Sub-Dirac operators, spectral Einstein functionals and the noncommutative residue

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Abstract

In this paper, we define the spectral Einstein functional associated with the sub-Dirac operator for manifolds with boundary. A proof of the Dabrowski-Sitarz-Zalecki type theorem for spectral Einstein functions associated with the sub-Dirac operator on four-dimensional manifolds with boundary is also given.

Keywords: Sub-Dirac operator; noncommutative residue; spectral Einstein functional.

1. Introduction

The noncommutative residue which is found in [1, 2] by M. Wodzicki and V. W. Guillemin plays a prominent role in noncommutative geometry. Recently Dabrowski etc. [3] obtained the metric and Einstein functionals by two vector fields and Laplace-type operators over vector bundles, giving an interesting example of the spinor connection and square of the Dirac operator. An eminent spectral scheme is the small-time asymptotic expansion of the (localised) trace of heat kernel [4, 5] that generates geometric objects on manifolds such as residue, scalar curvature, and other scalar combinations of curvature tensors. The theory has very rich structures both in physics and mathematics.

In [6], Connes used the noncommutative residue to derive a conformal 4-dimensional Polyakov action analogy. Connes proved that the noncommutative residue on a compact manifold M coincided with Dixmier's trace on pseudodifferential operators of order $-dimM$ [7]. Furthermore, Connes claimed the noncommutative residue of the square of the inverse of the Dirac operator was proportioned to the Einstein-Hilbert action in[7, 8]. Kastler provided a brute-force proof of this theorem in [9], while Kalau and Walze proved it in the normal coordinates system simultaneously in [10], which is called the Kastler-Kalau-Walze theorem now. Building upon the theory of the noncommutative reside introduced by Wodzicki, Fedosov etc. [11] constructed a noncommutative residue on the algebra of classical elements in Boutet de Monvel's calculus on a compact manifold with boundary of dimension $n > 2$. With elliptic pseudodifferential operators and noncommutative residue, it is natural way for investigating the Kastler-Kalau-Walze type theorem and operator-theoretic explanation of the gravitational action on manifolds with boundary. Concerning Dirac operators and signature operators, Wang performed computations of the noncommutative residue and successfully demonstrated the Kastler-Kalau-Walze type theorem for manifolds with boundaries [12–14].

Earlier Jean-Michel Bismut [15] proved a local index theorem for Dirac operators on a Riemannian manifold M associated with connections on TM which have non zero torsion. In [16], Ackermann and Tolksdorf proved a generalized version of the well-known Lichnerowicz formula for the square of the most general Dirac operator with torsion D_T on an even-dimensional spin manifold associated to a metric connection with torsion. Pfäffle and Stephan considered compact Riemannian spin manifolds without boundary equipped with orthogonal connections, and investigated the induced Dirac operators in [17]. Wang, Wang

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and Wu computed $\widetilde{wres}[\pi^+\tilde{\nabla}_X\tilde{\nabla}_Y(D_T^*D_T)^{-1}\circ\pi^+(D_T^*D_T)^{-1}]$ and $\widetilde{wres}[\pi^+\tilde{\nabla}_X\tilde{\nabla}_Y D_T^{-1}\circ\pi^+(D_T^*D_TD_T^*)]$ in [20], and provided an important reference for our paper calculation. In [25], in order to prove the Connes vanshing theorem, Liu and Zhang introduced the sub-Dirac operator. In [19] Wang and Wang defined lower dimensional volumes associated to sub-Dirac operators for foliations and compute these lower dimensional volumes. They also prove the Kastler-Kalau-Walze type theorems for foliations with or without boundary.

The purpose of this paper is to generalize the results in [3], [19], [20], and get spectral functionals associated with sub-Dirac operators on compact manifolds with boundary. For lower dimensional compact Riemannian manifolds with boundary, we compute the 4-dimensional residue of $\widetilde{\nabla}_X\widetilde{\nabla}_Y D_F^{-4}$ and get Dabrowski-Sitarz-Zalecki theorems.

2. Spectral functionals for the sub-Dirac operator

In this section, we shall restrict our attention to the sub-Dirac operators for foliations. Let (M,F) be a closed foliation and M has spin leave, g^F be a metric on F. Let g^{TM} be a metric on TM which restricted to g^F on F. Let $F^\perp$ be the orthogonal complement of F in TM with respect to g^{TM}. Then we have the following orthogonal splitting

$$TM = F \oplus F^\perp, \tag{2.1}$$

$$g^{TM} = g^F \oplus g^{F^\perp}, \tag{2.2}$$

where $g^{F^\perp}$ is the restriction of g^{TM} to $F^\perp$.

Let $P, P^\perp$ be the orthogonal projection from TM to $F, F^\perp$ respectively. Let ∇^{TM} be the Levi-Civita connection of g^{TM} and $\nabla^F(resp.\nabla^{F^\perp})$be the restriction of ∇^{TM} to $F(resp.F^\perp)$. Without loss of generality, we assume F is oriented, spin and carries a fixed spin structure. Furthermore, we assume $F^\perp$ is oriented and we do not assume that $\dim F$ and $\dim F^\perp$ are even. By assumption, we may write

$$\nabla^F = P\nabla^{TM}P, \tag{2.3}$$

$$\nabla^{F^\perp} = P^\perp\nabla^{TM}P^\perp. \tag{2.4}$$

Moreover, the connections $\nabla^F\left(\nabla^{F^\perp}\right)$ lift to $S(F)\left(\wedge\left(F^{\perp,\star}\right)\right)$ naturally denoted by $\nabla^{S(F)}\left(\nabla^{\wedge\left(F^{\perp,\star}\right)}\right)$. Then $S(F)\otimes\wedge\left(F^{\perp,\star}\right)$ carries the induced tensor product connection

$$\nabla^{S(F)\otimes\wedge\left(F^{\perp,\star}\right)} = \nabla^{S(F)}\otimes \mathrm{Id}_{\wedge(F^{\perp,\star})} + \mathrm{Id}_{S(F)}\otimes\nabla^{\wedge\left(F^{\perp,\star}\right)}. \tag{2.5}$$

Then we can define $S\in\Omega\left(T^*M\right)\otimes\Gamma(\mathrm{End}(TM))$,

$$\nabla^{TM} = \nabla^F + \nabla^{F^\perp} + S \tag{2.6}$$

For any $X\in\Gamma(TM), S(X)$ exchanges $\Gamma(F)$ and $\Gamma\left(F^\perp\right)$ and is skew-adjoint with respect to g^{TM} . Let $\{f_i\}_{i=1}^p$ be an oriented orthonormal basis of F , $\{h_s\}_{s=1}^q$ be an oriented orthonormal basis of $F^\perp$, we define

$$\widetilde{\nabla}^F = \nabla^{S(F)\otimes\wedge\left(F^{\perp,\star}\right)} + \frac{1}{2}\sum_{j=1}^p\sum_{s=1}^q < S(.)f_j, h_s > c\left(f_j\right)c\left(h_s\right) \tag{2.7}$$

where the vector bundle $F^\perp$ might well be non-spin.

Definition 2.1. *[19] Let D_F be the operator mapping from $\Gamma\left(S(F)\otimes\wedge\left(F^{\perp,\star}\right)\right)$ to itself defined by*

$$D_F = \sum_{i=1}^p c\left(f_i\right)\widetilde{\nabla}_{f_i}^F + \sum_{s=1}^q c\left(h_s\right)\widetilde{\nabla}_{h_s}^F. \tag{2.8}$$

From (2.19) in [25], we shall make use of the Bochner Laplacian $\triangle^F$ stating that

$$\triangle^F := -\sum_{i=1}^{p}\left(\widetilde{\nabla}_{f_i}\right)^2 - \sum_{s=1}^{q}\left(\widetilde{\nabla}_{h_s}\right)^2 + \widetilde{\nabla}_{\sum_{i=1}^{p}\nabla^{TM}_{f_i} f_i} + \widetilde{\nabla}_{\sum_{s=1}^{q}\nabla^{TM}_{h_s} h_s}. \tag{2.9}$$

Let r_M be the scalar curvature of the metric g^{TM} . Let $R^{F^\perp}$ be the curvature tensor of $F^\perp$. From theorem 2.3 in [25], we have the following Lichnerowicz formula for D_F.

Lemma 2.2. *[25] The following identity holds*

$$\begin{aligned} D_F^2 =& \triangle^F + \frac{r_M}{4} + \frac{1}{4}\sum_{i=1}^{p}\sum_{r,s,t=1}^{q}\left\langle R^{F^\perp}(f_i, h_r) h_t, h_s\right\rangle c(f_i) c(h_r) \widehat{c}(h_s) \widehat{c}(h_t) \\ &+ \frac{1}{8}\sum_{i,j=1}^{p}\sum_{s,t=1}^{q}\left\langle R^{F^\perp}(f_i, f_j) h_t, h_s\right\rangle c(f_i) c(f_j) \widehat{c}(h_s) \widehat{c}(h_t) \\ &+ \frac{1}{8}\sum_{s,t,r,u=1}^{q}\left\langle R^{F^\perp}(h_r, h_l) h_t, h_s\right\rangle c(h_r) c(h_u) \widehat{c}(h_s) \widehat{c}(h_t). \end{aligned} \tag{2.10}$$

Let us now turn to compute the specification of D_F^3.

