KUROKAWA-MIZUMOTO CONGRUENCE AND DIFFERENTIAL OPERATORS ON AUTOMORPHIC FORMS

NOBUKI TAKEDA

Abstract. We give the sufficient conditions for the vector-valued Kurokawa-Mizumoto congruence related to the Klingen-Eisenstein series to hold. And we give a reinterpretation for differential operators on automorphic forms by the representation theory.

1. Introduction

Let k, ν be even integers, and $f(z) = \sum_{n>0} a(n,f)\mathbf{e}(nz)$ be a (degree 1) normalized Hecke eigenform of weight $k+\nu$. Here $\mathbf{e}(x) := e^{2\pi i x}$ for $x \in \mathbb{C}$. Then, it is conjectured that for a sufficiently large prime ideal $\mathfrak{p}$ of Hecke field $\mathbb{Q}(f)$ for f associated with the special value of the L-function $L(k-1, f, \mathrm{St})$, there is a some degree 2 Siegel cusp form F of weight $\det^k \otimes \mathrm{Sym}^\nu$ and that the following congruence holds:

$$\lambda_F(p) = (1+p^{k-2})a(p,f) \mod \mathfrak{p}',$$

where $\lambda_F(p)$ is the eigenvalue of the Hecke operator $T(p)$ on F, and $\mathfrak{p}'$ is a prime ideal of $\mathbb{Q}(f,F) = \mathbb{Q}(f)\cdot\mathbb{Q}(F)$ lying above $\mathfrak{p}$. Let $[f]^{(k+\nu,k)}$ be a Klingen-Eisenstein lift of f to a degree 2, weight $\det^k \otimes \mathrm{Sym}^\nu$. Since $\lambda_{[f]^{(k+\nu,k)}} = (1+p^{k-2})a(p,f)$ due to Arakawa [2], This congruence is between a Klingen-Eisenstein lift and another Siegel modular form. Congruences of this type are proved initially by Kurokawa [19] for $k=20, \nu=0$, then by Mizumoto [20] for $k=22, \nu=0$, and by Satoh [23] for $\nu=2$, and sufficient conditions examined by Katsurada-Mizumoto [16] for $\nu=0$. We call the congruences of this type Kurokawa-Mizumoto congruences.

We give sufficient conditions for the vector-valued Kurokawa-Mizumoto congruence. This result proves Conjecture 10.6 of Begström, Faber and van der Geer [5].

Theorem (Theorem 5.10). *Let k,ν be positive even integers with $k \geq 4$, $f_{1,1} = f, \dots, f_{1,d_1}$ be a basis of $S_{k+\nu}(\Gamma_1)$ consist of normalized Hecke eigenforms, p be a prime number of $\mathbb{Q}$ and $A \in H_2(\mathbb{Z})_{>0}$ be a half-integral positive definite matrix of degree 2. Suppose that A and p satisfy the following conditions:*

(1) $\mathrm{ord}_p(\mathbb{L}(k-1,f,\mathrm{St})) =: \alpha > 0$,
(2) $\mathrm{ord}_p(\mathcal{C}_{4,k}(f)a(A,[f]^{(k+\nu,k)})) = 0$,
(3) $p \geq 2(k+\nu)-3$.

Then, there is a Hecke eigenform $G \in M_{\rho_2}(\Gamma_2)$ such that G is not a scalar multiple of $[f]^{(k+\nu,k)}$ and

$$[f]^{(k+\nu,k)} \equiv_{ev} G \mod \mathfrak{p}$$

for some prime ideal $\mathfrak{p} \mid p$ of $\mathbb{Q}(G)$. If $\mathrm{ord}_p(\gamma_1) = 0$, Condition (3) can be changed to Condition (3)':

(3)' $p \geq \max\{2k, k+\nu-2\}$.

If moreover $k \geq 6$ and p satisfy the following conditions:

(4) $\mathrm{ord}_p(\mathbb{L}(k-1, f_{1,i}, \mathrm{St})) \leq 0$ $(2 \leq i \leq d_1)$,

(5) *p is coprime with every $\mathfrak{A}(f_{r,i})$ $(1 \leq r \leq 2,\ 1 \leq i \leq d_r)$.*

there is a Hecke eigenform $G \in S_{\rho_2}(\Gamma_2)$ such that G is not a scalar multiple of $[f]^{(k+\nu,k)}$ and

$$[f]^{(k+\nu,k)} \equiv_{ev} G \mod \mathfrak{p}^{\alpha}$$

for some prime ideal $\mathfrak{p} \mid p$ of $\mathbb{Q}(G)$.

The former part of the proof of this theorem follows Atobe-Chida-Ibukiyama-Katsurada-Yamauchi's method [3], and the key to the proof of the main theorem is the following lemma [3, Lemma 6.10].

Lemma (Lemma 3.5)**.** *Let $F_1, \ldots, F_d$ be Hecke eigenforms in $M_{\mathbf{k}}(\Gamma_n)$ linearly independent over $\mathbb{C}$. Let K be the composite field $\mathbb{Q}(F_1), \ldots, \mathbb{Q}(F_d)$, $\mathcal{O}$ the ring of integers in K and $\mathfrak{p}$ a prime ideal of K. Let $G(Z) \in (M_{\rho_n}(\Gamma_n) \otimes V_{n,\mathbf{k}})(\mathcal{O}_{(\mathfrak{p})})$ and assume the following conditions*

(1) *G is expressed as*

$$G(Z) = \sum_{i=1}^{d} c_i F_i(Z)$$

with $c_i \in V_{n,\mathbf{k}}$.

(2) *$c_1 a(A, F_1) \in (V_{n,\mathbf{k}} \otimes V_{n,\mathbf{k}})(K)$ and $\mathrm{ord}_{\mathfrak{p}}(c_1 a(A, F_1)) < 0$ for some $A \in H_n(\mathbb{Z})$.*

Then there exists $i \neq 1$ such that

$$F_i \equiv_{ev} F_1 \mod \mathfrak{p}.$$

The latter part of the proof of the main theorem is due to Katsurada-Mizumoto's method [16].

To obtain the main theorem, we take as $G(Z)$ in this corollary a function constructed from the Siegel-Eisenstein series $E_{n,\mathbf{k}}(Z,s)$ and using the pullback formula (Theorem 4.16), we show that a decomposition of $G(Z)$ that satisfies the conditions of the corollary can be obtained. We also showed in Theorem 5.14, using Galois representations, the conditions under which G in Theorem 5.10 can be taken as a cusp form.

In the course of the proof, we will discuss using differential operators that preserve the automorphic properties, which are deeply investigated by Ibukiyama [10, 11]. In this paper, we also reinterpreted this differential operator by using Howe duality in representation theory.

Theorem (Theorem 4.12)**.** *Let F be a holomorphic automorphic form of weight $\frac{k}{2}\mathbb{1}_n$ for Γ_n and $D \in \Delta_{n,k}$ be a Young diagram such that $l(D) \leq \min\{n_1, \ldots, n_d\}$. We put $\partial_Z = \left(\frac{1+\delta_{i,j}}{2}\frac{\partial}{\partial z_{i,j}}\right)$. We denote by Res be the restriction of a function on $\mathbb{H}_n$ to $\mathbb{H}_{n_1} \times \cdots \times \mathbb{H}_{n_d}$*

Then for any $h \in \left(\left(\overset{d}{\underset{s=1}{\otimes}} \mathfrak{H}_{n_s,k}(D)\right)^{\mathrm{O}_k} \otimes \left(\overset{d}{\underset{s=1}{\otimes}} U_{\tau'_{n_s,k}(D)}\right)\right)^{\widetilde{K'}}$, $\mathrm{Res}\,(\Phi_h(\partial_Z) \cdot F)$ is a holomorphic automorphic form of weight $\bigotimes_{s=1}^{d} (\tau_{n_s,k}(D))$ for Γ_n.

This paper is organized as follows: In Section 2, We explain the Siegel modular form, the Hecke operator, and the terminology used in this paper. In Section 3, we define the Siegel operator and Klingen-Eisenstein series that raise and lower the degree of Siegel modular forms and review their properties. In Section 4, we state the results of differential operators that preserve the automorphic properties and give an interpretation by representation theory of the differential operator. Then, we explain the Pullback formula, which is the key to the proof of the main theorem. In Section 5, we prove the main theorem. In Section 6, we consider the conditions that appear in the main theorem (Theorem 5.10). In section 7, we give explicit examples of Kurokawa-Mizumoto congruences.

Notation. For a commutative ring R, we denote by $R^\times$ the unit group of R. We denote by $M_{m,n}(R)$ the set of $m \times n$ matrices with entries in R. In particular, we denote $M_n(R) := M_{n,n}(R)$. Let $\mathrm{GL}_n(R) \subset M_n(R)$ be a general linear group of degree n. For an element $X \in M_n(R)$, we denote by $X > 0$ (resp. $X \geq 0$) X is a positive definite matrix (resp. a non-negative definite matrix). For a subset $S \subset M_n(R)$, we denote by $S_{>0}$ (resp. $S_{\geq 0}$) the subset of positive definite (resp. non-negative definite) matrices in S. Let $\det^k$ be the 1 dimensional representation of multiplying k-square of determinant for $\mathrm{GL}_n(\mathbb{C})$, and Sym^ν be the degree ν symmetric tensor product representation of $\mathrm{GL}_2(\mathbb{C})$.

Let K be an algebraic field, and $\mathfrak{p}$ be a prime ideal of K. We denote by $K_\mathfrak{p}$ be a $\mathfrak{p}$-adic completion of K and by $\mathcal{O}_K$ the integer ring of K. We denote by $\mathrm{ord}_\mathfrak{p}(\cdot)$ the additive valuation of $K_\mathfrak{p}$ normalized so that $\mathrm{ord}_\mathfrak{p}(\varpi) = 1$ for a prime element ϖ of $K_\mathfrak{p}$. Let $p_\mathfrak{p}$ be the prime number such that $p_\mathfrak{p}\mathbb{Z} = \mathbb{Z} \cap \mathfrak{p}$.

If a group G acts on a set V then, we denote by V^G the G-invariant subspace of V.

2. Siegel Modular Forms

Let $\mathbb{H}_n$ be the Siegel upper half space of degree n, that is

$$\mathbb{H}_n = \{Z \in M_n(\mathbb{C}) | Z = {}^tZ = X + \sqrt{-1}Y,\ X, Y \in M_n(\mathbb{R}),\ Y > 0\}.$$

We put $\Gamma_n = \mathrm{Sp}_n(\mathbb{Z}) = \{M \in \mathrm{GL}_{2n}(\mathbb{Z}) | MJ_n{}^tM = J_n\}$, where $J_n = \begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix}$.

Let (ρ, V) be a polynomial representation of $\mathrm{GL}_n(\mathbb{C})$ on a finite-dimensional complex vector space V, and take a Hermitian inner product on V such that

$$(\rho(g)v, w) = (v, \rho({}^t\overline{g})w).$$

For a V-valued function F on $\mathbb{H}_n$, and $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$, put $j(g, Z) = CZ + D$, and

$$F|_\rho[g](Z) = \rho(j(g, Z))^{-1} F((AZ + B)(CZ + D)^{-1}) \quad (Z \in \mathbb{H}_n).$$

We say that F is a (level 1 holomorphic) Siegel modular form of weight ρ if F is a holomorphic V-valued function on $\mathbb{H}_n$ and $F|_\rho[g] = F$ for all $g \in \Gamma_n$. (When $n = 1$, another holomorphic condition is also needed.)

Let $H_n(\mathbb{Z})_{\geq 0}$ (resp. $H_n(\mathbb{Z})_{>0}$) be the set of half-integral non-negative definite (resp. positive definite) matrices of degree n. A modular form $F \in M_\rho(\Gamma_n)$ has the Fourier expansion

$$F(Z) = \sum_{T \in H_n(\mathbb{Z})_{\geq 0}} a(F, T)\mathbf{e}(\mathrm{tr}(TZ)),$$

where $a(F, T) \in V$, $\mathbf{e}(z) = \exp(2\pi\sqrt{-1}z)$, and tr is the trace of $M_n(\mathbb{C})$. If $a(F, T) = 0$ unless T is positive definite, we say that F is a (level 1 holomorphic) Siegel cusp form of weight (ρ, V). We denote by $M_\rho(\Gamma_n)$ (resp. $S_\rho(\Gamma_n)$) a complex vector space of all modular (resp.

cusp) forms of weight ρ. For $F, G \in M_\rho(\Gamma_n)$, we can define the Petersson inner product as

$$(F,G) = \int_D \left(\rho(\sqrt{Y})F(Z), \rho(\sqrt{Y})G(Z)\right) \det(Y)^{-n-1} dZ,$$

where $Y = \Im(Z)$, $\sqrt{Y}$ is a positive definite symmetric matrix such that $\sqrt{Y}^2 = Y$, and D is a Siegel domain on $\mathbb{H}_n$ for Γ_n. This integral converges if either F or G is a cusp form.

We fix a basis $\{v_1, \ldots v_r\}$ of the representation space V, which $(\rho|_{\mathrm{GL}_n(\mathbb{Z})}, \oplus_{i=1}^r \mathbb{Z}v_i)$ is a representation of $\mathrm{GL}_n(\mathbb{Z})$, and we write $V(\mathbb{Z}) = \oplus_{i=1}^r \mathbb{Z}v_i$. For a subring R of $\mathbb{C}$, we write $V(R) = V(\mathbb{Z}) \otimes_{\mathbb{Z}} R$. We denote $M_\rho(\Gamma_n)(R)$ by the set of all Modular form $F \in M_\rho(\Gamma_n)$ which Fourier coefficients $a(F,T)$ of F are in $V(R)$, and $S_\rho(\Gamma_n)(R)$ is defined in the same way.

Let K be a number field and $\mathcal{O}$ be the ring of integers in K. For a prime ideal $\mathfrak{p}$ of $\mathcal{O}$ and $a = \sum_{i=1}^r a_i v_i \in V(R)$, we define the order $\mathrm{ord}_{\mathfrak{p}}(a)$ of a respect to $\mathfrak{p}$ as

$$\mathrm{ord}_{\mathfrak{p}}(a) = \min_{i=1,\ldots,r} \mathrm{ord}_{\mathfrak{p}}(a_i).$$

We say that $\mathfrak{p}$ divides a if $\mathrm{ord}_{\mathfrak{p}}(a) > 0$ and write $\mathfrak{p}|a$. This order does not change when changing the basis by $GL_r(\mathbb{Z})$.

