta_1} \leq 1\} = \{y \in M_{\mathbb{R}} \,:\, \max_j \langle -v_j, y\rangle \leq 1\}, \tag{27}$$

and irreducible toric divisors correspond to facets of $P$

$$D_i := \{y \in P \,:\, \langle y, -v_i \rangle = 1\}. \tag{28}$$

Note that $P$ contains the origin in its interior. Also note that (26) is the standard convention since then lattice points of $P$ correspond to monomials via (33). Batyrev–Selivanova use $-P$ instead.

**Lemma 2.3.** *Let $P \subset M_{\mathbb{R}}$ be the polytope associated to a smooth toric variety $X$. Then $P$ is an integral lattice polytope, i.e.,* $\operatorname{Ver} P \subset M$.

*Proof.* By duality, if $u \in M_{\mathbb{R}}$ is a vertex of $P$ then

$$F = \{v \in Q \,:\, \langle u, -v \rangle = 1\} \subset N_{\mathbb{R}} \tag{29}$$

is a facet of $Q = -P^\circ \subset N_{\mathbb{R}}$. Since $X$ is smooth, the vertices of $F$ form a $\mathbb{Z}$-basis for $N$ by (25). Choose coordinates on $N$ associated to this $\mathbb{Z}$-basis, i.e., the vertices of $F$ are the standard basis vectors $e_1, \ldots, e_n$. Thus, the facet $F = \operatorname{co}\{e_1, \ldots, e_n\}$ is the standard $(n-1)$-simplex in $\mathbb{R}^n$ cut-out by the equation $\langle u, -v \rangle = 1$ where $u = (-1, \ldots, -1) \in M$, as desired.

Alternatively, if one does not wish to choose coordinates but rather work invariantly, denote by $\operatorname{Ver} F = \{f_1, \ldots, f_n\} \subset N$, and note $\operatorname{span}_{\mathbb{Z}} \operatorname{Ver} F = N$. Thus, any $v \in N$ can be written uniquely as $v = \sum_{i=1}^n a_i f_i$ (with $a_i \in \mathbb{Z}$), and so $u \in M_{\mathbb{R}} = \operatorname{Hom}(N, \mathbb{R})$ can be identified (recall (16)) with the map

$$N \ni v \mapsto -\sum_{i=1}^{n} a_i,$$

As $\sum_{i=1}^n a_i \in \mathbb{Z}$, this map actually belongs to $\operatorname{Hom}(N, \mathbb{Z}) = M$. $\square$

Another useful fact is a sort of maximum principle for convex polytopes, saying essentially that a convex function on a polytope achieves its maximum at some vertex (regardless of continuity).

**Lemma 2.4.** *Let $A$ be a convex polytope and $f : A \to \mathbb{R} \cup \{\infty\}$ a convex function. Then:*
*(i)* $\sup_A f = \sup_{\operatorname{Ver} A} f$.
*(ii) if $f$ is bounded on* $\operatorname{Ver} A$ *it is bounded on $A$ and its maximum is achieved in a vertex.*
*(iii) if $f$ attains its finite maximum on* $\operatorname{int} A$ *it is constant.*
*(iv) if $f$ attains its finite maximum on the relative interior of a face $F \subset A$ it is constant on $F$.*

*Proof.* Write $\operatorname{Ver} A = \{p_1, \ldots, p_\ell\}$ and assume $f(p_1) \leq \cdots \leq f(p_\ell)$. By convexity, $A = \operatorname{co} \operatorname{Ver} A$, and (11) implies that any $x \in A$ can be expressed as

$$x = \sum_{i=1}^{\ell} \lambda_i p_i, \quad \sum_{i=1}^{\ell} \lambda_i = 1, \quad \lambda \in [0,1]^\ell. \tag{30}$$

Then $f(x) \leq \sum_{i=1}^{\ell} \lambda_i f(p_i) \leq \sum_{i=1}^{\ell} \lambda_i f(p_\ell) = f(p_\ell)$, proving (i) and (ii).

To see (iii), again express any $x \in A$ using (30). Suppose that $x_{\text{int}} \in \operatorname{int} A$ achieves the (finite) maximum of $f$. Note that $x_{\text{int}} \in \operatorname{int} A$ means one has the representation (30) for $x_{\text{int}}$ for some $\lambda_{\text{int}} \in (0,1)^\ell$. Choose $\delta = \delta(x) \in (0,1)$ so that $\lambda_{\text{int}} - \delta\lambda \in \mathbb{R}^\ell_+$. Define

$$\lambda' := \frac{1}{1-\delta}\left(\lambda_{\text{int}} - \delta\lambda\right) \in \mathbb{R}^\ell_+.$$

Note that $\lambda'$ satisfies $\sum_{i=1}^{\ell} \lambda'_i = 1$, so letting $x' = \sum_{i=1}^{\ell} \lambda'_i p_i$ we have $x' \in A$ by (30). Also, $\delta x + (1-\delta)\, x' = x_{\text{int}}$. Thus, $\max_A f = f(x_{\text{int}}) \le \delta f(x) + (1-\delta)\, f(x') \le \max_A f$, forcing equality, i.e., $f(x) = \max_A f$ (since $\delta > 0$ and $\max_A f < \infty$), proving (iii). The proof of (iv) is identical by working on the polytope $F$. □

Finally, we recall Ehrhart's theorem on the polynomiality of the number of lattice points in dilations of lattice polytopes [15, 16], [21, Theorem 19.1]. Set

$$E_P(k) := |kP \cap M|, \quad k \in \mathbb{N}. \tag{31}$$

**Proposition 2.5.** *Let $M$ be a lattice and $P \subset M_{\mathbb{R}}$ be a lattice polytope of dimension $n$. Then,*

$$E_P(k) = \sum_{i=0}^{n} a_i k^i, \quad \textit{for any } k \in \mathbb{N},$$

*with $a_n = \operatorname{Vol}(P)$, and*

$$k \le k' \quad \Rightarrow \quad E_P(k) \le E_P(k'). \tag{32}$$

# 3 A natural equivariant Hermitian metric and volume form

In light of Lemma 4.2 below it makes sense to choose a convenient pair $(\mu, h)$ of a volume form and a Hermitian metric. In fact, we are free to choose such a pair for each $k$. The special feature of working with $L = -K_X$ is that in fact a volume form essentially doubles as a Hermitian metric, which is sometimes a bit confusing to keep track of in terms of notation, but is quite convenient for computations. This section serves to explain this choice $(\mu_k = h_k^{1/k}, h_k)$, see (39) and (41), originally due to Song [28, Lemma 4.3]. We emphasize that the Hermitian metric must additionally be chosen $G$-invariant in Definition 4.1, and this is confirmed for $h_k$ in Lemma 3.1.

Let $X$ be a toric Fano manifold and $P$ its associated polytope. There is a natural basis of the space of holomorphic sections $H^0(X, -kK_X)$ defined by the monimials $z^{ku}$ where $u \in P \cap \frac{1}{k}M$. That is, there exists an invariant frame $e$ over the open orbit such that

$$s_{k,u}(z) = z^{ku} e. \tag{33}$$

What does $e$ actually look like? This is most naturally expressed in terms of the *monomial basis*. When $k = 1$ and $u \in P \cap M$ [19, §4.3],

$$s_{1,u} = z^u \prod_{i=1}^{n} z_i \cdot \partial_{z_1} \wedge \cdots \wedge \partial_{z_n}.$$

In general, for any $k \in \mathbb{N}$ and $u \in P \cap \frac{1}{k}M$,

$$s_{k,u} = z^{ku} \left(\prod_{i=1}^{n} z_i\right)^k (\partial_{z_1} \wedge \cdots \wedge \partial_{z_n})^{\otimes k}. \tag{34}$$

In other words,

$$e = \left(\prod_{i=1}^{n} z_i\right)^k (\partial_{z_1} \wedge \cdots \wedge \partial_{z_n})^{\otimes k}. \tag{35}$$

Next, let us construct a canonical Hermitian metric $h_k$ on $-kK_X$. Since $-kK_X$ is very ample [19, p. 70], it is natural to pull-back the Fubini–Study Hermitian metric via the Kodaira embedding. It turns out that choosing the Kodaira embedding given by the monomial basis

$$\iota_k : X \ni z \mapsto [s_{k,u}(z)/e(z)]_{u\in P\cap\frac{1}{k}M} = [z^{ku}]_{u\in P\cap\frac{1}{k}M} \in \mathbb{P}^{E_P(k)-1}, \tag{36}$$

will yield the desired $h_k$; importantly, the resulting $h_k$ will be *torus-invariant*, smooth, and essentially transform computations on $X$ to $P$. To wit, the Fubini–Study metric on $\mathcal{O}(1) \to \mathbb{P}^{E_P(k)-1}$ is (where $E_P(k)$ is the number of lattice points in $kP$)

$$h_{FS}(Z_i, Z_j) := \frac{Z_i \bar{Z}_j}{\sum_\ell |Z_\ell|^2},$$

and we define

$$h_k := \iota_k^* h_{FS}. \tag{37}$$

Note that each homogeneous coordinate $Z_i \in H^0(\mathbb{P}^{E_P(k)-1}, \mathcal{O}(1))$ pulls-back via $\iota_k$ to one of the monomial sections $s_{k,u}$ (which one depends on the ordering for the elements of $P \cap \frac{1}{k}M$ chosen in (36)). Thus, to express $h_k$ it suffices to compute it on the monomial basis of $H^0(X, -kK_X)$:

$$h_k(s_{k,u_1}, s_{k,u_2})(z) = h_{FS}\big(s_{k,u_1}, s_{k,u_2}\big)(\iota_k(z)) = \frac{z^{ku_1}\overline{z^{ku_2}}}{\sum\limits_{u\in P\cap\frac{1}{k}M} |z^{ku}|^2}. \tag{38}$$

Comparing (34) and (38) means that $h_k$ can be written as

$$h_k = \frac{\left(dz^1 \wedge \overline{dz^1} \wedge \cdots \wedge dz^n \wedge \overline{dz^n}\right)^{\otimes k}}{\left(\prod\limits_{i=1}^{n} |z_i|^2\right)^k \sum\limits_{u\in P\cap\frac{1}{k}M} |z^{ku}|^2}. \tag{39}$$

In conclusion, $h_k^{1/k}$ is a smooth metric on $-K_X$ ($h_k$ being obtained as a pull-back of a smooth metric under the Kodaira *embedding*), hence it is a smooth volume form on $X$. To express this volume form, on the open orbit $(\mathbb{C}^*)^n = \mathbb{R}^n \times (S^1)^n$ consider the holomorphic coordinates

$$w_i := x_i/2 + \sqrt{-1}\theta_i = \log z_i \in \mathbb{C}^n. \tag{40}$$

In these coordinates then, this volume form, on the open orbit, is

$$\mu_k := h_k^{\frac{1}{k}} = \frac{dx_1 \wedge \cdots \wedge dx_n \wedge d\theta_1 \wedge \cdots \wedge d\theta_n}{\left(\sum\limits_{u\in P\cap M/k} e^{\langle ku, x\rangle}\right)^{\frac{1}{k}}}. \tag{41}$$

**Lemma 3.1.** *Let $G(H) \subseteq \operatorname{Aut} X$ (4) be a subgroup generated by $(S^1)^n$ and a subgroup $H$ of $\operatorname{Aut}(P)$. Then $h_k$ (39) is $G(H)$-invariant.*

*Proof.* From (41) it is evident that $h_k$ is independent of $(\theta_1, \dots, \theta_n)$, i.e., it is $(S^1)^n$-invariant. An automorphism $\sigma \in \operatorname{Aut} P \subseteq GL(M)$ can be represented (via choosing a basis for the lattice $M$) by a matrix in $GL(n, \mathbb{Z}) \cong GL(M)$. Since $\sigma$ preserves the polytope $P$, then $\det \sigma \in \{\pm 1\}$ ($\sigma$ could be orientation-reversing, e.g., in the case of a reflection). The induced action of $\sigma$ on the dual space $N_{\mathbb{R}}$ is naturally represented (via the pairing between $M$ and $N$) by the transpose matrix, that we denote by $\sigma^T$, and this action is actually coming from the $\mathbb{C}$-linear action of $\sigma^T$ on $N_{\mathbb{C}} \cong \mathbb{C}^n$ (40). Thus,

$$\sigma.(dx_1 \wedge \cdots \wedge dx_n) = d(x_1 \circ \sigma) \wedge \cdots \wedge d(x_n \circ \sigma) = \det(\sigma^T) dx_1 \wedge \cdots \wedge dx_n$$

(here $\sigma.$ denotes the action of $\sigma$ on forms, i.e., by pull-back), and $\sigma.(d\theta_1 \wedge \cdots \wedge d\theta_n) = \det(\sigma^T) d\theta_1 \wedge \cdots \wedge d\theta_n$. Since $(\det \sigma^T)^2 = 1$ it remains to consider the denominator of (41):

$$\begin{aligned} \sigma. \sum_{u \in P \cap M/k} e^{k\langle u, x\rangle} &= \sum_{u \in P \cap M/k} e^{k\langle u, \sigma^T . x\rangle} \\ &= \sum_{u \in P \cap M/k} e^{k\langle \sigma . u, x\rangle} \\ &= \sum_{u \in \sigma(P \cap M/k)} e^{k\langle u, x\rangle} \\ &= \sum_{u \in P \cap M/k} e^{k\langle u, x\rangle}, \end{aligned}$$

since $\sigma$ preserves both $P$ and $M/k$. In particular, $h_k$ is invariant under $\sigma$, concluding the proof. □

**Remark 3.2.** An alternative, more invariant, proof of Lemma 3.1 is as follows. By (37), $\sigma . h_k = (\iota_k \circ \sigma)^* h_{\mathrm{FS}}$. Now $\iota_k \circ \sigma$ induces the exact same Kodaira embedding if $\sigma \in (S^1)^n < G(H)$. So it suffices to consider $\sigma \in H$; then, since $H$ preserves $P \cap M/k$, one obtains the same Kodaira embedding up to permutation of the coordinates in $\mathbb{P}^{E_P(k)-1}$. Either way, one obtains the same pull-back of the Fubini–Study metric as can be from the definition of the Fubini–Study metric or directly from (39), i.e., $\sigma . h_k = h_k$.

# 4 An algebraic $\alpha_{k,G}$-invariant and a Demailly type result

## 4.1 Analytic definition

Analogous to the classical $\alpha_G$-invariant, Tian [31, p. 128] defined the $\alpha_{k,G}$-invariant. For this one restricts to a $G$-invariant subset of $\mathcal{H}_k$. To write the subset explicitly in terms of global Kähler potentials it is necessary to choose a continuous $G$-invariant Hermitian metric $h$ on $-kK_X$:

$$\mathcal{H}_k^G(h) := \left\{ \varphi = \frac{1}{k} \log \sum_i |s_i|_h^2 \, : \, \varphi \text{ is } G\text{-invariant, } \{s_i\} \text{ is a basis of } H^0(X, -kK_X) \right\} \subset C^\infty(X). \tag{42}$$

**Definition 4.1.** Let $G \subseteq \operatorname{Aut} X$ be a compact subgroup. Let $h$ be a fixed continuous $G$-invariant Hermitian metric on $-kK_X$, and $\mu$ a fixed continuous volume form on $X$. Then

$$\alpha_{k,G}(h, \mu) := \sup \left\{ c > 0 \, : \, \sup_{\varphi \in \mathcal{H}_k^G} \int_X e^{-c(\varphi - \sup \varphi)} d\mu < \infty \right\}.$$

**Lemma 4.2.** *Definition 4.1 does not depend on the choice of $h$ or $\mu$.*

For this reason we will simply denote the invariants by $\alpha_{k,G}$ from now on.