$$\begin{aligned} D_F^3 =& -\sum_l g^{tl}(x) c(\partial_t) g^{ij}\partial_i\partial_j\partial_l + \left[\sum_l g^{tl}(x) c(\partial_t)\right]\left\{\sum_{ij}\left(\partial_l g^{ij}\right)\partial_i\partial_j - 4\sum_{ij} g^{ij}\sigma_i\partial_l\partial_j \otimes Id_{\wedge(F^{\perp,\star})}\right. \\ &- Id_{S(F)}\otimes 4\sum_{ij} g^{ij}\tilde{\sigma}_i\partial_l\partial_j - 4\sum_{ij} g^{ij} s(\partial_j)\partial_l\partial_i + 2\sum_{ij} g^{ij}\sum_{kl}\Gamma^k_{ij}\partial_l\partial_k \\ &\left. - \left[\frac{1}{4}\sum_{s,t=1}^{p}\left\langle\nabla^F_{\partial_j} f_s, f_t\right\rangle c(f_s) c(f_t) + \frac{1}{4}\sum_{l,\alpha=1}^{q}\left\langle\nabla^{F^\perp}_{\partial_j} h_l, h_\alpha\right\rangle\left[c(h_l) c(h_\alpha) - \hat{c}(h_l)\hat{c}(h_\alpha)\right]\right]\sum_{i,j} g^{ij}\partial_i\partial_j\right\} \\ &+ \left[\sum_l g^{tl}(x) c(\partial_t)\right]\left\{-2\sum_{ij}\left(\partial_l g^{ij}\right)\sigma_i\partial_j \otimes Id_{\wedge(F^{\perp,\star})} - 2\sum_{ij} g^{ij}\left(\partial_l\sigma_i\right)\partial_j \otimes Id_{\wedge\left(F^{\perp,\star}\right)}\right. \\ &- Id_{S(F)\otimes 2}\left(\partial_l g^{ij}\right)\tilde{\sigma}_i\partial_j - Id_{S(F)}\otimes 2\sum_{ij} g^{ij}\left(\partial_l\tilde{\sigma}_i\right)\partial_j - 2\sum_{ij}\left(\partial_l g^{ij}\right) S(\partial_j)\partial_i \\ &\left. -2\sum_{ij} g^{ij}\left[\partial_l S(\partial_j)\right]\partial_i + \sum_{ij}\left(\partial_l g^{ij}\right)\sum_k\Gamma^k_{ij}\partial_k + \sum_{ij} g^{ij}\sum_k\left(\partial_l\Gamma^k_{ij}\right)\partial_k\right\} \\ &+ \left\{\frac{1}{4}\sum_{s,t=1}^{p}\left\langle\nabla^F_{\partial_j} f_s, f_t\right\rangle c(f_s) c(f_t) + \frac{1}{4}\sum_{l,\alpha=1}^{q}\left\langle\nabla^{F^\perp}_{\partial_j} h_l, h_\alpha\right\rangle\left[c(h_l) c(h_\alpha) - \hat{c}(h_l)\hat{c}(h_\alpha)\right]\right\} \\ &\times\left\{-\sum_{ij} g^{ij}\left[2\sigma_i\partial_j \otimes Id_{\wedge(F^{\perp,*})} + Id_{S(F)}\otimes 2\tilde{\sigma}_i\partial_j + S(\partial_j)\partial_i + S(\partial_i)\partial_j - \sum_k\Gamma^k_{ij}\partial_k\right] + \frac{r_M}{4}\right. \\ &+ \frac{1}{4}\sum_{i=1}^{p}\sum_{r,s,t=1}^{q}\left\langle R^{F^\perp}(f_i, h_r) h_t, h_s\right\rangle c(f_i) c(h_r)\widehat{c}(h_s)\widehat{c}(h_t) \\ &+ \frac{1}{8}\sum_{i,j=1}^{p}\sum_{s,t=1}^{q}\left\langle R^{F^\perp}(f_i, f_j) h_t, h_s\right\rangle c(f_i) c(f_j)\widehat{c}(h_s)\widehat{c}(h_t) \\ &\left. +\frac{1}{8}\sum_{s,t,r,u=1}^{q}\left\langle R^{F^\perp}(h_r, h_l) h_t, h_s\right\rangle c(h_r) c(h_u)\widehat{c}(h_s)\widehat{c}(h_t)\right\}. \end{aligned} \tag{2.11}$$

In order to get a Kastler-Kalau-Walze type theorem for foliations, Liu and Wang [26] considered the noncommutative residue of the $-n+2$ power of the sub-Dirac operator, and got the following Kastler-Kalau-Walze type theorem for foliations.

The following lemma of Dabrowski etc.'s Einstein functional play a key role in our proof of the Einstein functional . Let V, W be a pair of vector fields on a compact Riemannian manifold M, of dimension $n=2m$. Using the Laplace operator $\Delta=-(\sum_{j=1}^{n}\widetilde{\nabla}_{e_j}\widetilde{\nabla}_{e_j}-\widetilde{\nabla}_{\nabla^L_{e_j}e_j})+E$ acting on sections of a vector bundle $\overline{E}$ where $\widetilde{\nabla}$ is a connetion on $\overline{E}$, which may contain both some nontrivial connections and torsion, the spectral functionals over vector fields defined by

Lemma 2.3. *[3] The Einstein functional equal to*

$$Wres\big(\widetilde{\nabla}_V\widetilde{\nabla}_W\Delta^{-m}\big)=\frac{\upsilon_{n-1}}{6}2^m\int_M G(V,W)vol_g+\frac{\upsilon_{n-1}}{2}\int_M F(V,W)vol_g+\frac{1}{2}\int_M(\mathrm{Tr}E)g(V,W)vol_g,\quad(2.12)$$

where $G(V,W)$ denotes the Einstein tensor evaluated on the two vector fields, $F(V,W)=Tr(V_aW_bF_{ab})$ and F_{ab} is the curvature tensor of the connection T, $\mathrm{Tr}E$ denotes the trace of E and $\upsilon_{n-1}=\frac{2\pi^m}{\Gamma(m)}$.

The aim of this section is to prove the following.

Theorem 2.4. *For the Laplace (type) operator $\Delta_F=D_F^2$, the Einstein functional equal to*

$$Wres\big(\widetilde{\nabla}^F_V\widetilde{\nabla}^F_W(D^F)^{-2m}\big)=\frac{2^{\frac{p}{2}+q+1}\pi^{\frac{n}{2}}}{6\Gamma(\frac{p+q}{2})}\int_M G(V,W)dvol_g+2^{\frac{p}{2}+q-3}\int_M sg(V,W)dvol_M,\quad(2.13)$$

where s is the scalar curvature.

Proof. By the definition of connection $\widetilde{\nabla}^F$, we have

$$\begin{aligned}\widetilde{\nabla}^F_X=&\nabla_X^{S(F)\otimes(F^\perp,*)}+\frac{1}{2}\sum_{j=1}^{p}\sum_{s=1}^{q}\langle S(X)f_j,h_s\rangle c(f_j)c(h_s)\\=&X+\frac{1}{4}\sum_{j,l=1}^{p}\langle\nabla^F_Xf_j,f_l\rangle c(f_j)c(f_l)+\frac{1}{4}\sum_{s,t=1}^{q}\langle\nabla^{F^\perp}_Xh_s,h_t\rangle(c(h_s)c(h_t)-\hat{c}(h_s)\hat{c}(h_t))\\&+\frac{1}{2}\sum_{j=1}^{p}\sum_{s=1}^{q}\langle S(X)f_j,h_s\rangle c(f_j)c(h_s)\\:=&X+\overline{A}(X).\end{aligned}\quad(2.14)$$

Let $V=\sum_{a=1}^{n}V^ae_a$, $W=\sum_{b=1}^{n}W^be_b$, in view of that

$$F(V,W)=Tr(V_aW_bF_{ab})=\sum_{a,b=1}^{n}V_aW_bTr^{S(F)\otimes(F^\perp,*)}(F_{e_a,e_b}),\quad(2.15)$$

we obtain

$$F_{e_a,e_b}=e_a(\overline{A}(e_b))-e_b(\overline{A}(e_a))+\overline{A}(e_a)\overline{A}(e_b)-\overline{A}(e_b)\overline{A}(e_a)-\overline{A}([e_a,e_b]).\quad(2.16)$$

Also, straightforward computations yield

$$\begin{aligned}\mathrm{Tr}\big(e_a(\overline{A}(e_b))\big)=&\mathrm{Tr}\Big[e_a(\frac{1}{4}\sum_{j,l=1}^{p}\langle\nabla^F_Xf_j,f_l\rangle c(f_j)c(f_l)+\frac{1}{4}\sum_{s,t=1}^{q}\langle\nabla^{F^\perp}_Xh_s,h_t\rangle(c(h_s)c(h_t)-\hat{c}(h_s)\hat{c}(h_t))\\&+\frac{1}{2}\sum_{j=1}^{p}\sum_{s=1}^{q}\langle S(X)f_j,h_s\rangle c(f_j)c(h_s))]\\=&0,\end{aligned}\quad(2.17)$$

in the same vein

$$\mathrm{Tr}\overline{A}([e_a, e_b]) = 0, \tag{2.18}$$

so

$$F(V, W) = 0. \tag{2.19}$$

Let $D_F^2 = \Delta^F + E$, we have

$$\begin{aligned} E =& \frac{s}{4} + \frac{1}{4}\sum_{i=1}^{p}\sum_{r,s,t=1}^{q}\langle R^{F^\perp}(f_i, h_r)h_t, h_s\rangle c(f_i)c(h_r)\hat{c}(h_s)\hat{c}(h_t) \\ &+ \frac{1}{4}\sum_{i,j=1}^{p}\sum_{s,t=1}^{q}\langle R^{F^\perp}(f_i, f_j)h_t, h_s\rangle c(f_i)c(f_j)\hat{c}(h_s)\hat{c}(h_t) \\ &+ \frac{1}{4}\sum_{s,t,r,u=1}^{q}\langle R^{F^\perp}(h_r, h_u)h_t, h_s\rangle c(h_r)c(h_u)\hat{c}(h_s)\hat{c}(h_t), \end{aligned} \tag{2.20}$$

and

$$\mathrm{Tr}\langle R^{F^\perp}(f_i, h_r)h_t, h_s\rangle c(f_i)c(h_r)\hat{c}(h_s)\hat{c}(h_t) = 0, \tag{2.21}$$

$$\mathrm{Tr}\langle R^{F^\perp}(f_i, f_j)h_t, h_s\rangle c(f_i)c(f_j)\hat{c}(h_s)\hat{c}(h_t) = 0, \tag{2.22}$$

$$\mathrm{Tr}\langle R^{F^\perp}(h_r, h_u)h_t, h_s\rangle c(h_r)c(h_u)\hat{c}(h_s)\hat{c}(h_t) = 0. \tag{2.23}$$

Hence

$$\mathrm{Tr}E = \frac{s}{4}\mathrm{Tr}[Id] = \frac{s}{4}2^{\frac{p}{2}+q} = 2^{\frac{p}{2}+q-2}s, \tag{2.24}$$

which finishes the proof of Theorem 2.4. □

3. The residue for the sub-Dirac operator $\widetilde{\nabla}_X^F\widetilde{\nabla}_Y^F D_F^{-2}$ and D_F^{-2}

In this section, we compute the lower dimensional volume for 4-dimension compact manifolds with boundary and get a Dabrowski-Sitarz-Zalecki type formula in this case. Some basic knowledge such as boundary metric, frame, noncommutative residue of manifold with boundary and Boutet de Monvel's algebra can be referred to in [12], and we will not repeat it here. We will consider D_F^2. Since $[\sigma_{-4}(\widetilde{\nabla}_X^F\widetilde{\nabla}_Y^F D_F^{-2} \circ D_F^{-2})]|_M$ has the same expression as $[\sigma_{-4}(\widetilde{\nabla}_X^F\widetilde{\nabla}_Y^F D_F^{-2} \circ D_F^{-2})]|_M$ in the case of manifolds without boundary, so locally we can use Theorem 2.4 to compute the first term.

Corollary 3.1. *Let M be a 4-dimensional compact manifold without boundary and $\widetilde{\nabla}^F$ be an orthogonal connection. Then we get the volumes associated to $\widetilde{\nabla}_X^F\widetilde{\nabla}_Y^F D_F^{-2}$ and D_F^2 on compact manifolds without boundary*

$$Wres[\sigma_{-4}(\widetilde{\nabla}_X^F\widetilde{\nabla}_Y^F D_F^{-2} \circ D_F^{-2})] = \frac{8\pi^2}{3}\int_M G(X,Y)dvol_g + \int_M sg(X,Y)dvol_M, \tag{3.1}$$

where s is the scalar curvature.