We call a sequence of non-negative integers $\mathbf{k} = (k_1, k_2, \ldots)$ a dominant integral weight if $k_i \geq k_{i+1}$ for all i, and $k_i = 0$ for almost all i, and the biggest integer m such that $k_m \neq 0$ a depth $l(\mathbf{k})$ of $\mathbf{k}$. The set of dominant integral weights with depth less than or equal to n corresponds bijectively to the set of irreducible polynomial representations of $\mathrm{GL}_n(\mathbb{C})$. We denote the representation of $\mathrm{GL}_n(\mathbb{C})$ corresponds to a dominant integral weight $\mathbf{k}$ by $(\rho_{n,\mathbf{k}}, V_{n,\mathbf{k}})$, and write $M_{\mathbf{k}}(\Gamma_n) = M_{\rho_{n,\mathbf{k}}}(\Gamma_n)$ and $S_{\mathbf{k}}(\Gamma_n) = S_{\rho_{n,\mathbf{k}}}(\Gamma_n)$. When $\mathbf{k} = (k, k, \ldots, k)$ (i.e. $\rho_{\mathbf{k}} = \det^k$), we also write $M_k(\Gamma_n) = M_{\mathbf{k}}(\Gamma_n)$ and $S_k(\Gamma_n) = S_{\mathbf{k}}(\Gamma_n)$. For a dominant integral weight $\mathbf{k} = (k_1, \ldots, k_n)$ associated to a representation of $\mathrm{GL}_n(\mathbb{C})$ we define strong depth $s\text{-}l(\mathbf{k})$ of $\mathbf{k}$ as the biggest integer m such that $k_m > k_n$. if $\mathbf{k} = (k, \ldots, k)$, set $s\text{-}l(\mathbf{k}) = 0$.

Let $\mathcal{H}_n$ be the Hecke algebra over $\mathbb{Z}$ associated to the pair $(\Gamma_n, \ \mathrm{GSP}_n(\mathbb{Q}) \cap M_{2n}(\mathbb{Z}))$. For a subring R of $\mathbb{C}$ we write $\mathcal{H}_n(R) = \mathcal{H}_n \otimes_{\mathbb{Z}} R$. We define the action of Hecke algebra $\mathcal{H}_n(\mathbb{C})$ on $M_{\mathbf{k}}(\Gamma_n)$ as

$$T(F) = \nu(g)^{k_1 + \cdots + k_n - \frac{n(n+1)}{2}} \sum_{i=1}^r F|_{\rho_{n,\mathbf{k}}} g_i$$

for a modular $F \in M_{\mathbf{k}}(\Gamma_n)$ and an element $T = \Gamma_n g \Gamma_n = \bigsqcup_{i=1}^r \Gamma_n g_i$ (cosets decomposition) $\in \mathcal{H}_n(\mathbb{C})$, where $\nu(g) \in \mathbb{Q}$ is a similitude factor of $g \in \mathrm{GSP}_n(\mathbb{Q})$. Note that this definition does not depend on the representatives and it is well-defined.

For a positive rational number a and a positive integer m, we define a Hecke operator $[a]_n, T(m) \in \mathcal{H}_n$ as

$$[a]_n = \Gamma_n (a \cdot 1_n) \Gamma_n,$$

$$T(m) = \sum_{\substack{d_1, \ldots, d_n, e_1, \ldots, e_n \\ d_i | d_{i+1}, \ e_{i+1} | e_i \\ d_i e_i = m}} \Gamma_n \mathrm{diag}(d_1, \ldots, d_n, e_1, \ldots, e_n) \Gamma_n.$$

Then, $\mathcal{H}_n(\mathbb{Q})$ generated over $\mathbb{Q}$ by $T(p)$, $T(p^2)$, and $[p^{-1}]_n$ for all prime p, and $\mathcal{H}_n$ generated over $\mathbb{Z}$ by $T(p)$ and $T(p^2)$ for all prime p.

We say that a modular form $F \in M_{\mathbf{k}}(\Gamma_n)$ is Hecke eigenform if F is a common eigenfunction of all Hecke operators $T \in \mathcal{H}_n(\mathbb{C})$. For a Hecke eigenform $F \in M_{\mathbf{k}}(\Gamma_n)$ we call the field $\mathbb{Q}(F)$ generated by all Hecke eigenvalues over $\mathbb{Q}$ the Hecke field of F. It is well known that $\mathbb{Q}(F)$ is a finite totally real algebraic field. For ease, we write $\mathbb{Q}(F_1,\dots,F_m) = \mathbb{Q}(F_1)\cdots\mathbb{Q}(F_m)$ for $F_1,\dots,F_m \in M_{\mathbf{k}}(\Gamma_n)$.

Definition 2.1. Let $F, G \in M_{\mathbf{k}}(\Gamma_n)$ be Hecke eigenforms, and $\mathfrak{p}$ be prime ideal in $\mathbb{Q}(F,G)$. if $\lambda_F(T) \equiv \lambda_G(T) \mod \mathfrak{p}$ for all $T \in \mathcal{H}_n$, we say that F and G are Hecke congruent, and denote $F \equiv_{ev} G \mod \mathfrak{p}$.

Let $F \in M_{\mathbf{k}}(\Gamma_n)$ be a Hecke eigenform, and for a prime number p we take the Satake p-parameters $\alpha_0(p), \alpha_1(p), \dots, \alpha_n(p)$ of F so that

$$\alpha_0(p)^2\alpha_1(p)\cdots\alpha_n(p) = p^{k_1+\cdots+k_n-n(n+1)/2}.$$

We define standard L-function $L(s,F,\mathrm{St})$ by

$$L(s,F,\mathrm{St}) = \prod_p \left((1-\alpha_0(p)p^{-s}) \prod_{r=1}^{n} \prod_{1\le i_1<\cdots<i_r\le n} (1-\alpha_0(p)\alpha_{i_1}(p)\cdots\alpha_{i_r}(p)p^{-s}) \right).$$

For a Hecke eigenform $F \in S_k(\Gamma_n)$, we define $\mathbb{L}(s,F,\mathrm{St})$ by

$$\mathbb{L}(s,F,\mathrm{St}) = \Gamma_{\mathbb{C}}(s)\prod_{i=1}^{n}\Gamma_{\mathbb{C}}(s+k-i)\frac{L(s,F,\mathrm{St})}{(F,F)}.$$

Proposition 2.2 ([21, Appendix A])**.** *Let F be a Hecke eigenform in $S_k(\Gamma_n)(\mathbb{Q}(F))$. We set $n_0 = 3$ if $n \ge 5$ with $n \equiv 4$ and $n_0 = 1$ otherwise. Let m be a positive integer $n_0 \le m \le k-n$ such that $m \equiv n \mod 2$. Then $\mathbb{L}(m,F,\mathrm{St}) \in \mathbb{Q}(F)$.*

3. Siegel Operator and Klingen-Eisenstein series

In this section, let $\mathbf{k} = (k_1, k_2, \dots)$ be a dominant integral weight with depth m and strong depth m', and $n \ge r > 0$ be positive integers such that $n \ge m$ and $r \ge m'$. In addition, put $\mathbf{k}' = (k_1,\dots,k_r)$.

For a modular form $F \in M_{\mathbf{k}}(\Gamma_n)$ and we define the Siegel operator $\Phi = \Phi^n_r$ as

$$\Phi(F)(Z_1) = \lim_{y\to\infty} F\left(\begin{pmatrix} Z_1 & O \\ O & \sqrt{-1}y\cdot 1_{n-r}\end{pmatrix}\right) = \sum_{T_1\in H_r(\mathbb{Z})_{\ge 0}} a\left(F, \begin{pmatrix} T_1 & O \\ O & O\end{pmatrix}\right)\mathbf{e}(\mathrm{tr}(T_1Z_1)).$$

Then, $\Phi(F)$ belongs to $M_{\mathbf{k}'}(\Gamma_r)$.

Let $\Delta_{n,r}$ be the subgroup of Γ_n defined by

$$\Delta_{n,r} = \left\{ \begin{pmatrix} * & * \\ O_{n-r,n+r} & * \end{pmatrix} \in \Gamma_n \right\}.$$

We define the Klingen-Eisenstein series $[F]^{\mathbf{k}}(Z,s) = [F]^{\mathbf{k}}_{\mathbf{k}'}(Z,s)$ of $F \in M_{\mathbf{k}'}(\Gamma_r)$ as

$$[F]^{\mathbf{k}}(Z,s) = \sum_{g\in\Delta_{n,r}\backslash\Gamma_n} \left(\frac{\det\Im(Z)}{\det\Im(\mathrm{pr}^n_r(Z))}\right)^s F(\mathrm{pr}^n_r(Z))\,|_{\rho_{n,\mathbf{k}}} g \quad (Z\in\mathbb{H}_n, s\in\mathbb{C}),$$

where $\mathrm{pr}_r^n(Z) = Z_1$ for $Z = \begin{pmatrix} Z_1 & Z_{12} \\ {}^tZ_{12} & Z_2 \end{pmatrix}$ with $Z_1 \in \mathbb{H}_r, Z_2 \in \mathbb{H}_{n-r}, Z_{12} \in M_{r,n-r}(\mathbb{C})$ and define the Siegel-Eisenstein series $E_{n,\mathbf{k}}(Z,s)$ of weight $\mathbf{k}$ with respect to Γ_n as

$$E_{n,\mathbf{k}}(Z,s) = \sum_{g \in \Delta_{n,0} \backslash \Gamma_n} (\det \Im(Z))^s |_{\rho_{n,\mathbf{k}}} g \quad (Z \in \mathbb{H}_n, s \in \mathbb{C}).$$

The holomorphy of the Klingen-Eisenstein series and the Siegel-Eisenstein series holds as follows, which is due to Shimura [24] and [3, Proposition 2.1.].

Proposition 3.1 ([3, Proposition 2.1.]). *Let k be a positive even integer.*

(1) *Suppose that $k \geq \dfrac{n+1}{2}$ and that neither $k = \dfrac{n+2}{2} \equiv 2 \mod 4$ nor $k = \dfrac{n+3}{2} \equiv 2 \mod 4$. Then $E_{n,k}(Z) = E_{n,k}(Z,0)$ belongs to $M_k(\Gamma_n)$.*

(2) *Let $\mathbf{k} = (\underbrace{k+\nu, \ldots, k+\nu}_{m}, \underbrace{k, \ldots, k}_{n-m})$ such that $\nu \geq 0$ and $k > \dfrac{3m}{2} + 1$ and let f be a Hecke eigenform in $S_{k+\nu}(\Gamma_m)$. Then $[f]^{\mathbf{k}}(Z,s)$ can be continued meromorphically to the whole s-plane as a function of s, and holomorphic at $s = 0$. Moreover suppose that $k > \dfrac{n+m+3}{2}$. Then $[f]^{\mathbf{k}}(Z) = [f]^{\mathbf{k}}(Z,0)$ belongs to $M_{\mathbf{k}}(\Gamma_n)$.*

Consider the case $n = 2$, $r = 1$. In this case, the correspondence between the Siegel operator and the Klingen-Eisenstein series has been investigated by Andrianov [1] and Arakawa [2].

Let $N_{(k+\nu,k)}(\Gamma_2)$ be the orthogonal complement of $S_{(k+\nu,k)}(\Gamma_2)$ in $M_{(k+\nu,k)}(\Gamma_2)$ for the Petersson inner product. Then, the following isomorphism holds.

Proposition 3.2 ([2, Proposition 1.3.]). *Let k, ν be even integers with $k > 4$, $\nu > 0$.*

Then the space $N_{(k+\nu,k)}(\Gamma_2)$ is isomorphic to $S_{k+\nu}(\Gamma_1)$, and the isomorphism is given via the Siegel operator Φ_1^2 and the Klingen-Eisenstein series $[\ \cdot\ ]^{(k+\nu,k)}$.

The following facts about the Hecke eigenvalues are known.

Proposition 3.3 ([2, Proposition 3.2.]). *Let $F \in N_{(k+\nu,k)}(\Gamma_2)$ and put $\Phi(F)(z) = f(z) \in S_{k+\nu}(\Gamma_1)$. Then, F is a Hecke eigenform for $\mathcal{H}_2$, if and only if f is a common eigenform for $\mathcal{H}_1$.*

In this situation, set $T(m)F = \lambda_F(m)F$ and $T(m)f = \lambda_f(m)f$ $(m = 1, 2, \ldots)$. Then, for any prime p, we have

$$\begin{cases} \lambda_F(p) = (1 + p^{k-2})\lambda_f(p), \\ \lambda_F(p^2) = (1 + p^{k-2} + p^{2k-4})\lambda_f(p^2) + (p-1)p^{2k+\nu-4}. \end{cases}$$

Now we will state the conjecture about Kurokawa-Mizumoto congruence.

Conjecture 3.4. *Let k, ν be even integers with $k > 4$, $\nu > 0$.*

Let $f(z) = \sum_{n>0} a(n,f)\mathbf{e}(nz) \in S_{k+\nu}(\Gamma_1)$ be a normalized Hecke eigenform (i.e. a Hecke eigenform with $a(1,f) = 1$), and suppose that a large prime $\mathfrak{p}$ of $\mathbb{Q}$ divides $L(k-1, f, \mathrm{St})$. Then, there exist a Hecke eigenform $F \in S_{(k+\nu,k)}(\Gamma_2)$, and a prime ideal $\mathfrak{p}'|\mathfrak{p}$ in $\mathbb{Q}(F)$ such that

$$F \equiv_{ev} [f]^{(k+\nu,k)} \mod \mathfrak{p}'.$$

In particular, for all primes p

$$\lambda_F(p) \equiv_{ev} (1+p^{k-2})\lambda_f(p) \mod \mathfrak{p}'.$$

The key to the proof of the main theorem is the following lemma [3, Lemma 6.10], which is proved in the same way as [15, Lemma 5.1].

Lemma 3.5. *Let $F_1, \dots, F_d$ be Hecke eigenforms in $M_{\mathbf{k}}(\Gamma_n)$ linearly independent over $\mathbb{C}$. Let K be the composite field $\mathbb{Q}(F_1), \dots, \mathbb{Q}(F_d)$, $\mathcal{O}$ the ring of integers of K and $\mathfrak{p}$ a prime ideal of K. Let $G(Z) \in (M_{\rho_n}(\Gamma_n) \otimes V_{n,\mathbf{k}})(\mathcal{O}_{(\mathfrak{p})})$ and assume the following conditions*

(1) *G is expressed as*

$$G(Z) = \sum_{i=1}^{d} c_i F_i(Z)$$

with $c_i \in V_{n,\mathbf{k}}$.