*Proof.* Since $X$ is compact, any two continuous volume forms are uniformly bounded and hence define the same $L^1$ spaces. Next, given two continuous $G$-invariant Hermitian metrics $h$ and $\tilde{h}$ on $-kK_X$ there is an isomorphism from $\mathcal{H}_k^G(h)$ to $\mathcal{H}_k^G(\tilde{h})$ given by $\varphi \mapsto \varphi + \frac{1}{k}\log\frac{\tilde{h}}{h}$. Observe that $\frac{1}{k}\log\frac{\tilde{h}}{h}$ is (again by compactness of $X$) a uniformly bounded function on $X$. Hence, for a fixed $c > 0$,

$$\sup_{\varphi\in\mathcal{H}_k^G(h)}\int_X e^{-c(\varphi-\sup\varphi)}d\mu < \infty \quad \Leftrightarrow \sup_{\varphi\in\mathcal{H}_k^G(\tilde{h})}\int_X e^{-c(\varphi-\sup\varphi)}d\mu < \infty,$$

as desired. □

## 4.2 Algebraic definition

Consider a (complex) non-zero vector subspace $V$ of $H^0(X, kL)$. Associated to it is the (not necessarily complete) linear system $|V| := \mathbb{P}V \subset |kL| := \mathbb{P}H^0(X, kL)$ [20, p. 137].

**Lemma 4.3.** *Let $V$ be vector subspace of $H^0(X, kL)$ of dimension $p > 0$. For any basis $\nu_1, \ldots, \nu_p \in H^0(X, kL)$ of $V$, the number*

$$\sup\left\{c > 0\ :\ \left(\sum_{j=1}^{p}|\nu_j(z)|^2\right)^{-c} \text{ is locally integrable on } X\right\}$$

*is the same.*

*Proof.* Let $\{\nu_1^{(\ell)}, \ldots, \nu_p^{(i)}\}$, $\ell \in \{1,2\}$, be two bases for $V \in H^0(X, kL)$. Let $A \in GL(p, \mathbb{C})$ be the change-of-basis matrix, i.e., $\nu_j^{(2)} = A_j^i \nu_i^{(1)}$. Observe that $A^H A$ is a positive Hermitian matrix-valued on $X$, and denote its eigenvalues $0 < \lambda_1 \le \cdots \le \lambda_p$. Denote $\nu^{(\ell)}(z) := (\nu_1^{(\ell)}(z), \ldots, \nu_p^{(\ell)}(z)) \in \mathbb{C}^p$ and $|\nu^{(\ell)}(z)|^2 = \sum_{i=1}^p |\nu_i^{(\ell)}(z)|^2$. Then,

$$\frac{|\nu^{(2)}(z)|^2}{|\nu^{(1)}(z)|^2} = \frac{|A\nu^{(1)}(z)|^2}{|\nu^{(1)}(z)|^2} \in [\lambda_1, \lambda_p],$$

Thus, $|\nu^{(2)}|^2$ is locally integrable if and only if $|\nu^{(1)}|^2$ is. □

Thus, define the log canonical threshold of the linear system $|V|$ by

$$\mathrm{lct}|V| := \sup\left\{c > 0\ :\ \left(\sum_j |\nu_j(z)|^2\right)^{-c} \text{ is locally integrable on } X\right\}. \tag{43}$$

When $L = -K_X$, there is an $\mathrm{Aut}\,X$ action on $H^0(X, -kK_X)$ for every $k \in \mathbb{N}$. Demailly [12, Theorem A.3, (A.1)] noted that then $\alpha_G$-invariants (for compact subgroup $G \subset \mathrm{Aut}\,X$) can be algebraically computed as

$$\alpha_G = \inf_{k\in\mathbb{N}} k \inf_{\substack{|V|\subset|-kK_X|\\ V^G=V\neq 0}} \mathrm{lct}|V|, \tag{44}$$

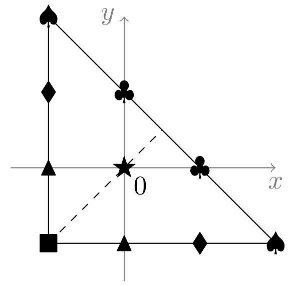


Figure 2: The six orbits $O_1^{(1)}, \ldots, O_6^{(1)}$ of the action of the group generated by the reflection about $y = x$ on the polytope corresponding to $\mathbb{P}^2$ with $k = 1$.

where

$$V^G := \{v \in V \ : \ g.v \in V \quad \forall g \in G\}.$$

Note that in (44) $V \neq 0$ ranges over all $G$-invariant vector subspaces of $H^0(X, -kK_X)$ (i.e., of any positive dimension). For example, $\mathrm{lct}|-kK_X| = \infty$. From the definition, if $V_1 \subset V_2$ are two such subspaces it suffices to compute $\mathrm{lct}|V_1|$ since

$$\mathrm{lct}|V_1| \leq \mathrm{lct}|V_2|. \tag{45}$$

It is thus natural to define the algebraic counterpart of the $\alpha_{k,G}$-invariant as follows.

**Definition 4.4.** Let $X$ be a Fano manifold and $G \subset \mathrm{Aut}\, X$ be a compact subgroup of the automorphism group. Define

$$\mathrm{glct}_{k,G} := k \inf_{\substack{|V| \subset |-kK_X| \\ V^G = V}} \mathrm{lct}|V|. \tag{46}$$

## 4.3 Characterization of the equivariant Bergman spaces

First, we show that in the toric setting the space $\mathcal{H}_k^{G(H)}$ consists of Kähler potentials induced by Kodaira embeddings of multiples of monomials sections, with the norming constants constant along orbits of $H$.

Denote by

$$O_1^{(k)}, \ldots, O_N^{(k)}, \tag{47}$$

the orbits of $H$ in $k^{-1}M \cap P$ (see Figure 2 for an example).

**Lemma 4.5.** *Let $\{s_{k,u}\}_{u \in k^{-1}M \cap P}$ be the monomial basis* (33) *of $H^0(X, -kL)$. Let $G(H) \subseteq \mathrm{Aut}\, X$* (4) *be a subgroup generated by $(S^1)^n$ and a subgroup $H$ of $\mathrm{Aut}(P)$. Then (recall (42) and (47)),*

$$\mathcal{H}_k^{G(H)} := \mathcal{H}_k^{G(H)}(h_k) = \left\{ k^{-1} \log \sum_{i=1}^N \lambda_i \sum_{u \in O_i^{(k)}} |s_{k,u}|^2_{h_k} \ : \ \lambda_i > 0 \right\}.$$

*Proof.* Recall the natural basis $\{s_{k,u}\}_{u \in P \cap M/k}$ defined by the lattice monomials (33). Given any other basis $\{s_i\}$ for $H^0(X, -kK_X)$, let $A \in GL(E_P(k), \mathbb{C})$ be the change-of-basis matrix, so

$$s_i = A_i^u s_{k,u}, \quad i \in \{1, \ldots, E_P(k)\}$$

(we use the Einstein summation convention).

Let $\varphi \in \mathcal{H}_k^{G(H)}$ and let $\{s_i\}$ be the associated basis (42). Expanding $\varphi$ in terms of the monomials,

$$\begin{aligned} e^{k\varphi} = \sum_i |s_i|^2_{h_k} &= \sum_{u,u' \in P\cap M/k} \sum_{i,j=1}^{E_P(k)} A_i^u \overline{A_j^{u'}} \langle s_{k,u}, s_{k,u'} \rangle_{h_k} \\ &=: |s_{k,0}|^2_{h_k} \sum_{u,u' \in P\cap M/k} c_{u,u'} z^{ku} \bar{z}^{ku'} \end{aligned}$$

for some coefficients $\{c_{u,u'}\}_{u,u' \in P\cap M/k}$. We claim that $\{z^u \bar{z}^{u'}\}_{u,u' \in M}$ are linearly independent. To see that, suppose

$$f(z) = \sum_{u,u' \in M} c_{u,u'} z^u \bar{z}^{u'} = 0$$

with all but finitely many coefficients being 0. We may assume for any $c_{u,u'} \neq 0$, $u, u'$ lie in the positive orthant, by multiplying by $\prod_i |z_i|^2$ to some sufficiently large power. Then

$$c_{u,u'} = \frac{1}{u!u'!} \left. \frac{\partial^{|u|+|u'|}}{\partial^u z \partial^{u'} \bar{z}} \right|_{z=0} f(z) = 0,$$

proving the claim. Now, the subgroup $(S^1)^n < G(H)$ acts on the open orbit $(\mathbb{C}^*)^n$ by

$$(\beta_1, \ldots, \beta_n).(z_1, \ldots, z_n) = (e^{\sqrt{-1}\beta_1} z_1, \ldots, e^{\sqrt{-1}\beta_n} z_n). \tag{48}$$

So by $(S^1)^n$-invariance and Lemma 3.1,

$$\beta.e^{k\varphi} = |s_{k,0}|^2_{h_k} \sum_{u,u' \in P\cap M/k} c_{u,u'} e^{\sqrt{-1}\langle \beta, k(u-u') \rangle} z^{ku} \bar{z}^{ku'} = |s_{k,0}|^2_{h_k} \sum_{u,u' \in P\cap M/k} c_{u,u'} z^{ku} \bar{z}^{ku'} = e^{k\varphi},$$

for any $\beta \in (S^1)^n = (\mathbb{R}/2\pi\mathbb{Z})^n$. By comparing the coefficients and using the claim we just demonstrated, it follows that for $e^{k\varphi}$ to be $(S^1)^n$-invariant we must have $c_{u,u'} = 0$ whenever $u \neq u'$ (the converse is also true, of course).

Moreover, we also require $e^{k\varphi}$ to be invariant under the action of $H$, i.e., $c_{u,u} = c_{\sigma u, \sigma u}$ for any $\sigma \in H$. In conclusion, let $O_1^{(k)}, \ldots, O_N^{(k)}$ be the orbits of the action of $H$ on $P \cap M/k$. Then

$$e^{k\varphi} = \sum_{i=1}^N \lambda_i \sum_{u \in O_i^{(k)}} |s_{k,u}|^2_{h_k},$$

for some coefficients $\{\lambda_i\}_{i=1}^N$. Thus (42) simplifies to

$$\mathcal{H}_k^{G(H)}(h) = \left\{ \varphi = \frac{1}{k} \log \sum_{i=1}^N \lambda_i \sum_{u \in O_i^{(k)}} |s_{k,u}|^2_{h_k} \, : \, \lambda_i > 0 \right\},$$

as claimed. ∎

### 4.4 Replacing a basis of sections by an orbit of a section

The next result generalizes [27, Proposition 2.1] to the equivariant setting using Lemma 4.5.

**Proposition 4.6.** *For a subgroup $G(H) \subseteq \operatorname{Aut} X$ (4) generated by $(S^1)^n$ and a subgroup $H$ of* $\operatorname{Aut}(P)$*,*

$$\alpha_{k,G(H)} = \sup \left\{ c > 0 \, : \, \int_X \Big( \sum_{\sigma \in H} |s_{k,\sigma u}|^2_{h_k} \Big)^{-\frac{c}{k}} d\mu < \infty, \ \forall\, u \in P \cap M/k \right\}. \tag{49}$$

*Proof. Step 1: estimate a basis-type element using the worst orbit present in its expansion.* Assuming $\lambda_N = \max_{i=1,\dots,N} \lambda_i$, we have

$$\sup \varphi \le \sup \left( \frac{1}{k} \log \sum_{i=1}^{N} \lambda_N \sum_{u \in O_i^{(k)}} |s_{k,u}|^2_{h_k} \right) = \frac{1}{k} \log \lambda_N,$$

since $1/h_k = \sum_{u \in P \cap M/k} |s_{k,u}|^2$. If $c > 0$ is such that for each $i \in \{1, \dots, N\}$,

$$\int_X \left( \sum_{u \in O_i^{(k)}} |s_{k,u}|^2_{h_k} \right)^{-\frac{c}{k}} d\mu \le C_c,$$

then for $\varphi \in \mathcal{H}_k^{G(H)}$ (using Lemma 4.5),

$$\begin{aligned} \int_X e^{-c(\varphi - \sup \varphi)} d\mu &= e^{c \sup \varphi} \int_X \left( \sum_{i=1}^{N} \lambda_i \sum_{u \in O_i^{(k)}} |s_{k,u}|^2_{h_k} \right)^{-\frac{c}{k}} d\mu \\ &\le e^{\frac{c}{k} \log \lambda_N} \int_X \left( \lambda_N \sum_{u \in O_N^{(k)}} |s_{k,u}|^2_{h_k} \right)^{-\frac{c}{k}} d\mu \le C_c. \end{aligned}$$

*Step 2: estimate an orbit-type element using an approximation by degenerating basis-type elements.*

Conversely, assume $\alpha > 0$ is such that for any $\varphi \in \mathcal{H}_k^{G(H)}$,

$$\int_X e^{-c(\varphi - \sup \varphi)} d\mu \le C_c.$$

Since, for any $\ell = 1, \dots, N$,

$$\left( \sum_{u \in O_\ell^{(k)}} |s_{k,u}|^2_{h_k} \right)^{-\frac{c}{k}} = \lim_{\lambda_i \to 0, i \ne \ell} \left( \sum_{i=1}^{N} \lambda_i \sum_{u \in O_i^{(k)}} |s_{k,u}|^2_{h_k} \right)^{-\frac{c}{k}},$$

where $\lambda_\ell = 1$, by Fatou's Lemma [17, Lemma 2.18],

$$\int_X \left( \sum_{u \in O_\ell^{(k)}} |s_{k,u}|^2_{h^k} \right)^{-\frac{c}{k}} d\mu \le \liminf_{\lambda_i \to 0, i \ne \ell} e^{-\alpha \sup \varphi_\lambda} e^{\alpha \sup \varphi_\lambda} \int_X \left( \sum_{i=1}^{N} \lambda_i \sum_{u \in O_i^{(k)}} |s_{k,u}|^2_{h_k} \right)^{-\frac{\alpha}{k}} d\mu$$

$$
\begin{aligned}
&= \left(\sup \sum_{u\in O_\ell^{(k)}} |s_{k,u}|^2_{h_k}\right)^{-\frac{c}{k}} \liminf_{\lambda_i\to 0, i\neq \ell} \int_X e^{-c(\varphi_\lambda - \sup\varphi_\lambda)} d\mu \\
&\leq \left(\sup \sum_{u\in O_\ell^{(k)}} |s_{k,u}|^2_{h_k}\right)^{-\frac{c}{k}} C_c,
\end{aligned}
$$

where

$$
\varphi_\lambda = \frac{1}{k}\log \sum_{i=1}^N \lambda_i \sum_{u\in O_i^{(k)}} |s_{k,u}|^2_{h_k}.
$$

Thus, we have shown that

$$
\begin{aligned}
\alpha_{k,G(H)} &= \sup\left\{ c>0\,:\, \int_X \left(\sum_{u\in O_i^{(k)}} |s_{k,u}|^2_{h_k}\right)^{-\frac{c}{k}} d\mu < \infty,\ \forall\ i\right\} \qquad (50)\\
&= \sup\left\{ c>0\,:\, \int_X \left(\sum_{\sigma\in H} |s_{k,\sigma u}|^2_{h_k}\right)^{-\frac{c}{k}} d\mu < \infty,\ \forall\ u\in P\cap M/k\right\},
\end{aligned}
$$

proving (49). ☐

**Corollary 4.7.** *Fix $k\in\mathbb{N}$ and let $O_1^{(k)},\ldots,O_N^{(k)}$ be the orbits of the action of $H$ on $k^{-1}M\cap P$. Then*

$$
\alpha_{k,G(H)} = \min_{1\leq i\leq N} \sup\left\{ c>0\,:\, \int_X \left(\sum_{u\in O_i^{(k)}} |s_{k,u}|^2_{h_k}\right)^{-\frac{c}{k}} d\mu < \infty\right\}.
$$

*Proof.* By (50),

$$
\begin{aligned}
\alpha_{k,G(H)} &= \sup\left\{ c>0\,:\, \int_X \left(\sum_{u\in O_i^{(k)}} |s_{k,u}|^2_{h_k}\right)^{-\frac{c}{k}} d\mu < \infty,\ \forall\ i\right\}\\
&= \min_{1\leq i\leq N} \sup\left\{ c>0\,:\, \int_X \left(\sum_{u\in O_i^{(k)}} |s_{k,u}|^2_{h_k}\right)^{-\frac{c}{k}} d\mu < \infty\right\}.
\end{aligned}
$$

☐

We can now answer affirmatively Problem 1.2 in our setting.