Let p_1, p_2 be nonnegative integers and $p_1+p_2\leq n$, denote by $\sigma_l(\widetilde{A})$ the l-order symbol of an operator $\widetilde{A}$, an application of (3.5) and (3.6) in [12] shows that

Definition 3.2. *Spectral Einstein functional associated the sub-Dirac operator of manifolds with boundary is defined by*

$$vol_n^{\{p_1,p_2\}}M := \widetilde{Wres}[\pi^+(\widetilde{\nabla}_X^F\widetilde{\nabla}_Y^F(D_F^2)^{-p_1})\circ\pi^+(D_F^{-2})^{p_2}], \tag{3.2}$$

where $\pi^+(\widetilde{\nabla}_X^F\widetilde{\nabla}_Y^F(D_F^2)^{-p_1})$, $\pi^+(D_F^{-2})^{p_2}$ are elements in Boutet de Monvel's algebra [13].

For the sub-Dirac operator $\widetilde{\nabla}_X^F\widetilde{\nabla}_Y^F D_F^{-2}$ and D_F^{-2}, denote by $\sigma_l(\widetilde{A})$ the l-order symbol of an operator $\widetilde{A}$. An application of (2.1.4) in [12] shows that

$$\begin{aligned}&\widetilde{Wres}[\pi^+(\widetilde{\nabla}_X^F\widetilde{\nabla}_Y^F(D_F^{-2})^{p_1})\circ\pi^+(D_F^2)^{-p_2}]\\&=\int_M\int_{|\xi|=1}\mathrm{tr}_{S(F)\otimes\wedge(F^\perp,*)}[\sigma_{-n}(\widetilde{\nabla}_X^F\widetilde{\nabla}_Y^F(D_F^2)^{-p_1}\circ(D_F^2)^{-p_2}]\sigma(\xi)\mathrm{d}x+\int_{\partial M}\Phi,\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}\Phi=&\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j,k=0}^{\infty}\sum\frac{(-i)^{|\alpha|+j+k+l}}{\alpha!(j+k+1)!}\mathrm{tr}_{S(F)\otimes\wedge(F^\perp,*)}[\partial_{x_n}^j\partial_{\xi'}^\alpha\partial_{\xi_n}^k\sigma_r^+(\widetilde{\nabla}_X^F\widetilde{\nabla}_Y^F(D_F^2)^{-p_1})(x',0,\xi',\xi_n)\\&\times\partial_{x_n}^\alpha\partial_{\xi_n}^{j+1}\partial_{x_n}^k\sigma_l((D_F^2)^{-p_2})(x',0,\xi',\xi_n)]\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x',\end{aligned} \tag{3.4}$$

and the sum is taken over $r-k-|\alpha|+\ell-j-1=-n, r\leq -p_1, \ell\leq -p_2$.

So we only need to compute $\int_{\partial M}\Phi$. Recall the definition of the sub-Dirac operator D_F in [19].

$$D_F=\sum_{i=1}^{p}c(f_i)\widetilde{\nabla}_{f_i}+\sum_{s=1}^{q}c(h_s)\widetilde{\nabla}_{h_s}, \tag{3.5}$$

where $c(f_i), c(h_s)$ denotes the Clifford action.

We define

$$\begin{aligned}\widetilde{\nabla}_X^F :=&X+\frac{1}{2}\sum_{j=1}^{p}\sum_{s=1}^{q}\langle S(X)f_j,h_s\rangle c(f_j)c(h_s)+\frac{1}{4}\sum_{j,l=1}^{p}\langle\nabla_X^F f_j,f_l\rangle c(f_j)c(f_l)\\&+\frac{1}{4}\sum_{s,t=1}^{p}\langle\nabla_X^{F^\perp}h_s,h_t\rangle[c(h_s)c(h_t)-\hat{c}(h_s)\hat{c}(h_t)].\end{aligned} \tag{3.6}$$

which is a connection on $S(F)\otimes\wedge(F^\perp,*)$.

Set

$$\begin{aligned}&A(X)=\frac{1}{2}\sum_{j=1}^{p}\sum_{s=1}^{q}\langle S(X)f_j,h_s\rangle c(f_j)c(h_s);\quad M(X)=\frac{1}{4}\sum_{j,l=1}^{p}\langle\nabla_X^F f_j,f_l\rangle c(f_j)c(f_l);\\&N(X)=\frac{1}{4}\sum_{s,t=1}^{p}\langle\nabla_X^{F^\perp}h_s,h_t\rangle[c(h_s)c(h_t)-\hat{c}(h_s)\hat{c}(h_t)].\end{aligned} \tag{3.7}$$

Let $\widetilde{\nabla}_X^F=X+A(X)+M(X)+N(X)$, and $\widetilde{\nabla}_Y^F=Y+A(Y)+M(Y)+N(Y)$, by (2.11), we obtain

$$\begin{aligned}\widetilde{\nabla}_X^F\widetilde{\nabla}_Y^F=&(X+A(X)+M(X)+N(X))(Y+A(Y)+M(Y)+N(Y))\\=&XY+M(Y)X+N(Y)X+A(Y)X+M(X)Y+N(X)Y+A(X)Y\\&+X[M(Y)]+X[N(Y)]+X[A(Y)]+M(X)M(Y)+M(X)N(Y)+M(X)A(Y)\\&+N(X)M(Y)+N(X)N(Y)+N(X)A(Y)+A(X)M(Y)+A(X)N(Y)+A(X)A(Y),\end{aligned} \tag{3.8}$$

where $X=\Sigma_{j=1}^{n}X_j\partial_{x_j}, Y=\Sigma_{l=1}^{n}Y_l\partial_{x_l}$.

Let $g^{ij}=g(dx_i,dx_j)$, $\xi=\sum_k \xi_j dx_j$ and $\nabla^L_{\partial_i}\partial_j=\sum_k\Gamma_{ij}^k\partial_k$, we get

$$\sigma_k=-\frac{1}{4}\sum_{k,l}\omega_{k,l}\left(\partial_k\right)c\left(f_k\right)c\left(f_l\right);\quad \tilde{\sigma}_k=\frac{1}{4}\sum_{r,t}\omega_{r,t}\left(\partial_k\right)\left[\bar{c}\left(h_r\right)\bar{c}\left(h_t\right)-c\left(h_r\right)c\left(h_t\right)\right];$$
$$\xi^j=g^{ij}\xi_i;\quad \Gamma^k=g^{ij}\Gamma_{ij}^k;\quad \sigma^j=g^{ij}\sigma_i;\quad \tilde{S}(\partial i)=g^{ij}S(\partial i). \tag{3.9}$$

Then we have the following lemmas.

Lemma 3.3. *The following identities hold:*

$$\begin{aligned}\sigma_0(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y)=&X[M(Y)]+X[N(Y)]+X[A(Y)]+M(X)M(Y)+M(X)N(Y)+M(X)A(Y)\\&+N(X)M(Y)+N(X)N(Y)+N(X)A(Y)+A(X)M(Y)+A(X)N(Y)+A(X)A(Y);\end{aligned} \tag{3.10}$$

$$\begin{aligned}\sigma_1(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y)=&\sqrt{-1}\sum_{j,l=1}^{n}X_j\frac{\partial_{Y_l}}{\partial_{x_j}}\xi_l+\sqrt{-1}\sum_{j=1}^{n}[M(Y)+N(Y)+A(Y)]X_j\xi_j\\&+\sqrt{-1}\sum_{l=1}^{n}[M(X)+N(X)+A(X)]Y_l\xi_l;\end{aligned} \tag{3.11}$$

$$\sigma_2(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y)=-\sum_{j,l=1}^{n}X_jY_l\xi_j\xi_l. \tag{3.12}$$

Lemma 3.4. *[19] The following identities hold:*

$$\sigma_{-1}(D_F^{-1})=\frac{ic(\xi)}{|\xi|^2}; \tag{3.13}$$

$$\sigma_{-2}(D_F^{-1})=\frac{c(\xi)\sigma_0(D_F)c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}\sum_j c(\mathrm{d}x_j)[\partial_{x_j}(c(\xi))|\xi|^2-c(\xi)\partial_{x_j}(|\xi|^2)]; \tag{3.14}$$

$$\sigma_{-2}\left(D_F^{-2}\right)=|\xi|^{-2}; \tag{3.15}$$

$$\begin{aligned}\sigma_{-3}\left(D_F^{-2}\right)=&-\sqrt{-1}|\xi|^{-4}\xi_k\Big(\Gamma^k-2\sigma^k\otimes\mathrm{Id}_{\wedge(F^{\perp,\star})}-\mathrm{Id}_{S(F)}\otimes2\tilde{\sigma}^k\\&-\frac{1}{2}\sum_{i,j=1}^{2p}\sum_{s=1}^{q}\left\langle\nabla^{TM}_{\partial_k}f_j,h_s\right\rangle c\left(f_j\right)c\left(h_s\right)-\frac{1}{2}\sum_{s,t=1}^{q}\sum_{i=1}^{2p}\left\langle\nabla^{TM}_{\partial_k}f_j,h_s\right\rangle c\left(f_j\right)c\left(h_s\right)\Big)\\&-\sqrt{-1}|\xi|^{-6}2\xi^j\xi_\alpha\xi_\beta\partial_jg^{\alpha\beta}.\end{aligned} \tag{3.16}$$

where,

$$\begin{aligned}\sigma_0\left(D_F\right)=&-\frac{1}{4}\sum_{i,k,l}\omega_{k,l}\left(f_i\right)c\left(f_i\right)c\left(f_k\right)c\left(f_l\right)\otimes\mathrm{Id}_{\wedge(F^{\perp,\star})}\\&-\frac{1}{4}\sum_{s,k,l}\omega_{k,l}\left(f_i\right)c\left(f_k\right)c\left(f_l\right)c\left(h_s\right)\otimes\mathrm{Id}_{\wedge(F^{\perp,\star})}\\&+\mathrm{Id}_{S(F)}\otimes\frac{1}{4}\sum_{i,r,t}\omega_{r,t}\left(f_i\right)c\left(f_i\right)\left[\bar{c}\left(h_r\right)\bar{c}\left(h_t\right)-c\left(h_r\right)c\left(h_t\right)\right]\\&+\mathrm{Id}_{S(F)}\otimes\frac{1}{4}\sum_{s,r,t}\omega_{r,t}\left(h_s\right)c\left(h_s\right)\left[\bar{c}\left(h_r\right)\bar{c}\left(h_t\right)-c\left(h_r\right)c\left(h_t\right)\right]\\&+\frac{1}{2}\sum_{i,j=1}^{p}\sum_{s=1}^{q}\left\langle\nabla^{TM}_{f_i}f_j,h_s\right\rangle c\left(f_i\right)c\left(f_j\right)c\left(h_s\right)+\frac{1}{2}\sum_{s,t=1}^{q}\sum_{i=1}^{p}\left\langle\nabla^{TM}_{h_s}h_t,f_i\right\rangle c\left(h_s\right)c\left(h_t\right)c\left(f_i\right).\end{aligned}$$

Lemma 3.5. *The following identities hold:*

$$\sigma_0(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-2}) = -\sum_{j,l=1}^{n} X_jY_l\xi_j\xi_l|\xi|^{-2}; \tag{3.17}$$

$$\begin{aligned}\sigma_{-1}(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-2}) =&\sigma_2(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y)\sigma_{-3}(D_F^{-2}) + \sigma_1(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y)\sigma_{-2}((D_F^{-2}) \\ &+\sum_{j=1}^{n}\partial_{\xi_j}\big[\sigma_2(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y)\big]D_{x_j}\big[\sigma_{-2}((D_F^{-2})\big].\end{aligned} \tag{3.18}$$

Now we need to compute $\int_{\partial M}\Phi$. When $n=p+q=4$, then $\mathrm{tr}_{S(F)\otimes(F^{\perp},*)}[\mathtt{id}]=8$, the sum is taken over $r+l-k-j-|\alpha|=-3,\ \ r\leq 0,\ \ l\leq -2$, then we have the following five cases:

case a) I) $r=0,\ l=-2,\ k=j=0,\ |\alpha|=1$.