(2) *$c_1 a(A, F_1) \in (V_{n,\mathbf{k}} \otimes V_{n,\mathbf{k}})(K)$ and $\mathrm{ord}_{\mathfrak{p}}(c_1 a(A, F_1)) < 0$ for some $A \in H_n(\mathbb{Z})$.*

Then there exists $i \neq 1$ such that

$$F_i \equiv_{ev} F_1 \mod \mathfrak{p}.$$

4. Differential Operators

Let $n = n_1 + \cdots + n_d \geq 2$ be a positive integer with $n_d \geq \cdots \geq n_1 \geq 1$. We embed $\mathbb{H}_{n_1} \times \cdots \times \mathbb{H}_{n_d}$ in $\mathbb{H}_n$ and embed $\Gamma_{n_1} \times \cdots \times \Gamma_{n_d}$ in Γ_n diagonally. Let $\mathbf{k}$ be a dominant integral weight with $l(\mathbf{k}) \leq n_1$ and $(\rho_{n_i,\mathbf{k}}, V_i)$ be the irreducible representation of $\mathrm{GL}_{n_i}(\mathbb{C})$ associated to $\mathbf{k}$. For irreducible representations (ρ_i, V_i) of $\mathrm{GL}_{n_i}(\mathbb{C})$, a $V_1 \otimes \cdots \otimes V_d$-valued function $f(Z_1, \dots, Z_d)$ on $\mathbb{H}_{n_1} \times \cdots \times \mathbb{H}_{n_d}$ and $g_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \in \mathrm{Sp}_{n_i}(\mathbb{R})$, we put

$$f|_{\rho_1 \otimes \cdots \otimes \rho_d}[g_1, \dots, g_d] = \rho_1(C_1 Z_1 + D_1)^{-1} \otimes \cdots \otimes \rho_d(C_d Z_d + D_d)^{-1} f(g_1 Z_1, \dots, g_d Z_d).$$

We consider $V_{\mathbf{k},n_1,\dots,n_d} := V_{\mathbf{k},n_1} \otimes \cdots \otimes V_{\mathbf{k},n_d}$-valued differential operators $\mathbb{D}$ on scalar-valued functions of $\mathbb{H}_n$, satisfying Condition (A) below on automorphic properties:

Condition (A). We fix k and $\mathbf{k}$. For any holomorphic function F on $\mathbb{H}_n$ and any $(g_1, \dots, g_d) \in \mathrm{Sp}_{n_1}(\mathbb{R}) \times \cdots \times \mathrm{Sp}_{n_d}(\mathbb{R}) \subset \mathrm{Sp}_n(\mathbb{R})$, the operator $\mathbb{D}$ satisfies

$$\mathrm{Res}(\mathbb{D}(F|_k[(g_1, \dots, g_d)])) = (\mathrm{Res}\ \mathbb{D}(F))|_{\det^k \rho_{n_1,\mathbf{k}} \otimes \cdots \otimes \det^k \rho_{n_d,\mathbf{k}}}[g_1, \dots, g_d]$$

where Res means the restriction of a function on $\mathbb{H}_n$ to $\mathbb{H}_{n_1} \times \cdots \times \mathbb{H}_{n_d}$.

We will give another interpretation of this differential operator by a representation theory. The proof techniques in this chapter are based on the proofs in Ban [4].

4.1. **Howe duality.** Let $G_n = \mathrm{Sp}_n(\mathbb{R}) = \{M \in \mathrm{GL}_{2n}(\mathbb{Z}) | M J_n {}^t M = J_n\}$ be the symplectic group and $\widetilde{G_n} = \mathrm{Mp}_n(\mathbb{R})$ be the metaplectic group, which is the double cover of G_n. G_n acts on $\mathbb{H}_n$ in the same way that Γ_n does. Let K_n be the stabilizer of $\sqrt{-1} \in \mathbb{H}_n$ in G_n. Then, K_n is a maximal compact subgroup of G_n and isomorphic to the unitary group $\mathrm{U}_n(\mathbb{C})$, which is given by $\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A - \sqrt{-1}B$. We take a maximal compact subgroup $\widetilde{K_n}$ of $\widetilde{G_n}$ by the inverse image of K_n to $\widetilde{G_n}$.

We put $\mathfrak{g}_n = \mathrm{Lie}(\widetilde{G_n})$, $\mathfrak{k}_n = \mathrm{Lie}(\widetilde{K_n})$ and let $\mathfrak{g}_n = \mathfrak{k}_n \oplus \mathfrak{p}_n$ be the Cartan decomposition. We put

$$\kappa_{i,j} = \mathfrak{c}\begin{pmatrix} e_{i,j} & 0 \\ 0 & -e_{j,i} \end{pmatrix}\mathfrak{c}^{-1}, \quad \pi^+_{i,j} = \mathfrak{c}\begin{pmatrix} 0 & e_{i,j}+e_{j,i} \\ 0 & 0 \end{pmatrix}\mathfrak{c}^{-1}, \quad \text{and} \quad \pi^-_{i,j} = \mathfrak{c}\begin{pmatrix} 0 & 0 \\ e_{i,j}+e_{j,i} & 0 \end{pmatrix}\mathfrak{c}^{-1},$$

where $\mathfrak{c} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & 1 \end{pmatrix} \in \mathrm{M}_{2n}(\mathbb{C})$ and $e_{i,j} \in \mathrm{M}_{n,n}(\mathbb{C})$ is the matrix whose only non-zero entry is 1 in (i,j)-component. $\{\kappa_{i,j}\}$ is a basis of $\mathfrak{k}_{n,\mathbb{C}}$. Let $\mathfrak{p}_n^+$ (resp. $\mathfrak{p}_n^-$) be the $\mathbb{C}$-span of $\{\pi^+_{i,j}\}$ (resp. $\{\pi^-_{i,j}\}$) in $\mathfrak{g}_{n,\mathbb{C}}$. Then $\mathfrak{p}_n^+ \oplus \mathfrak{p}_n^- = \mathfrak{p}_{n,\mathbb{C}}$.

Definition 4.1. Let $L_{n,k} = \mathbb{C}[\mathrm{M}_{n,k}]$ be the space of polynomials in the entries of (n,k)-matrix $X = (X_{i,j})$ over $\mathbb{C}$.

(1) we define the $(\mathfrak{g}_{n,\mathbb{C}}, \widetilde{K_n})$-module structure $l_{n,k}$ on $L_{n,k}$ as follows:

$$l_{n,k}(\kappa_{i,j}) = \sum_{s=1}^{k} X_{i,s}\frac{\partial}{\partial X_{j,s}} + \frac{k}{2}\delta_{i,j},$$

$$l_{n,k}(\pi^+_{i,j}) = \sqrt{-1}\sum_{s=1}^{k} X_{i,s}X_{j,s},$$

$$l_{n,k}(\pi^-_{i,j}) = \sqrt{-1}\sum_{s=1}^{k} \frac{\partial^2}{\partial X_{i,s}\partial X_{j,s}}.$$

For $(g,\epsilon) \in \widetilde{\mathrm{U}_n(\mathbb{C})} \cong \widetilde{K_n}$ $(g \in \mathrm{U}_n(\mathbb{C}), \epsilon \in \{\pm 1\})$ and $f(X) \in L_{n,k}$, we define

$$l_{n,k}((g,\epsilon))f(X) = \epsilon^k f({}^t gX)$$

(2) we define the left action of the orthogonal group O_k on $L_{n,k}$ by the right transition.

This representation $(l_{n,k}, L_{n,k})$ is well-defined and we call it the Weil representation.

For a representation (λ, V_λ) in the unitary dual $\widehat{\mathrm{O}_k}$ of O_k, we put $L_{n,k}(\lambda) = \mathrm{Hom}_{\mathrm{O}_k}(V_\lambda, L_{n,k})$ and induce $(\mathfrak{g}_{n,\mathbb{C}}, \widetilde{K_n})$-module structure from that of $L_{n,k}$ to it. We denote by $L(\tau)$ the unitary lowest weight $(\mathfrak{g}_{n,\mathbb{C}}, \widetilde{K_n})$-module with lowest $\widetilde{K_n}$-type τ. Let (τ, U_τ) be the highest weight module of $\widetilde{K_n}$ with highest module τ and (λ, V_λ) be the highest weight module of O_k with highest module λ.

Remark 4.2. It is not always uniquely determined by the highest weight alone. The specific expression follows from Howe [9, §3.6.2].

The following symbols are provided to represent the decomposition of $L_{n,k}$.

Definition 4.3. Let $\Delta_{n,k}$ be the set of Young diagrams $D = (\mu_1, \mu_2, \ldots)$ whose depth $l(D)$ satisfies $l(D) \le \min\{n,k\}$ and if $l(D) > k/2$ then $\mu_j = 1$ $(k - l(D) < j \le l(D))$. We put $\mathbb{1}_n = (\underbrace{1,\ldots,1}_{n})$. For $D \in \Delta_{n,k}$, we define

$$\tau_{n,k}(D) = (\mu_1 + \tfrac{k}{2}, \ldots, \mu_n + \tfrac{k}{2}), \qquad \lambda_k(D) = \begin{cases} (\mu_1,\ldots,\mu_k) & (l(D) \le k/2) \\ (\mu_1,\ldots,\mu_{k-l(D)}) & (l(D) > k/2) \end{cases}.$$

Theorem 4.4. (1) *We have $L_{n,k}(\lambda) \neq 0$ if and only if $\lambda = \lambda_k(D)$ for some $D \in \Delta_{n,k}$.*

(2) *The lowest* $\widetilde{K_n}$*-type of* $L_{n,k}(\lambda_k(D))$ *is* $\tau_{n,k}(D)$.
(3) *Under the joint action of* $(\mathfrak{g}_{n,\mathbb{C}}, \widetilde{K_n}) \times \mathrm{O}_k$*, we have*
$$L_{n,k} \cong \bigoplus_{D \in \Delta_{n,k}} L(\tau_{n,k}(D)) \boxtimes V_{\lambda_k(D)}.$$

These results are proved by Kashiwara-Vergne [13], and Howe [9]. From this, we get correspondence between the highest weights of $\mathrm{GL}_n(\mathbb{C})$ and those of O_k, which is called Howe duality.

We fix positive integers $n_1, \ldots, n_d$ and set $n = n_1 + \cdots + n_d$. We embed $\widetilde{G_{n_1}} \times \cdots \times \widetilde{G_{n_d}}$ (resp. $\mathfrak{g}_{n_1,\mathbb{C}} \oplus \cdots \oplus \mathfrak{g}_{n_d,\mathbb{C}}$, $\widetilde{K_{n_1}} \times \cdots \times \widetilde{K_{n_d}}$) diagonally into $\widetilde{G_n}$ (resp. $\mathfrak{g}_{n,\mathbb{C}}$, $\widetilde{K_n}$). We denote its image by $\widetilde{G'_n}$ (resp. $\mathfrak{g}'_{\mathbb{C}}$, $\widetilde{K'_n}$). We denote by $X^{(s)}$ the indeterminate of L_{n_s}. Then, we can easily check that the $\mathbb{C}$-isomorphism
$$\bigotimes_{s=1}^{d} L_{n_s,k} \cong L_{n,k}$$
given by $X^{(s)}_{i,j} \mapsto X_{n_1+\cdots+n_{s-1}+i,j}$ is the isomorphism as $(\mathfrak{g}'_{\mathbb{C}}, \widetilde{K'}) \times {\mathrm{O}_k}^d$-module. We identify $\bigotimes_{s=1}^{d}(l_{n_s,k}, L_{n_s,k})$ with $(l_{n,k}, L_{n,k})$.

4.2. **Pluriharmonic polynomials.**

Definition 4.5. If polynomial $f(X) \in L_{n,k}$ satisfies
$$l_k(\pi^-_{i,j})f = \sum_{s=1}^{k} \frac{\partial^2 f}{\partial X_{i,s} \partial X_{j,s}} = 0 \text{ for any } i, j \in \{1, \ldots, n\},$$
we say that $f(X)$ is pluriharmonic polynomial for O_k.
We denote the set of all pluriharmonic polynomials for O_k in $L_{n,k}$ by $\mathfrak{H}_{n,k}$.

The following proposition is given by Kashiwara-Vergne [13].

Proposition 4.6. (1) $L_{n,k} = L_{n,k}^{\mathrm{O}_k} \cdot \mathfrak{H}_{n,k}$.
(2) $L_{n,k}^{\mathrm{O}_k}$ *is the subspace* $\mathbb{C}[Z^{(n)}]$ *of polynomials in the entries of* (n, n)*-symmetric matrix* $Z^{(n)} = X^t X$.
(3) *Under the joint action of* $\widetilde{K_n} \times \mathrm{O}_k$*, we have*
$$\mathfrak{H}_{n,k} \cong \bigoplus_{D \in \Delta_{n,k}} U_{\tau_{n,k}(D)} \boxtimes V_{\lambda_k(D)}.$$

We denote by $\mathfrak{H}_{n,k}(D)$ the subspace of $\mathfrak{H}_{n,k}$ corresponding to $U_{\tau_{n,k}(D)} \boxtimes V_{\lambda(D)}$ under this isomorphism.

Lemma 4.7.
$$\mathrm{Hom}_{({\mathfrak{g}'}_{\mathbb{C}}, \widetilde{K'})}\left(\overset{d}{\underset{s=1}{\otimes}} L(\tau_{n_s,k}(D)), L(\tfrac{k}{2}\mathbb{1}_n)\right) \cong \left(\left(\overset{d}{\underset{s=1}{\otimes}} \mathfrak{H}_{n_s,k}(D)\right)^{\mathrm{O}_k} \otimes \left(\overset{d}{\underset{s=1}{\otimes}} U^*_{\tau_{n_s,k}(D)}\right)\right)^{\widetilde{K'}}.$$

Proof. We note that $L_{n,k}(\emptyset) = L_{n,k}^{\mathrm{O}_k} \cong L(\frac{k}{2}\mathbb{1}_n)$ by Theorem 4.4. We have

$$\begin{aligned}\mathrm{Hom}_{(\mathfrak{g}'_{\mathbb{C}},\widetilde{K'})}\left(\bigotimes_{s=1}^{d} L(\tau_{n_s,k}(D)), L_{n,k}\right) &\cong \bigotimes_{s=1}^{d}\mathrm{Hom}_{(\mathfrak{g}_s,\widetilde{K_s})}\left(L(\tau_{n_s,k}(D)), L_{n_s,k}\right)\\ &\cong \bigotimes_{s=1}^{d}\mathrm{Hom}_{\widetilde{K_s}}\left(U_{\tau_{n_s,k}(D)}, \mathfrak{H}_{n_s,k}\right)\\ &\cong \bigotimes_{s=1}^{d}\left(\mathfrak{H}_{n_s,k}\otimes U^*_{\tau_{n_s,k}(D)}\right)^{\widetilde{K_s}}\\ &\cong \left(\left(\bigotimes_{s=1}^{d}\mathfrak{H}_{n_s,k}\right)\otimes\left(\bigotimes_{s=1}^{d}U^*_{\tau_{n_s,k}(D)}\right)\right)^{\widetilde{K'}}.\end{aligned}$$

Restricting to the O_k-invariant subspace gives the desired isomorphism. ☐

There is a natural injection

$$\begin{aligned}\left(\bigotimes_{s=1}^{d}\mathfrak{H}_{n_s,k}(D)\right)^{\mathrm{O}_k}\otimes\left(\bigotimes_{s=1}^{d}U^*_{\tau_{n_s,k}(D)}\right) &\hookrightarrow \left(\bigotimes_{s=1}^{d}L_{n_s,k}(D)\right)^{\mathrm{O}_k}\otimes\left(\bigotimes_{s=1}^{d}U^*_{\tau_{n_s,k}(D)}\right)\\ &\cong L_{n,k}^{\mathrm{O}_k}\otimes\left(\bigotimes_{s=1}^{d}U^*_{\tau_{n_s,k}(D)}\right)\\ &\cong \mathbb{C}[Z^{(n)}]\otimes\left(\bigotimes_{s=1}^{d}U^*_{\tau_{n_s,k}(D)}\right).\end{aligned}$$

We denote the image of $h \in \left(\bigotimes_{s=1}^{d}\mathfrak{H}_{n_s,k}(D)\right)^{\mathrm{O}_k}\otimes\left(\bigotimes_{s=1}^{d}U^*_{\tau_{n_s,k}(D)}\right)$ by $\Phi_h(Z^{(n)})$.