*Proof of Proposition 1.3.* Let $\left|V_{O_i^{(k)}}\right|$ denote the linear system generated by $\{(s_{k,u})\ :\ \ u\in O_i^{(k)}\}$ (recall (47)). By (43) and (49),

$$
\alpha_{k,G(H)} = k\inf_i \operatorname{lct}\left|V_{O_i^{(k)}}\right|.
$$

It remains to show

$$\operatorname{glct}_{k,G(H)} = k \inf_i \operatorname{lct}\left|V_{O_i^{(k)}}\right|.$$

By (45), it suffices to restrict to irreducible $G(H)$-invariant linear systems (cf. [23, p. 146]). Now, any $(S^1)^n$-invariant linear system $|V|$ is spanned by monomials (see Lemma 4.8 below). By irreducibility, this means $|V|$ is spanned by the monomials coming from some $H$-orbit $O_i$. i.e., $|V| = \left|V_{O_i^{(k)}}\right|$. □

**Lemma 4.8.** *Let $V \subseteq H^0(X, -kK_X)$ be a complex vector subspace invariant under $(S^1)^n$. Then there exists a unique subset $\mathcal{F} \subset P \cap M/k$ such that*

$$V = \operatorname{span}\{s_{k,u}\}_{u\in\mathcal{F}}.$$

*Proof.* It suffices to prove that any section in $V$ is generated by monomials in $V$. That is, given any section

$$s = \sum_{u\in P\cap M/k} a_u s_{k,u} \in V,$$

if $a_u \neq 0$, then $s_{k,u} \in V$.

By $(S^1)^n$-invariance, for any $\beta \in (S^1)^n$ and $s \in V$ also $\beta.s \in V$ (recall (48)), i.e.,

$$\beta.s = \sum_{u\in P\cap M/k} a_u \beta.s_{k,u} = \sum_{u\in P\cap M/k} a_u e^{\sqrt{-1}\langle\beta,ku\rangle} s_{k,u} \in V.$$

Thus for any $u_0 \in P \cap M/k$,

$$\begin{aligned}
\int_{(S^1)^n} e^{-\sqrt{-1}\langle\beta,ku_0\rangle}\beta.s\, d\beta &= \sum_{u\in P\cap M/k} a_u s_{k,u} \int_{(S^1)^n} e^{\sqrt{-1}\langle\beta,ku-ku_0\rangle} d\beta \\
&= \sum_{u\in P\cap M/k} a_u s_{k,u} \cdot (2\pi)^n \delta_{u-u_0} \\
&= (2\pi)^n a_{u_0} s_{k,u_0} \in V.
\end{aligned}$$

If $a_{u_0} \neq 0$, then $s_{k,u_0} \in V$. This completes the proof. □

**Corollary 4.9.**

$$\alpha_{G(H)} = \inf_k \alpha_{k,G(H)} = \sup\left\{c > 0 \,:\, \int_X \left(\sum_{\sigma\in H} |s_{k,\sigma u}|^2_{h_k}\right)^{-\frac{c}{k}} d\mu < \infty,\ \forall\, u \in P\cap M/k,\ k\in\mathbb{N}\right\}. \tag{51}$$

*Proof.* By Demailly's theorem (44) [12, (A.1)], (46), and Proposition 1.3,

$$\alpha_{G(H)} = \inf_{k\in\mathbb{N}} \operatorname{glct}_{k,G(H)} = \inf_{k\in\mathbb{N}} \alpha_{k,G(H)}.$$

By (49),

$$\begin{aligned}
\alpha_{G(H)} &= \inf_{k\in\mathbb{N}} \sup\left\{c > 0 \,:\, \int_X \left(\sum_{\sigma\in H} |s_{k,\sigma u}|^2_{h_k}\right)^{-\frac{c}{k}} d\mu < \infty,\ \forall\, u \in P\cap M/k\right\} \\
&= \sup\left\{c > 0 \,:\, \int_X \left(\sum_{\sigma\in H} |s_{k,\sigma u}|^2_{h_k}\right)^{-\frac{c}{k}} d\mu < \infty,\ \forall\, u \in P\cap M/k,\ k\in\mathbb{N}\right\}.
\end{aligned}$$

□

**Remark 4.10.** Note that here it would not have been enough to invoke Tian's theorem [31, Proposition 6.1], and it was necessary to invoke Demailly's theorem [12, (A.1)]. The reason is that the $\alpha_{k,G}$-invariants are defined by requiring certain integrals to be merely finite, i.e., bounded, but with a constant possibly depending on $k$. Tian's theorem says that $\mathcal{H}^G$ is approximated by $\mathcal{H}_k^G$, but in approximating $\varphi \in \mathcal{H}^G$ by a sequence $\varphi_k \in \mathcal{H}_k^G$ it might happen that the integrals $\int_X e^{-c(\varphi_k - \sup \varphi_k)}\omega^n$ blow up as $k$ tends to infinity. Demailly precludes that from happening by using the Demailly–Kollár lower semi-continuity of complex singularity exponents.

# 5 The case of divisible $k$

We ultimately improve on the results of this section, but we include them since they serve to emphasize the difficulties that still need to be dealt with (see Remark 5.6).

**Proposition 5.1.** *Let $G(H) \subseteq \operatorname{Aut} X$ (4) be a subgroup generated by $(S^1)^n$ and a subgroup $H$ of* $\operatorname{Aut}(P)$. *For $k, \ell \in \mathbb{N}$, $\alpha_{k,G(H)} \geq \alpha_{k\ell,G(H)}$.*

*Proof.* Since $h_{k\ell}$ and $h_k^\ell$ (38) are both smooth metrics on $-k\ell K_X$, by compactness of $X$, they are equivalent. We also use the following lemma.

**Lemma 5.2.** *Let $a_1, \ldots, a_N \geq 0$. Then for $\ell \in \mathbb{N}$,*

$$\sum_{i=1}^N a_i^\ell \leq \left(\sum_{i=1}^N a_i\right)^\ell \leq N^{\ell-1} \sum_{i=1}^N a_i^\ell.$$

*Proof.* The first inequality follows by expanding $(\sum_{i=1}^N a_i)^\ell$. For the second inequality, consider the function $f(x) := x^\ell$ on $[0, +\infty)$. Since $f$ is convex, by Jensen's inequality,

$$f\left(\frac{1}{N}\sum_{i=1}^N a_i\right) \leq \frac{1}{N}\sum_{i=1}^N f(a_i),$$

i.e., $\left(\sum_{i=1}^N a_i\right)^\ell \leq N^{\ell-1} \sum_{i=1}^N a_i^\ell.$ □

By Proposition 4.6, and since $M/k\ell \subset M/k$,

$$\begin{aligned}
\alpha_{k\ell,G(H)} &= \sup\left\{c>0 \,:\, \int_X \Big(\sum_{\sigma\in H} |s_{k\ell,\sigma u}|^2_{h_{k\ell}}\Big)^{-\frac{c}{k\ell}} d\mu < \infty,\ \forall\, u \in P \cap M/k\ell\right\} \\
&\leq \sup\left\{c>0 \,:\, \int_X \Big(\sum_{\sigma\in H} |s_{k\ell,\sigma u}|^2_{h_{k\ell}}\Big)^{-\frac{c}{k\ell}} d\mu < \infty,\ \forall\, u \in P \cap M/k\right\} \\
&= \sup\left\{c>0 \,:\, \int_X \Big(\sum_{\sigma\in H} \left|s_{k,\sigma u}^{\otimes\ell}\right|^2_{h_k^\ell}\Big)^{-\frac{c}{k\ell}} d\mu < \infty,\ \forall\, u \in P \cap M/k\right\} \\
&= \sup\left\{c>0 \,:\, \int_X \Big(\sum_{\sigma\in H} |s_{k,\sigma u}|^{2\ell}_{h_k}\Big)^{-\frac{c}{k\ell}} d\mu < \infty,\ \forall\, u \in P \cap M/k\right\} \\
&= \sup\left\{c>0 \,:\, \int_X \Big(\sum_{\sigma\in H} |s_{k,\sigma u}|^2_{h_k}\Big)^{-\frac{c}{k}} d\mu < \infty,\ \forall\, u \in P \cap M/k\right\}
\end{aligned}$$

$$= \alpha_{k,G(H)},$$

where Lemma 5.2 was invoked in the penultimate equality. $\square$

**Proposition 5.3.** *Let $G(H) \subseteq \operatorname{Aut} X$ (4) be a subgroup generated by $(S^1)^n$ and a subgroup $H$ of* $\operatorname{Aut}(P)$. *There exists $k_0 \in \mathbb{N}$ such that $\alpha_{K,G(H)} = \alpha_{G(H)}$ for all $K$ divisible by $k_0$.*

**Remark 5.4.** In fact, the proof will show that $k_0$ is determined by the fan as follows: let $u_0 \in P^H$ attain $\sup_{u \in P^H} \max_i \langle u, v_i \rangle$. Since $P^H$ is a convex polytope $u_0$ will be a vertex of $P^H$, i.e., cut out by $P^H$ and the supporting hyperplanes of $P$ containing $u_0$. The equations defining $P^H$ are determined by $H \subset GL(M)$ hence are linear equations with integer coefficients. So are the equations cutting out $\partial P$. It follows that $u_0$ is a rational point, i.e., $u_0 \in M/k_0$ for some $k_0$. Let $k_0$ be the smallest such positive integer. The fact that $u_0$ is a rational point is originally due to Song [28, p. 1257] who proved that $\alpha_{k_0,G(\operatorname{Aut} P)} = \alpha_{G(\operatorname{Aut} P)}$.

*Proof.* We can further simplify (51). Recall (6). By the geometric-arithmetic mean inequality,

$$\left( \frac{1}{|H|} \sum_{\sigma \in H} |s_{k,\sigma u}|^2_{h_k} \right)^{-\frac{\alpha}{k}} \leq \prod_{\sigma \in H} |s_{k,\sigma u}|_{h_k}^{-\frac{2\alpha}{|H|k}} = \left| s_{|H|k,\pi_H(u)} \right|_{h_{|H|k}}^{-\frac{2\alpha}{|H|k}}$$
$$= \left( \frac{1}{|H|} \sum_{\sigma \in H} \left| s_{|H|k,\sigma\pi_H(u)} \right|^2_{h_{|H|k}} \right)^{-\frac{\alpha}{|H|k}},$$

with the last equality since $\pi_H(u)$ is fixed by $H$ (recall (5)–(6)) so each term in the sum is identical. Therefore the supremum in (51) is unchanged if restricted to those $u$ fixed by $H$. Thus, using Lemma 5.5 (proven below), (51) simplifies to (recall (5))

$$\begin{aligned}
\alpha_{G(H)} &= \sup \left\{ c > 0 \ : \ \int_X |s_{k,u}|_{h_k}^{-\frac{2c}{k}} \, d\mu < \infty, \ \forall \ u \in P^H \cap \frac{1}{k} M, \ k \in \mathbb{N} \right\} \\
&= \inf \left\{ k \cdot \operatorname{lct}(s_{k,u}) \ : \ \ u \in P^H \cap \frac{1}{k} M, \ k \in \mathbb{N} \right\} \\
&= \inf_{k \in \mathbb{N}} \inf_{u \in P^H \cap \frac{1}{k} M} \frac{1}{\max_i \langle u, v_i \rangle + 1} \\
&= \inf_{u \in P^H} \frac{1}{\max_i \langle u, v_i \rangle + 1} \\
&= \frac{1}{\max_i \langle u_0, v_i \rangle + 1},
\end{aligned}$$

where we used the notation of Remark 5.4. By that same Remark, $u_0 \in P^H \cap \frac{1}{K} M$ whenever $K \in \mathbb{N}$ is divisible by $k_0$. In particular, by (49), and using Lemma 5.5 again,

$$\begin{aligned}
\alpha_{K,G(H)} &\leq \sup \left\{ c > 0 \ : \ \int_X |s_{K,u_0}|_{h_K}^{-\frac{2c}{K}} \, d\mu < \infty \right\} \\
&= K \cdot \operatorname{lct}(s_{K,u_0}) \\
&= \min_i \frac{1}{\langle u_0, v_i \rangle + 1} \\
&= \alpha_{G(H)}.
\end{aligned}$$

Since by definition $\alpha_{G(H)} \leq \alpha_{K,G(H)}$, equality is achieved and $\alpha_{K,G(H)} = \alpha_{G(H)}$ for all $K$ divisible by $k_0$. $\square$

**Lemma 5.5.** *For $u \in P \cap k^{-1}M$, $\mathrm{lct}(s_{k,u}) = \dfrac{1}{k}\dfrac{1}{1+\max_i\langle u, v_i\rangle}$ (recall (20) and (33)).*

*Proof.* This is well-known (see, e.g., [6, Corollary 7.4], [23, Theorem 4.1]). It is also a consequence of Proposition 6.8 proven below (put $\mathcal{F} = \{u\}$). ☐

**Remark 5.6.** According to Proposition 5.3, the sequence $\{\alpha_{k,G(H)}\}_{k\in\mathbb{N}}$ is constant (equal to $\alpha_{G(H)}$) along the subsequence $k_0, 2k_0, \ldots$. Proposition 5.3 does not yield information for all $k \in \mathbb{N}$ unless $k_0 = 1$, and examples show (see §8) that oftentimes $k_0 > 1$. Moreover, even though the sequence also satisfies $\alpha_{k,G(H)} \geq \alpha_{k\ell,G(H)} \geq \alpha_{G(H)}$ for any $k, \ell \in \mathbb{N}$ (Proposition 5.1), these facts combined are still not enough to conclude Tian's conjecture without further work. For instance, the sequence

$$a_k := \begin{cases} \alpha_{G(H)}, & k \text{ even}, \\ \alpha_{G(H)} + k^{-1}, & k \text{ odd}, \end{cases}$$

satisfies these requirements for $k_0 = 2$ (and also satisfy $\lim_k a_k = \alpha_{G(H)}$). Perhaps a more natural sequence that satisfies all the requirements and even for the $k_0$ of Remark 5.4 is

$$a_k := \inf\left\{k \cdot \mathrm{lct}(s_{k,u}) \;:\; u \in P^H \cap \frac{1}{k}M\right\},$$

with the proof of Proposition 5.3 showing that $a_{k_0\ell} = a_{k_0} = \alpha_{G(H)}$ for $\ell \in \mathbb{N}$ but possibly $a_k > \alpha_{G(H)}$ for $k$ not divisible by $k_0$. Thus, we are led to develop more refined estimates that are the topic of the next section.