By (3.4), we get

$$\Phi_1 = -\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{Tr}[\partial^{\alpha}_{\xi'}\pi^+_{\xi_n}\sigma_0(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-2})\times\partial^{\alpha}_{x'}\partial_{\xi_n}\sigma_{-2}(D_F^{-2})](x_0)d\xi_n\sigma(\xi')dx'. \tag{3.19}$$

By Lemma 2.2 in [13], for $i<n$, then

$$\partial_{x_i}\sigma_{-2}(D_F^{-2})(x_0) = \partial_{x_i}(|\xi|^{-2})(x_0) = -\frac{\partial_{x_i}(|\xi|^2)(x_0)}{|\xi|^4} = 0, \tag{3.20}$$

so $\Phi_1=0$.

case a) II) $r=0,\ l=-2,\ k=|\alpha|=0,\ j=1$.

By (3.4), we get

$$\Phi_2 = -\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{Tr}[\partial_{x_n}\pi^+_{\xi_n}\sigma_0(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-2})\times\partial^2_{\xi_n}\sigma_{-2}(D_F^{-2})](x_0)d\xi_n\sigma(\xi')dx'. \tag{3.21}$$

By Lemma 3.3, we have

$$\partial^2_{\xi_n}\sigma_{-2}(D_F^{-2})(x_0) = \partial^2_{\xi_n}(|\xi|^{-2})(x_0) = \frac{6\xi_n^2-2}{(1+\xi_n^2)^3}. \tag{3.22}$$

It follows that

$$\partial_{x_n}\sigma_0(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-2})(x_0) = \partial_{x_n}(-\sum_{j,l=1}^{n}X_jY_l\xi_j\xi_l|\xi|^{-2}) = \frac{\sum_{j,l=1}^{n}X_jY_l\xi_j\xi_lh'(0)|\xi'|^2}{(1+\xi_n^2)^2}. \tag{3.23}$$

By integrating formula, we obtain

$$\begin{aligned}\pi^+_{\xi_n}\partial_{x_n}\sigma_0(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-2})(x_0) =& \partial_{x_n}\pi^+_{\xi_n}\sigma_0(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y(D_F^{-2}) \\ =& -\frac{i\xi_n+2}{4(\xi_n-i)^2}\sum_{j,l=1}^{n-1}X_jY_l\xi_j\xi_lh'(0) - \frac{i\xi_n}{4(\xi_n-i)^2}X_nY_nh'(0) \\ &-\frac{i}{4(\xi_n-i)^2}\sum_{j=1}^{n-1}X_jY_n\xi_j - \frac{i}{4(\xi_n-i)^2}\Sigma_{l=1}^{n-1}X_nY_l\xi_l.\end{aligned} \tag{3.24}$$

We note that $i<n,\ \ \int_{|\xi'|=1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2d+1}}\sigma(\xi')=0$, so we omit some items that have no contribution for

computing **case a) II)**. From (3.32) and (3.34), we obtain

$$
\begin{aligned}
&\mathrm{Tr}[\partial_{x_n}\pi^+_{\xi_n}\sigma_0(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-2})\times\partial^2_{\xi_n}\sigma_{-2}(D_F^{-2})](x_0)\\
&=4[\frac{i-3i\xi_n^2}{(\xi_n-i)^4(\xi_n+i)^3}+\frac{1-3\xi_n^2}{(\xi_n-i)^5(\xi_n+i)^3}]\sum_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l h'(0)\\
&+4[\frac{i-3i\xi_n^2}{(\xi_n-i)^4(\xi_n+i)^3}-\frac{1-3\xi_n^2}{(\xi_n-i)^5(\xi_n+i)^3}]X_nY_nh'(0)\\
&+4\frac{(1-3\xi_n^2)i}{(\xi_n-i)^5(\xi_n+i)^3}\sum_{j=1}^{n-1}X_jY_n\xi_j+4\frac{(1-3\xi_n^2)i}{(\xi_n-i)^5(\xi_n+i)^3}\sum_{l=1}^{n-1}X_nY_l\xi_l. \qquad (3.25)
\end{aligned}
$$

Therefore, we get

$$
\Phi_2=\left(\frac{5}{24}\sum_{j=1}^{n-1}X_jY_j-\frac{1}{8}X_nY_n\right)h'(0)\pi\Omega_3dx', \qquad (3.26)
$$

where Ω_3 is the canonical volume of S^2.

case a) III) $r=0,\ l=-2,\ j=|\alpha|=0,\ k=1.$

By (3.4), we get

$$
\begin{aligned}
\Phi_3&=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{Tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_0(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-2})\times\partial_{\xi_n}\partial_{x_n}\sigma_{-2}(D_F^{-2})](x_0)d\xi_n\sigma(\xi')dx'\\
&=\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{Tr}[\partial^2_{\xi_n}\pi^+_{\xi_n}\sigma_0(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-2})\times\partial_{x_n}\sigma_{-2}(D_F^{-2})](x_0)d\xi_n\sigma(\xi')dx'. \qquad (3.27)
\end{aligned}
$$

By Lemma 4.4, we have

$$
\partial_{x_n}\sigma_{-2}(D_F^{-2})(x_0)|_{|\xi'|=1}=-\frac{h'(0)}{(1+\xi_n^2)^2}. \qquad (3.28)
$$

An easy calculation gives

$$
\begin{aligned}
\pi^+_{\xi_n}\sigma_0(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y(D_F^{-2})(x_0)|_{|\xi'|=1}&=\frac{i}{2(\xi_n-i)}\sum_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l-\frac{i}{2(\xi_n-i)}X_nY_n\\
&-\frac{1}{2(\xi_n-i)}\sum_{j=1}^{n-1}X_jY_n\xi_j-\frac{1}{2(\xi_n-i)}\sum_{l=1}^{n-1}X_nY_l\xi_l. \qquad (3.29)
\end{aligned}
$$

Also, straightforward computations yield

$$
\begin{aligned}
\partial^2_{\xi_n}\pi^+_{\xi_n}\sigma_0(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y(D_F^{-2})(x_0)|_{|\xi'|=1}&=\frac{i}{(\xi_n-i)^3}\sum_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l-\frac{1}{(\xi_n-i)^3}X_nY_n\\
&-\frac{1}{(\xi_n-i)^3}\sum_{j=1}^{n-1}X_jY_n\xi_j-\frac{1}{(\xi_n-i)^3}\sum_{l=1}^{n-1}X_nY_l\xi_l. \qquad (3.30)
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
\Phi_3&=\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{Tr}[\partial^2_{\xi_n}\pi^+_{\xi_n}\sigma_0(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y(D_F^{-2})\times\partial_{x_n}\sigma_{-2}(D_F^{-2})](x_0)d\xi_n\sigma(\xi')dx'\\
&=\left(-\frac{5}{24}\sum_{j=1}^{n-1}X_jY_j+\frac{5}{8}X_nY_n\right)h'(0)\pi\Omega_3dx'. \qquad (3.31)
\end{aligned}
$$

case b) $r=0,\ l=-3,\ k=j=|\alpha|=0.$

By (3.4), we get

$$\begin{aligned}\Phi_4&=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{Tr}[\pi^+_{\xi_n}\sigma_0(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-2})\times\partial_{\xi_n}\sigma_{-3}(D_F^{-2})](x_0)d\xi_n\sigma(\xi')dx'\\&=i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{Tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_0(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-2})\times\sigma_{-3}(D_F^{-2})](x_0)d\xi_n\sigma(\xi')dx'.\end{aligned}\tag{3.32}$$

By Lemma 3.4, we have

$$\begin{aligned}\sigma_{-3}\left(D_F^{-2}\right)(x_0)|_{|\xi'|=1}&=\frac{-2ih'(0)\xi_n}{(1+\xi_n^2)^3}-\frac{i}{(1+\xi_n^2)^2}\Bigg(\frac{3}{2}h'(0)\xi_n+\frac{1}{2}\sum_{k,l}\omega_{k,l}\left(\partial^k\right)(x_0)c\left(f_k\right)c\left(f_l\right)\otimes\mathrm{Id}_{\wedge(F^{\perp,\star})}\\&-\mathrm{Id}_{S(F)}\otimes\frac{1}{2}\sum_{r,t}\omega_{r,t}\left(\partial^k\right)(x_0)\left[\hat{c}\left(h_r\right)\hat{c}\left(h_t\right)-c\left(h_r\right)c\left(h_t\right)\right]-\sum_{i,j=1}^{p}\sum_{s=1}^{q}\left\langle\nabla^{TM}_{\partial^k}f_j,h_s\right\rangle c\left(f_j\right)c\left(h_s\right)\Bigg).\end{aligned}\tag{3.33}$$

$$\begin{aligned}\partial_{\xi_n}\pi^+_{\xi_n}\sigma_0(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-2})(x_0)|_{|\xi'|=1}&=-\frac{i}{2(\xi_n-i)^2}\sum_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l+\frac{i}{2(\xi_n-i)^2}X_nY_n\\&+\frac{1}{2(\xi_n-i)^2}\sum_{j=1}^{n-1}X_jY_n\xi_j+\frac{1}{2(\xi_n-i)^2}\sum_{l=1}^{n-1}X_nY_l\xi_l.\end{aligned}\tag{3.34}$$

We note that $i<n,\ \int_{|\xi'|=1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2d+1}}\sigma(\xi')=0$, so we omit some items that have no contribution for computing **case b)**. By the trace identity $\mathrm{tr}(AB)=\mathrm{tr}(BA),\mathrm{tr}(A\otimes B)=\mathrm{tr}(A)\cdot\mathrm{tr}(B)$ and the relation of the Clifford action, we have

$$\mathrm{Tr}\left[c\left(f_k\right)c\left(f_l\right)\right]=-\delta_k^l2^{\frac{p}{2}},\tag{3.35}$$

$$\mathrm{Tr}\left[\hat{c}\left(h_r\right)\hat{c}\left(h_t\right)-c\left(h_r\right)c\left(h_t\right)\right]=2\delta_r^t2^q.\tag{3.36}$$

Then, we have

$$\begin{aligned}&\mathrm{Tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_0(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-2})\times\sigma_{-3}D_F^{-2})](x_0)\\&\sim[-\frac{8\xi_n}{(\xi_n-i)^5(\xi_n+i)^3}-\frac{6\xi_n}{(\xi_n-i)^4(\xi_n+i)^2}]h'(0)\sum_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l\\&\quad+[\frac{8\xi_n}{(\xi_n-i)^5(\xi_n+i)^3}+\frac{6\xi_n}{(\xi_n-i)^4(\xi_n+i)^2}]h'(0)X_nY_n.\end{aligned}\tag{3.37}$$