4.3. **Holomorphic automorphic forms.**

Definition 4.8. Let (τ, U_τ) be an irreducible unitary representation of $\widetilde{K_n}$ and $\widetilde{\Gamma_n}$ be a discrete subgroup of $\widetilde{G_n}$. Then, a holomorphic automorphic form of type τ for $\widetilde{\Gamma_n}$ is a U_τ^*-valued C^∞-function ϕ on $\widetilde{G_n}$ which satisfies the following conditions:

(1) $\phi(\gamma g k) = \tau^*(k)^{-1}\phi(g)$ for $k \in \widetilde{K_n}$ and $\gamma \in \widetilde{\Gamma_n}$,
(2) ϕ is annihilated by the right derivation of $\mathfrak{p}^-$,
(3) ϕ is of moderate growth.

We denote the space consisting of all holomorphic automorphic forms of type τ for $\widetilde{\Gamma_n}$ by $\left[C^\infty_{\mathrm{mod}}(\widetilde{\Gamma_n}\backslash\widetilde{G_n})\otimes U_\tau^*\right]^{\widetilde{K_n},\mathfrak{p}^-=0}$.

Note that $\mathrm{Hom}_{\widetilde{K_n}}\left(U_\tau, C^\infty_{\mathrm{mod}}(\widetilde{\Gamma_n}\backslash\widetilde{G_n})\right) \cong \left[C^\infty_{\mathrm{mod}}(\widetilde{\Gamma_n}\backslash\widetilde{G_n})\otimes U_\tau^*\right]^{\widetilde{K_n}}$, We can obtain the following well-known isomorphism.

Proposition 4.9. *We have*

$$\mathrm{Hom}_{(\mathfrak{g}_{n,\mathbb{C}},\widetilde{K_n})}\left(L(\tau), C^\infty_{\mathrm{mod}}(\widetilde{\Gamma_n}\backslash\widetilde{G_n})\right) \cong \left[C^\infty_{\mathrm{mod}}(\widetilde{\Gamma_n}\backslash\widetilde{G_n})\otimes U_\tau^*\right]^{\widetilde{K_n},\mathfrak{p}^-=0}.$$

Under this isomorphism, we denote by $I_F \in \mathrm{Hom}_{(\mathfrak{g}_{n,\mathbb{C}},\widetilde{K_n})}\left(L(\tau), C^\infty_{\mathrm{mod}}(\widetilde{\Gamma_n}\backslash\widetilde{G_n})\right)$ the corresponding homomorphism of $F \in \left[C^\infty_{\mathrm{mod}}(\widetilde{\Gamma_n}\backslash\widetilde{G_n}) \otimes U^*_\tau\right]^{\widetilde{K_n},\mathfrak{p}^-=0}$. Let $\widetilde{\Gamma}'$ be the image of $\widetilde{\Gamma_{n_1}} \times \cdots \times \widetilde{\Gamma_{n_d}}$ in $\widetilde{G_n}$.

Theorem 4.10. *Let F be a holomorphic automorphic form of type $\frac{k}{2}\mathbb{1}_n$ for $\widetilde{\Gamma_n}$ and $D \in \Delta_{n,k}$ be a Young diagram such that $l(D) \leq \min\{n_1, \ldots, n_d\}$. We put $\pi_n^+ = (\pi^+_{i,j}) \in M_n(\mathfrak{p}^+)$. We denote by* Res *be the pullback of the functions on $\widetilde{G_n}$ by the diagonal embedding $\widetilde{G_{n_1}} \times \cdots \times \widetilde{G_{n_d}} \hookrightarrow \widetilde{G_n}$.*

Then for any $h \in \left(\left(\bigotimes_{s=1}^d \mathfrak{H}_{n_s,k}(D)\right)^{\mathrm{O}_k} \otimes \left(\bigotimes_{s=1}^d U^*_{\tau_{n_s,k}(D)}\right)\right)^{\widetilde{K'}}$, $\mathrm{Res}\,(\Phi_h(\pi_n^+)\cdot F)$ *is a holomorphic automorphic form of type $\bigotimes_{s=1}^d(\tau_{n_s,k}(D))$ for $\widetilde{\Gamma_n}$.*

Proof. We fix an isomorphism $U^*_{\frac{k}{2}\mathbb{1}_n} \cong \mathbb{C}$. Let $I_F : L(\frac{k}{2}\mathbb{1}_n) \to C^\infty_{\mathrm{mod}}(\widetilde{\Gamma_n}\backslash\widetilde{G_n})$ be the corresponding $(\mathfrak{g}_{n,\mathbb{C}}, \widetilde{K_n})$-homomorphism of holomorphic automorphic form F. Multiplying the isomorphism $U^*_{\frac{k}{2}\mathbb{1}_n} \cong \mathbb{C}$ by a non-zero constant if necessarily, we assume that $I_F(1) = F$. We put

$$h = \sum_i h_i \otimes w_i^* \qquad \left(h_i \in \left(\bigotimes_{s=1}^d \mathfrak{H}_{n_s,k}(D)\right)^{\mathrm{O}_k},\ w_i^* \in \bigotimes_{s=1}^d U^*_{\tau_{n_s,k}(D)}\right).$$

Then, we denote by $I_h : \bigotimes_{s=1}^d L(\tau_{n_s,k}(D)) \to L(\frac{k}{2}\mathbb{1}_n)$ the corresponding $(\mathfrak{g}'_{\mathbb{C}}, \widetilde{K'})$-homomorphism of h in the isomorphism of Lemma 4.7. For $w \in \bigotimes_{s=1}^d U_{\tau_{n_s,k}(D)} \subset \bigotimes_{s=1}^d L(\tau_{n_s,k}(D))$, we have

$$I_h(w) = \sum_i \langle w, w_i^*\rangle h_i \in \left(\bigotimes_{s=1}^d \mathfrak{H}_{n_s,k}(D)\right)^{\mathrm{O}_k} \subset L^{\mathrm{O}_k}_{n,k} \cong L(\tfrac{k}{2}\mathbb{1}_n).$$

On the other hand, by the definition of Weil representation, we have

$$h_i = \Phi_{h_i}(Z^{(n)}) = \Phi_{h_i}(-\sqrt{-1}\pi_n^+)\cdot 1 = (-\sqrt{-1})^m \Phi_{h_i}(\pi_n^+)\cdot 1,$$

where m is a degree of Φ_{h_i}. (Note that nonzero elements of $\mathfrak{H}_{n_s,k}(D)$ consist of the same degree homogeneous polynomial.)

Therefore,

$$\begin{aligned}
I_F \circ I_h(w) &= I_F\left(\textstyle\sum_i \langle w, w_i^*\rangle h_i\right)\\
&= (-\sqrt{-1})^m I_F\left(\textstyle\sum_i \langle w, w_i^*\rangle \Phi_{h_i}(\pi_n^+)\cdot 1\right)\\
&= (-\sqrt{-1})^m \textstyle\sum_i \langle w, w_i^*\rangle \Phi_{h_i}(\pi_n^+)\cdot I_F(1)\\
&= (-\sqrt{-1})^m \textstyle\sum_i \langle w, w_i^*\rangle \Phi_{h_i}(\pi_n^+)\cdot F\\
&= \langle w, (-\sqrt{-1})^m \left(\textstyle\sum_i w_i^* \otimes \Phi_{h_i}\right)(\pi_n^+)\cdot F\rangle\\
&= \langle w, (-\sqrt{-1})^m \Phi_h(\pi_n^+)\cdot F\rangle.
\end{aligned}$$

By Proposition 4.9, the $(\mathfrak{g}'_{n,\mathbb{C}}, \widetilde{K'_n})$-homomorphism $\mathrm{Res} \circ I_F \circ I_h : L(\tau_{n,k}(D)) \to C^\infty_{\mathrm{mod}}(\widetilde{\Gamma'}\backslash\widetilde{G'_n})$ corresponds to $\mathrm{Res}\left((-\sqrt{-1})^m \Phi_h(\pi_n^+) \cdot F\right)$, which is the holomorphic automorphic form of type $\bigotimes_{s=1}^d (\tau_{n_s,k}(D))$ for $\widetilde{\Gamma'}$. From the above, we have that $\mathrm{Res}\,(\Phi_h(\pi_n^+) \cdot F)$ is the holomorphic automorphic form of type $\bigotimes_{s=1}^d (\tau_{n_s,k}(D))$ for $\widetilde{\Gamma'}$. $\square$

We define the action of $\widetilde{G_n}$ on $\mathbb{H}_n$ to be through G_n and we also denoted The complexification of the representation τ of $\widetilde{K_n} \cong \widetilde{U_n}$ by the same symbol.

For a representation (τ, U_τ) of $\widetilde{K_n}$, we define the representation $(\tau', U_{\tau'}\ (= U_\tau))$ by $\tau'(g) = \tau({}^t g^{-1})$. There is an isomorphism of $\tau^* \cong \tau'$ as representations. For $f \in M_\tau(\Gamma_n)$, we define a U_τ-valued C^∞-function ϕ_f on $\widetilde{G_n}$ by

$$\phi_f(g) = (f|_{\tau'} g)(\sqrt{-1}) = \tau({}^t j(g, \sqrt{-1})) f(g \cdot \sqrt{-1}) \quad \text{for} \quad g \in \widetilde{G_n}.$$

Proposition 4.11. *The above correspondence $f \mapsto \phi_f$ gives the isomorphism*

$$M_\tau(\Gamma_n) \xrightarrow{\sim} \left[C^\infty_{\mathrm{mod}}(\widetilde{\Gamma_n}\backslash\widetilde{G_n}) \otimes U_{\tau'}\right]^{\widetilde{K_n}, \mathfrak{p}^- = 0}.$$

By the above isomorphism and Theorem 4.10, We have the following theorem.

Theorem 4.12. *Let F be a holomorphic automorphic form of weight $\frac{k}{2}\mathbb{1}_n$ for Γ_n and $D \in \Delta_{n,k}$ be a Young diagram such that $l(D) \leq \min\{n_1, \ldots, n_d\}$. We put $\partial_Z = \left(\frac{1+\delta_{i,j}}{2} \frac{\partial}{\partial z_{i,j}}\right)$. We denote by* Res *be the restriction of a function on $\mathbb{H}_n$ to $\mathbb{H}_{n_1} \times \cdots \times \mathbb{H}_{n_d}$*

Then for any $h \in \left(\left(\bigotimes_{s=1}^d \mathfrak{H}_{n_s,k}(D)\right)^{\mathrm{O}_k} \otimes \left(\bigotimes_{s=1}^d U_{\tau'_{n_s,k}(D)}\right)\right)^{\widetilde{K'}}$, $\mathrm{Res}\,(\Phi_h(\partial_Z) \cdot F)$ *is a holomorphic automorphic form of weight $\bigotimes_{s=1}^d (\tau_{n_s,k}(D))$ for Γ_n.*

Before the proof, we provide some notations and a lemma.

Definition 4.13. For a holomorphic function f on $\mathbb{H}_n$ and a representation (τ, U_τ) of $\widetilde{U_n}$, we define the function $\tilde{f}$ on $\widetilde{G_n}$ and the representation $(\widetilde{\tau}, U_\tau)$ of $\widetilde{G_n}$ as follow:

$$\tilde{f}(g) = f(g \cdot \sqrt{-1}) \qquad g \in \widetilde{G_n},$$
$$\tilde{\tau}((g_0, \epsilon)) = \tau((j(g_0, \sqrt{-1}), \epsilon)) \qquad g_0 \in G_n,\ \epsilon \in \{\pm 1\}.$$

For $Z = X + \sqrt{-1}Y \in \mathbb{H}_n$, we put $g^{(n)}_{Z,0} = \begin{pmatrix} \sqrt{Y} & X\sqrt{Y}^{-1} \\ 0 & \sqrt{Y}^{-1} \end{pmatrix} \in G_n$. We define the section of $\widetilde{G_n} \ni g \mapsto g \cdot \sqrt{-1} \in \mathbb{H}_n$ by

$$Z \mapsto (g^{(n)}_{Z,0}, (\det(\sqrt{Y})^{\frac{1}{2}})) := g^{(n)}_Z.$$

We have

$$j(g^{(n)}_{Z,0}, \sqrt{-1}) = \sqrt{Y}^{-1} \qquad g^{(n)}_Z \cdot \sqrt{-1} = Z.$$

For $D = (\mu_1, \mu_2, \ldots) \in \Delta_{n,k}$, we denote by $\rho_{n,D}$ the representation of $\widetilde{GL_n(\mathbb{C})}$ with a dominant integral weight D. (Then, $\tau_{n,k}(D) = \det^{\frac{k}{2}} \otimes \rho_{n,D}$.)

Lemma 4.14 (Ban [4]). *For a holomorphic function f on $\mathbb{H}_n$ and a representation (τ, U_τ) of $\widetilde{U_n}$, we have*

$$(\pi^+_{i,j}\tilde{f})(g) = 4({}^t\tilde{\omega}(g)\cdot \widetilde{\partial_Z f}(g)\cdot \tilde{\omega}(g))_{i,j},$$
$$(\pi^+_{i,j}\tilde{\tau}')(g) = -\sqrt{-1}\tilde{\tau}'(g)\cdot d\tau((e_{i,j}+e_{j,i})\cdot \overline{\tilde{\omega}(g)}^{-1}\cdot \tilde{\omega}(g)),$$

where $\tilde{\omega}(g) = {}^t j(g_0, \sqrt{-1})^{-1}$ for $g = (g_0, \epsilon) \in \widetilde{G_n}$.

