

## ON THE EXTERIOR PRODUCT OF HÖLDER DIFFERENTIAL FORMS

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ABSTRACT. We introduce a complex of cochains,  $\alpha$ -fractional charges ( $0 < \alpha \leq 1$ ), whose regularity is between that of De Pauw-Moonens-Pfeffer's charges and that of Whitney's flat cochains. We show that  $\alpha$ -Hölder differential forms and their exterior derivative can be realized as  $\alpha$ -fractional charges, and that it is possible to define the exterior product between an  $\alpha$ -fractional and a  $\beta$ -fractional charge, under the condition that  $\alpha + \beta > 1$ . This construction extends the Young integral in arbitrary dimension and codimension.

## CONTENTS

1. Introduction	1
2. Preliminaries	2
3. Charges in middle dimension	5
4. $\alpha$ -Fractionality	9
5. A Littlewood-Paley type decomposition of fractional charges	15
6. Main result	16
References	19

## 1. INTRODUCTION

The Young integral has become a classical object: given two  $\alpha$ - and  $\beta$ -Hölder continuous functions  $f$  and  $g$  on the interval  $[0, 1]$ , the convergence of the Riemann-Stieltjes integral  $\int_0^1 f dg$  is assured when  $\alpha + \beta > 1$ . R. Züst proposed a higher-dimensional extension in [9, Section 3], introducing a Riemann-Stieltjes type integral to handle expressions like

$$\int_{[0,1]^d} f dg_1 \wedge \cdots \wedge dg_d \quad (1)$$

where  $f, g_1, \dots, g_d$  are Hölder functions of exponents  $\alpha_0, \alpha_1, \dots, \alpha_d$  satisfying  $\alpha_0 + \cdots + \alpha_d > d$ . This condition is proven to be sharp, as counterexamples are provided in the critical case [9, 3.2]. Furthermore, in [1], an extension of the Züst integral is proposed, allowing the integrator to be a Hölder charge. This generalization also enabled to define pathwise integrals with respect to stochastic processes, such as the fractional Brownian sheet with sufficient regularity [2].

This goal of this article is to define integrals of Hölder differential forms, such as those depicted in (1), in any codimension, allowing the domain of integration to be a normal current. In continuation of [1], we are led to consider the more general formalism of charges in middle dimension, as introduced by Th. De Pauw, L. Moonens and W. Pfeffer in [3], to represent the generalized differential forms Hölder forms must be. An  $m$ -charge is a linear functional on the space of normal currents that satisfies a certain continuity condition. Equivalently, the representation theorem of charges [3, Theorem 6.1] states that

$m$ -charges are the sums  $\omega + d\eta$  of a continuous  $m$ -form and the weak exterior derivative of an  $(m - 1)$ -form. The action on normal currents is given by the formula

$$(\omega + d\eta)(T) = \int_{\mathbb{R}^d} \langle \omega(x), \vec{T}(x) \rangle d\|T\|(x) + \int_{\mathbb{R}^d} \langle \eta(x), \vec{\partial T}(x) \rangle d\|\partial T\|(x).$$

Of course, there is no meaningful definition of the exterior product of two general charges. This obstacle, which is essentially of distributional nature, prevents us from defining integrals as in (1) in general.

The situation stands in stark contrast to the complex of locally flat cochains introduced by H. Whitney, see [8, 5]. These cochains are the continuous linear functionals on the space of flat  $m$ -chains. The main result of the theory is Wolfe's representation theorem, which asserts that locally flat  $m$ -cochains are in one-to-one correspondence with locally flat  $m$ -forms:  $L_{\text{loc}}^\infty$  differential  $m$ -forms with  $L_{\text{loc}}^\infty$  weak exterior derivative. Consequently, this allows for a pointwise definition of the exterior product of two locally flat cochains.

However, charges do not possess the sufficient regularity, and it is evident that the Hölder forms appearing in (1) cannot be assigned a pointwise meaning. Besides, the De Pauw-Moonens-Pfeffer representation theorem only allows to define a cup product at the cohomology level, see [3, Lemma 8.2]. Thus, to develop a full exterior calculus apparatus for charges, it is necessary to add some regularity assumptions. We propose the following definition: an  $m$ -charge  $\omega$  over  $\mathbb{R}^d$  is said to be  $\alpha$ -fractional whenever, for every compact  $K \subseteq \mathbb{R}^d$ , one can find  $C_K \geq 0$  such that

$$|\omega(T)| \leq C_K \mathbf{N}(T)^{1-\alpha} \mathbf{F}(T)^\alpha$$

for every normal current  $T$  supported in  $K$ . This definition is designed so that  $\alpha$ -Hölder continuous differential forms and their exterior derivatives can be represented as fractional charges. In the zero-codimensional case, it coincides with the Hölder charges described in [2]. When  $\alpha = 1$  we recover the definition of Whitney's locally flat cochains.