Therefore, we get

$$\begin{aligned}\Phi_4&=i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{Tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_0(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-2})\times\sigma_{-3}D_F^{-2})](x_0)d\xi_n\sigma(\xi')dx'\\&=\left(\frac{11}{24}\sum_{j=1}^{n-1}X_jY_j-\frac{11}{8}X_nY_n\right)h'(0)\pi\Omega_3dx'.\end{aligned}\tag{3.38}$$

case c) $r=-1,\ \ell=-2,\ k=j=|\alpha|=0.$

By (3.4), we get

$$\Phi_5=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{Tr}[\pi^+_{\xi_n}\sigma_{-1}(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_YD_F^{-2})\times\partial_{\xi_n}\sigma_{-2}(D_F^{-2})](x_0)d\xi_n\sigma(\xi')dx'. \tag{3.39}$$

By Lemma 4.4, we have

$$\partial_{\xi_n}\sigma_{-2}(D_F^{-2})(x_0)|_{|\xi'|=1}=-\frac{2\xi_n}{(\xi_n^2+1)^2}. \tag{3.40}$$

Since

$$\begin{aligned}\sigma_{-1}(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_YD_F^{-2}))(x_0)|_{|\xi'|=1}=&\sigma_2(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y)\sigma_{-3}(D_F^{-2})+\sigma_1(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y)\sigma_{-2}(D_F^{-2})\\&+\sum_{j=1}^n\partial_{\xi_j}\big[\sigma_2(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y)\big]D_{x_j}\big[\sigma_{-2}(D_F^{-2})\big].\end{aligned} \tag{3.41}$$

Explicit representation the first item of (3.41),

$$\begin{aligned}&\sigma_2(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y)\sigma_{-3}(D_F^{-2})(x_0)|_{|\xi'|=1}\\&=-\sum_{j,l=1}^nX_jY_l\xi_j\xi_l\times\Bigg\{\frac{-2ih'(0)\xi_n}{(1+\xi_n^2)^3}-\frac{i}{(1+\xi_n^2)^2}\Bigg(\frac{3}{2}h'(0)\xi_n+\frac{1}{2}\sum_{k,l}\omega_{k,l}\left(\partial^k\right)(x_0)c\left(f_k\right)c\left(f_l\right)\otimes\mathrm{Id}_{\wedge(F^{\perp,\star})}\\&-\mathrm{Id}_{S(F)}\otimes\frac{1}{2}\sum_{r,t}\omega_{r,t}\left(\partial^k\right)(x_0)\left[\hat{c}\left(h_r\right)\hat{c}\left(h_t\right)-c\left(h_r\right)c\left(h_t\right)\right]-\sum_{i,j=1}^p\sum_{s=1}^q\left\langle\nabla^{TM}_{\partial^k}f_j,h_s\right\rangle c\left(f_j\right)c\left(h_s\right)\Bigg)\Bigg\}.\end{aligned} \tag{3.42}$$

Explicit representation the second item of (3.41),

$$\begin{aligned}&\sigma_1(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y)\sigma_{-2}(D_F^{-2})(x_0)|_{|\xi'|=1}\\=&\Big(\sqrt{-1}\sum_{j,l=1}^nX_j\frac{\partial_{Y_l}}{\partial_{x_j}}\xi_l+\sqrt{-1}\sum_j[M(Y)+N(Y)+A(Y)]X_j\xi_j\\&+\sqrt{-1}\sum_l[M(X)+N(X)+A(X)]Y_l\xi_l\Big)\times|\xi|^{-2}.\end{aligned} \tag{3.43}$$

Explicit representation the third item of (3.41),

$$\begin{aligned}&\sum_{j=1}^n\partial_{\xi_j}\big[\sigma_2(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y)\big]D_{x_j}\big[\sigma_{-2}(D_F^{-2})\big](x_0)|_{|\xi'|=1}\\=&\sum_{j=1}^n\partial_{\xi_j}\big[-\sum_{j,l=1}^nX_jY_l\xi_j\xi_l\big](-\sqrt{-1})\partial_{x_j}\big(|\xi|^{-2}\big])\\=&\sum_{j=1}^n\sum_{l=1}^n\sqrt{-1}(x_jY_l+x_lY_j)\xi_l\partial_{x_j}(|\xi|^{-2}).\end{aligned} \tag{3.44}$$

We note that $i<n$, $\int_{|\xi'|=1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2d+1}}\sigma(\xi')=0$, so we omit some items that have no contribution for computing **case c)**. Also, straightforward computations yield

$$\begin{aligned}&\mathrm{Tr}[\pi^+_{\xi_n}\sigma_{-1}(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_YD_F^{-2})\times\partial_{\xi_n}\sigma_{-2}D_F^{-2})](x_0)|_{|\xi'|=1}\\&=\sum_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l\times\frac{18i\xi_n-10\xi_n^2}{(\xi_n-i)^5(\xi_n+i)^2}h'(0)+X_nY_n\times\big[\frac{-20i\xi_n^3-36\xi_n^2+12i\xi_n}{(\xi_n-i)^5(\xi_n+i)^2}+\frac{8\xi_n}{(\xi_n-i)^4(\xi_n+i)^2}\big]h'(0),\end{aligned} \tag{3.45}$$

Hence in this case,

$$\begin{aligned}\Phi_5 =& -i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{Tr}[\pi^+_{\xi_n}\sigma_{-1}(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-2})\times\partial_{\xi_n}\sigma_{-2}D_F^{-2})](x_0)d\xi_n\sigma(\xi')dx'\\ =& -\left(\frac{2}{3}\sum_{j=1}^{n-1}X_jY_j+\frac{5}{8}X_nY_n\right)h'(0)\pi\Omega_3dx'.\end{aligned}\tag{3.46}$$

Now Φ is the sum of the cases (a), (b) and (c). Let $X=X^T+X_n\partial_n,\ Y=Y^T+Y_n\partial_n$, then we have $\sum_{j=1}^{n-1}X_jY_j(x_0)=g(X^T,Y^T)(x_0)$.Therefore, we get

$$\Phi=\sum_{i=1}^{5}\Phi_i=-\left[\frac{5}{24}g(X^T,Y^T)+\frac{3}{2}X_nY_n\right]h'(0)\pi\Omega_3dx'.\tag{3.47}$$

Then we obtain following theorem

Theorem 3.6. *Let M be a 4-dimensional compact foliated manifold with boundary and spin leave , $\widetilde{\nabla}$ be an orthogonal connection with torsion. Then we get the volumes associated to $\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-2}$ and D_F^{-2}*

$$\begin{aligned}&\widetilde{\mathrm{Wres}}[\pi^+(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-2})\circ\pi^+(D_F^2)]\\ =&\frac{8\pi^2}{3}\int_M G(X,Y)dvol_g+\int_M sg(X,Y)dvol_M-\int_{\partial M}\left[\frac{5}{24}g(X^T,Y^T)+\frac{3}{2}X_nY_n\right]h'(0)\pi\Omega_3dx'.\end{aligned}\tag{3.48}$$

where s is the scalar curvature.

By [13], we have the extrinsic curvature $K=-\frac{3}{2}h'(0)$, so when $X_n=0$ or $Y_n=0$ on boundary, we have

Corollary 3.7.

$$\begin{aligned}&\widetilde{\mathrm{Wres}}[\pi^+(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-2})\circ\pi^+(D_F^2)]\\ =&\frac{8\pi^2}{3}\int_M G(X,Y)dvol_g+\int_M sg(X,Y)dvol_M+\frac{5}{36}\int_{\partial M}g(X,Y)_{\partial M}Kdvol_{\partial M}.\end{aligned}\tag{3.49}$$

So we have

Definition 3.8. *The spectral Einstein functional for manifolds with boundary*

$$E_H:=\int_M G(X,Y)dvol_g+c_0\int_{\partial M}g(X,Y)_{\partial M}Kdvol_{\partial M},\tag{3.50}$$

where c_0 is a constant.

4. The residue for sub-Dirac operators $\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-1}$ and D_F^{-3}

In this section, we compute the 4-dimension volume for sub-Dirac operators $\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-1}$ and D_F^{-3} . Since $[\sigma_{-4}(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-1}\circ D_F^{-3})]|_M$ has the same expression as $[\sigma_{-4}(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-1}\circ D_F^{-3})]|_M$ in the case of manifolds without boundary, so locally we can use Theorem 2.4 to compute the first term.

Corollary 4.1. *Let M be a four dimensional compact manifold without boundary and $\widetilde{\nabla}$ be an orthogonal connection with torsion. Then we get the volumes associated to $\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-1}$ and D_F^{-3} on compact manifolds without boundary*

$$\begin{aligned}&Wres[\sigma_{-4}(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-1}\circ D_F^{-3})]\\ =&\frac{8\pi^2}{3}\int_M G(X,Y)dvol_g+\int_M sg(X,Y)dvol_M,\end{aligned}\tag{4.1}$$

where s is the scalar curvature.

Similar definition3.2 ,we have

$$\widetilde{\Phi}=\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j,k=0}^{\infty}\sum\frac{(-i)^{|\alpha|+j+k+l}}{\alpha!(j+k+1)!}\mathrm{tr}_{S(F)\otimes\wedge(F^{\perp},*)}[\partial_{x_n}^j\partial_{\xi'}^{\alpha}\partial_{\xi_n}^k\sigma_r^+(\nabla_X^F\nabla_Y^F(D_F)^{-p_1})(x',0,\xi',\xi_n)$$
$$\times\partial_{x_n}^{\alpha}\partial_{\xi_n}^{j+1}\partial_{x_n}^k\sigma_l((D_F^3)^{-p_2})(x',0,\xi',\xi_n)]\mathtt{d}\xi_n\sigma(\xi')\mathtt{d}x',\tag{4.2}$$

From lemma 3.3 and lemma 3.4, we have

Lemma 4.2. *The following identities hold:*

$$\sigma_1(\widetilde{\nabla}_X^F\widetilde{\nabla}_Y^F D_F^{-1})=-\sqrt{-1}\sum_{j,l=1}^{n}X_jY_l\xi_j\xi_lc(\xi)|\xi|^{-2};\tag{4.3}$$

$$\begin{aligned}\sigma_0(\widetilde{\nabla}_X^F\widetilde{\nabla}_Y^F D_F^{-1})=&\sigma_2(\widetilde{\nabla}_X^F\widetilde{\nabla}_Y^F)\sigma_{-2}(D_F^{-1})+\sigma_1(\widetilde{\nabla}_X^F\widetilde{\nabla}_Y^F)\sigma_{-1}(D_F^{-1})\\&+\sum_{j=1}^{n}\partial_{\xi_j}\big[\sigma_2(\widetilde{\nabla}_X^F\widetilde{\nabla}_Y^F)\big]D_{x_j}\big[\sigma_{-1}(D_F^{-1})\big].\end{aligned}\tag{4.4}$$

Write

$$D_x^{\alpha}\quad=(-i)^{|\alpha|}\partial_x^{\alpha};\ \sigma(D_F^3)=p_3+p_2+p_1+p_0;\ \sigma(D_F^{-3})=\textstyle\sum_{j=3}^{\infty}q_{-j}.\tag{4.5}$$