# 6 Estimating singularities associated to orbits

## 6.1 Real singularity exponents and support functions

In this subsection we develop a key new technical estimate that expresses real singularity exponents associated to collections of toric monomials in terms of support functions.

**Definition 6.1.** For non-empty finite sets $\mathcal{F}, \mathcal{U} \subseteq \mathbb{R}^n$ and $k \in \mathbb{N}$,

$$c_k(\mathcal{F},\mathcal{U}) := \sup\left\{c \in (0,1) \;:\; \int_{\mathbb{R}^n} \frac{\left(\sum_{u\in\mathcal{F}} e^{\langle ku,x\rangle}\right)^{-\frac{c}{k}}}{\left(\sum_{u\in\mathcal{U}} e^{\langle ku,x\rangle}\right)^{\frac{1-c}{k}}} dx < \infty\right\}.$$

**Proposition 6.2.** *For $\mathcal{F},\mathcal{U} \subseteq \mathbb{R}^n$ non-empty finite sets with $0 \in \mathrm{int}\,\mathrm{co}\,\mathcal{U}$ (recall (10)),*

$$\begin{aligned} c_k(\mathcal{F},\mathcal{U}) &= \sup\left\{c \in (0,1) \;:\; 0 \in (1-c)\,\mathrm{co}\,\mathcal{U} + c\,\mathrm{co}\,\mathcal{F}\right\} \\ &= \sup\left\{c \in (0,1) \;:\; \left(-\frac{c}{1-c}\,\mathrm{co}\,\mathcal{F}\right) \cap \mathrm{co}\,\mathcal{U} \neq \emptyset\right\} > 0. \end{aligned}$$

*In particular, $c_k(\mathcal{F},\mathcal{U})$ is independent of $k$.*

*Proof.* Using polar coordinates $x = r\nu$ and $dx = r^{n-1}dr \wedge dS(\nu)$ with $r \in \mathbb{R}_+$, $\nu \in S^{n-1}(1) := \{x \in \mathbb{R}^n \,:\, |x| = 1\}$,

$$\int_{\mathbb{R}^n} \frac{\left(\sum_{u\in\mathcal{F}} e^{\langle ku,x\rangle}\right)^{-\frac{c}{k}}}{\left(\sum_{u\in\mathcal{U}} e^{\langle ku,x\rangle}\right)^{\frac{1-c}{k}}} dx = \int_{S^{n-1}(1)} f_c(\nu)\,dS(\nu), \tag{52}$$

where $f_c : S^{n-1}(1) \to \mathbb{R}_+$ is defined by

$$f_c(\nu) := \int_0^\infty \frac{\left(\sum_{u\in\mathcal{F}} e^{kr\langle u,\nu\rangle}\right)^{-\frac{c}{k}}}{\left(\sum_{u\in\mathcal{U}} e^{kr\langle u,\nu\rangle}\right)^{\frac{1-c}{k}}} r^{n-1} dr.$$

Notice that for any finite set $\mathcal{A} \subset \mathbb{R}^n$ and $r \in \mathbb{R}_+$ (recall (12)),

$$e^{krh_{\mathcal{A}}(\nu)} \le \sum_{u\in\mathcal{A}} e^{kr\langle u,\nu\rangle} \le |\mathcal{A}| e^{krh_{\mathcal{A}}(\nu)}.$$

Hence for $c \in (0,1)$,

$$\frac{|\mathcal{F}|^{-\frac{c}{k}}}{|\mathcal{U}|^{\frac{1-c}{k}}} e^{-krg_c(\nu)} \le \frac{\left(\sum_{u\in\mathcal{F}} e^{kr\langle u,\nu\rangle}\right)^{-\frac{c}{k}}}{\left(\sum_{u\in\mathcal{U}} e^{kr\langle u,\nu\rangle}\right)^{\frac{1-c}{k}}} \le e^{-krg_c(\nu)}, \tag{53}$$

where

$$\begin{aligned} g_c(x) &:= c \max_{u\in\mathcal{F}} \langle u,x\rangle + (1-c)\max_{u\in\mathcal{U}}\langle u,x\rangle \\ &= h_{c\mathcal{F}+(1-c)\mathcal{U}}(x) \\ &= c \max_{u\in\operatorname{co}\mathcal{F}} \langle u,x\rangle + (1-c)\max_{u\in\operatorname{co}\mathcal{U}}\langle u,x\rangle \\ &= ch_{\operatorname{co}\mathcal{F}}(x) + (1-c)h_{\operatorname{co}\mathcal{U}}(x) = h_{c\operatorname{co}\mathcal{F}+(1-c)\operatorname{co}\mathcal{U}}(x), \end{aligned} \tag{54}$$

where $A + B := \{x + y : x \in A, y \in B\}$ is the Minkowski sum.

We need the following property of support functions.

**Claim 6.3.** *Let $\mathcal{A} \subset \mathbb{R}^n$ be a nonempty finite set. Then $h_{\mathcal{A}}|_{S^{n-1}(1)} > 0$ if and only if $0 \in \operatorname{int}\operatorname{co}\mathcal{A}$.*

*Proof.* Note first that

$$h_{\mathcal{A}} = h_{\operatorname{co}\mathcal{A}} : \tag{55}$$

$h_{\mathcal{A}} \le h_{\operatorname{co}\mathcal{A}}$ since $\mathcal{A} \subset \operatorname{co}\mathcal{A}$, while if $h_{\operatorname{co}\mathcal{A}}(y) = \langle a(y), y\rangle$ for some $a(y) \in \operatorname{co}\mathcal{A}$, and writing $a(y) = \sum_{i=1}^{|\mathcal{A}|} \lambda_i a_i$ with $\sum_{i=1}^{|\mathcal{A}|} \lambda_i = 1$ and $\lambda_i \ge 0$, where $\mathcal{A} = \{a_i\}$ yields $h_{\operatorname{co}\mathcal{A}}(y) = \sum_{i=1}^{|\mathcal{A}|} \lambda_i \langle a_i, y\rangle \le \sum_{i=1}^{|\mathcal{A}|} \lambda_i h_{\mathcal{A}}(y) = h_{\mathcal{A}}(y)$.

Next, if $0 \in \operatorname{int}\operatorname{co}\mathcal{A}$ then $\epsilon B_2^n \subset \operatorname{co}\mathcal{A}$ where $B_2^n := \{x \in \mathbb{R}^n : |x| \le 1\}$ and $h_{\operatorname{co}\mathcal{A}} \ge h_{\epsilon B_2^n} = \epsilon h_{B_2^n} = \epsilon$ when restricted to $S^{n-1}(1)$.

Conversely, if $0 \notin \operatorname{int}\operatorname{co}\mathcal{A}$ then by convexity of $\operatorname{co}\mathcal{A}$ there exists a hyperplane $H$ passing through a boundary point of $\operatorname{co}\mathcal{A}$ so that $\operatorname{co}\mathcal{A}$ lies on one side of it and 0 lies on the other side of it. If $\nu \in S^{n-1}(1)$ is normal to $H$ and points toward the side not containing $\operatorname{co}\mathcal{A}$ then $h_{\operatorname{co}\mathcal{A}}(\nu) \le 0$. $\square$

**Claim 6.4.** *Let $\mathcal{A} \subset \mathbb{R}^n$ be a nonempty finite set. Then $h_{\mathcal{A}}|_{S^{n-1}(1)}$ is somewhere negative if and only if $0 \in \operatorname{int}(\mathbb{R}^n \setminus \operatorname{co}\mathcal{A}) = \mathbb{R}^n \setminus \operatorname{co}\mathcal{A}$.*

*Proof.* Suppose $0 \notin \operatorname{int}(\mathbb{R}^n \setminus \operatorname{co}\mathcal{A}) = \mathbb{R}^n \setminus \operatorname{co}\mathcal{A}$. Equivalently $0 \in \operatorname{co}\mathcal{A}$. Then $h_{\mathcal{A}} \ge h_{\{0\}} = 0$ so $h_{\mathcal{A}}$ is nowhere negative.

Conversely, if $0 \notin \operatorname{co}\mathcal{A}$ then by convexity of $\operatorname{co}\mathcal{A}$ there exists a hyperplane $H$ passing through the origin so that $\operatorname{co}\mathcal{A}$ lies *strictly* on one side. Let $\nu \in S^{n-1}(1)$ be a unit normal to $H$ that points toward the side not containing $\operatorname{co}\mathcal{A}$. By the strictness mentioned above and compactness of $\operatorname{co}\mathcal{A}$, for some small $\epsilon > 0$, $\operatorname{co}\mathcal{A} + \epsilon\nu$ still lies on the same side of $H$ as $\operatorname{co}\mathcal{A}$. Hence, as in the proof of Claim 6.3, $h_{\operatorname{co}\mathcal{A}+\epsilon\nu}(\nu) \le 0$. In other words,

$$h_{\operatorname{co}\mathcal{A}}(\nu) = h_{\operatorname{co}\mathcal{A}+\epsilon\nu}(\nu) - \epsilon|\nu|^2 \le -\epsilon,$$

as claimed. $\square$

**Corollary 6.5.** *For $\mathcal{F}, \mathcal{U} \subseteq \mathbb{R}^n$ non-empty finite sets with $0 \in \operatorname{int} \operatorname{co} \mathcal{U}$ (recall (10)),*

$$\sup \left\{ c \in (0,1) \, : \, \left( -\frac{c}{1-c} \operatorname{co} \mathcal{F} \right) \cap \operatorname{co} \mathcal{U} \neq \emptyset \right\} = \begin{cases} 1, & 0 \in \operatorname{co} \mathcal{F}, \\ \left[ 1 - \min\limits_{S^{n-1}(1)} \dfrac{h_{\mathcal{F}}}{h_{\mathcal{U}}} \right]^{-1}, & 0 \notin \operatorname{co} \mathcal{F}. \end{cases} \tag{56}$$

*Proof.* By assumption $0 \in \operatorname{int} \operatorname{co} \mathcal{U}$, so by Claim 6.3, $h_{\mathcal{U}}|_{S^{n-1}(1)} > 0$. By (54),

$$g_c = h_{\operatorname{co} \mathcal{U}} \cdot \left( 1 - c \left( 1 - \frac{h_{\operatorname{co} \mathcal{F}}}{h_{\operatorname{co} \mathcal{U}}} \right) \right). \tag{57}$$

*Case 1:* $0 \in \operatorname{co} \mathcal{F}$. If $0 \in \operatorname{co} \mathcal{F}$, i.e., $h_{\operatorname{co} \mathcal{F}} \geq 0$, then $g_c \geq h_{\operatorname{co} \mathcal{U}} \cdot (1-c) > 0$ for any $c \in (0,1)$. By (57) and Claim 6.3 then $0 \in \operatorname{int}(c \operatorname{co} \mathcal{F} + (1-c) \operatorname{co} \mathcal{U}) \subseteq c \operatorname{co} \mathcal{F} + (1-c) \operatorname{co} \mathcal{U}$ and so (56) holds in this case.

*Case 2:* $0 \notin \operatorname{co} \mathcal{F}$. By assumption and Claim 6.4, $h_{\operatorname{co} \mathcal{F}} < 0$ somewhere, by continuity/compactness let $\nu_0$ be a minimizer of the function

$$\frac{h_{\operatorname{co} \mathcal{F}}}{h_{\operatorname{co} \mathcal{U}}}$$

on $S^{n-1}(1)$. In particular,

$$\frac{h_{\operatorname{co} \mathcal{F}}(\nu_0)}{h_{\operatorname{co} \mathcal{U}}(\nu_0)} = \min_{S^{n-1}(1)} \frac{h_{\mathcal{F}}}{h_{\mathcal{U}}} < 0.$$

Set

$$c(\mathcal{F}, \mathcal{U}) := \left[ 1 - \frac{h_{\operatorname{co} \mathcal{F}}(\nu_0)}{h_{\operatorname{co} \mathcal{U}}(\nu_0)} \right]^{-1} = \left[ 1 - \min_{S^{n-1}(1)} \frac{h_{\mathcal{F}}}{h_{\mathcal{U}}} \right]^{-1}.$$

First, if $c \in (0, c(\mathcal{F}, \mathcal{U}))$, for all $\nu \in S^{n-1}(1)$,

$$\begin{aligned} g_c(\nu) &= h_{\operatorname{co} \mathcal{U}}(\nu) \left( 1 - c \left( 1 - \frac{h_{\operatorname{co} \mathcal{F}}(\nu)}{h_{\operatorname{co} \mathcal{U}}(\nu)} \right) \right) \\ &\geq h_{\operatorname{co} \mathcal{U}}(\nu) \left( 1 - c \left( 1 - \frac{h_{\operatorname{co} \mathcal{F}}(\nu_0)}{h_{\operatorname{co} \mathcal{U}}(\nu_0)} \right) \right) \\ &= h_{\operatorname{co} \mathcal{U}}(\nu) \left( 1 - \frac{c}{c(\mathcal{F}, \mathcal{U})} \right) > 0. \end{aligned}$$

Thus, by (54) and Claim 6.3, $0 \in \operatorname{int}(c \operatorname{co} \mathcal{F} + (1-c) \operatorname{co} \mathcal{U}) \subseteq c \operatorname{co} \mathcal{F} + (1-c) \operatorname{co} \mathcal{U}$; in particular,

$$\left( - \frac{c}{1-c} \operatorname{co} \mathcal{F} \right) \cap \operatorname{co} \mathcal{U} \neq \emptyset.$$

So,

$$\sup \left\{ c \in (0,1) \, : \, \left( - \frac{c}{1-c} \operatorname{co} \mathcal{F} \right) \cap \operatorname{co} \mathcal{U} \neq \emptyset \right\} \geq c(\mathcal{F}, \mathcal{U}).$$

Second, if $c \in (c(\mathcal{F}, \mathcal{U}), 1)$,

$$\begin{aligned} g_c(\nu_0) &= h_{\operatorname{co} \mathcal{U}}(\nu_0) \left( 1 - c \left( 1 - \frac{h_{\operatorname{co} \mathcal{F}}(\nu_0)}{h_{\operatorname{co} \mathcal{U}}(\nu_0)} \right) \right) \\ &= h_{\operatorname{co} \mathcal{U}}(\nu_0) \left( 1 - \frac{c}{c(\mathcal{F}, \mathcal{U})} \right) < 0. \end{aligned}$$

Thus, by Claim 6.4, $0 \in \mathbb{R}^n \setminus (c \operatorname{co} \mathcal{F} + (1-c) \operatorname{co} \mathcal{U})$, equivalently,

$$\Big( -\frac{c}{1-c} \operatorname{co} \mathcal{F} \Big) \cap \operatorname{co} \mathcal{U} = \emptyset,$$

and

$$\sup \Big\{ c \in (0,1) \, : \, \Big( -\frac{c}{1-c} \operatorname{co} \mathcal{F} \Big) \cap \operatorname{co} \mathcal{U} \neq \emptyset \Big\} \le c(\mathcal{F}, \mathcal{U}).$$

Thus, also in this case (56) holds. □

The following corollary follows from the proof of Corollary 6.5.