In particular, for $Z = X + \sqrt{-1}Y$, we have

$$(\pi^+_{i,j}\tilde{f})(g_Z) = 4(\sqrt{Y}\cdot \widetilde{\partial_Z f}(g_Z)\cdot \sqrt{Y})_{i,j},$$
$$(\pi^+_{i,j}\tilde{\tau}')(g_Z) = -\sqrt{-1}\tilde{\tau}'(g_Z)\cdot d\tau((e_{i,j}+e_{j,i})).$$

Proof of Theorem 4.12. From Theorem 4.10, $\mathrm{Res}\,(\Phi_h(\pi^+_n)\cdot\phi_F)$ is a holomorphic automorphic form of type $\bigotimes_{s=1}^d(\tau_{n_s,k}(D))$ for $\widetilde{\Gamma'_n}$. By Proposition 4.11, there exists a holomorphic automorphic form f of weight $\bigotimes_{s=1}^d(\tau_{n_s,k}(D))$ for $\widetilde{\Gamma'_n}$ such that $\phi_f = \mathrm{Res}\,(\Phi_h(\pi^+_n)\cdot\phi_F)$. Since

$$\phi_f(g^{(n_1)}_{Z_1},\ldots,g^{(n_d)}_{Z_d}) = (\mathop{\otimes}_{s=1}^{d}\det(\sqrt{Y_s})^{\frac{k}{2}}(\rho_{n_s,D}(\sqrt{Y_s})))f(Z_1\ldots,Z_d),$$

we have

$$f(Z_1,\ldots,Z_d) = (\mathop{\otimes}_{s=1}^{d}\det(\sqrt{Y_s})^{-\frac{k}{2}}\rho^{-1}_{n_s,D}(\sqrt{Y_s}))\mathrm{Res}\left(\Phi_h(\pi^+_n)\cdot\phi_F\right)(g^{(n_1)}_{Z_1},\ldots,g^{(n_d)}_{Z_d}).$$

Now we consider $\pi^+_{i,j}$'s action on ϕ_F. Note that $\phi_F(g) = (\widetilde{\det^{\frac{k}{2}}\cdot \widetilde{F}})(g)$ using the notations above.

Under the isomorphism $\mathbb{H}_n\times\widetilde{K_n}\cong\widetilde{G_n}$, we regard the function ϕ_F as the function in $Z = X+\sqrt{-1}Y\in\mathbb{H}_n$, $\sqrt{Y}$ and $k\in\widetilde{K_n}$. We can easily check that $d\tau((e_{i,j}+e_{j,i})\cdot\overline{\tilde{\omega}(g)}^{-1}\cdot\tilde{\omega}(g))$ is invariant under the left transition by g_Z for any $Z\in\mathbb{H}_n$ and this could be regarded as a function on $\widetilde{K_n}$. Since the right derivations by $\pi^+_{i,j}$ commutes with the left transition by g_Z, the derivated function of $d\tau((e_{i,j}+e_{j,i})\cdot\overline{\tilde{\omega}(g)}^{-1}\cdot\tilde{\omega}(g))$ could also be regarded as a function on $\widetilde{K_n}$.

From the Lemma 4.14, we can easily check that the highest degree part of $\Phi_h(\pi^+_n)\cdot\phi_F$ in $\sqrt{Y}$ is $4^m\det(\sqrt{Y})^{\frac{k}{2}}\cdot\Phi_h(\sqrt{Y}\partial_Z\sqrt{Y})\cdot\phi_F$, where m is a degree of Φ_F.

Since $h\in\left(\left(\mathop{\otimes}_{s=1}^{d}\mathfrak{H}_{n_s,k}(D)\right)^{\mathrm{O}_k}\otimes\left(\mathop{\otimes}_{s=1}^{d}U_{\tau'_{n_s,k}(D)}\right)\right)^{\widetilde{K'}}$, we have

$$\mathrm{Res}(4^m\det(\sqrt{Y})^{\frac{k}{2}}\cdot\Phi_h(\sqrt{Y}\partial_Z\sqrt{Y})\cdot\phi_F) = 4^m(\mathop{\otimes}_{s=1}^{d}\det(\sqrt{Y_s})^{\frac{k}{2}}(\rho_{n_s,D}(\sqrt{Y_s})))\mathrm{Res}(\Phi_h(\partial_Z)\cdot\phi_F).$$

Thus we may denote $\mathrm{Res}\,(\Phi_h(\pi^+_n)\cdot\phi_F)$ by

$$\mathrm{Res}\left(\Phi_h(\pi^+_n)\cdot\phi_F\right)(g_1,\ldots,g_n) = 4^m(\mathop{\otimes}_{s=1}^{d}\det(\sqrt{Y_s})^{\frac{k}{2}}(\rho_{n_s,D}(\sqrt{Y_s})))\mathrm{Res}(\Phi_h(\partial_Z)\cdot\phi_F)$$
$$+(\mathop{\otimes}_{s=1}^{d}\det(\sqrt{Y_s})^{\frac{k}{2}})R,$$

where $R = R(Z_1, \ldots, Z_d, \sqrt{Y_1}, \ldots, \sqrt{Y_d}, k_1, \ldots, k_d)$ is a function with a degree strictly lower than that of $\bigotimes_{s=1}^{d} \rho_{n_s,D}(\sqrt{Y_s})$ in $(\sqrt{Y_1}, \ldots, \sqrt{Y_d})$. Then we have,

$$f = 4^m \mathrm{Res}(\Phi_h(\partial_Z) \cdot \phi_F) + (\bigotimes_{s=1}^{d} \rho_{n_s,D}(\sqrt{Y_s})^{-1})R.$$

On the other hand, Since f is a holomorphic function, $R = 0$. Therefore, $\mathrm{Res}(\Phi_h(\partial_Z) \cdot \phi_F) = 4^{-m} f$ is a holomorphic automorphic form of weight $\bigotimes_{s=1}^{d} (\tau_{n_s,k}(D))$ for $\widetilde{\Gamma_n}$. $\square$

Proposition 4.15 (Ibukiyama [10])**.** *If $d = 2$ and $4k \geq n = n_1 + n_2$, then there exists a differential operator satisfying condition (A) and it is unique up to a constant.*

A more detailed algorithm for the construction of the differential operators is examined by Ibukiyama [11].

4.4. **Pullback formula.** We take a differential operator $\mathbb{D}_{\mathbf{k}} = \mathbb{D}_{\mathbf{k},n_1,n_2}$ on $\mathbb{H}_n$ satisfying Condition (A) for k and $\det^k \rho_{n_1,\mathbf{k}} \otimes \det^k \rho_{n_2,\mathbf{k}}$. For an integer r such that $l(\mathbf{k}) \leq r$, we put $\rho_r = \det^k \rho_{r,\mathbf{k}}$. For a Hecke eigenform $f \in S_{\rho_r}(\Gamma_r)$, we define $D(s, f)$ by

$$D(s, f) = \zeta(s)^{-1} \prod_{i=1}^{r} \zeta(2s - 2i)^{-1} L(s - r, f, \mathrm{St}).$$

The next theorem is due to Kozima [18].

Theorem 4.16 (pullback formula)**.** *Let $\mathbf{k} = (\nu, \ldots, \nu, 0, \ldots, 0)$ with even integer ν, and $n = n_1 + n_2 \geq 2$ be a positive integer with $n_2 \geq n_1 \geq 1$. Let $s \in \mathbb{C}$ such that $2\Re(s) + k > n + 1$. Then for any Hecke eigenform $f \in S_{\rho_{n_1}}(\Gamma_{n_1})$ we have*

$$\left(f, \mathbb{D}_{\mathbf{k},n_1,n_2} E_{n,k}\left(\begin{pmatrix} * & O \\ O & -\overline{W} \end{pmatrix}, \bar{s} \right) \right) = c(s, \rho_{n_1}) D(2s + k, f) \, [f]_{\rho_{n_1}}^{\rho_{n_2}} (W, s),$$

where $c(s, \rho_{n_1})$ is a function of s depending on ρ_{n_1} but not on ρ_{n_2}.

4.5. **The case of $l(\mathbf{k}) = 1$.** In the case $\mathbf{k} = (\nu, 0, 0, \ldots)$ (i.e. $l(\mathbf{k}) = 1$), it is well-known that the differential operator $\mathbb{D}_{k,\nu;n_1,n_2} = \mathbb{D}_{k,\nu}$ is given using the Gegenbauer polynomial.

Definition 4.17. We define the polynomial $P_{d,\nu}(s, m)$ by

$$\frac{1}{(1 - 2st + mt^2)^{(d-2)/2}} = \sum_{\nu=0}^{\infty} P_{d,\nu}(s, m) t^{\nu}.$$

These polynomials are called Gegenbauer polynomials.

The Gegenbauer polynomial $P_{d,\nu}(s, m)$ can be written concretely as follow:

$$P_{d,\nu}(s, m) = \sum_{\mu=0}^{[\nu/2]} (-1)^{\mu} \frac{(d/2 - 1)_{\nu-\mu}}{(\nu - 2\mu)! \mu!} (2s)^{\nu - 2\mu} m^{\mu},$$

where $(x)_\mu = x(x + 1) \cdots (x + \mu - 1)$ and $[\ \cdot\ ]$ is the Gauss symbol.

For $Z \in \mathbb{H}_n$, we write $Z = \begin{pmatrix} Z_1 & Z_{12} \\ {}^tZ_{12} & Z_2 \end{pmatrix}$ $(Z_1 \in \mathbb{H}_{n_1},\ Z_2 \in \mathbb{H}_{n_2},\ Z_{12} \in M_{n_1,n_2}(\mathbb{C}))$ and $Z_m = (z_{i,j}^{(m)})$, $Z_{12} = (z_{i,j}^{(12)})$. We define the $n_m \times n_m$ matrix of partial differential operator $\dfrac{\partial}{\partial Z_m}$ by

$$\frac{\partial}{\partial Z_m} = \left(\frac{1+\delta_{i,j}}{2} \frac{\partial}{\partial z_{i,j}^{(m)}} \right)$$

and the $n_1 \times n_2$ matrix of partial differential operator $\dfrac{\partial}{\partial Z_{12}}$ by

$$\frac{\partial}{\partial Z_{12}} = \left(\frac{1}{2} \frac{\partial}{\partial z_{i,j}^{(12)}} \right).$$

Proposition 4.18. *Let $\mathbf{k} = (\nu, 0, 0, \dots)$ be a dominant integral weight and k be a non-negative integer. We take $u = (u_1, \dots, u_{n_1})$ and $v = (v_1, \dots, v_{n_2})$ to be the variables in the representation spaces of $\rho_{n_1,\mathbf{k}}$ and $\rho_{n_2,\mathbf{k}}$ as the symmetric tensor representations. Then,*

$$\mathbb{D}_{k,\nu} = P_{2k,\nu}\left(u \frac{\partial}{\partial Z_{12}} {}^t v, \left(u \frac{\partial}{\partial Z_1} {}^t u \right) \left(v \frac{\partial}{\partial Z_2} {}^t v \right) \right)$$

satisfies Condition (A) for $\det^k$ and $\det^k \rho_{n_1,\mathbf{k}} \otimes \det^k \rho_{n_2,\mathbf{k}}$.

We shall determine $c_{k,\nu,1} := c(0, \rho_1)$ in Theorem 4.16. Let $\mathring{\mathfrak{D}}_{1,k}^{\nu}$ be a differential operator in [6]. From [6, Theorem3.1], we have

$$\left(f, \mathring{\mathfrak{D}}_{1,k}^{\nu} E_{2,k} \left(\begin{pmatrix} * & O \\ O & -\overline{W} \end{pmatrix}, \bar{s} \right) \right) = \widetilde{c_1} D(k,f) f(W),$$

where

$$\widetilde{c_1} = (-1)^{(k+\nu)/2} \cdot 2^{3-2(k+\nu)} \cdot \pi \cdot \frac{\Gamma(k+\nu-1/2)\Gamma(k+\nu-1)}{\Gamma(k)\Gamma(k-1/2)}.$$

By Corollary 4.15, there exist a constant $d_{k,\nu}$ such that $\mathbb{D}_{\mathbf{k},1} = d_{k,\nu} \cdot \mathring{\mathfrak{D}}_{1,k}^{\nu}$.

By [15], taking $Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix} \in \mathbb{H}_2$, we have

$$\mathring{\mathfrak{D}}_{1,k}^{\nu}(z_{12}^{\nu}) = (-1)^{\nu} \frac{\nu}{2} \prod_{\mu=1}^{\nu} (k+\nu-1-\mu/2).$$

Using the Gegenbauer polynomial, we have also

$$\mathbb{D}_{\mathbf{k},1,1}(z_{12}^{\nu}) = \frac{(k+\nu-2)!}{2^{\nu}(k-2)!}.$$

Therefore,

$$d_{k,\nu} = \frac{(k+\nu-2)!}{(k-2)!\nu!} \cdot \prod_{\mu=1}^{\nu} \frac{1}{k+\nu-1-\mu/2}$$

$$c_{k,\nu,1} = (-1)^{(k+\nu)/2} \cdot 2^{3-2(k+\nu)} \cdot \pi \cdot \frac{((k+\nu-2)!)^2}{(k-1)!(k-2)!\nu!} \prod_{\mu=2}^{\nu} \frac{2(k+\nu)-1-2\mu}{2(k+\nu)-2-\mu}.$$

Using the exact pullback formula given by Ibukiyama [12], we have

$$c_{k,\nu,2} := c(0,\rho_2) = (-1)^{(2k+\nu)/2}\cdot 2^{7-2(k+\nu)}\cdot \pi^3 \times \frac{((k+\nu-2)!)^2}{(k-1)(k-2)((k-2)!)^2\nu!(2k+\nu-3)}\prod_{\mu=2}^{\nu}\frac{2(k+\nu)-1-2\mu}{2(k+\nu)-2-\mu}.$$

Thus, in particular, the following holds.

Corollary 4.19. *If a prime p satisfies $p \geq 2(k+\nu)-3$, then $\pi^{-1}c_{k,\nu,1}$ and $\pi^{-3}c_{k,\nu,2}$ are a p-unit rational number.*

5. Kurokawa-Mizumoto congruence

Let k, ν, n_1, and n_2 be positive integers such that k,ν are even and $1 \leq n_1 \leq n_2 \leq 2$. For a dominant integral weight $\mathbf{k} = (\nu, 0, 0, \ldots)$, we define $\rho_n = \det^k \rho_{n,\mathbf{k}}$ as in the previous chapters. Let $f_{r,1},\ldots,f_{r,d_r}$ be a orthogonal basis of $S_{\rho_r}(\Gamma_r)$ consisting of Hecke eigenforms. By Taylor [25, Lemma 2.1], we can assume that$f_{r,1},\ldots,f_{r,d_r}$ are elements in $S_{\rho_r}(\Gamma_r)(\mathbb{Z})$.