By the composition formula of pseudodifferential operators, we have

$$\begin{aligned}1=\sigma(D_F^3\circ D_F^{-3})&=\sum_{\alpha}\frac{1}{\alpha!}\partial_{\xi}^{\alpha}[\sigma(D_F^3)D_x^{\alpha}[\sigma(D_F^{-3})]\\&=(p_3+p_2+p_1+p_0)(q_{-3}+q_{-4}+q_{-5}+\cdots)\\&\quad+\sum_j(\partial_{\xi_j}p_3+\partial_{\xi_j}p_2++\partial_{\xi_j}p_1+\partial_{\xi_j}p_0)(D_{x_j}q_{-3}+D_{x_j}q_{-4}+D_{x_j}q_{-5}+\cdots)\\&=p_3q_{-3}+(p_3q_{-4}+p_2q_{-3}+\sum_j\partial_{\xi_j}p_3D_{x_j}q_{-3})+\cdots,\end{aligned}\tag{4.6}$$

so

$$q_{-3}=p_3^{-1};\ q_{-4}=-p_3^{-1}[p_2p_3^{-1}+\sum_j\partial_{\xi_j}p_3D_{x_j}(p_-3^{-1})].\tag{4.7}$$

Hence by Lemma 4.1 in [19] and (3.9), we have

Lemma 4.3. *The symbol of the sub-Dirac operator:*

$$\sigma_{-3}(D_F^{-3})=\sqrt{-1}c(\xi)|\xi|^{-4};\tag{4.8}$$

$$\begin{aligned}\sigma_{-4}(D_F^{-3})=&\frac{c(\xi)\sigma_2(D_F^3)c(\xi)}{|\xi|^8}+\frac{\sqrt{-1}c(\xi)}{|\xi|^8}\bigg(|\xi|^4c(dx_n)\partial_{x_n}c(\xi')\\&-2h'(0)c(\mathrm{d}x_n)c(\xi)+2\xi_nc(\xi)\partial x_nc(\xi')+4\xi_nh'(0)\bigg),\end{aligned}\tag{4.9}$$

where,

$$\begin{aligned}\sigma_2(D_F^3)=&c(\mathrm{d}x_l)\partial_l(g^{i,j})\xi_i\xi_j+2c(\xi)\big[2\sigma^i\otimes\mathrm{Id}_{\wedge(F^{\perp},\star)}+\mathrm{Id}_{S(F)}\otimes2\tilde{\sigma}_i-\Gamma^k\\&+2\tilde{S}(\partial i)+(M(X)+N(X))|\xi|^2\big]\xi_k.\end{aligned}\tag{4.10}$$

Now we need to compute $\int_{\partial M}\widetilde{\Phi}$. When $n=4$, then $\mathrm{tr}_{S(F)\otimes\wedge(F^{\perp,\star})}[\mathtt{id}]=8$, the sum is taken over $r+l-k-j-|\alpha|=-3,\ \ r\leq 0,\ \ l\leq -2$, then we have the following five cases:

case a) I) $r=1,\ l=-2,\ k=j=0,\ |\alpha|=1$.

By (4.2), we get

$$\widetilde{\Phi}_1=-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{Tr}[\partial^{\alpha}_{\xi'}\pi^{+}_{\xi_n}\sigma_1(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-1})\times\partial^{\alpha}_{x'}\partial_{\xi_n}\sigma_{-3}(D_F^{-3})](x_0)d\xi_n\sigma(\xi')dx'. \tag{4.11}$$

By Lemma 2.2 in [13], for $i<n$, then

$$\partial_{x_i}\sigma_{-3}(D_F^{-3})(x_0)=\partial_{x_i}(\sqrt{-1}c(\xi)|\xi|^{-4})(x_0)=\sqrt{-1}\frac{\partial_{x_i}c(\xi)}{|\xi|^4}(x_0)+\sqrt{-1}\frac{c(\xi)\partial_{x_i}(|\xi|^4)}{|\xi|^8}(x_0)=0, \tag{4.12}$$

so $\widetilde{\Phi}_1=0$.

case a) II) $r=1,\ l=-3,\ k=|\alpha|=0,\ j=1$.

By (4.2), we get

$$\widetilde{\Phi}_2=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{Tr}[\partial_{x_n}\pi^{+}_{\xi_n}\sigma_1(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-1})\times\partial^2_{\xi_n}\sigma_{-3}(D_F^{-1})](x_0)d\xi_n\sigma(\xi')dx'. \tag{4.13}$$

By Lemma 4.3, we have

$$\partial^2_{\xi_n}\sigma_{-3}(D_F^{-3})(x_0)=\partial^2_{\xi_n}(c(\xi)|\xi|^{-4})(x_0)=\sqrt{-1}\frac{(20\xi_n^2-4)c(\xi')+12(\xi_n^3-\xi_n)c(\mathrm{d}x_n)}{(1+\xi_n^2)^4}, \tag{4.14}$$

and

$$\begin{aligned}\partial_{x_n}\sigma_1(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-1})(x_0)&=\partial_{x_n}(-\sqrt{-1}\sum_{j,l=1}^{n}X_jY_l\xi_j\xi_l c(\xi)|\xi|^{-2})\\&=-\sqrt{-1}\sum_{j,l=1}^{n}X_jY_l\xi_j\xi_l\left[\frac{\partial_{x_n}c(\xi')}{1+\xi_n^2}-\frac{c(\xi)h'(0)|\xi'|^2}{(1+\xi_n^2)^2}\right].\end{aligned} \tag{4.15}$$

We note that $i<n,\ \int_{|\xi'|=1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2d+1}}\sigma(\xi')=0$, so we omit some items that have no contribution for computing **case a) II)**. Then there is the following formula

$$\begin{aligned}&\mathrm{Tr}[\partial_{x_n}\pi^{+}_{\xi_n}\sigma_1(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-1})\times\partial^2_{\xi_n}\sigma_{-3}(D_F^{-3})](x_0)\\&=\sum_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l h'(0)\frac{-8(3i\xi_n^2+2\xi_n-i)}{(\xi_n-i)^5(\xi_n+i)^4}\\&+X_nY_nh'(0)\left[\frac{-16i(5\xi_n^2-1)+48(\xi_n^3-\xi_n)}{(\xi_n-i)^5(\xi_n+i)^4}+\frac{8(5\xi_n^2-1)+24i(\xi_n^3-\xi_n)}{(\xi_n-i)^6(\xi_n+i)^4}\right].\end{aligned} \tag{4.16}$$

Therefore, we get

$$\begin{aligned}\widetilde{\Phi}_2=&\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\left\{\sum_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l h'(0)\left[\frac{-8(3i\xi_n^2+2\xi_n-i)}{(\xi_n-i)^5(\xi_n+i)^4}\right]\right.\\&\left.+X_nY_nh'(0)\left[\frac{-16i(5\xi_n^2-1)+48(\xi_n^3-\xi_n)}{(\xi_n-i)^5(\xi_n+i)^4}+\frac{8(5\xi_n^2-1)+24i(\xi_n^3-\xi_n)}{(\xi_n-i)^6(\xi_n+i)^4}\right]\right\}d\xi_n\sigma(\xi')dx'\\=&\left(\frac{5}{16}\sum_{j=1}^{n-1}X_jY_j+\frac{1}{16}X_nY_n\right)h'(0)\pi\Omega_3dx',\end{aligned} \tag{4.17}$$

where Ω_3 is the canonical volume of S^2.

case a) III) $r=1,\ l=-3,\ j=|\alpha|=0,\ k=1.$

By (4.2), we get

$$\widetilde{\Phi}_3=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{Tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_1(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_YD_F^{-1})\times\partial_{\xi_n}\partial_{x_n}\sigma_{-3}(D_F^{-3})](x_0)d\xi_n\sigma(\xi')dx'$$
$$=\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{Tr}[\partial^2_{\xi_n}\pi^+_{\xi_n}\sigma_1(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_YD_F^{-1})\times\partial_{x_n}\sigma_{-3}(D_F^{-3})](x_0)d\xi_n\sigma(\xi')dx'. \tag{4.18}$$

By Lemma 4.3, we have

$$\partial_{x_n}\sigma_{-3}(D_F^{-3})(x_0)|_{|\xi'|=1}=\frac{i\partial_{x_n}[c(\xi')]}{(1+\xi_n^2)^4}-\frac{2ih'(0)c(\xi)|\xi'|^2_{g^{\partial M}}}{(1+\xi_n^2)^6}. \tag{4.19}$$

And we have

$$\partial^2_{\xi_n}\pi^+_{\xi_n}\sigma_1(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_YD_F^{-1})=-\frac{c(\xi')+ic(\mathrm{d}x_n)}{(\xi_n-i)^3}\sum_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l+\frac{c(\xi')+ic(\mathrm{d}x_n)}{(\xi_n-i)^3}X_nY_n$$
$$-\frac{ic(\xi')-c(\mathrm{d}x_n)}{(\xi_n-i)^3}\sum_{j=1}^{n-1}X_jY_n\xi_j-\frac{ic(\xi')-c(\mathrm{d}x_n)}{(\xi_n-i)^3}\sum_{l=1}^{n-1}X_nY_l\xi_l. \tag{4.20}$$

We note that $i<n,\ \int_{|\xi'|=1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2d+1}}\sigma(\xi')=0$, so we omit some items that have no contribution for computing **case a) III)**, then

$$\mathrm{Tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_1(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_YD_F^{-1})\times\partial_{\xi_n}\partial_{x_n}\sigma_{-3}(D_F^{-3})](x_0)|_{|\xi'|=1}$$
$$=\Big[\frac{4i}{(\xi_n-i)^5(\xi_n+i)^2}+\frac{16}{(\xi_n-i)^5(\xi_n+i)^3}\Big]h'(0)\sum_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l$$
$$-\Big[\frac{4i}{(\xi_n-i)^5(\xi_n+i)^2}+\frac{16}{(\xi_n-i)^5(\xi_n+i)^3}\Big]h'(0)X_nY_n. \tag{4.21}$$

Therefore, we get

$$\widetilde{\Phi}_3=\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\Bigg(\Big[\frac{4i}{(\xi_n-i)^5(\xi_n+i)^2}+\frac{16}{(\xi_n-i)^5(\xi_n+i)^3}\Big]h'(0)\sum_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l$$
$$-\Big[\frac{4i}{(\xi_n-i)^5(\xi_n+i)^2}+\frac{16}{(\xi_n-i)^5(\xi_n+i)^3}\Big]h'(0)X_nY_n\Bigg)d\xi_n\sigma(\xi')dx'$$
$$=\left(-\frac{25}{48}\sum_{j=1}^{n-1}X_jY_j+\frac{25}{16}X_nY_n\right)h'(0)\pi\Omega_3dx'. \tag{4.22}$$

case b) $r=0,\ l=-3,\ k=j=|\alpha|=0.$

By (4.2), we get

$$\widetilde{\Phi}_4=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{Tr}[\pi^+_{\xi_n}\sigma_0(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_YD_F^{-1})\times\partial_{\xi_n}\sigma_{-3}(D_F^{-3})](x_0)d\xi_n\sigma(\xi')dx'. \tag{4.23}$$