The main accomplishment of this paper is the definition of the exterior product between an  $\alpha$ -fractional charge  $\omega$  and a  $\beta$ -fractional charge  $\eta$ , under the Young type condition that  $\alpha + \beta > 1$ . The resulting charge is also fractional, with the fractional exponent being  $\alpha + \beta - 1$ . Our methods are inspired by tools from harmonic analysis and bear resemblance to the Fourier approach to integration developed by M. Gubinelli, P. Imkeller, and N. Perkowski in [7]. Our principal tool is a Littlewood-Paley type decomposition result for fractional charges, that constitutes a generalization (over  $\mathbb{R}^d$  only) of the decomposition of Hölder functions described in [6, Appendix B, 2.6]. By introducing the Littlewood-Paley decompositions of  $\omega$  and  $\eta$ , it is then possible to split  $\omega \wedge \eta$  formally into two paraproducts, the existence of which is easier to establish.

The paper is structured into six sections, each with self-explanatory titles. Section 3 provides a self-contained introduction to charges in middle dimension, focusing solely on the results pertinent to this paper.

## 2. PRELIMINARIES

2.1 (Notations). — Throughout the article, the ambient space will be  $\mathbb{R}^d$ , with  $d \geq 1$ . It is equipped with the Lebesgue outer measure, denoted  $\mathcal{L}^d$ . A measurable subset of  $\mathbb{R}^d$  always refers to a set that is Lebesgue-measurable.

For a function  $f: X \rightarrow E$  defined on a locally closed subset  $X \subseteq \mathbb{R}^d$  with values in a normed space  $(E, \|\cdot\|)$ , a subset  $Y \subseteq X$  and  $0 < \alpha \leq 1$ , we define the extended real numbers

$$\begin{aligned} \|f\|_{\infty, Y} &:= \sup \{ \|f(y)\| : y \in Y \} \\ \text{Lip}^\alpha(f; Y) &:= \sup \left\{ \frac{\|f(y_1) - f(y_2)\|}{|y_1 - y_2|^\alpha} : y_1, y_2 \in Y \text{ and } y_1 \neq y_2 \right\} \end{aligned}$$

In addition, we write  $\|f\|_\infty = \|f\|_{\infty, X}$  and  $\text{Lip}^\alpha(f) = \text{Lip}^\alpha(f; X)$ . When  $\alpha = 1$ , we may of course write  $\text{Lip}$  instead of  $\text{Lip}^\alpha$ .

The space of  $\alpha$ -Hölder continuous maps is written  $\text{Lip}^\alpha(X; E)$ . A function  $f: X \rightarrow E$  is locally  $\alpha$ -Hölder continuous whenever  $\text{Lip}^\alpha(f; K) < \infty$  for all compact  $K \subseteq X$ , and the space of such functions is denoted  $\text{Lip}_{\text{loc}}^\alpha(X; E)$ . When  $E = \mathbb{R}$  we abbreviate  $\text{Lip}^\alpha(X)$  or  $\text{Lip}_{\text{loc}}^\alpha(X)$ .

For a compact subset  $K \subseteq \mathbb{R}^d$  and  $\varepsilon > 0$ , we define the tubular closed neighborhood  $K_\varepsilon = \{x \in \mathbb{R}^d : \text{dist}(x, K) \leq \varepsilon\}$ . The reader will encounter quite frequently the expression  $K_1$ , which is a specific case of this notation.

We will work within the setting of Federer-Fleming's currents. For an in-depth exploration of this subject, we refer the reader to [4]. In this preliminary part, our focus will be on defining common notations, highlighting those that deviate from [4], and revisiting a few definitions.