Theorem 5.1 (Ibukiyama [12])**.** *Let k be a integer such that $k \geq \max\left(\frac{n_1+n_2+1}{2}, 4, \frac{n_2+6}{2}\right)$ and neither $k = \frac{n_1+n_2+2}{2} \equiv 2 \mod 4$ nor $k = \frac{n_1+n_2+3}{2} \equiv 2 \mod 4$. Then we have*

$$\mathbb{D}_{\mathbf{k},n_1,n_2}E_{n_1+n_2,k}\begin{pmatrix} Z & O \\ O & W\end{pmatrix} = \sum_{r=1}^{\min\{n_1,n_2\}} c_{k,\nu,r}\sum_{j=1}^{d_r}\frac{D(k,f_{r,j})}{(f_{r,j},f_{r,j})}\left[f_{r,j}\right]^{\rho_{n_1}}(Z)\left[f_{r,j}\right]^{\rho_{n_2}}(W).$$

Now we define the functions as follows:

$$Z(n,k) = \zeta(1-k)\prod_{j=1}^{[n/2]}\zeta(1+2j-2k),$$

$$\widetilde{E_{n,k}}(Z) = Z(n,k)E_{n,k}(Z),$$

$$\mathcal{E}(Z_1,Z_2) = \mathcal{E}_{k,\nu,n_1,n_2}(Z_1,Z_2) := \nu!(2\pi\sqrt{-1})^{-\nu}\mathbb{D}_{\nu,n_1,n_2}\widetilde{E_{n_1+n_2,k}}\begin{pmatrix} Z_1 & O \\ O & Z_2\end{pmatrix}.$$

For $f \in S_k(\Gamma_1)$, we put

$$\mathbb{L}(s,f,\mathrm{St}) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s+k-1)\frac{L(s,f,\mathrm{St})}{(F,F)},$$

$$\mathcal{C}_{m,k}(f) = \frac{Z(m,k)}{Z(2,k)}\mathbb{L}(k-1,f,\mathrm{St}).$$

For $F \in S_{(k+\nu,k)}(\Gamma_2)$, we put

$$\mathbb{L}(s,F,\mathrm{St}) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s+k+\nu-1)\Gamma_{\mathbb{C}}(s+k-2)\frac{L(s,F,\mathrm{St})}{(F,F)},$$

$$\mathcal{C}_{m,k}(F) = \frac{Z(m,k)}{Z(4,k)}\mathbb{L}(k-2,F,\mathrm{St}).$$

Proposition 5.2 (Kozima [17]). *Let $k, \nu \in 2\mathbb{Z}_{\geq 0}$ and $k \geq 2n+2$. Let $f \in S_{\det^k \otimes \mathrm{Sym}^\nu}(\Gamma_n)(\mathbb{Q}(f))$ be an Hecke eigenform. Let $m \in \mathbb{Z}$ be such that $1 \leq m \leq k-n$ and $m \equiv n \mod 2$. We assume that $n \equiv 3 \mod 4$ if $m = 1$. Then we have*

$$\frac{L(m, f, \mathrm{St})}{\pi^{nk+l+m(n+1)-n(n+1)/2}(f, f)} \in \mathbb{Q}(f).$$

By Proposition 2.2 and Proposition 5.2, we obtain the following corollary.

Corollary 5.3. (1) *For a Hecke eigenform $f \in S_k(\Gamma_1)(\mathbb{Q}(f))$, we have*

$$\mathcal{C}_{4,k}(f) \in \mathbb{Q}(f).$$

(2) *For a Hecke eigenform $F \in S_{(k+\nu,k)}(\Gamma_2)(\mathbb{Q}(f))$ with $k \geq 6$, we have*

$$\mathcal{C}_{4,k}(F) \in \mathbb{Q}(F).$$

From the above theorem, the following proposition follows.

Proposition 5.4. *Let $1 \leq n_1 \leq n_2 \leq 2$ and $k \geq 4$. Then*

$$\mathcal{E}_{k,\nu,n_1,n_2}(Z_1, Z_2) = \sum_{r=1}^{\min\{n_1,n_2\}} \gamma_r \sum_{j=1}^{d_r} \mathcal{C}_{n_1+n_2,k}(f_{r,j}) \left[f_{r,j}\right]^{\rho_{n_1}}(Z_1) \left[f_{r,j}\right]^{\rho_{n_2}}(Z_2),$$

where γ_r is a p-unit rational number for $r = 1, 2$ and any prime p with $p \geq 2(k+\nu) - 3$.

Proof. We only need to determine the γ_r for $r = 1, 2$. we have

$$\begin{aligned} c_{k,\nu,1} Z(2,k) \frac{D(k, f_j)}{(f_j, f_j)} &= \frac{\zeta(1-k)\zeta(3-2k)}{\zeta(k)\zeta(2k-2)} c_{k,\nu,1} \cdot \frac{L(k-1, f_j, \mathrm{St})}{(f_j, f_j)} \\ &= (-1)^{k/2+1} \frac{\Gamma_{\mathbb{C}}(k)\Gamma_{\mathbb{C}}(2k-2)}{\Gamma_{\mathbb{C}}(k-1)\Gamma_{\mathbb{C}}(2k+\nu-2)} c_{k,\nu,1} \cdot \mathbb{L}(k-1, f_j, \mathrm{St}) \\ &= \frac{(k-1)(-1)^{k/2+1}(2\pi)^{\nu-1}}{(2k+\nu-3)_\nu} c_{k,\nu,1} \cdot \mathbb{L}(k-1, f_j, \mathrm{St}). \end{aligned}$$

Therefore, we may take γ as

$$\begin{aligned} \gamma_1 &= \nu! \cdot (2\pi\sqrt{-1})^{-\nu} \cdot \frac{(k-1)(-1)^{k/2+1}(2\pi)^{\nu-1}}{(2k+\nu-3)_\nu} \cdot c_{k,\nu,1} \\ &= -2^{2-2(k+\nu)} \prod_{\mu=1}^{\nu} \frac{(k+\nu-1-\mu)^2(2k+2\nu-1-2\mu)}{(2k+\nu-2-\mu)(2k+2\nu-2-\mu)}. \end{aligned}$$

By the same calculation, we have

$$\gamma_2 = 2^{4-2(k+\nu)} \prod_{\mu=1}^{\nu} \frac{(k+\nu-1-\mu)^2(2k+2\nu-1-2\mu)}{(2k+\nu-2-\mu)(2k+2\nu-2-\mu)}.$$

Thus, the proposition follows from Theorem 5.1. □

We write $\mathcal{E}(Z_1, Z_2)$ as

$$\mathcal{E}(Z_1, Z_2) = \sum_{N \in H_{n_2}(\mathbb{Z})_{\geq 0}} g^{(n_1)}_{(k,\nu,n_1,n_2),N}(Z_1)(\mathrm{tr}(NZ_2)).$$

Then, $g^{n_1}_N = g^{(n_1)}_{(k,\nu,n_1,n_2),N} \in M_{\rho_{n_1}}(\Gamma_{n_1}) \otimes V_{n_2,\mathbf{k}}$. From the proposition, we get the following corollary.

Corollary 5.5. *For $N \in H_{n_2}(\mathbb{Z})_{>0}$ and $1 \le n_1 \le n_2 \le 2$, we have*
$$g_N^{n_1}(Z_1) = \sum_{r=1}^{\min\{n_1,n_2\}} \gamma_r \sum_{j=1}^{d_r} \mathcal{C}_{n_1+n_2,k}(f_{r,j}) \left[f_{r,j}\right]^{\rho_{n_1}}(Z_1) a\left(N, [f_{r,j}]^{\rho_{n_2}}\right).$$

About the rationality of these functions, the following propositions hold. These can be proved in the same way as [3].

Corollary 5.6. *We have*
$$g^{(n_1)}_{(k,\nu,n_1,n_2),N}(Z_1) \in (M_{\rho_{n_1}}(\Gamma_{n_1}) \otimes V_{n_2,\mathbf{k}})(\mathbb{Q}).$$
Moreover, if $p > \max\{2k, k+\nu-2\}$, then
$$g^{(n_1)}_{(k,\nu,n_1,n_2),N}(Z_1) \in (M_{\rho_{n_1}}(\Gamma_{n_1}) \otimes V_{n_2,\mathbf{k}})(\mathbb{Z}_{(p)}).$$

Proposition 5.7. *If $f \in S_{k+\nu}(\Gamma_1)(\mathbb{Q})$, then $[f]^{(k+\nu,k)} \in S_{(k+\nu,k)}(\Gamma_2)(\mathbb{Q})$.*

Now we define the integral ideal $\mathfrak{A}(f)$ of $\mathbb{Q}(f)$ for a Hecke eigenform $f \in S_{(k_1,\dots,k_n)}(\Gamma_n)(\mathbb{Q}(f))$ with $k_n > n$ and state the integrality lemma in [22].

We put $V = \bigoplus_\tau \mathbb{C} f^\tau$, where τ runs over all embeddings of $\mathbb{Q}(f)$ into $\mathbb{C}$. Let $V^\perp$ be the orthogonal complement of V in $S_k(\Gamma_n)$. Let $\nu(f)$ (resp. $\kappa(f)$) be the exponent of the finite abelian group $S_k(\Gamma_n)(\mathbb{Z})/(V(\mathbb{Z}) \oplus V^\perp(\mathbb{Z}))$ (resp. $\mathcal{O}_{\mathbb{Q}(f)}/\mathbb{Z}[\lambda_f(T) | T \in \mathcal{H}_n]$). we put
$$\mathfrak{A}(f) = \kappa(f)\nu(f)\mathfrak{d}(\mathbb{Q}(f)),$$
where $\mathfrak{d}(\mathbb{Q}(f))$ be the different of $\mathbb{Q}(f)/\mathbb{Q}$.

Lemma 5.8 (Integrality lemma [22])**.** *Let $f \in S_k(\Gamma_n)(\mathbb{Q}(f))$be a Hecke eigenform with $n \in \mathbb{Z}_{>0}$ and even integer k such that $k \ge \frac{3}{2}(n+1)$. Suppose that f has a Fourier coefficient which is equal to 1. Let K be an algebraic number field. Then for any $g \in S_k(\Gamma_n)(\mathcal{O}_K)$ we have*
$$\frac{(f,g)}{(f,f)} \in \mathfrak{A}(f)^{-1} \cdot \mathcal{O}_{K\cdot\mathbb{Q}(f)}$$

Using the integrality lemma (Lemma 5.8) for a normalized Hecke eigenform $f \in S_{k+\nu}(\Gamma_1)$ and $\mathrm{pr}(g^{(1)}_{(k,\nu,1,2),N}(Z_1))$, where $\mathrm{pr} : M_{k+\nu}(\Gamma_n) \to S_{k+\nu}(\Gamma_n)$ be the orthogonal projection, we have the following proposition.

Proposition 5.9. *Let k, ν be positive even integers with $k \ge 4$, $f \in S_{k+\nu}(\Gamma_1)$ be a normalized Hecke eigenform. For any prime p with $p > \max\{2k, k+\nu-2\}$ and $A \in H_2(\mathbb{Z})_{>0}$, we have*
$$a(A, [f]^{\rho_2}) \in V_{2,\mathbf{k}}\left(\frac{Z(4,k)}{\gamma_1 \mathcal{C}_{4,k}(f)} \mathfrak{A}(f)^{-1} \cdot \mathbb{Z}_{(p)}\right)$$

Theorem 5.10. *Let k, ν be positive even integers with $k \ge 4$, $f_{1,1} = f, \dots, f_{1,d_1}$ be a basis of $S_{k+\nu}(\Gamma_1)$ consist of normalized Hecke eigenforms, p be a prime number of $\mathbb{Q}$ and $A \in H_2(\mathbb{Z})_{>0}$ be a half-integral positive definite matrix of degree 2. Suppose that A and p satisfy the following conditions:*

(1) $\mathrm{ord}_p(\mathbb{L}(k-1, f, \mathrm{St})) =: \alpha > 0$,
(2) $\mathrm{ord}_p(\mathcal{C}_{4,k}(f) a(A, [f]^{(k+\nu,k)})) = 0$,
(3) $p \ge 2(k+\nu) - 3$.

Then, there is a Hecke eigenform $G \in M_{\rho_2}(\Gamma_2)$ such that G is not a scalar multiple of $[f]^{(k+\nu,k)}$ and

$$[f]^{(k+\nu,k)} \equiv_{ev} G \mod \mathfrak{p}$$

for some prime ideal $\mathfrak{p} \mid p$ of $\mathbb{Q}(G)$. If $\mathrm{ord}_p(\gamma_1) = 0$, Condition (3) can be changed to Condition (3)':

(3)' $p \geq \max\{2k, k+\nu-2\}$.

If moreover $k \geq 6$ and p satisfy the following conditions:

(4) $\mathrm{ord}_p(\mathbb{L}(k-1, f_{1,i}, \mathrm{St})) \leq 0$ $(2 \leq i \leq d_1)$,
(5) *p is coprime with every $\mathfrak{A}(f_{r,i})$ $(1 \leq r \leq 2,\ 1 \leq i \leq d_r)$.*

there is a Hecke eigenform $G \in S_{\rho_2}(\Gamma_2)$ such that G is not a scalar multiple of $[f]^{(k+\nu,k)}$ and

$$[f]^{(k+\nu,k)} \equiv_{ev} G \mod \mathfrak{p}^{\alpha}$$

for some prime ideal $\mathfrak{p} \mid p$ of $\mathbb{Q}(G)$.

Remark 5.11. Before the proof, We make a few comments on the conditions of the theorem.

- condition (3) in the main theorem could be loosened a bit more. In fact, when $(k,\nu) = (6,12)$ and $p = 13$, numerical calculations estimate that there will be a congruence.
- Whether conditions (1) and (2) are valid when there is a congruence is a delicate question. It has been suggested that this question is connected to the Bloch-Kato conjecture and is not easily proven.
- It is known by Katsurada-Mizumoto [16] that there is an example in the case of scalar values where the congruence disappears when conditions (2) do not hold, even if conditions (1) and (3) hold. It is unknown that if there are similar examples for vector valued cases.

Proof. From the assumptions,

$$\begin{aligned} &\mathrm{ord}_p\left(\zeta(3-2k)\ \gamma_1\ \mathcal{C}_{4,k}(f) a(A, [f]^{(k+\nu,k)}) a(A, [f]^{(k+\nu,k)})\right) \\ = &\mathrm{ord}_p\left(\gamma_1 \cdot \frac{\mathcal{C}_{4,k}(f) a(A, [f]^{(k+\nu,k)}) \cdot \mathcal{C}_{4,k}(f) a(A, [f]^{(k+\nu,k)})}{\mathbb{L}(k-1, f, St)}\right) = -\alpha < 0. \end{aligned}$$

On the other hand, by Von Staudt–Clausen theorem, $\mathrm{ord}_p(\zeta(3-2k)) = \mathrm{ord}_p\left(\frac{B_{2k-2}}{2k-2}\right) \geq 0$. Thus, the Lemma 3.5, Corollary 5.5 and Corollary 5.6 give the first part of the theorem.