**Corollary 6.6.** *For $\mathcal{F}, \mathcal{U} \subseteq \mathbb{R}^n$ non-empty finite sets with $0 \in \operatorname{int} \operatorname{co} \mathcal{U}$ (recall (10)), let*

$$\begin{aligned} c(\mathcal{F}, \mathcal{U}) : &= \sup \Big\{ c \in (0,1) \, : \, \Big( -\frac{c}{1-c} \operatorname{co} \mathcal{F} \Big) \cap \operatorname{co} \mathcal{U} \neq \emptyset \Big\} \\ &= \sup \big\{ c \in (0,1) \, : \, 0 \in (1-c) \operatorname{co} \mathcal{U} + c \operatorname{co} \mathcal{F} \big\}. \end{aligned}$$

*Recall* (54). *If $c < c(\mathcal{F}, \mathcal{U})$, then $g_c > 0$; if $c > c(\mathcal{F}, \mathcal{U})$, then $g_c < 0$ somewhere.*

We can now complete the proof of Proposition 6.2. Set

$$c(\mathcal{F}, \mathcal{U}) := \sup \Big\{ c \in (0,1) \, : \, \Big( -\frac{c}{1-c} \operatorname{co} \mathcal{F} \Big) \cap \operatorname{co} \mathcal{U} \neq \emptyset \Big\}.$$

By assumption $0 \in \operatorname{int} \operatorname{co} \mathcal{U}$, thus $c(\mathcal{F}, \mathcal{U}) > 0$. We consider two cases.

*Case 1:* $c \in (0, c(\mathcal{F}, \mathcal{U}))$. By the proof of Corollary 6.5, $g_c|_{S^{n-1}(1)} > 0$. By continuity/compactness it attains its minimum $g_c(\hat{\nu}) > 0$. Therefore,

$$f_c(\nu) \le \int_0^\infty e^{-krg_c(\nu)} r^{n-1} dr \le \int_0^\infty e^{-krg_c(\hat{\nu})} r^{n-1} dr < \infty, \tag{58}$$

so by (52), $c_k(\mathcal{F}, \mathcal{U}) \ge c$, i.e., $c_k(\mathcal{F}, \mathcal{U}) \ge c(\mathcal{F}, \mathcal{U})$.

*Case 2:* $c \in (c(\mathcal{F}, \mathcal{U}), 1)$. By the proof of Corollary 6.5, $g_c(\nu_0) < 0$ for some open neighborhood $U \subset S^{n-1}(1)$. Thus, by (53), for some constant $C = C(c, k, \mathcal{F}, \mathcal{U}) > 0$,

$$f_c(\nu) \ge C \int_0^\infty e^{-krg_c(\nu)} r^{n-1} dr = \infty, \quad \text{for } \nu \in U,$$

so $c_k(\mathcal{F}, \mathcal{U}) \le c$, i.e., $c_k(\mathcal{F}, \mathcal{U}) \le c(\mathcal{F}, \mathcal{U})$.

In conclusion, $c_k(\mathcal{F}, \mathcal{U}) = c(\mathcal{F}, \mathcal{U})$ and Proposition 6.2 is proved. □

## 6.2 Complex singularity exponents

Now, we go back to our setting of estimating complex singularity exponents, where $P$ was a reflexive polytope coming from a fan. For any $k \in \mathbb{N}$ and a non-empty subset $\mathcal{F} \subseteq P \cap M/k$ set

$$c_k(\mathcal{F}) := \sup \left\{ c \in (0,1) \, : \, \int_X \left( \sum_{u \in \mathcal{F}} |s_{k,u}|^2_{h_k} \right)^{-\frac{c}{k}} d\mu < \infty \right\}. \tag{59}$$

**Remark 6.7.** Consider

$$\widetilde{c_k}(\mathcal{F}) := \sup\left\{c \in (0,\infty) \,:\, \int_X \left(\sum_{u\in\mathcal{F}} |s_{k,u}|^2_{h_k}\right)^{-\frac{c}{k}} d\mu < \infty\right\}.$$

Then, by definition (43), $\widetilde{c_k}(\mathcal{F}) = k\,\mathrm{lct}\big|\,\mathrm{span}\{s_{k,u}\}_{u\in\mathcal{F}}\big|$. However, Proposition 6.2 would not be applicable then. The point is that for the proof of Tian's conjecture we need to compute equivariant *global* log canonical thresholds and there will always be 'admissible' invariant linear series for which $k$ times the log canonical threshold is at most 1 (i.e., the "worst" ones will not have $\widetilde{c_k} > 1$), i.e., for which $c_k = \widetilde{c_k}$. This is essentially because our $H$ are linear groups so $\{0\}$ is always an $H$-orbit and $\mathrm{lct}(s_{k,0}) = 1/k = c_k(\{0\}) = \widetilde{c_k}(\{0\})$ (by Lemma 5.5).

**Proposition 6.8.** *For any $k \in \mathbb{N}$ and a non-empty subset $\mathcal{F} \subseteq P \cap M/k$,*

$$c_k(\mathcal{F}) = \sup\left\{c \in (0,1) \,:\, P \cap \Big(-\frac{c}{1-c}\,\mathrm{co}\,\mathcal{F}\Big) \neq \emptyset\right\} > 0.$$

*In particular, it is independent of $k$.*

*Proof.* By (41),

$$\int_X \left(\sum_{u\in\mathcal{F}} |s_{k,u}|^2_{h_k}\right)^{-\frac{c}{k}} d\mu = (2\pi)^n \int_{\mathbb{R}^n} \frac{\left(\sum_{u\in\mathcal{F}} e^{\langle ku,x\rangle}\right)^{-\frac{c}{k}}}{\left(\sum_{u\in P\cap M/k} e^{\langle ku,x\rangle}\right)^{\frac{1-c}{k}}} dx.$$

By Proposition 6.2 with $\mathcal{U} = P \cap M/k$,

$$c_k(\mathcal{F}) = \sup\left\{c \in (0,1) \,:\, P \cap \Big(-\frac{c}{1-c}\,\mathrm{co}\,\mathcal{F}\Big) \neq \emptyset\right\} > 0,$$

since $\mathrm{co}\,\mathcal{U} = P$ (as the vertices of $P$ belong to $M/k$ for all $k \in \mathbb{N}$). □

**Remark 6.9.** Although not needed for our analysis, it might be of independent interest to understand whether the supremum in (59) (and in the definition of $\alpha_{k,G(H)}$ (49)) is attained. In other words, is (recall the notation of Corollary 6.6)

$$\int_{S^{n-1}(1)} f_c(\nu) dS(\nu) \tag{60}$$

finite for $c = c(\mathcal{F},\mathcal{U})$? By (52)–(53) it is equivalent to consider this question for the integral

$$\int_{S^{n-1}(1)} \int_0^\infty e^{-krg_c(\nu)} r^{n-1} dr dS(\nu) = \int_{\mathbb{R}^n} e^{-kh_{(1-c)\,\mathrm{co}\,\mathcal{U}+c\,\mathrm{co}\,\mathcal{F}}(x)} dx. \tag{61}$$

A classical formula in convex geometry says that whenever $K \subset \mathbb{R}^n$ is a compact convex set with the origin in its interior, then $n!|K^\circ| = \int_{\mathbb{R}^n} e^{-h_K(y)} dy$ (for a detailled proof see [4, (4.2)]). In fact, it is possible to show that when $0 \in \partial K$ the formula still holds in the sense that both sides are equal to $\infty$ (this is related to, but stronger than, the classical fact that $0 \in \mathrm{int}\, K$ if and only if $K^\circ$ is bounded [25, Corollary 14.5.1]). To summarize, both (60) and (61) are infinite when $c = c(\mathcal{F},\mathcal{U})$. We remark that one can also use this point of view to give an alternative, but equivalent, proof of Corollary 6.6.

## 6.3 Group orbits and singularities

Consider the orbit-averaging map $\pi_H$ (6), taking a point to the average of its image under $H$. In particular, points on the same orbit have the same image. Let $M_{\mathbb{R}}^H$ be the subspace of fixed points of $H$ in $M_{\mathbb{R}}$. Then $\pi_H|_{M_{\mathbb{R}}^H} = id_{M_{\mathbb{R}}^H}$, and $M_{\mathbb{R}}^H$ is the image of $\pi_H$. Note that unless $H$ is trivial, this is a proper subspace of $\mathbb{M}_{\mathbb{R}}$, so $\pi_H$ is a projection from $M_{\mathbb{R}}$ to the invariant subspace $M_{\mathbb{R}}^H$, justifying the notation.

Recall (5),

$$P^H := P \cap M_{\mathbb{R}}^H,$$

Since $P$ is convex, $\pi_H(P) \subseteq P$. Hence $\pi_H(P) \subseteq P^H$. On the other hand, $P^H = \pi_H(P^H) \subseteq \pi_H(P)$. This implies

$$P^H = \pi_H(P). \tag{62}$$

Then (recall (23))

$$P^H = \pi_H(P) = \pi_H(\mathrm{co}(\mathrm{Ver}\, P)) = \mathrm{co}(\pi_H(\mathrm{Ver}\, P)).$$

In particular,

$$\mathrm{Ver}\, P^H \subseteq \pi_H(\mathrm{Ver}\, P). \tag{63}$$

Note that equality does not hold in general (for instance, consider $P = [-1, 1]^2$ and $H$ the reflection group about a diagonal).

**Lemma 6.10.** *Fix $u_0 \in M_{\mathbb{R}}^H$, and let $\mathcal{F}$ be a non-empty $H$-invariant convex subset of some fiber $\pi_H^{-1}(u_0)$ of $\pi_H$. Then $\mathcal{F} \cap P \neq \emptyset$ if and only if $u_0 \in P$.*

*Proof.* If $\mathcal{F} \cap P \neq \emptyset$, then $\pi_H(\mathcal{F} \cap P) \neq \emptyset$. By (62), $\pi_H(\mathcal{F} \cap P) \subset \pi_H(P) = P^H \subset P$, and since $\emptyset \neq \mathcal{F} \subset \pi_H^{-1}(u_0)$, then $\pi_H(\mathcal{F} \cap P) \subset \pi_H(\mathcal{F}) = \{u_0\}$. Thus, $\pi_H(\mathcal{F} \cap P) \subset \{u_0\} \cap P$ and for this to be nonempty it follows that $u_0 \in P$.

For the converse, since $\mathcal{F}$ is $H$-invariant and convex, $\pi_H(\mathcal{F}) \subset \mathcal{F}$. Additionally, as before $\emptyset \neq \mathcal{F} \subset \pi_H^{-1}(u_0)$ implies $\pi_H(\mathcal{F}) = \{u_0\}$. Thus $u_0 \in \mathcal{F}$. Therefore, if $u_0 \in P$ then actually $u_0 \in \mathcal{F} \cap P$. ☐

## 6.4 Proof of Tian's conjecture

We can now prove our main result.

*Proof of Theorem 1.4.* Fix $k \in \mathbb{N}$ and let $O_1^{(k)}, \ldots, O_N^{(k)}$ be the orbits of the action of $H$ on $P \cap M/k$. Recall that $\pi_H$ maps an orbit to a singleton,

$$\{o_i^{(k)}\} := \pi_H(O_i^{(k)}).$$

Note that $O_i^{(k)} \subset \pi_H^{-1}(o_i^{(k)})$ and hence (as the action of $H$ and hence also $\pi_H$ is linear) also $\mathrm{co}\, O_i^{(k)} \subset \pi_H^{-1}(o_i^{(k)})$. Moreover, $\mathrm{co}\, O_i^{(k)}$ is convex and $H$-invariant. By Corollary 4.7, Proposition 6.8, and Lemma 6.10,

$$\begin{aligned}
\alpha_{k,G(H)} &= \min_{1 \leq i \leq N} c(O_i^{(k)}) \\
&= \min_{1 \leq i \leq N} \sup \left\{ c \in (0,1) \,:\, \left( -\frac{c}{1-c}\, \mathrm{co}\, O_i^{(k)} \right) \cap P \neq \emptyset \right\} \\
&= \min_{1 \leq i \leq N} \sup \left\{ c \in (0,1) \,:\, -\frac{c}{1-c} o_i^{(k)} \in P \right\}
\end{aligned}$$

$$= \sup \left\{ c \in (0,1) \,:\, -\frac{c}{1-c}\{o_1^{(k)}, \ldots, o_N^{(k)}\} \subset P \right\}.$$

Since

$$\{o_1^{(k)}, \ldots, o_N^{(k)}\} = \pi_H \left( \bigcup_{i=1}^{N} O_i^{(k)} \right) = \pi_H \left( k^{-1} M \cap P \right),$$

then

$$\begin{aligned}
\alpha_{k,G(H)} &= \sup \left\{ c \in (0,1) \,:\, -\frac{c}{1-c}\pi_H(k^{-1}M \cap P) \subset P \right\} \\
&= \sup \left\{ c \in (0,1) \,:\, -\frac{c}{1-c}\operatorname{co}(\pi_H(k^{-1}M \cap P)) \subset P \right\} \\
&= \sup \left\{ c \in (0,1) \,:\, -\frac{c}{1-c}\pi_H(\operatorname{co}(k^{-1}M \cap P)) \subset P \right\} \\
&= \sup \left\{ c \in (0,1) \,:\, -\frac{c}{1-c}\pi_H(P) \subset P \right\} \\
&= \sup \left\{ c \in (0,1) \,:\, -\frac{c}{1-c}P^H \subset P \right\} \\
&= \sup \left\{ c \in (0,1) \,:\, -\frac{c}{1-c}\operatorname{Ver} P^H \subset P \right\} \\
&= \min_{u \in \operatorname{Ver} P^H} \sup \left\{ c \in (0,1) \,:\, -\frac{c}{1-c}u \in P \right\}.
\end{aligned} \tag{64}$$

Also, by (63),

$$\begin{aligned}
\alpha_{k,G(H)} &= \sup \left\{ c \in (0,1) \,:\, -\frac{c}{1-c}\operatorname{Ver} P^H \subset P \right\} \\
&\geq \sup \left\{ c \in (0,1) \,:\, -\frac{c}{1-c}\pi_H(\operatorname{Ver} P) \subset P \right\} \\
&\geq \sup \left\{ c \in (0,1) \,:\, -\frac{c}{1-c}\pi_H(P) \subset P \right\} \\
&= \alpha_{k,G(H)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\alpha_{k,G(H)} &= \sup \left\{ c \in (0,1) \,:\, -\frac{c}{1-c}\pi_H(\operatorname{Ver} P) \subset P \right\} \\
&= \min_{u \in \pi_H(\operatorname{Ver} P)} \sup \left\{ c \in (0,1) \,:\, -\frac{c}{1-c}u \in P \right\}.
\end{aligned} \tag{65}$$

Recall (27),

$$P = \bigcap_{j=1}^{d} \{x \in M_{\mathbb{R}} \mid \langle x, -v_j \rangle \leq 1\}.$$

For any point $u$ and $c \in (0,1)$,

$$-\frac{c}{1-c}u \in P \quad \text{if and only if} \quad c \leq \min_j \frac{1}{\langle u, v_j \rangle + 1} \quad \text{for all } j \in \{1, \ldots, d\}.$$

In sum, combining (64) and (65) implies (7). In particular, $\alpha_{k,G(H)}$ is independent of $k \in \mathbb{N}$ and, by Corollary 4.9, is equal to $\alpha_{G(H)}$. This completes the proof of Theorem 1.4. $\square$

## 7 Grassmannian Tian invariants

The following definition is due to Tian [32, (6.1)].