The spaces of  $m$ -vectors and  $m$ -covectors are  $\wedge_m \mathbb{R}^d$  and  $\wedge^m \mathbb{R}^d$ , and they are respectively given the mass and comass norm described in [4, 1.8.1]. The bracket notation  $\langle \cdot, \cdot \rangle$  will be reserved for the duality between  $m$ -covectors and  $m$ -vectors. The canonical basis of  $\mathbb{R}^d$  is  $e_1, \dots, e_d$ . We write  $\Lambda(d, m)$  the set of strictly increasing maps  $\{1, \dots, d\} \rightarrow \{1, \dots, m\}$ . We introduce the dual bases of  $\wedge_m \mathbb{R}^d$  and  $\wedge^m \mathbb{R}^d$

$$\begin{aligned} e_I &= e_{i_1} \wedge \dots \wedge e_{i_m} \\ dx_I &= dx_{i_1} \wedge \dots \wedge dx_{i_m} \end{aligned}$$

for  $I \in \Lambda(d, m)$ .

All our currents will be defined on  $\mathbb{R}^d$  and have typically dimension  $m$ , that is, they will belong to  $\mathcal{D}_m(\mathbb{R}^d)$ , the topological dual of the space  $\mathcal{D}^m(\mathbb{R}^d)$  of compactly supported smooth  $m$ -forms, see [4, 4.1.7]. The boundary, the mass and the normal mass of  $T \in \mathcal{D}_m(\mathbb{R}^d)$  are  $\partial T$ ,  $\mathbf{M}(T)$  and  $\mathbf{N}(T)$ . If  $\omega$  is a smooth  $k$ -form and  $k \leq m$ , the current  $T \llcorner \omega \in \mathcal{D}_{m-k}(\mathbb{R}^d)$  is defined by  $T \llcorner \omega(\eta) = T(\omega \wedge \eta)$  for  $\eta \in \mathcal{D}^{m-k}(\mathbb{R}^d)$ . If  $\xi: \mathbb{R}^d \rightarrow \wedge_m \mathbb{R}^d$  is a locally integrable  $m$ -vectorfield,  $\mathcal{L}^d \wedge \xi$  is the  $m$ -current that sends  $\omega \in \mathcal{D}^m(\mathbb{R}^d)$  to  $\int_{\mathbb{R}^d} \langle \omega, \xi \rangle$ . Whenever  $E \subseteq \mathbb{R}^d$  is measurable, we denote  $\llbracket E \rrbracket \in \mathcal{D}_d(\mathbb{R}^d)$  the zero-codimensional current defined by  $\llbracket E \rrbracket(\omega) = \int_E \langle e_1 \wedge \dots \wedge e_d, \omega \rangle$ . If  $x \in \mathbb{R}^d$ , the 0-current  $\llbracket x \rrbracket \in \mathcal{D}_0(\mathbb{R}^d)$  is defined by  $\llbracket x \rrbracket(\omega) = \omega(x)$ .

We recall that normal currents and flat chains, as defined in [4] have compact supports. As such, we may at times evaluate such currents against smooth forms that are not compactly supported, with no warning.

The spaces of normal  $m$ -currents and flat  $m$ -chains are denoted  $\mathbf{N}_m(\mathbb{R}^d)$  and  $\mathbf{F}_m(\mathbb{R}^d)$ , following customary notation. For any subset  $X \subseteq \mathbb{R}^d$ , we write

$$\begin{aligned} \mathbf{N}_m(X) &= \{T \in \mathbf{N}_m(\mathbb{R}^d) : \text{spt } T \subseteq X\} \\ \mathbf{F}_m(X) &= \{T \in \mathbf{F}_m(\mathbb{R}^d) : \text{spt } T \subseteq X\} \end{aligned}$$

We define the flat norm of a normal current  $T \in \mathbf{N}_m(\mathbb{R}^d)$  in a way which departs from Federer's exposure:

$$\begin{aligned} \mathbf{F}(T) &= \sup \{T(\omega) : \omega \in \mathcal{D}^m(\mathbb{R}^d) \text{ and } \max \{\|\omega\|_\infty, \|\text{d}\omega\|_\infty\} \leq 1\} \\ &= \inf \{\mathbf{M}(S) + \mathbf{M}(T - \partial S) : S \in \mathbf{N}_{m+1}(\mathbb{R}^d)\}. \end{aligned}$$