Let $f_{2,1}, \ldots, f_{2,d_2}$ be a orthogonal basis of $S_{\rho_2}(\Gamma_2)(\mathbb{Q})$ consisting of Hecke eigenforms. this set is also a orthogonal basis of $S_{\rho_2}(\Gamma_2)$ by [25].

From Corollary 5.5 and Corollary 5.6, we have

$$\sum_{r=1}^{2} \gamma_r \sum_{j=1}^{d_r} \mathcal{C}_{4,k}(f_{r,j}) \left[f_{r,j}\right]^{\rho_2}(Z_1) a(A, [f_{r,j}]^{\rho_2}) \equiv 0 \mod \mathbb{Z}_{(p)}.$$

Here the congruence is understood to be the system of congruences for Fourier coefficients. Under the conditions (4)-(5), by using Proposition 5.9 , we can calculate as above to obtain

$$\mathrm{ord}_p\left(\gamma_1 \mathcal{C}_{4,k}(f_{1,j}) \left[f_{1,j}\right]^{\rho_2}(Z_1) a(A, [f_{1,j}]^{\rho_2})\right) \geq 0$$

for any integer i with $2 \le j \le d_1$. Thus we have

$$\gamma_1 \mathcal{C}_{4,k}(f)\,[f]^{\rho_2}\,(Z_1) a\left(A, [f]^{\rho_2}\right) + \gamma_2 \sum_{j=1}^{d_2} \mathcal{C}_{4,k}(f_{2,j}) f_{2,j}(Z_1) a(A, f_{2,j}) \equiv 0 \mod \mathbb{Z}_{(p)}. \tag{5.1}$$

Let $\{v_1, \dots, v_{\nu+1}\}$ be a fixed basis of $V_{2,(k+\nu,k)}$ and put

$$a(A, f_{2,j}) = a_{j,1}v_1 + \cdots + a_{j,\nu+1}v_{\nu+1},$$

where $a_{j,i} \in \mathbb{Q}$. Multiplying each $f_{2,j}$ by an element of $\mathbb{Q}^\times$ and renumber the subscripts if necessary, we can assume that

$$\begin{cases} a_{j,1} = 1 & (1 \le j \le d'), \\ a_{j,1} = 0 & (d'+1 \le j \le d_2). \end{cases}$$

and

$$\mathrm{ord}_{\mathfrak{P}}(\gamma_2 \mathcal{C}_{4,k}(f_{2,1}) a(A, f_{2,1})^2) = \mathrm{ord}_{\mathfrak{P}}(\gamma_2 \mathcal{C}_{4,k}(f_{2,1})) \le -\alpha \tag{5.2}$$

since

$$\mathrm{ord}_p\left(\gamma_2 \sum_{j=1}^{d_2} \mathcal{C}_{4,k}(f_{2,j}) a(A, f_{2,j})^2\right) = \mathrm{ord}_p\left(\gamma_1\ \mathcal{C}_{4,k}(f) a(A, [f]^{\rho_2})^2\right) = -\alpha.$$

We note that $\mathcal{C}_{4,k}(f_{2,j}) a(A, f_{2,j})^2$ remains unchanged if $f_{2,j}$ is replaced by $\gamma f_{2,j}$ with any $\gamma \in \mathbb{C}$.

For any $T \in \mathcal{H}_n$ we act $T - \lambda_{[f]^{\rho_2}}(T)$ on the both sides of (5.1), we have

$$H(Z_1) := \gamma_2 \sum_{j=1}^{d_2} \left(\lambda_{f_{2,j}}(T) - \lambda_{[f]^{\rho_2}}(T)\right) \mathcal{C}_{4,k}(f_{2,j}) f_{2,j}(Z_1) a(A, f_{2,j}) \equiv 0 \mod \mathbb{Z}_{(p)},$$

since T preserve the p-integrality of the Fourier coefficients. By [25], we can take p-unit u such that

$$uH \in S_{(k+\nu,k)}(\Gamma_2)(\mathbb{Z}).$$

Lemma 5.12. *Let $F \in S_{(k+\nu,k)}(\Gamma_2)(\mathbb{Q}(F))$ be a Hecke eigenform. Suppose that F has a Fourier coefficient $a(N, F) = a_1 v_1 + \cdots + a_{\nu+1} v_{\nu+1}$ with $a_i = 1$ for some i. Then for any $G \in S_{(k+\nu,k)}(\Gamma_2)(\mathbb{Z})$, we have*

$$\frac{(F, G)}{(F, F)} \in \mathfrak{A}(F)^{-1}.$$

Proof. This proof can be done in the same way as for the scalar valued case [22]. ☐

Applying Lemma 5.12 with $F = f_{2,1}$, $G = uH$, we have

$$u\gamma_2 \left(\lambda_{f_{2,1}}(T) - \lambda_{[f]^{\rho_2}}(T)\right) \mathcal{C}_{4,k}(f_{2,1}) a(A, f_{2,1}) \in \mathfrak{A}(f_{2,1})^{-1}.$$

Therefore (5.2) gives

$$\lambda_{f_{2,1}}(T) \equiv \lambda_{[f]^{\rho_2}}(T) \mod \mathfrak{p}^\alpha$$

for any prime $\mathfrak{p} \mid p$ in $\mathbb{Q}(f_{2,1})$ since $\mathfrak{A}(f_{2,1})$ is coprime by the condition (5). This completes the proof of the main theorem. ☐

In the rest of this chapter, we will examine the conditions that the G in Theorem 5.10 is a cusp form by using Chenevier-Lannes's method [7].

Let $\mathcal{O}$ be the ring of integers in an algebraic number field K, and we take $\mathfrak{p}$ be a maximal ideal of $\mathcal{O}$. Let $A_{\mathfrak{p}}$ be a Grothendieck ring of finite-dimensional continuos representations of

$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ with coefficients in $\mathcal{O}/\mathfrak{p}$ unramified outside $\mathfrak{p}$. Let $\mathcal{S}$ be the set of isomorphism classes of the simple representations of $A_{\mathfrak{p}}$. For $H = \sum_{S\in\mathcal{S}} n_S S$ ($n_S \in \mathbb{Z}$), we set

$$\|H\| = \sum_{S\in\mathcal{S}} |n_S| \dim S.$$

Let $\chi_{\mathfrak{p}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_1(K_{\mathfrak{p}})$ be the cyclotomic character and $\overline{\chi_{\mathfrak{p}}}$ be the mod $\mathfrak{p}$ representation of $\chi_{\mathfrak{p}}$. The subscript $\mathfrak{p}$ may be omitted if there is no confusion.

Lemma 5.13. *Let j be an integer. If an element $H \in A_{\mathfrak{p}}$ satisfies $(1+\overline{\chi}^i)H = 0$, then $i\|H\|$ is divisible by $(p_{\mathfrak{p}} - 1)$.*

Proof. This lemma for $i = 1$ has been proved by Chenevier and Lannes [7]. The general case can be proved in the same way, but for readers' convenience we give a proof.

Let $C_{\overline{\chi}}$ be a cyclic subgroup of $A_{\mathfrak{p}}^{\times}$ generated by $\overline{\chi}$. For $S \in \mathcal{S}$, we denote by $\Omega(S)$ the orbit of S under the action of $C_{\overline{\chi}}$, and let $m_i(S)$ be the least integer $k \geq 1$ such that we have $\overline{\chi}^{ik} S = S$.

We fix an element $S \in \mathcal{S}$. we put $d = m_1(S)/m_i(S) \in \mathbb{Z}$. We consider $1+\overline{\chi}^i$ as an endomorphism in $\mathrm{End}(\mathbb{Z}[\Omega(S)])$. It is easy to see that when $m_i(S)$ is odd, we have $\mathrm{Ker}(1+\overline{\chi}^i) = 0$, and when $m_i(S)$ is even, $\mathrm{Ker}(1+\overline{\chi}^i)$ is generated by $\{(1-\overline{\chi}^i + \cdots - \overline{\chi}^{(m_i(S)-1)i})\overline{\chi}^j S\}_{j=0}^{j=d-1}$. Let $\mathcal{S}_i$ be the subset of $\mathcal{S}$ consisting of an element $S \in \mathcal{S}$ such that $m_i(S)$ is even.

From the above discussion, if $(1+\overline{\chi}^i)H = 0$, then H can be denoted as

$$H = \sum_{S\in\mathcal{S}_i} \sum_{j=1}^{d} n_{S,j}(1-\overline{\chi}^i + \cdots - \overline{\chi}^{(m_i(S)-1)i})\overline{\chi}^j S,$$

where $n_{S,j} \in \mathbb{Z}$. By the definition, we have

$$\|H\| = \sum_{j=1}^{d}(\sum_{j=1}^{d} |n_{S,j}|) m_i(S) \dim S.$$

On the other hand, $im_i(S)\dim S$ is divisible by $(p_{\mathfrak{p}} - 1)$. In fact, we have

$$\det S = \det((\overline{\chi}^i)^{m_i(S)} S) = \overline{\chi}^{im_i(S)\dim S} \det S$$

and the order of $\overline{\chi} \in A_{\mathfrak{p}}^{\times}$ is $p_{\mathfrak{p}} - 1$. Therefore, we have $i\|H\|$ is divisible by $(p_{\mathfrak{p}} - 1)$. □

Theorem 5.14. *We consider under the conditions (1) $\sim$ (3) of Theorem 5.10. If $f \in S_{k+\nu}(\Gamma_1)$ is not Hecke congruent with $f_{1,i}$ ($i = 2, \ldots, d_1$) respectively, and if $4(k-2)$ is not divided by $(p-1)$, then G in Theorem 5.10 is a cusp form in $S_{(k+\nu,k)}(\Gamma_2)$.*

Proof. From Proposition 3.2, it is enough to show that if $[f_1]^{(k+\nu,k)} \equiv_{ev} [f_2]^{(k+\nu,k)} \mod p$ for Hecke eigenforms $f_1, f_2 \in S_{k+\nu}(\Gamma_1)$, then $f_1 \equiv_{ev} f_2 \mod p$.

Let $\rho_{f_{1,i}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Q}_p)$ be the Galois representation attached to the spin L functions of $f_{1,i}$ and $\overline{\rho_{f_{1,i}}}$ be the mod p representation of $\rho_{f_{1,i}}$. If $[f_1]^{(k+\nu,k)} \equiv_{ev} [f_2]^{(k+\nu,k)} \mod p$, we have

$$(1+\overline{\chi}^{-(k-2)})\overline{\rho_{f_1}} = (1+\overline{\chi}^{-(k-2)})\overline{\rho_{f_2}}$$

in A_p by Proposition 3.3. Hence, applying Lemma 5.13 to $i = k-2$ and $H = \overline{\rho_{f_1}} - \overline{\rho_{f_2}}$, we see that $\overline{\rho_{f_1}} = \overline{\rho_{f_2}}$, since $\|H\| = 4$ or 0. This implies that $\lambda_{f_1}(p) \equiv \lambda_{f_2}(p) \mod q'$ for any prime $q \neq p$. This congruence is obviously true for $q = p$. □

6. Applications

In this section, we consider the conditions that appear in Theorem 5.10. We give a way to compute the special value $\mathbb{L}(k-1, f, \mathrm{St})$ of L-function appearing in condition (1) by the Peterson inner product and give the necessary conditions for condition (2).

6.1. **Condition (1).**

For $r, N \in \mathbb{Z}_{\geq 0}$, we define $H(r,N)$ by

$$H(r,N) = \begin{cases} \zeta(1-2r) & (N=0) \\ L(1-r, \left(\frac{-N}{\cdot}\right)) & (N>0,\ N \equiv 0,3 \mod 4) \end{cases}$$

as in [8]. Let $P_{k,r}(t,m)$ be the Gegenbauer polynomial defined in Definition 4.17.

Theorem 6.1 (Cohen [8], Theorem 6.2)**.** *Let r, k be positive integers such that $3 \leq r \leq k-1$, r is odd and k is even, and set*

$$C_{k,r}(z) = \sum_{m=0}^{\infty} \left(\sum_{\substack{t \in \mathbb{Z} \\ t^2 \leq 4m}} P_{2r+2,k-r-1}(t/2, m) H(r, 4m - t^2) \right) q^m \quad (z \in \mathbb{H}_1).$$

Then, $C_{k,r} \in M_k(\Gamma_1)$. Moreover, if $r < k-1$, $C_{k,r} \in S_k(\Gamma_1)$.

Petersson inner product of $C_{k,r}(z)$ and other Hecke eigenform holds information on the special value of the L-function, which has been investigated by Zagier [26].

Theorem 6.2 (Zagier [26], Theorem 2)**.** *Let r, k be positive integers such that $3 \leq r \leq k-1$, r is odd and k is even. For any a Hecke eigenform $f \in S_k(\Gamma_1)$,*

$$(f, C_{k,r}) = -\frac{(r+k-2)!(k-2)!}{(k-r-1)!} \cdot \frac{1}{4^{r+k-2}\pi^{2r+k-1}} L(r, f, \mathrm{St}).$$

Thus, for a Hecke eigenform $f \in S_{k+\nu}(\Gamma_1)$,

$$(f, C_{k+\nu,k-1}) = -\frac{(2k+\nu-3)!(k+\nu-2)!}{\nu!} \cdot \frac{1}{4^{2k+\nu-3}\pi^{3k+\nu-3}} \mathrm{L(k-1,f,St)},$$

and we find

$$\begin{aligned} \mathbb{L}(k-1, f, \mathrm{St}) &= \Gamma_{\mathbb{C}}(k-1)\Gamma_{\mathbb{C}}(2k+\nu-2) \frac{L(k-1, f, \mathrm{St})}{(f,f)} \\ &= -\frac{\nu!(k-2)!}{(k+\nu-2)!} \cdot 2^{k+\nu-3} \cdot \frac{(f, C_{k+\nu,k-1})}{(f,f)}. \end{aligned}$$

6.2. **Condition (2).**

We define a Hecke operator $T^{(m)}$ for $m = p_1 \cdots p_r$ (prime decomposition) by

$$T^{(m)} = T(p_1) \cdots T(p_r).$$

Note that if $p_1, \ldots, p_r$ are different from each other, $T^{(m)} = T(m)$. We write

$$g_N(z) = g^{(N)}_{(k,\nu,1,2)}(z) = \sum_{n \in \mathbb{Z}_{>0}} \epsilon_{k,\nu}(n, N) q^n,$$

and

$$g_N | T^{(m)}(z) = \sum_{n \in \mathbb{Z}_{>0}} \epsilon_{k,\nu}(m, n, N) q^n.$$

Let $\{f_j\}_{j=1}^d$ be a basis of $S_{k+\nu}(\Gamma_1)$ consist of normalized Hecke eigenforms, and we set $f_j|T^{(m)} = \lambda_{j,m} f_j$. By Corollary 5.5, we get the following proposition.