**Definition 7.1.** Let $X$ be a Fano manifold and $G \subseteq \mathrm{Aut}\, X$ a compact subgroup. For $k, m \in \mathbb{N}$, define (recall (43))

$$\alpha_{k,m,G} := k \inf_{\substack{|V| \subset |-kK_X| \\ \dim V = m \\ V^G = V}} \mathrm{lct}|V|,$$

with the convention $\inf \emptyset = \infty$.

Equivalently, these invariants are the minimum of the function $V \mapsto k\mathrm{lct}|V|$ over the Grassmannian restricted to the subset of $G$-invariant $m$-dimensional subspaces of $H^0(X, -kK_X)$. This subset can be, in some situations, even a finite set (see Remark 7.9). By Remark 7.2 below, these invariants generalize the invariants $\alpha_{k,G}$. Note also that Definition 7.1 is an *algebraic* one. Also, $\alpha_{k,E_P(k),G} = \infty$ (recall (31)) so it is *not* equal to $\alpha_{k,G}$ from Definition 4.1.

In the toric setting it is natural to consider Conjecture 1.5 for the invariants $\alpha_{k,m,(S^1)^n}$. For these invariants we show that Conjecture 1.5 does not quite hold, and we give a precise breakdown of when it does in terms of a natural convex geometric criterion. In contrast with previous sections, we do not work with the additional symmetry corresponding to the groups $G(H)$ (4). The reason for that is that even in simple situations $\alpha_{k,m,G(H)}$ will not be well-behaved: see Example 8.6.

**Remark 7.2.** Putting $H = \{\mathrm{id}\}$ in Proposition 4.6, notice that $\alpha_{k,(S^1)^n} = \alpha_{k,1,(S^1)^n}$. In this sense, $\alpha_{k,m,(S^1)^n}$ is a generalization of the $\alpha_{k,(S^1)^n}$-invariants.

First, we show, as a rather immediate consequence of our work, an explicit formula for the invariants of Definition 7.1.

**Corollary 7.3.** *Let $X$ be toric Fano with associated polytope $P$ (27). For $k, m \in \mathbb{N}$ (recall (59)),*

$$\alpha_{k,m,(S^1)^n} = \min_{\substack{\mathcal{F} \subseteq P \cap M/k \\ |\mathcal{F}| = m}} c_k(\mathcal{F}).$$

*Proof.* Fix $k, m \in \mathbb{N}$. Let $V \subseteq H^0(X, -kK_X) = \mathrm{span}\{s_u\}_{u \in P \cap M/k}$ be a subspace invariant under $(S^1)^n$. By Lemma 4.8, there exists a unique $\mathcal{F} \subseteq P \cap M/k$ such that

$$V = \mathrm{span}\, \{s_u\}_{u \in \mathcal{F}}.$$

Note that $|\mathcal{F}| = \dim V$. By (43) and (59),

$$\alpha_{k,m,(S^1)^n} = k \min_{\substack{\mathcal{F} \subseteq P \cap M/k \\ |\mathcal{F}| = m}} \mathrm{lct} \left| \mathrm{span}\, \{s_u\}_{u \in \mathcal{F}} \right| = \min_{\substack{\mathcal{F} \subseteq P \cap M/k \\ |\mathcal{F}| = m}} c_k(\mathcal{F}).$$

∎

**Lemma 7.4.** *Let $K \subset \mathbb{R}^n$ be a compact convex set with $0 \in \mathrm{int}\, K$. For any set $S \subset \mathbb{R}^n$,*

$$\inf_S \| \cdot \|_K = \inf \{\lambda \geq 0 \,:\, S \cap \lambda K \neq \emptyset\}.$$

*Proof.* If $S \cap \lambda K \neq \emptyset$ for some $\lambda \geq 0$, pick $x \in S \cap \lambda K$. Since $x \in \lambda K$, by (13), $\|x\|_K \leq \lambda$. Therefore $\inf_{x \in S} \|x\|_K \leq \inf\{\lambda \geq 0 \,:\, S \cap \lambda K \neq \emptyset\}$.

On the other hand, for any $x \in S$, $S \cap \|x\|_K K \supseteq \{x\} \neq \emptyset$ by (15). So $\inf_{x \in S} \|x\|_K \geq \inf\{\lambda \geq 0 \,:\, S \cap \lambda K \neq \emptyset\}$. ∎

Proposition 6.8 can be reformulated in terms of near-norms:

**Corollary 7.5.** *For $k \in \mathbb{N}$ and a non-empty set $\mathcal{F} \subseteq P \cap M/k$ (recall (59)),*

$$c_k(\mathcal{F}) = \frac{1}{1 + \min\limits_{\operatorname{co}\mathcal{F}} \|\cdot\|_{-P}}.$$

*Proof.* By Proposition 6.8 and Lemma 7.4,

$$\begin{aligned}
c_k(\mathcal{F}) &= \sup\left\{c \in (0,1) \,:\, P \cap \left(-\frac{c}{1-c}\operatorname{co}\mathcal{F}\right) \neq \emptyset\right\} \\
&= \sup\left\{c \in (0,1) \,:\, \operatorname{co}\mathcal{F} \cap \frac{1-c}{c}(-P) \neq \emptyset\right\} \\
&= \sup\left\{\frac{1}{1+a} \,:\, a \in (0,\infty),\ \operatorname{co}\mathcal{F} \cap (-aP) \neq \emptyset\right\} \\
&= \left[1 + \inf\left\{a \in (0,\infty) \,:\, \operatorname{co}\mathcal{F} \cap (-aP) \neq \emptyset\right\}\right]^{-1} \\
&= \left[1 + \inf_{\operatorname{co}\mathcal{F}} \|\cdot\|_{-P}\right]^{-1} = \left[1 + \min_{\operatorname{co}\mathcal{F}} \|\cdot\|_{-P}\right]^{-1},
\end{aligned}$$

using compactness of $\operatorname{co}\mathcal{F}$. □

Combining Corollaries 7.3 and 7.5 yields an explicit formula for $\alpha_{k,m,(S^1)^n}$.

**Corollary 7.6.** *Let $X$ be toric Fano with associated polytope $P$ (27). For $k, m \in \mathbb{N}$,*

$$\alpha_{k,m,(S^1)^n} = \left[1 + \max_{\substack{\mathcal{F} \subseteq P \cap M/k \\ |\mathcal{F}| = m}} \min_{\operatorname{co}\mathcal{F}} \|\cdot\|_{-P}\right]^{-1}.$$

**Remark 7.7.** For $m = 1$, Corollary 7.6 reads

$$\alpha_{k,(S^1)^n} = \frac{1}{1 + \max\limits_{u \in P \cap M/k} \|u\|_{-P}}.$$

Being a near-norm, the function $\|\cdot\|_{-P}$ satisfies the triangle inequality and is positively 1-homogeneous, hence it is convex. By Lemma 2.3, the vertex set of $P$ is in the integral lattice $M$, hence also contained in $M/k$ for all $k \in \mathbb{N}$. This combined with Lemma 2.4 gives,

$$\alpha_{k,(S^1)^n} = \frac{1}{1 + \max\limits_{\operatorname{Ver} P} \|\cdot\|_{-P}} = \frac{1}{1 + \max\limits_{P} \|\cdot\|_{-P}}. \tag{66}$$

This is, of course, a special case of Theorem 1.4 (proved in §6), derived here in slightly different notation (using the near-norm instead of the support function). Thus,

$$\alpha_{k,(S^1)^n} = \alpha_{(S^1)^n} = \alpha, \tag{67}$$

where the first equality follows from (66) and Demailly's result [12, (A.1)], and the last equality follows by comparing Theorem 1.4 with Blum–Jonsson's formula [6, (7.2)].

**Proposition 7.8.** *Let $X$ be toric Fano with associated polytope $P$ (27). For $k \in \mathbb{N}$ and $m' > m$,*

$$\alpha_{k,m',(S^1)^n} \geq \alpha_{k,m,(S^1)^n}. \tag{68}$$

*In particular,*

$$\alpha_{k,m,(S^1)^n} \geq \alpha_{k,1,(S^1)^n} = \alpha_{k,(S^1)^n} = \alpha.$$

*Equality holds if and only if there is $\mathcal{F} \subseteq P \cap M/k$ with $|\mathcal{F}| = m$ satisfying*

$$\|\cdot\|_{-P}\Big|_{\mathrm{co}\,\mathcal{F}} = \max_P \|\cdot\|_{-P}\,. \tag{69}$$

**Remark 7.9.** For a general Fano $X$, it does not follow from Definition 7.1 that $\alpha_{k,m',G} \geq \alpha_{k,m,G}$, i.e., there may be no monotonicity in $m$. For instance, there might be no $m$-dimensinal $G$-invariant subspaces. Thanks to Lemma 4.8 the situation for $G = (S^1)^n$ is particularly simple: there are finitely-many $m$-dimensional $(S^1)^n$-invariant subspaces (in fact, the Grassmannian of $m$-dimensional subspaces has exactly $\binom{E_P(k)}{m}$ fixed points by the $(S^1)^n$-action), and moreover every $(S^1)^n$-invariant subspace consists of 1-dimensional blocks.

*Proof.* Pick $\mathcal{F}' \subseteq P \cap M/k$ that computes $\alpha_{k,m',(S^1)^n}$ (such a subset exists since $P \cap M/k$ is a finite set). That is, $|\mathcal{F}'| = m'$ and $\alpha_{k,m',(S^1)^n} = \big[1 + \min_{\mathrm{co}\,\mathcal{F}'} \|\cdot\|_{-P}\big]^{-1}$ (recall Corollary 7.6). For $\mathcal{F}$ be a subset of $\mathcal{F}'$ with $|\mathcal{F}| = m$, Corollary 7.6 implies

$$\alpha_{k,m,(S^1)^n} = \frac{1}{1 + \max\limits_{\substack{\mathcal{A}\subseteq P\cap M/k \\ |\mathcal{A}|=m}} \min\limits_{\mathrm{co}\,\mathcal{A}} \|\cdot\|_{-P}} \leq \frac{1}{1 + \min\limits_{\mathrm{co}\,\mathcal{F}} \|\cdot\|_{-P}} \leq \frac{1}{1 + \min\limits_{\mathrm{co}\,\mathcal{F}'} \|\cdot\|_{-P}} = \alpha_{k,m',(S^1)^n},$$

proving (68).

Next, suppose $\alpha_{k,m,(S^1)^n} = \alpha$, i.e. (recall (66)–(67)),

$$\max_{\substack{\mathcal{F}\subseteq P\cap M/k \\ |\mathcal{F}|=m}} \min_{\mathrm{co}\,\mathcal{F}} \|\cdot\|_{-P} = \max_P \|\cdot\|_{-P}\,. \tag{70}$$

For any $\mathcal{F} \subseteq P$, convexity of $P$ (recall (27)) implies $\mathrm{co}\,\mathcal{F} \subseteq P$. Hence,

$$\min_{\mathrm{co}\,\mathcal{F}} \|\cdot\|_{-P} \leq \max_P \|\cdot\|_{-P}\,.$$

The equality (70) holds if and only if there is $\mathcal{F} \subseteq P \cap M/k$ such that $|\mathcal{F}| = m$ and

$$\min_{\mathrm{co}\,\mathcal{F}} \|\cdot\|_{-P} = \max_P \|\cdot\|_{-P}\,,$$

which forces $\|\cdot\|_{-P}$ to be the constant $\max_P \|\cdot\|_{-P}$ on $\mathrm{co}\,\mathcal{F}$. ∎

## 7.1 Proof of Theorem 1.6 and the intuition behind it

Theorem 1.4 resolved Tian's classical stabilization Problem 1.1 in the affirmative. This corresponds to the case $m = 1$ of Conjecture 1.5. Next we show that Conjecture 1.5 is only partially true for $m \geq 2$.

*Proof of Theorem 1.6.* Let $\mathcal{F} \subset P \cap M/k$ with $|\mathcal{F}| = m$. By convexity of $P$ and as $m \geq 2$, $\mathrm{co}\,\mathcal{F}$ contains an interval in $P$. In particular, $\mathcal{F}$ contains a point of $P \setminus \mathrm{Ver}\,P$. Thus, if $(*_P)$ holds then (69) does not hold (for any such $\mathcal{F}$), and Proposition 7.8 implies (8).

Next, suppose $(*_P)$ fails (and now we relax to $m \geq 1$). Let $\hat{x} \in P \setminus \mathrm{Ver}\,P$ achieve the maximum of $\|\cdot\|_{-P}$ on $P$. Express $\hat{x}$ as a convex combination of a *subset* of vertices of $P$, $\{p_1, \ldots, p_\ell\} \subset \mathrm{Ver}\,P$, namely,

$$\hat{x} = \sum_{i=1}^{\ell} \hat{\lambda}_i p_i, \quad \sum_{i=1}^{\ell} \hat{\lambda}_i = 1, \quad \hat{\lambda} \in (0,1)^\ell. \tag{71}$$

Note that necessarily
$$\ell > 1. \tag{72}$$
We claim that
$$\|x\|_{-P} = \max_{x\in P} \|\cdot\|_{-P}, \quad \text{for any } x \in \operatorname{co}\{p_i\}_{i=1}^{\ell}.$$
Indeed, letting
$$P' := \operatorname{co}\{p_i\}_{i=1}^{\ell} \subset P,$$
the claim follows from $\hat{x} \in \operatorname{int} P' \subset P$ and Lemma 2.4 (iii).