By Lemma 4.3, we obtain

$$\partial_{\xi_n}\sigma_{-3}(D_F^{-3})(x_0)|_{|\xi'|=1}=\frac{ic(\mathrm{d}x_n)}{(1+\xi_n^2)^2}-\frac{4\sqrt{-1}\xi_nc(\xi)}{(1+\xi_n^2)^3}. \tag{4.24}$$

By Lemma 4.2, we have

$$\begin{aligned}\sigma_0(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-1}) =&\sigma_2(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y)\sigma_{-2}(D_F^{-1})+\sigma_1(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y)\sigma_{-1}(D_F^{-1})\\ &+\sum_{j=1}^{n}\partial_{\xi_j}\left[\sigma_2(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y)\right]D_{x_j}\left[\sigma_{-1}(D_F^{-1})\right]\\ &:=A+B+C.\end{aligned}\tag{4.25}$$

(1) Explicit representation the first item of (4.25)

$$\begin{aligned}&\sigma_2(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y)\sigma_{-2}(D_F^{-1})(x_0)|_{|\xi'|=1}=-\sum_{j,l=1}^{n}X_jY_l\xi_j\xi_l\sigma_{-2}(D_F^{-1})(x_0)|_{|\xi'|=1}\\ &=-\sum_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l\sigma_{-2}(D_F^{-1})(x_0)|_{|\xi'|=1}-X_nY_n\xi_n^2\sigma_{-2}(D_F^{-1})(x_0)|_{|\xi'|=1}\\ &-\sum_{j=1}^{n-1}X_jY_n\xi_j\xi_n\sigma_{-2}(D_F^{-1})(x_0)|_{|\xi'|=1}-\sum_{l=1}^{n-1}X_nY_l\xi_n\xi_l\sigma_{-2}(D_F^{-1})(x_0)|_{|\xi'|=1},\end{aligned}\tag{4.26}$$

we let

$$Q_1=\sum_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l\sigma_{-2}(D_F^{-1})(x_0)|_{|\xi'|=1},\quad Q_2=X_nY_n\xi_n^2\sigma_{-2}(D_F^{-1})(x_0)|_{|\xi'|=1}.\tag{4.27}$$

By the trace identity $\mathrm{tr}(AB)=\mathrm{tr}(BA),\mathrm{tr}(A\otimes B)=\mathrm{tr}(A)\cdot\mathrm{tr}(B)$ and the relation of the Clifford action, we have

$$\mathrm{Tr}\left[\sigma_0(D_F)c(h_q)\right]=4\sum_{i=1}^{p+q}\left\langle\nabla^{TM}_{e_i}e_i,h_q\right\rangle=-4\sum_{i=1}^{n-1}\left\langle e_j,\nabla^{TM}_{e_j}\partial_{x_n}\right\rangle:=-4div_{\partial M}(\partial x_n),\tag{4.28}$$

$$\mathrm{Tr}\left[\sigma_0(D_F)c(\xi')\right]=4\sum_{i=1}^{p+q}\left\langle\nabla^{TM}_{e_i}e_i,\xi_i\right\rangle=0.\tag{4.29}$$

Then

$$\begin{aligned}&\mathrm{Tr}[\pi^+_{\xi_n}Q_1\times\partial_{\xi_n}\sigma_{-3}(D_F^{-3})]|_{|\xi'|=1}\\ =&-\sum_{j,l=1}^{n-1}X_jY_l\xi_j\xi_lh'(0)\left[\frac{3\xi_n^3-6i\xi_n^2-5\xi_n+2i}{(\xi_n-i)^5(\xi_n+i)^3}-\frac{3\xi_n^4-9i\xi_n^3-17\xi_n^2+15i\xi_n+4}{(\xi_n-i)^6(\xi_n+i)^3}\right]\\ &-\sum_{j,l=1}^{n-1}X_jY_l\xi_j\xi_lh'(0)\frac{3\xi_n-i}{2(\xi_n-i)^4(\xi_n+i)^3}\mathrm{Tr}\left[i\sigma_0(D_F)c(\mathrm{d}x_n)+\sigma_0(D_F)c(\xi')\right]\\ =&-2\sum_{j,l=1}^{n-1}X_jY_l\xi_j\xi_lh'(0)\frac{3\xi_n-i}{(\xi_n-i)^5(\xi_n+i)^3}+2\sum_{j,l=1}^{n-1}X_jY_l\xi_j\xi_lh'(0)div_{\partial M}(\partial x_n)\frac{3i\xi_n+1}{(\xi_n-i)^4(\xi_n+i)^3},\end{aligned}\tag{4.30}$$

and

$$
\begin{aligned}
&\operatorname{Tr}[\pi^+_{\xi_n} Q_2 \times \partial_{\xi_n}(D_F^{-3})]|_{|\xi'|=1}\\
=&-X_nY_nh'(0)\left[\frac{3\xi_n^2-5i\xi_n}{(\xi_n-i)^4(\xi_n+i)^3}+\frac{3\xi_n^4+6i\xi_n^3+11\xi_n^2-6i\xi_n}{(\xi_n-i)^6(\xi_n+i)^3}\right]\\
&-X_nY_nh'(0)\frac{6i\xi_n^2+5\xi_n-i}{2(\xi_n-i)^4(\xi_n+i)^3}\operatorname{Tr}\left[i\sigma_0(D_F)c(\mathrm{d}x_n)+\sigma_0(D_F)c(\xi')\right]\\
=&-X_nY_nh'(0)\frac{6\xi_n^4-5i\xi_n^3-2\xi_n^2-i\xi_n}{(\xi_n-i)^6(\xi_n+i)^3}-2X_nY_nh'(0)div_{\partial M}(\partial x_n)\frac{6\xi_n^2-5i\xi_n-1}{(\xi_n-i)^4(\xi_n+i)^3}.
\end{aligned}\tag{4.31}
$$

We have,

$$
\begin{aligned}
&-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{Tr}[\pi^+_{\xi_n}(Q_1+Q_2)\times\partial_{\xi_n}\sigma_{-3}(D_F^{-3})](x_0)d\xi_n\sigma(\xi')dx'\\
=&-\left(\frac{5}{16}\sum_{j=1}^{n-1}X_jY_j+\frac{5}{16}X_nY_n\right)h'(0)\pi\Omega_3dx'+\left(\frac{1}{3}\sum_{j=1}^{n-1}X_jY_j+\frac{1}{2}X_nY_n\right)div_{\partial M}(\partial x_n)h'(0)\pi\Omega_3dx'.
\end{aligned}\tag{4.32}
$$

(2) Explicit representation the second item of (4.25)

$$
\begin{aligned}
&\sigma_1(\widetilde{\nabla}_X^F\widetilde{\nabla}_Y^F)\sigma_{-1}(D_F^{-1})(x_0)|_{|\xi'|=1}\\
=&\Big[\sqrt{-1}\sum_{j,l=1}^n X_j\frac{\partial_{Y_l}}{\partial_{x_j}}\xi_l+\sqrt{-1}\sum_j\big(M(Y)+N(Y)+A(Y)\big)X_j\xi_j\\
&+\sqrt{-1}\sum_l\big(M(X)+N(X)+A(X)\big)Y_l\xi_l\Big]\frac{\sqrt{-1}c(\xi)}{|\xi|^2};
\end{aligned}\tag{4.33}
$$

We note that $i<n,\ \int_{|\xi'|=1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2d+1}}\sigma(\xi')=0$, then

$$
\begin{aligned}
&\operatorname{Tr}\left(\pi^+_{\xi_n}\Big(\sigma_1(\widetilde{\nabla}_X^F\widetilde{\nabla}_Y^F)\sigma_{-1}(D_F^{-1})\Big)\times\partial_{\xi_n}\sigma_{-3}(D^{-3})\right)(x_0)\\
=&\frac{4(3\xi_n-i)}{(1+\xi_n^2)^3}\sum_{j,l=1}^n X_j\frac{\partial_{Y_l}}{\partial_{x_j}}\xi_l,
\end{aligned}\tag{4.34}
$$

so we have

$$
\begin{aligned}
&-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{Tr}[\pi^+_{\xi_n}\sigma_1(\widetilde{\nabla}_X\widetilde{\nabla}_Y)\sigma_{-1}(D^{-1})\times\partial_{\xi_n}\sigma_{-3}(D^{-3})](x_0)d\xi_n\sigma(\xi')dx'\\
=&\frac{3i}{2}X(Y_n)\Omega_3dx',
\end{aligned}\tag{4.35}
$$

where $X(Y_n)$ is $\sum_{j=1}^n X_j\frac{\partial_{Y_n}}{\partial_{x_j}}$.

(3) Explicit representation the third item of (4.25)

$$
\sum_{j=1}^n\sum_{|\alpha|=1}\frac{1}{\alpha!}\partial_\xi^{|\alpha|}\big[\sigma_2(\widetilde{\nabla}_X^F\widetilde{\nabla}_Y^F)\big]D_{x_j}\big[\sigma_{-1}(D_F^{-1})\big](x_0)|_{|\xi'|=1}=\sum_{j=1}^n\sum_{l=1}^n\sqrt{-1}(x_jY_l+x_lY_j)\xi_l\partial_{x_j}(\frac{\sqrt{-1}c(\xi)}{|\xi|^2}).\tag{4.36}
$$

We note that $i<n,\ \int_{|\xi'|=1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2d+1}}\sigma(\xi')=0$, then

$$
\begin{aligned}
&\operatorname{Tr}\left(\pi^+_{\xi_n}\Big(\sum_{j=1}^n\sum_{|\alpha|=1}\frac{1}{\alpha!}\partial_\xi^{|\alpha|}\big[\sigma_2(\widetilde{\nabla}_X^F\widetilde{\nabla}_Y^F)\big]D_{x_j}\big[\sigma_{-1}(D_F^{-1})\big]\Big)\times\partial_{\xi_n}\sigma_{-3}(D^{-3})\right)(x_0)|_{|\xi'|=1}\\
=&4X_nY_nh'(0)\frac{3\xi_n^2-i\xi_n}{(\xi_n-i)^4(\xi_n+i)^3},
\end{aligned}\tag{4.37}
$$

Substituting (4.37) into (4.23) yields

$$
\begin{aligned}
&-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{Tr}\Big[\pi^{+}_{\xi_n}\Big(\sum_{j=1}^{n}\sum_{\alpha}\frac{1}{\alpha!}\partial^{\alpha}_{\xi}\big[\sigma_2(\widetilde{\nabla}_X\widetilde{\nabla}_Y)\big]D^{\alpha}_{x}\big[\sigma_{-1}(D^{-1})\big]\Big)\times\partial_{\xi_n}\sigma_{-3}(D^{-3})\Big](x_0)d\xi_n\sigma(\xi')dx'\\
&=\frac{1}{2}X_nY_n\pi h'(0)\Omega_3dx'. \qquad (4.38)
\end{aligned}
$$