Proposition 6.3. *We have*

$$\epsilon_{k,\nu}(m,n,N) = \gamma \sum_{j=1}^{d} \lambda_{j,m} \mathcal{C}_{3,k}(f_j) a(n,f_j) \overline{a(N,[f_j]^{(k+\nu,k)})}.$$

Note that $\mathcal{C}_{4,k}(f) = \zeta(3-2k)\mathcal{C}_{3,k}(f)$, the following propositions follow by a simple calculation.

Proposition 6.4. *For $N \in \mathbb{H}_2(\mathbb{Z})_{>0}$, we define $e_m = \epsilon_{k,\nu}(m,1,N)$. Let $f_1 = f, \dots, f_d$ be a basis of $S_{k+\nu}(\Gamma_1)$ consist of Hecke eigenforms. For $m_1, \dots, m_d \in \mathbb{Z}_{>0}$, we set $\Delta = \Delta(m_1, \dots, m_d) = \det(\lambda_{j,m_j})$. Then,*

$$\Delta \gamma \mathcal{C}_{4,k}(f) \overline{a(N,[f]^{\rho_2})} = \zeta(3-2k) \begin{vmatrix} e_{m_1} & \lambda_{2,m_1} & \dots & \lambda_{d,m_1} \\ \vdots & \vdots & \ddots & \vdots \\ e_{m_d} & \lambda_{2,m_d} & \dots & \lambda_{d,m_d} \end{vmatrix}.$$

Corollary 6.5. *Assume that a prime ideal $\mathfrak{p}$ of $\mathbb{Q}$ satisfies $\mathfrak{p} > \max\{2k, k+\nu-2\}$ and γ is $\mathfrak{p}$-integer. Suppose that $\mathfrak{p}$ divides neither $\zeta(3-2k)$ nor $\begin{vmatrix} e_{m_1} & \lambda_{2,m_1} & \dots & \lambda_{d,m_1} \\ \vdots & \vdots & \ddots & \vdots \\ e_{m_d} & \lambda_{2,m_d} & \dots & \lambda_{d,m_d} \end{vmatrix}$. Then*

$$\operatorname{ord}_{\mathfrak{p}} \left(\mathcal{C}_{4,k}(f) \overline{a(N,[f]^{\rho_2})} \right) \leq 0.$$

Proof. It follows from Proposition 6.4 since Δ and γ are $\mathfrak{p}$-integers. ☐

Fourier coefficients $a(T, \widetilde{E_{n,k}})$ of the Siegel-Eisenstein Series $\widetilde{E_{n,k}}(Z)$ are considered by Katsurada [14]. The remaining part of this chapter will summarize the results of the study.

We define $\chi_p(a)$ and a polynomial $\gamma_p(B,X)$ for $a \in \mathbb{Q}_p^\times$ and a nondegenerate matrix $B \in H_n(\mathbb{Z}_p)$ by

$$\chi_p(a) = \begin{cases} 1 & (\mathbb{Q}_p(\sqrt{a}) = \mathbb{Q}_p) \\ -1 & (\mathbb{Q}_p(\sqrt{a})/\mathbb{Q}_p \text{ is quadratic unramified}) \\ 0 & (\mathbb{Q}_p(\sqrt{a})/\mathbb{Q}_p \text{ is quadratic ramified}) \end{cases},$$

$$\gamma_p(B,X) = \begin{cases} (1-X)\prod_{i=1}^{n/2}(1-p^{2i}X^2)(1-p^{n/2}\chi_p\left((-1)^{n/2}\det B\right)X)^{-1} & (n \text{ is even}) \\ (1-X)\prod_{i=1}^{(n-1)/2}(1-p^{2i}X^2) & (n \text{ is odd}) \end{cases}.$$

For $B \in H_n(\mathbb{Z})$ with n even, let $\mathfrak{d}_B$ be the discriminant of $\mathbb{Q}(\sqrt{(-1)^{n/2}\det B})/\mathbb{Q}$ and $\chi_B = (\frac{\mathfrak{d}_B}{\cdot})$ be the Kronecker character.

Let $b_p(B,s)$ be the local Siegel series for an element $B \in H_n(\mathbb{Z}_p)$. We define the polynomial $F_p(B,X) \in \mathbb{Z}[X]$ with constant term 1 by

$$b_p(B,s) = \gamma_p(B,p^{-s})F_p(B,p^{-s}).$$

For $T \in H_n(\mathbb{Z}_p)\backslash\{0\}$ (resp. $T \in H_n(\mathbb{Z})_{\geq 0}\backslash\{0\}$), there exists a nondegenerate matrix $\tilde{T} \in H_m(\mathbb{Z}_p)$ (resp. $\tilde{T} \in H_n(\mathbb{Z})_{>0}$) such that T is similar to $\begin{pmatrix} \tilde{T} & O \\ O & O \end{pmatrix}$ over $\mathbb{Z}_p$ (resp. $\mathbb{Z}$). Using this $\tilde{T}$, define $F_p^*(T,X) \in \mathbb{Z}[X]$ for a matrix $T \in H_n(\mathbb{Z}_p)$ by $F_p^*(T,X) = F_p(\tilde{T},X)$, and χ_T^* for $T \in H_n(\mathbb{Z})_{>0}$ with even rank by $\chi_T^* = \chi_{\tilde{T}}$.

Proposition 6.6. *Let $k \in 2\mathbb{Z}$. Assume that $k \geq (n+1)/2$ and that neither $k = (n+2)/2 \equiv 2 \mod 4$ nor $k = (n+3)/2 \equiv 2 \mod 4$. Then for $T \in H_n(\mathbb{Z})_{\geq 0}$ of rank m, we have*

$$a(T, \widetilde{E_{n,k}}) = 2^{[(m+1)/2]} \prod_{p \mid \det(2\tilde{T})} F_p^*(T, p^{k-m-1})$$
$$\times \begin{cases} \prod_{i=m/2+1}^{[n/2]} \zeta(1+2i-2k) L(1+m/2-k, \chi_T^*) & (m \text{ is even}) \\ \prod_{i=(m+1)/2}^{[n/2]} \zeta(1+2i-2k) & (m \text{ is odd}) \end{cases}.$$

Here we make the convention $F_p^*(T, p^{k-m-1}) = 1$ and $L(1+m/2-k, \chi_T^*) = \zeta(1-k)$ if $m = 0$.

We take a variables x, y of $\rho_{(k+\nu,k)} := \det^k \otimes \mathrm{Sym}^\nu$. (Then, $\{x^\nu, x^{\nu-1}y, \ldots, y^\nu\}$ is a basis of $\det^k \otimes \mathrm{Sym}^\nu$.) We put $v = {}^t(x,y)$, and $r(n,N,R) = \mathrm{rank} \begin{pmatrix} n & R/2 \\ {}^tR/2 & N \end{pmatrix} \in \{2,3\}$. We can easily find the following proposition.

Proposition 6.7.

$$\epsilon_{k,\nu}(n,N)(v) = \sum_{R \in M_{1,2}(\mathbb{Z})} a\left(T_{(n,N,R)}, \widetilde{E_{3,k}}\right) \cdot P_{2k,\nu}(R/2 \cdot v, n {}^tvNv),$$

where $T_{(n,N,R)} = \begin{pmatrix} n & R/2 \\ {}^tR/2 & N \end{pmatrix}$ and $P_{2k,\nu}$ is the Gegenbauer polynomial.

7. Examples

7.1. $(k,\nu) = (14,2)$**.** It is known that $\dim S_{16}(\Gamma_1) = \dim S_{(16,14)}(\Gamma_2) = 1$, and we take normalized Hecke eigenforms $\Delta_{16} \in S_{16}(\Gamma_1)$ and $\Delta_{16,14} \in S_{(16,2)}(\Gamma_2)$. We can calculate that

$$\mathbb{L}(13, \Delta_{16}, \mathrm{St}) = \frac{2^{20} \cdot 3^4 \cdot 373}{7}.$$

Therefore 373 is the only prime that satisfies the condition(3) in Theorem5.10. Let $n = 1, N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we get

$$\epsilon_{14,2}\left(1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = -5291173154072 \neq 0 \mod 373,$$
$$\gamma = -\frac{91}{2147483648} \neq 0 \mod 373.$$

By theorem5.10 we prove the following theorem.

Theorem 7.1.

$$\Delta_{16,14} \equiv_{ev} [\Delta_{16}]^{(16,14)} \mod 373.$$

In particular,

$$\lambda_{\Delta_{16,14}}(p) \equiv (1+p^{12}) \lambda_{\Delta_{16}}(p) \mod 373.$$

7.2. $(k,\nu) = (8,8)$**.** As previous subsection, dim $S_{16}(\Gamma_1) = \dim S_{(16,8)}(\Gamma_2) = 1$, and we take normalized Hecke eigenforms $\Delta_{16,8} \in S_{(16,8)}(\Gamma_2)$. We can calculate that

$$\mathbb{L}(7, \Delta_{16}, \mathrm{St}) = \frac{2^{15} \cdot 23^2}{11 \cdot 13}.$$

Therefore 23 is the only prime that satisfies the condition(3) in Theorem5.10. Let $n = 1, N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we get

$$\epsilon_{8,8}\left(1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = -46666368 \neq 0 \mod 23,$$

$$\gamma = -\frac{945945}{2143483648} \neq 0 \mod 23.$$

Noting that dim $S_{16}(\Gamma_1) = 1$, by a simple improvement of Lemma 3.5 and theorem 5.10, we can prove the following theorem.

Theorem 7.2.

$$\Delta_{16,8} \equiv_{ev} [\Delta_{16}]^{(16,8)} \mod 23^2.$$

In particular,

$$\lambda_{\Delta_{16,8}}(p) \equiv (1 + p^6)\lambda_{\Delta_{16}}(p) \mod 23^2.$$

8. Acknowledgment

The author would like to thank T. Ikeda for his great guidance and support as my supervisor, and H. Katsurada for his many suggestions and comments.

References

[1] A. N. Andrianov. Euler products corresponding to siegel modular forms of genus 2. *Russian Mathematical Surveys*, 29(3):45, jun 1974.

[2] T. Arakawa. Vector valued siegel's modular forms of degree two and the associated andrianov l-functions. *Manuscripta mathematica*, 44(1):155–185, 1983.

[3] H. Atobe, M. Chida, T. Ibukiyama, H. Katsurada, and T. Yamauchi. Harder's conjecture i. *Journal of the Mathematical Society of Japan*, 75(4):1339–1408, 2023.

[4] K. Ban. On rankin-cohen-ibukiyama operators for automorphic forms of several variables. *Commentarii mathematici Universitatis Sancti Pauli*, 55(2):149–171, 2006.

[5] J. Bergström, C. Faber, and G. van der Geer. Siegel modular forms of degree three and the cohomology of local systems. *Selecta Math. (N.S.)*, 20(1):83–124, 2014.

[6] S. Böcherer and C.-G. Schmidt. p-adic measures attached to siegel modular forms. In *Annales de l'institut Fourier*, volume 50, pages 1375–1443, 2000.

[7] G. Chenevier and J. Lannes. Automorphic forms and even unimodular lattices. *Ergebnisse der Mathematik und ihrer Grenzgebiete*, 3, 2019.

[8] H. Cohen. Sums involving the values at negative integers of L-functions of quadratic characters. *Math. Ann.*, 217(3):271–285, 1975.

[9] R. Howe. Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond. In *The Schur lectures (1992) (Tel Aviv)*, volume 8 of *Israel Math. Conf. Proc.*, pages 1–182. Bar-Ilan Univ., Ramat Gan, 1995.

[10] T. Ibukiyama. On differential operators on automorphic forms and invariant pluri-harmonic polynomials. *Comment. Math. Univ. St. Paul.*, 48(1):103–118, 1999.

[11] T. Ibukiyama. Generic differential operators on Siegel modular forms and special polynomials. *Selecta Math. (N.S.)*, 26(5):Paper No. 66, 50, 2020.

[12] T. Ibukiyama. Differential operators, exact pullback formulas of Eisenstein series, and Laplace transforms. *Forum Math.*, 34(3):685–710, 2022.

[13] M. Kashiwara and M. Vergne. On the Segal-Shale-Weil representations and harmonic polynomials. *Invent. Math.*, 44(1):1–47, 1978.

[14] H. Katsurada. An explicit formula for Siegel series. *Amer. J. Math.*, 121(2):415–452, 1999.

[15] H. Katsurada. Congruence of Siegel modular forms and special values of their standard zeta functions. *Math. Z.*, 259(1):97–111, 2008.

[16] H. Katsurada and S. Mizumoto. Congruences for Hecke eigenvalues of Siegel modular forms. *Abh. Math. Semin. Univ. Hambg.*, 82(2):129–152, 2012.

[17] N. Kozima. On special values of standard L-functions attached to vector valued Siegel modular forms. *Kodai Math. J.*, 23(2):255–265, 2000.

[18] N. Kozima. Pullback formula for vector valued siegel modular forms and its applications. *arXiv preprint arXiv:2109.11753*, 2021.

[19] N. Kurokawa. Congruences between Siegel modular forms of degree two. *Proc. Japan Acad. Ser. A Math. Sci.*, 55(10):417–422, 1979.

[20] S. Mizumoto. Congruences for eigenvalues of Hecke operators on Siegel modular forms of degree two. *Math. Ann.*, 275(1):149–161, 1986.

[21] S. Mizumoto. Poles and residues of standard L-functions attached to Siegel modular forms. *Math. Ann.*, 289(4):589–612, 1991.

[22] S. Mizumoto. On integrality of Eisenstein liftings. *Manuscripta Math.*, 89(2):203–235, 1996.

[23] T. Satoh. On certain vector valued Siegel modular forms of degree two. *Math. Ann.*, 274(2):335–352, 1986.

[24] G. Shimura. *Arithmeticity in the theory of automorphic forms*, volume 82 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2000.

[25] R. L. Taylor. *On congruences between modular forms*. ProQuest LLC, Ann Arbor, MI, 1988. Thesis (Ph.D.)–Princeton University.

[26] D. Zagier. Modular forms whose Fourier coefficients involve zeta-functions of quadratic fields. In *Modular functions of one variable, VI (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976)*, volume Vol. 627 of *Lecture Notes in Math.*, pages 105–169. Springer, Berlin-New York, 1977.

Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan

Email address: `takeda.nobuki.72z@st.kyoto-u.ac.jp`