Now, the lattice polytope $P' := \operatorname{co}\{p_i\}_{i=1}^{\ell}$ has positive dimension by (72). Equivalently, the Ehrhart polynomial of $P'$ has degree at least 1. Hence, by Proposition 2.5, $|P' \cap M/k| \geq m$ for all sufficiently large $k$. Pick $\mathcal{F} \subseteq P' \cap M/k \subseteq P \cap M/k$ such that $|\mathcal{F}| = m$. Then $\operatorname{co}\mathcal{F} \subseteq P'$. By Proposition 7.8, (9) follows. The case $m = 1$ was already treated in Theorem 1.4 but also from this proof we see that any $k \in \mathbb{N}$ works for $m = 1$. $\square$

The proof of Theorem 1.6 and condition $(*_P)$ have a pleasing geometric interpretation. Geometrically, by Proposition 7.8, $\alpha_{k,m,(S^1)^n} = \alpha$ if and only if there is a subset $\mathcal{F} \subseteq P \cap M/k$ such that $|\mathcal{F}| = m$ and $\operatorname{co}\mathcal{F}$ lies entirely in the level set
$$\left\{\|\cdot\|_{-P} = \max_P \|\cdot\|_{-P}\right\} \subseteq \partial P \tag{73}$$
Recall that any convex subset of the boundary of a polytope must lie in a single face. Hence $\mathcal{F}$ must be a subset of a face $F$ of $P$. We may assume $F$ is the minimal face that contains $\mathcal{F}$, i.e., $\mathcal{F}$ intersects the relative interior of $F$. In particular, $\|\cdot\|_{-P}|_F$ attains its maximum in the interior of $F$ and so $F$ has to lie entirely in the level set (73) by Lemma 2.4. Therefore $\alpha_{k,m,(S^1)^n} = \alpha$ if and only if there is a subset $\mathcal{F} \subseteq P \cap M/k$ such that $|\mathcal{F}| = m$ and $\mathcal{F}$ is a subset of a face that lies in the level set (73), or equivalently, there is a face $F$ that lies in the level set (73), and $|F \cap M/k| \geq m$.

Notice that the level set $\{\|\cdot\|_{-P} = \lambda\}$ is the dilation $\lambda\partial(-P)$, and (73) is the largest dilation that intersects $P$ (by Lemma 7.4); see Figure 1. Combining all the above then, $(*_P)$ states that
$$P \text{ intersects } \max\nolimits_P \|\cdot\|_{-P}\partial(-P) \text{ only at vertices.}$$
If $(*_P)$ holds, then any such face $F$ consists of a single lattice point. In particular, $|F \cap M/k| = 1$ for any $k$, and $\alpha_{k,m,(S^1)^n}$ stabilizes if and only if $m = 1$. On the other hand, if $(*_P)$ fails, then there is a face $F$ of positive dimension that lies in the level set (73). Since $|F \cap M/k|$ increases to infinity, for any $m$, we can find $k_0$ such that $|F \cap M/k| \geq m$ for $k \geq k_0$, i.e., $\alpha_{k,m,(S^1)^n}$ stabilizes in $k$.

## 7.2 A Demailly result for Grassmannian invariants

The next result was obtained independently by Li–Zhu [24, Proposition 4.1].

**Proposition 7.10.** *Let $X$ be toric Fano with associated polytope $P$ (27). For $m \in \mathbb{N}$,*
$$\lim_{k\to\infty} \alpha_{k,m,(S^1)^n} = \alpha.$$

*Proof.* By continuity, given $\epsilon > 0$ there exists an open set $U_\epsilon \subset P$ such that
$$\|\cdot\|_{-P}\Big|_{U_\epsilon} > \max_P \|\cdot\|_{-P} - \epsilon. \tag{74}$$
Fix $\delta > 0$ sufficiently small so that there is a closed cube $C_\delta \subset U_\epsilon$ with edge length $\delta$. Now, for $k \geq K$, $|C_\delta \cap M/k| \geq \lfloor K\delta\rfloor^n$. Given $m \in \mathbb{N}$, choose $K$ so $\lfloor K\delta\rfloor^n \geq m$. Then for each $k \geq K$

there exists $\mathcal{F} \subset C_\delta \cap M/k \subset P \cap M/k$ such that $|\mathcal{F}| = m$. Since $\mathcal{F} \subset C_\delta$ and $C_\delta$ is convex, $\operatorname{co}\mathcal{F} \subset C_\delta \subset U_\epsilon$. By (74),

$$\min_{\operatorname{co}\mathcal{F}} \|\cdot\|_{-P} \geq \min_{C_\delta} \|\cdot\|_{-P} > \max_P \|\cdot\|_{-P} - \epsilon.$$

So by Corollary 7.6 and (66)–(67),

$$\alpha_{k,m,(S^1)^n} \leq \frac{1}{1 + \min\limits_{\operatorname{co}\mathcal{F}} \|\cdot\|_{-P}} < \frac{1}{1 + \max\limits_P \|\cdot\|_{-P} - \epsilon} = \frac{\alpha}{1 - \epsilon\alpha},$$

for any $k$ such that $\lfloor k\delta \rfloor \geq \sqrt[n]{m}$. Hence $\limsup_{k\to\infty} \alpha_{k,m,(S^1)^n} \leq \frac{\alpha}{1-\epsilon\alpha}$. On the other hand, by Proposition 7.8, $\liminf_{k\to\infty} \alpha_{k,m,(S^1)^n} \geq \alpha$. Since $\epsilon > 0$ is arbitrary the proof is complete. □

# 8 Examples

This section illustrates Theorems 1.4 and 1.6 in the case of toric del Pezzo surfaces.

**Example 8.1.** Let $v_1 = (1,0), v_2 = (0,1), v_3 = (-1,-1)$. Then $\mathbb{P}^2 = X(\Delta)$ for

$$\Delta = \{\{0\}, \mathbb{R}_+v_1, \mathbb{R}_+v_2, \mathbb{R}_+v_3, \mathbb{R}_+v_1 + \mathbb{R}_+v_2, \mathbb{R}_+v_1 + \mathbb{R}_+v_3, \mathbb{R}_+v_2 + \mathbb{R}_+v_3\}$$

and $P = \{-v_1, -v_2, -v_3\}^\circ = \{y \in M_\mathbb{R} : \langle y, -v_i\rangle \leq 1, i \in \{1,2,3\}\} \subset M_\mathbb{R}$ is depicted in Figure 3. Then $\operatorname{Aut} P \cong D_6 = S_3$, the dihedral/cyclic group of order 6 consisting of the permutations of the 3 vertices, or the 3 homogeneous coordinates $\mathbb{P}^2$, as in Figure 2. For $H = \operatorname{Aut} P$, $P^H$ is the origin, so by Theorem 1.4, $\alpha_{k,G(H)} = \alpha_{G(H)} = 1$. When $H$ is the trivial group then $P^H = P$ and Theorem 1.4 gives $\alpha_{k,G(H)} = \alpha_{G(H)} = \frac{1}{3}$, with minimum achieved at any vertex of $P$. When $H$ is the subgroup $\mathbb{Z}_2$ given by reflection about, say, $y = x$ (i.e., generated by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ associated to switching two of the homogeneous coordinates of $\mathbb{P}^2$), $P^H$ is the line segment $\operatorname{co}\{(-1,-1), (\frac{1}{2}, \frac{1}{2})\}$ and $\alpha_{k,G(H)} = \alpha_{G(H)} = \frac{1}{3}$, with minimum achieved at $(-1,-1)$. When $H$ is the subgroup of order 3, $P^H = \{0\}$, and $\alpha_{k,G(H)} = \alpha_{G(H)} = 1$,

**Example 8.2.** Let $P$ be the polytope for $\mathbb{P}^2$ blown up at one point, i.e., $v_1, v_2, v_3$ as above and $v_4 = (1,1)$. Then $\operatorname{Aut} P \cong \mathbb{Z}_2$, as shown in Figure 3. There are therefore two choices for $H$. When $H = \operatorname{Aut} P$, $P^H$ is the line segment $\operatorname{co}\{(-\frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\}$. By Theorem 1.4, $\alpha_{k,G(H)} = \alpha_{G(H)} = \frac{1}{2}$, with minimum achieved at $\pm(\frac{1}{2}, \frac{1}{2})$. When $H$ is the trivial group then $P^H = P$ and Theorem 1.4 gives $\alpha_{k,G(H)} = \alpha_{G(H)} = \frac{1}{3}$, with minimum achieved at either of the vertices $(-1,2)$ or $(2,-1)$.

**Example 8.3.** Let $P$ be the polytope for $\mathbb{P}^2$ blown up at two points, i.e., $v_1, v_2, v_3$ as above and $v_4 = (-1,0), v_5 = (0,-1)$. Again $\operatorname{Aut} P \cong \mathbb{Z}_2$, as shown in Figure 3. For $H = \operatorname{Aut} P$, $P^H$ is the line segment $\operatorname{co}\{(-1,-1), (\frac{1}{2}, \frac{1}{2})\}$. By Theorem 1.4, $\alpha_{k,G} = \alpha_G = \frac{1}{3}$, with minimum achieved at $(-1,-1)$. When $H$ is the trivial group then $P^H = P$ and Theorem 1.4 gives $\alpha_{k,G(H)} = \alpha_{G(H)} = \frac{1}{3}$, with minimum achieved still at $(-1,-1)$.

**Example 8.4.** Let $P$ be the polytope for $\mathbb{P}^2$ blown up at three non-colinear points, i.e., $v_1, \ldots, v_5$ as in the previous example and $v_6 = (1,1)$. Now $\operatorname{Aut} P \cong D_{12}$, the dihedral group of order 12 consisting of the cyclic permutations of the 6 vertices and the 6 reflections, as in Figure 3. For $H = \operatorname{Aut} P$, $P^H$ is the origin, so by Theorem 1.4, $\alpha_{k,G(H)} = \alpha_{G(H)} = 1$. When $H$ is the trivial group then $P^H = P$ and Theorem 1.4 gives $\alpha_{k,G(H)} = \alpha_{G(H)} = \frac{1}{2}$, with minimum achieved at any vertex of $P$. When $H$ is the subgroup $\mathbb{Z}_2$ given by reflection about $y = x$, $P^H$ is the line segment

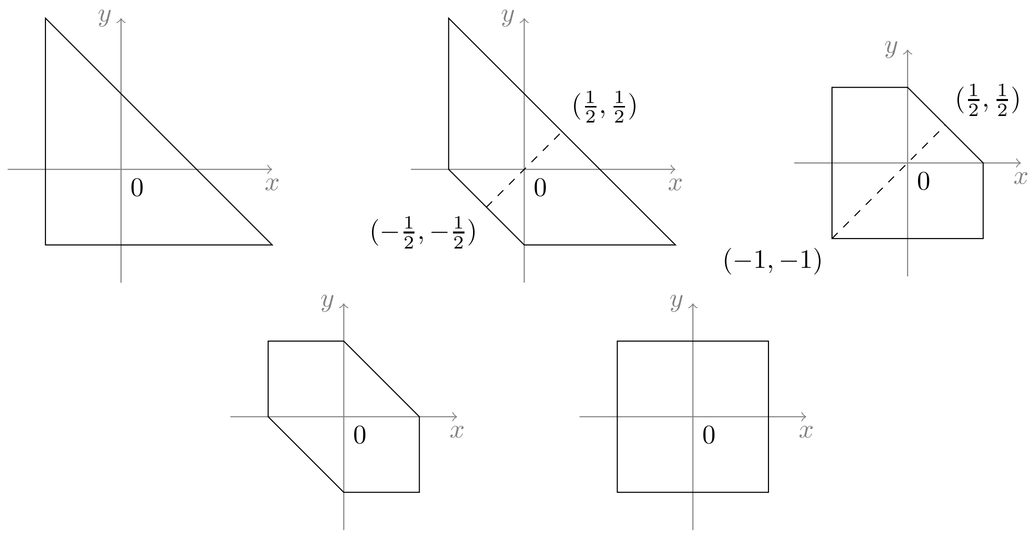


Figure 3: The polytope $P$ for del Pezzo surfaces, namely $\mathbb{P}^2$ blown up at no more than 3 generically positioned points, and $\mathbb{P}^1 \times \mathbb{P}^1$. For the two K-unstable examples, the automorphism group $H = \mathrm{Aut}(P)$ is generated by the reflection about $y = x$, and $P^H$ is the intersection of $P$ with this reflection axis. For the other three examples, the automorphism group $H = \mathrm{Aut}(P)$ is the dihedral group associated to the polygon $P$, and $P^H = \{0\}$.

$\mathrm{co}\{(-\frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\}$ and $\alpha_{k,G(H)} = \alpha_{G(H)} = \frac{1}{2}$, with minimum achieved at $\pm(\frac{1}{2}, \frac{1}{2})$. When $H$ is the subgroup $\mathbb{Z}_2$ given by reflection about $y = -x$, $P^H = \mathrm{co}\{(-1, 1), (1, -1)\}$, and $\alpha_{k,G(H)} = \alpha_{G(H)} = \frac{1}{2}$, with minimum achieved at $(\pm 1, \mp 1)$. Same for the other reflections. When $H$ contains a cyclic permutation, or more than one reflection, $P^H$ is the origin and $\alpha_{k,G(H)} = \alpha_{G(H)} = 1$.

**Example 8.5.** Let $P$ be the polytope for $\mathbb{P}^1 \times \mathbb{P}^1$, i.e., with $v_1, v_2$ as above and $v_3 = (-1, 0)$ and $v_4 = (0, -1)$, so $P = [-1, 1]^2$ and $\mathrm{Aut}\, P \cong D_8$, the dihedral group of order 8 generated by 4 reflections shown in Figure 3. Similar to previous examples, we obtain $\alpha_{k,G(H)} = \alpha_{G(H)} = 1$ whenever $H$ contains a cyclic permutation or more than one reflection. Otherwise it equals 1/2.

**Example 8.6.** Let $P$ be the polytope for $X = \mathbb{P}^2$ from Example 8.1. Recall that $\mathrm{Aut}\, P = S_3$. Let $H = A_3$, the alternating group of order 3, generated by a cyclic permutation of the three vertices. The group is represented by $\{I, A, A^2\}$ where

$$A = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

All orbits of $H$ on $M \cong \mathbb{Z}^2$ have cardinality 3 except the orbit of the origin, that has cardinality 1. Any $G(H)$-invariant subspace of $H^0(X, -kK_X)$ is spanned by a collection of monomials indexed by a union $\cup_{i=1}^{\ell} O_i^{(k)}$ of $H$-orbits on $kP \cap M$. This forces the subspace to have dimension $m = 3\ell$ or $m = 3(\ell - 1) + 1$. Thus, $m \equiv_3 0, 1$ (see Figure 4 for the case $k = 1$). As a result, if $m \equiv_3 2$ then $\alpha_{k,m,G(H)} = \infty$. For this reason we only consider in §7 the case $H = \{\mathrm{id}\}$.