The proof of the above equality is similar to [4, 4.1.12]. From the first equality, it is clear that  $\mathbf{N}$  and  $\mathbf{M}$  are lower semicontinuous with respect to  $\mathbf{F}$ . Note that, if  $T$  is supported in a compact set  $K$ , the flat norm we just defined may differ from

$$\begin{aligned} \mathbf{F}_K(T) &= \sup \{T(\omega) : \omega \in \mathcal{D}^m(\mathbb{R}^d) \text{ and } \max \{\|\omega\|_{\infty, K}, \|\text{d}\omega\|_{\infty, K}\} \leq 1\} \\ &= \inf \{\mathbf{M}(S) + \mathbf{M}(T - \partial S) : S \in \mathbf{N}_{m+1}(K)\}. \end{aligned}$$

However, if  $K$  is a 1-Lipschitz retract of  $\mathbb{R}^d$ , it is clear that  $\mathbf{F}(T) = \mathbf{F}_K(T)$ . Moreover, this assumption implies that  $\mathbf{F}_m(K)$  is the  $\mathbf{F}$ -closure of  $\mathbf{N}_m(K)$  within  $\mathcal{D}_m(\mathbb{R}^d)$ .

Finally, the letter  $C$  will refer generally to a constant, that may vary from line to line.

The construction of charges ultimately relies on the Federer-Fleming's compactness theorem of normal currents in flat norm. The following version uses the flat norm  $\mathbf{F}$  (rather than  $\mathbf{F}_K$  as in [4, 4.2.17(1)]). It can be easily deduced from the original version. Alternatively, it is possible to reproduce the arguments in the proof of Federer-Fleming, as was done in [3, Theorem 4.2].

**2.2. THEOREM (Compactness).** — *Let  $K \subseteq \mathbb{R}^d$  be compact. For all  $c \geq 0$ , the ball  $\{T \in \mathbf{N}_m(K) : \mathbf{N}(T) \leq c\}$  is  $\mathbf{F}$ -compact.*

**2.3 (Convolution of currents).** — In this article, convolutions will play an important role in regularizing charges. First, we need to recall how convolution works at the level of currents. The convolution of a current  $T \in \mathcal{D}_m(\mathbb{R}^d)$  with a function  $\phi \in C_c^\infty(\mathbb{R}^d)$  is defined by

$$(T * \phi)(\omega) = T(\check{\phi} * \omega) \quad \text{for all } \omega \in \mathcal{D}^m(\mathbb{R}^d)$$

where  $\check{\phi}(x) = \phi(-x)$  for all  $x \in \mathbb{R}^d$ . It is clear that  $T * \phi \in \mathcal{D}_m(\mathbb{R}^d)$ .

Throughout the article, we fix a  $C^\infty$  function  $\Phi: \mathbb{R}^d \rightarrow \mathbb{R}$  with compact support in the closed unit ball of  $\mathbb{R}^d$ , that is nonnegative and such that  $\int_{\mathbb{R}^d} \Phi = 1$ . For the sake of simplicity, we additionally assume that  $\Phi$  is even. For all  $\varepsilon > 0$ , we define  $\Phi_\varepsilon(x) = \varepsilon^{-d} \Phi(\varepsilon^{-1}x)$ . Below we compile several useful facts concerning  $T * \Phi_\varepsilon$ , when  $T \in \mathbf{N}_m(\mathbb{R}^d)$ .

**2.4. PROPOSITION.** — *There is  $C \geq 0$  such that, for all  $T \in \mathbf{N}_m(\mathbb{R}^d)$  and  $\varepsilon > 0$ ,*

- (A)  $\text{spt}(T * \Phi_\varepsilon) \subseteq (\text{spt } T)_\varepsilon$ ;
- (B)  $\mathbf{M}(T * \Phi_\varepsilon) \leq \mathbf{M}(T)$ ;
- (C)  $\partial T * \Phi_\varepsilon = \partial(T * \Phi_\varepsilon)$ ,  $\mathbf{N}(T * \Phi_\varepsilon) \leq \mathbf{N}(T)$  and  $\mathbf{F}(T * \Phi_\varepsilon) \leq \mathbf{F}(T)$ ;
- (D)  $\mathbf{M}(T * \Phi_\varepsilon) \leq C\varepsilon^{-1}\mathbf{F}(T)$  and  $\mathbf{N}(T * \Phi_\varepsilon) \leq C\varepsilon^{-1}\mathbf{F}(T)$  if  $\varepsilon \leq 1$ ;
- (E)  $\mathbf{F}(T - T * \Phi_\varepsilon) \leq \varepsilon \mathbf{N}(T)$ .

*Proof.* (A) is immediate.

(B). This is because  $\|\omega * \Phi_\varepsilon\|_\infty \leq \|\omega\|_\infty$  for all  $\omega \in \mathcal{D}^m(\mathbb{R}^d)$ .