Summing up (1), (2) and (3) leads to the desired equality

$$
\begin{aligned}
\widetilde{\Phi}_4=&\left(-\frac{5}{16}\sum_{j=1}^{n-1}X_jY_j+\frac{3}{16}X_nY_n\right)h'(0)\pi\Omega_3dx'+\frac{3i}{2}X(Y_n)\Omega_3dx'\\
&+\left(\frac{1}{3}\sum_{j=1}^{n-1}X_jY_j+\frac{1}{2}X_nY_n\right)div_{\partial M}(\partial x_n)h'(0)\pi\Omega_3dx'. \qquad (4.39)
\end{aligned}
$$

case c) $r=1,\ \ell=-4,\ k=j=|\alpha|=0.$
By (4.2), we get

$$
\begin{aligned}
\widetilde{\Phi}_5=&-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{Tr}[\pi^{+}_{\xi_n}\sigma_1(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_YD_F^{-1})\times\partial_{\xi_n}\sigma_{-4}D_F^{-3})](x_0)d\xi_n\sigma(\xi')dx'\\
=&\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{Tr}[\partial_{\xi_n}\pi^{+}_{\xi_n}\sigma_1(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_YD_F^{-1})\times\sigma_{-4}D_F^{-3})](x_0)d\xi_n\sigma(\xi')dx'. \qquad (4.40)
\end{aligned}
$$

By Lemma 4.3, we have

$$
\begin{aligned}
\sigma_{-4}(D_F^{-3})(x_0)|_{|\xi'|=1}=&\frac{c(\xi)\sigma_2(D_F^3)c(\xi)}{(\xi_n^2+1)^4}+\frac{ic(\xi)}{(\xi_n^2+1)^4}\big[|\xi|^4c(dx_n)\partial_{x_n}c(\xi')-2h'(0)c(dx_n)c(\xi)\\
&+2\xi_nc(\xi)\partial_{x_n}c(\xi')+4\xi_nh'(0)\big], \qquad (4.41)
\end{aligned}
$$

where

$$
\begin{aligned}
\sigma_2(D_F^3)=&\sum_{i,j,l}c(d_{x_l})(\partial_lg^{ij})+2c(\xi)\big[2\sigma^i\otimes\mathrm{Id}_{\wedge(F^{\perp,\star})}+\mathrm{Id}_{S(F)}\otimes2\tilde{\sigma}^k-\Gamma^k+2\widetilde{S}(\partial_i)\big]\xi_k\\
&+\big[M(X)+N(X)\big]|\xi|^2, \qquad (4.42)
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_{\xi_n}\pi^{+}_{\xi_n}\sigma_1(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_YD_F^{-1})(x_0)|_{|\xi'|=1}=&\frac{c(\xi')+ic(\mathrm{d}x_n)}{2(\xi_n-i)^2}\sum_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l-\frac{c(\xi')+ic(\mathrm{d}x_n)}{2(\xi_n-i)^2}X_nY_n\\
&+\frac{ic(\xi')-c(\mathrm{d}x_n)}{2(\xi_n-i)^2}\sum_{j=1}^{n-1}X_jY_n\xi_j+\frac{ic(\xi')-c(\mathrm{d}x_n)}{2(\xi_n-i)^2}\sum_{l=1}^{n-1}X_nY_l\xi_l. \qquad (4.43)
\end{aligned}
$$

We note that $i<n,\ \int_{|\xi'|=1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2d+1}}\sigma(\xi')=0$, so we omit some items that have no contribution for

computing **case c)**. Also, straightforward computations yield

$$
\begin{aligned}
&\mathrm{tr}\left[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-1})\times\sigma_{-4}(D_F^{-3})\right](x_0)|_{|\xi'|=1}\\
&=\sum_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l h'(0)\left[\frac{2i\xi_n+2}{(\xi_n-i)^4(\xi_n+i)^2}+\frac{(24+4i)\xi_n^2+(4-32i)\xi_n-8}{(\xi_n-i)^6(\xi_n+i)^4}-\frac{2\xi_n+2i\xi_n^2}{(\xi_n-i)^5(\xi_n+i)^3}\right]\\
&\quad-X_nY_nh'(0)\left[\frac{2i\xi_n+2}{(\xi_n-i)^4(\xi_n+i)^2}+\frac{(24+4i)\xi_n^2+(4-32i)\xi_n-8}{(\xi_n-i)^6(\xi_n+i)^4}-\frac{2\xi_n+2i\xi_n^2}{(\xi_n-i)^5(\xi_n+i)^3}\right]\\
&\quad+\sum_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l\xi_k\xi_t g(\mathrm{d}x^t,h_k)\left[\frac{2(i-1)}{(\xi_n-i)^5(\xi_n+i)^3}+\frac{(i-1)}{4(\xi_n-i)^4(\xi_n+i)^2}\right]h'(0)\\
&\quad+X_nY_n\xi_k\xi_t g(\mathrm{d}x^t,h_k)\left[\frac{2(1-i)}{(\xi_n-i)^5(\xi_n+i)^3}+\frac{(1-i)}{4(\xi_n-i)^4(\xi_n+i)^2}\right]h'(0).
\end{aligned}
\tag{4.44}
$$

We get

$$
\widetilde{\Phi}_5=\left[\frac{129-44i}{320}\sum_{j=1}^{n-1}X_jY_j-\frac{245-26i}{96}X_nY_n\right]h'(0)\pi\Omega_3dx'.
\tag{4.45}
$$

Let $X=X^T+X_n\partial_n,\ Y=Y^T+Y_n\partial_n$, then we have $\sum_{j=1}^{n-1}X_jY_j(x_0)=g(X^T,Y^T)(x_0)$. Now Φ is the sum of the cases (a), (b) and (c). Combining with the five cases, this yields

$$
\begin{aligned}
\widetilde{\Phi}=\sum_{i=1}^{5}\widetilde{\Phi}_i=&-\left[\frac{113+132i}{960}g(X^T,Y^T)+\frac{71-26i}{96}X_nY_n\right]h'(0)\pi\Omega_3dx'+\frac{3i}{2}X(Y_n)\Omega_3dx'\\
&+\left[\frac{1}{3}g(X^T,Y^T)+\frac{1}{2}X_nY_n\right]div_{\partial M}(\partial x_n)h'(0)\pi\Omega_3dx'.
\end{aligned}
\tag{4.46}
$$

So, we are reduced to prove the following.

Theorem 4.4. *Let M be a 4-dimensional compact filiated manifold with boundary and spin leave, $\widetilde{\nabla}$ be an orthogonal connection with torsion. Then we get the spectral Einstein functional associated to $\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-1}$ and D_F^{-3} on M*

$$
\begin{aligned}
&\widetilde{\mathrm{Wres}}[\pi^+(\widetilde{\nabla}^F_X\widetilde{\nabla}^F_Y D_F^{-1})\circ\pi^+(D_F^{-3})]\\
&=\frac{8\pi^2}{3}\int_M G(X,Y)dvol_g+\int_M sg(X,Y)dvol_M\\
&-\int_{\partial M}\left[\frac{113+132i}{960}g(X^T,Y^T)+\frac{71-26i}{96}X_nY_n\right]h'(0)\pi\Omega_3dx'+\int_{\partial M}\frac{3i}{2}X(Y_n)\Omega_3dx'\\
&+\int_{\partial M}\left[\frac{1}{3}g(X^T,Y^T)+\frac{1}{2}X_nY_n\right]div_{\partial M}(\partial x_n)h'(0)\pi\Omega_3dx'.
\end{aligned}
\tag{4.47}
$$

where s is the scalar curvature.

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References

[1] M. Wodzicki.: Non-commutative residue I, Lecture Notes in Math., Springer, New York, Vol. 1289 320-399, (1987).

[2] V. W. Guillemin.: A new proof of Weyl's formula on the asymptotic distribution of eigenvalues. Adv. Math. 55, no. 2, 131-160, (1985).

[3] Dabrowski L, Sitarz A, Zalecki P. Spectral Metric and Einstein Functionals. Advances in Mathematics, 427, 109128, (2023).

[4] P. B. Gilkey.: Invariance theory, the Heat equation, and the Atiyah-Singer Index theorem. Inc., USA, (1984).

[5] H. Figueroa, J. Gracia-Bondía, and J. Várilly.: Elements of Noncommutative Geometry. Birkhäuser Boston, (2001).

[6] A. Connes.: Quantized calculus and applications. XIth International Congress of Mathematical Physics(Paris,1994), Internat Press, Cambridge, MA, 15-36, (1995).

[7] A. Connes.: The action functinal in Noncommutative geometry. Comm. Math. Phys. 117, 673-683, (1988).

[8] A. Connes, J. Lott.: Particle models and Non-commutative geometry. Nucl. Phys. B Proc.Supp 18B, 29-47, (1990).

[9] D. Kastler.: The Dirac Operator and Gravitation. Comm. Math. Phys. 166, 633-643, (1995).

[10] W. Kalau and M. Walze.: Gravity, Noncommutative geometry and the Wodzicki residue. J. Geom. Physics. 16, 327-344,(1995).

[11] B. V. Fedosov, F. Golse, E. Leichtnam, E. Schrohe.: The noncommutative residue for manifolds with boundary. J. Funct. Anal. 142, 1-31, (1996).

[12] Y. Wang.: Diffential forms and the Wodzicki residue for Manifolds with Boundary. J. Geom. Physics. 56, 731-753, (2006).

[13] Y. Wang.: Gravity and the Noncommutative Residue for Manifolds with Boundary. Letters in Mathematical Physics. 80, 37-56, (2007).

[14] Y. Wang.: Lower-Dimensional Volumes and Kastler-kalau-Walze Type Theorem for Manifolds with Boundary . Commun. Theor. Phys. Vol 54, 38-42, (2010).

[15] J. M. Bismut.: A local index theorem for non Kähler manifolds. Math. Ann. 284, 681-699 (1989).

[16] T. Ackermann and J. Tolksdorf.: A generalized Lichnerowicz formula, the Wodzicki residue and gravity. J. Geom. Physics. 19, 143-150,(1996).

[17] F. Pfäffle and C. A. Stephan.: On gravity, torsion and the spectral action principle. J. Funct. Anal. 262, 1529-1565,(2012).

[18] W. Greub, S. Halperin and R.Vanstone.: Connections, Curvature and Cohomology, Vo.1 (Academic press, New York) (1976).

[19] J. Wang and Y. Wang.: Noncommutative residue and sub-Dirac operators, J. Math. Phys. 54, 012501 (2013).

[20] J. Wang, Y. Wang, T. Wu, Dirac operators with torsion, spectral Einstein functionals and the noncommutative residue. J. Math. Phys. 64, 102505(2023).

[21] Y. Yu.: The Index Theorem and The Heat Equation Method, Nankai Tracts in Mathematics-Vol.2, World Scientific Publishing, (2001).

[22] G. Grubb, E. Schrohe.: Trace expansions and the noncommutative residue for manifolds with boundary. J. Reine Angew. Math. 536, 167-207, (2001).

[23] S. Alexandrov and D. Vassilevich.: Heat kernel for nonminimal operators on a Kähler manifold. J. Math. Phys. 37, 5715-5718, (1996).

[24] B. Opozda.: Bochner¡¯s technique for statistical structures. Ann. Glob. Anal. Geom. 48(4):357-395, (2015).

[25] K. Liu, W. Zhang, Adiabatic limits and foliations. In: Topology, Geometry, and Algebra: Interactions and New Directions, Contemp Math, Vol. 279. Providence, RI: American Mathematical Society, 195C208, (2001).

[26] K. Liu ,Y. Wang.: Adiabatic limits, vanishing theorems and the noncommutative residue. Science in China Series A: Mathematics. 52,12, 2699-2713,(2009).