**Example 8.7.** Let $P$ be the polytope for $X = \mathbb{P}^2$ from Example 8.1. We compute the obstruction $(*_P)$, i.e., compute $\mathrm{argmax}_P \| \cdot \|_{-P} = \mathrm{argmax}_P h_Q = \mathrm{argmax}_P h_{\Delta_1}$. Recall $\Delta_1 = \{v_1, v_2, v_3\} = \{(1, 0), (0, 1), (-1, -1)\}$. Let $y = (y_1, y_2)$ be coordinates on $P$. Then,

$$h_{v_1}(y) = y_1, \quad h_{v_2}(y) = y_2, \quad h_{v_3}(y) = -y_1 - y_2,$$

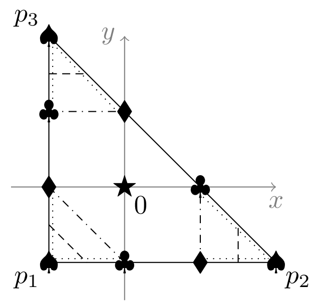


Figure 4: The four orbits $O_1^{(1)}, \ldots, O_4^{(1)}$ of the action of the cyclic group of order 3 (generated by cyclic permutation of the homogeneous coordinates of $\mathbb{P}^2$) on the polytope corresponding to $\mathbb{P}^2$ with $k = 1$. The dashed-dotted lines depict the sets on which $h_{\Delta_1}$ is constant equal to 1 and that compute $\alpha_{1,2,(S^1)^2} = 1/2$. These sets are the three components of $-\partial P \cap P$. The dotted lines depict the sets on which $h_{\Delta_1}$ is not constant but whose minimum is 1 and that also compute $\alpha_{1,2,(S^1)^2} = 1/2$. The dashed lines depict the sets on which $h_{\Delta_1}$ is constant equal to 3/2 and that compute $\alpha_{2,2,(S^1)^2} = 2/5$. These sets are the three components of $-\frac{3}{2}\partial P \cap P$.

so

$$h_{\Delta_1}(y) = \max\{y_1, y_2, -y_1 - y_2\}.$$

By Lemma 2.4 $\mathrm{Ver}\, P \cap \mathrm{argmax}_P h_{\Delta_1} \neq \emptyset$. Note,

$$\mathrm{Ver}\, P = \{p_1, p_2, p_3\} = \{(-1,-1), (2,-1), (-1,2)\}.$$

So

$$\max_P h_Q = \max\{h_Q(p_1), h_Q(p_2), h_Q(p_3)\} = \max\{2,2,2\} = 2,$$

and $\mathrm{Ver}\, P \subset \mathrm{argmax}_P h_{\Delta_1}$. Suppose that $h_Q(y) = 2$. Then a computation shows that $y \in \mathrm{Ver}\, P$. Thus $\mathrm{Ver}\, P = \mathrm{argmax}_P h_{\Delta_1}$. By Theorem 1.6, $\alpha_{k,m,(S^1)^n} > \alpha = \frac{1}{3}$ for all $k \in \mathbb{N}$ and $m \geq 2$.

Let us compute $\alpha_{1,2,(S^1)^n}$. Let $\mathcal{F} = \{f_1, f_2\} \subset P \cap M$. By Corollary 7.6 it suffices to consider 'minimal' $\mathcal{F}$ in the sense that for no other $\mathcal{F}'$, $\mathrm{co}\, \mathcal{F}' \subset \mathrm{co}\, \mathcal{F}$. So for instance, we do not need to consider $\{p_1, p_2\}$ but instead

$$\{p_1, (0,-1)\}, \{(0,-1), (1,-1)\}, \{(1,-1), p_2\}.$$

In fact, $\min_{\mathrm{co}\{p_1,(0,-1)\}} h_Q = h_Q(0,-1) = 1$ while $\min_{\{p_1,p_2\}} h_Q = h_Q(1/2,-1) = 1/2$.

Also, again by Corollary 7.6, we do not need to consider any $\mathcal{F}$ such that $0 \in \mathrm{co}\, \mathcal{F}$ since on such an interval the support function attains its absolute minimum, i.e., vanishes.

Next, observe that $h_{\Delta_1} = 1$ on the vertices of the hexagon

$$P_1 := \mathrm{co}\{\pm(1,0), \pm(0,1), \pm(-1,1)\}.$$

By Lemma 2.4,

$$h_{\Delta_1}|_{P_1} \leq h_{\Delta_1}|_{\mathrm{Ver}\, P_1} = 1.$$

Thus (by Corollary 7.6) we do not need to consider any $\mathcal{F}$ such that $\mathrm{co}\, \mathcal{F}$ intersects $P_1$, as such an $\mathcal{F}$ will satisfy $\min_{\mathrm{co}\, \mathcal{F}} h_Q \leq 1$. Since every $\mathcal{F}$ intersects $P_1$ it follows that $\alpha_{1,2,(S^1)^2} = 1/2$ by Corollary 7.6. See Figure 4.

By the same reasoning we can compute $\alpha_{k,2,(S^1)^2}$ using the hexagons

$$\begin{aligned} P_k := &\operatorname{co}\Big\{\Big(2-\frac{1}{k},\frac{1}{k}-1\Big),\Big(2-\frac{1}{k},-1\Big),\Big(-1+\frac{1}{k},-1\Big),\Big(-1,-1+\frac{1}{k}\Big),\\ &\qquad\Big(-1,2-\frac{1}{k}\Big),\Big(-1+\frac{1}{k},2-\frac{1}{k}\Big)\Big\}\\ =&\frac{1}{k}\operatorname{co}\Big\{\Big(2k-1,1-k\Big),\Big(2k-1,-k\Big),\Big(-k+1,-k\Big),\Big(-k,-k+1\Big),\\ &\qquad\Big(-k,2k-1\Big),\Big(-k+1,2k-1\Big)\Big\}\subset P\cap M/k. \end{aligned}$$

By Lemma 2.4,

$$h_{\Delta_1}|_{P_k}\le h_{\Delta_1}|_{\operatorname{Ver}P_k}=2-1/k.$$

Since every $\mathcal{F}$ intersects $P_k$ it follows that

$$\alpha_{k,2,(S^1)^2}=\Big[1+2-\frac{1}{k}\Big]^{-1}=\frac{k}{3k-1}$$

by Corollary 7.6. Note that $\lim_k\frac{k}{3k-1}=1/3=\alpha$ (recall Example 8.1 and (67)) in accordance with Proposition 7.10 and that this is a monotone sequence in accordance with Proposition 7.8.

The next result should be well-known but we could not find a reference.

**Lemma 8.8.** *Let $X$ be toric Fano. Let $P\subset M_{\mathbb{R}}$ (see (17), (26)) be the polytope associated to $(X,-K_X)$. Then $\alpha\le 1/2$, and $P$ is centrally symmetric (i.e., $P=-P$) if and only if $\alpha=1/2$. If $P$ is not centrally symmetric then $\alpha\le 1/3$.*

*Proof.* If $P\neq -P$, then there is $u_0\in P\setminus -P$. By (27), $-P=\{y\in M_{\mathbb{R}}:\max_j\langle v_j,y\rangle\le 1\}$. Thus $\langle u_0,v_{i_0}\rangle>1$ for some $v_{i_0}\in\Delta_1\subset N$ (recall (20)). Recalling Theorem 1.4 and (67), or [6, Corollary 7.16],

$$\alpha=\min_{u\in P}\min_i\frac{1}{1+\langle u,v_i\rangle}\le\frac{1}{1+\langle u_0,v_{i_0}\rangle}<\frac{1}{2}.$$

In fact, by convexity one may take $u_0\in\operatorname{Ver}P\setminus -P$, so $u_0\in M$ by Lemma 2.3 and hence $\mathbb{Z}\ni\langle u_0,v_{i_0}\rangle\ge 2$, i.e., $\alpha\le 1/3$.

If $P=-P$, by (66) and (67),

$$\alpha=\frac{1}{1+\max\limits_P\|\cdot\|_{-P}}=\frac{1}{1+\max\limits_P\|\cdot\|_P}=\frac{1}{1+1}=\frac{1}{2}$$

by (13) and (15). ∎

**Example 8.9.** The purpose of this paragraph is to show that when $n=2$, condition $(*_P)$ holds for a Delzant polytope $P$ if and only if $P$ is not centrally symmetric, i.e., $P\neq -P$. Alternatively, in $n=2$, Conjecture 1.5 holds if and only if $\alpha=1/2$, by Lemma 8.8. However, in dimension $n\ge 3$ this is no longer the case: for instance $\mathbb{P}^2\times\mathbb{P}^1$ has a non-centrally symmetric polytope $P$ but condition $(*_P)$ fails.

When $n=2$, the centrally symmetric Delzant polytopes are associated to $\mathbb{P}^2$ blown-up at the three non-colinear points, and $\mathbb{P}^1\times\mathbb{P}^1$ (Examples 8.4–8.5). Recall the description of $(*_P)$ given in §7.1. Since $P=-P$, the intersection of $P$ and $\max_P\|\cdot\|_{-P}\partial(-P)$ is precisely all of $\partial P$, i.e., $(*_P)$ fails.

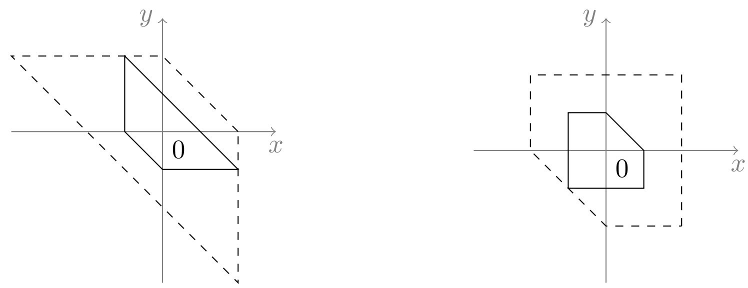


Figure 5: The polytope $P$ for $\mathbb{P}^2$ blown up 1 or 2 points, and the level set $\{\|\cdot\|_{-P} = 2\}$.

Thus, it remains to check $\mathbb{P}^2$ blown-up at up to two points. We have already showed that $(*_P)$ holds for (the non-centrally symmetric) $P$ coming from $\mathbb{P}^2$ (Example 8.7). It thus remains to check the remaining two cases. The polytope for the 1-point blow-up is a subset of the one for $\mathbb{P}^2$ by chopping a corner. Thus the intersection of $P$ and $\max_P \|\cdot\|_{-P}\partial(-P) = -2\partial P$ is still just the two vertices $(-1, 2)$ and $(2, -1)$. For the 2-point blow-up the intersection of $P$ and $\max_P \|\cdot\|_{-P}\partial(-P) = -2\partial P$ is just the vertex $(-1, -1)$, concluding the proof. See Figure 5.

# References


[1] H. Ahmadinezhad, I. Cheltsov, J. Schicho, On a conjecture of Tian, Math. Z., 288 (2018), 217–241

[2] V. V. Batyrev, E. N. Selivanova, Einstein–Kahler metrics on symmetric toric Fano manifolds, J. reine angew. Math., 519 (1999), 225–236.

[3] A. Cannas da Silva, Symplectic toric manifolds, in: Symplectic geometry of integrable Hamiltonian systems (Barcelona, 2001), Adv. Courses Math. CRM Barcelona, 2003, 85–173.

[4] B. Berndtsson, V. Mastrantonis, Y.A. Rubinstein, $L^p$-polarity, Mahler volumes, and the isotropic constant, preprint, 2023, arxiv:2304.14363.

[5] C. Birkar, Singularities of linear systems and boundedness of Fano varieties, Annals of Math., 193 (2022), 347–405.

[6] H. Blum, M. Jonsson, Thresholds, valuations, and K-stability, Adv. Math., 365 (2020), 107062.

[7] E. Calabi, The variation of K"ahler metrics. I. The structure of the space; II. A minimum problem, Bull. Amer. Math. Soc. 60 (1954), 167–168.

[8] E. Calabi, On geodesics in a function space of Kähler metrics, oral communication in: International Congress on Differential Geometry in memory of Alfred Gray, September 18–23, 2000, Universidad del Pais Vasco, Bilbao, Spain.

[9] D. Catlin, The Bergman Kernel and a Theorem of Tian, in: Analysis and geometry in several complex variables (G. Komatsu et al., Eds.) Birkhäuser, 1999, 1–23.

[10] I.A. Cheltsov, Log-canonical thresholds of del Pezzo surfaces, Geom. Funct. Anal.. 18 (2008), 1118–1144.

[11] I.A. Cheltsov, Y.A. Rubinstein, Asymptotically log Fano varieties, Adv. Math. 285 (2015), 1241–1300.

[12] I. A. Cheltsov, K. A. Shramov, Log-canonical thresholds for nonsingular Fano threefolds, with an appendix by J.-P. Demailly, Russian Math. Surveys, 63 (2008), 859–958.

[13] T. Delcroix, Alpha-invariant of toric line bundles, Ann. Polonici Math. 114 (2015), 13–27.

[14] T. Delcroix, Log canonical thresholds on group compactifications, Alg. Geom. 4 (2017), 203–220.

[15] E. Ehrhart, Sur un problème de géométrie diophantienne linéaire I. Polyèdres et réseaux, J. reine angew. Math. 226 (1967) 1–29.

[16] E. Ehrhart, Démonstration de la loi de réciprocité pour un polyèdre entier, C. R. Acad. Sci. Paris Sér. A-B 265 (1967) A5–A7.

[17] G.B. Folland, Real Analysis: Modern Techniques and Applications (2nd Ed.), Wiley, 1999.

[18] K. Fujita, Y. Odaka, On the K-stability of Fano varieties and anticanonical divisors, Tohoku Math. J. (2), 70(4) (2018), 511–521.

[19] W. Fulton, Introduction to toric varieties, Princeton Univ. Press, 1993.

[20] P. Griffiths, J. Harris, Principles of algebraic geometry, John Wiley & Sons, 1978.

[21] P. M. Gruber, Convex and Discrete Geometry, Springer, 2007.

[22] C. Jin, Y.A. Rubinstein, Asymptotics of stability thresholds on toric Fanos, (with an appendix by Y. Liu), preprint, 2024.

[23] H. Li, Y. Shi, Y. Yao, A criterion for the properness of the K-energy in a general Kähler class, Math. Ann., 361 (2015), 135–156.

[24] Y. Li, X. Zhu, Tian $\alpha^{\hat{K}}_{m,k}$-invariants on group compactifications, Math. Z. 298 (2021), 231–259.

[25] T.R. Rockafellar, Convex analysis, Princeton University Press, 1970.

[26] Y.A. Rubinstein, G. Tian, K. Zhang, Basis divisors and balanced metrics, J. reine angew. Math. 778 (2021), 171–218.

[27] Y. Shi, On the $\alpha$-invariants of cubic surfaces with Eckardt points, Adv. Math. 225, (2010), 1285–1307.

[28] J. Song, The $\alpha$-invariant on toric Fano manifolds, Amer. J. Math. 127 (2005), 1247–1259.

[29] G. Tian, On Kähler–Einstein metrics on certain Kähler manifolds with $c_1(M) > 0$, Invent. Math. 89 (1987), 225–246.

[30] G. Tian, Kähler metrics on algebraic manifolds, Ph.D. Thesis, Harvard University, 1988.

[31] G. Tian, On a set of polarized Kähler metrics on algebraic manifolds, J. Differential Geom. 32 (1990), 99–130.

[32] G. Tian, On one of Calabi's problems, Proc. Symp. Pure Math. 52 (1991), Part 2, 543–556.

[33] G. Tian, Existence of Einstein Metrics on Fano Manifolds, in: Metric and Differential Geometry, Springer, 2012, 119–159.

[34] S.-T. Yau, Nonlinear analysis in geometry, *L'Énseignement Mathématique* 33 (1987), 109–158.

[35] S. Zelditch, Szegö kernels and a theorem of Tian, Int. Math. Res. Notices (1998), 317–331.

[36] K. Zhang, Continuity of delta invariants and twisted Kähler–Einstein metrics, Adv. Math. 388 (2021).

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