(C). The first part comes from the identity  $d(\omega * \Phi_\varepsilon) = d\omega * \Phi_\varepsilon$ , valid for any  $\omega \in \mathcal{D}^m(\mathbb{R}^d)$ . Hence  $\mathbf{M}(\partial(T * \Phi_\varepsilon)) = \mathbf{M}(\partial T * \Phi_\varepsilon) \leq \mathbf{M}(\partial T)$ . This easily implies that  $\mathbf{N}(T * \Phi_\varepsilon) \leq \mathbf{N}(T)$  and  $\mathbf{F}(T * \Phi_\varepsilon) \leq \mathbf{F}(T)$ .

(D). Let  $S \in \mathbf{N}_{m+1}(\mathbb{R}^d)$ . We apply the identity

$$\partial S = - \sum_{k=1}^d \frac{\partial S}{\partial x_k} \lrcorner dx_k$$

to the current  $S * \Phi_\varepsilon$ , which yields

$$\partial(S * \Phi_\varepsilon) = - \sum_{k=1}^d \left( S * \frac{\partial \Phi_\varepsilon}{\partial x_k} \right) \lrcorner dx_k$$

From this, we obtain  $\mathbf{M}(\partial S * \Phi_\varepsilon) \leq C\varepsilon^{-1}\mathbf{M}(T)$ .

Thereafter, we decompose  $T = (T - \partial S) + \partial S$ . Applying (B) and the inequality from the preceding paragraph,

$$\mathbf{M}(T * \Phi_\varepsilon) \leq \mathbf{M}((T - \partial S) * \Phi_\varepsilon) + \mathbf{M}(\partial S * \Phi_\varepsilon) \leq \frac{C}{\varepsilon} (\mathbf{M}(T - \partial S) + \mathbf{M}(S)).$$

Taking the infimum on the right-hand side, as  $S$  ranges over  $\mathbf{N}_{m+1}(\mathbb{R}^d)$ , we obtain the first result. Then,

$$\mathbf{N}(T * \Phi_\varepsilon) = \mathbf{M}(T * \Phi_\varepsilon) + \mathbf{M}(\partial T * \Phi_\varepsilon) \leq \frac{C}{\varepsilon} \mathbf{F}(T).$$

(E). For any  $z \in \mathbb{R}^d$ , let  $\tau_z : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be the translation by  $z$ , and  $\tau_{z\#}T$  the pushforward current [4, 4.1.7]. Let  $\omega \in \mathcal{D}^m(\mathbb{R}^d)$ . By [4, 4.1.18],

$$\begin{aligned}
 (T - T * \Phi_\varepsilon)(\omega) &= T(\omega) - \int_{\mathbb{R}^d} \Phi_\varepsilon(z) (\tau_{z\#}T)(\omega) \, dz \\
 &= \int_{\mathbb{R}^d} \Phi_\varepsilon(z) (T - \tau_{z\#}T)(\omega) \, dz \\
 &\leq \int_{\mathbb{R}^d} \Phi_\varepsilon(z) \mathbf{F}(T - \tau_{z\#}T) \max\{\|\omega\|_\infty, \|d\omega\|_\infty\} \, dz \\
 &\leq \int_{\mathbb{R}^d} \Phi_\varepsilon(z) |z| \mathbf{N}(T) \max\{\|\omega\|_\infty, \|d\omega\|_\infty\} \, dz \\
 &\leq \varepsilon \mathbf{N}(T) \max\{\|\omega\|_\infty, \|d\omega\|_\infty\}. \quad \square
 \end{aligned}$$

### 3. CHARGES IN MIDDLE DIMENSION

3.1 (Charges in middle dimension). — In this section, we provide a self-contained introduction to charges in middle dimension and establish their fundamental properties. This notion was pioneered by De Pauw, Moonens and Pfeffer in [3].

An  **$m$ -charge** over a compact set  $K \subseteq \mathbb{R}^d$  is a linear map  $\omega : \mathbf{N}_m(K) \rightarrow \mathbb{R}$  that satisfies one of the following equivalent continuity properties:

- (A)  $\omega(T_n) \rightarrow 0$  for any bounded sequence  $(T_n)$  in  $\mathbf{N}_m(K)$  that converges in flat norm to 0;
- (B) the restriction of  $\omega$  to the unit ball of  $\mathbf{N}_m(K)$  is **F**-continuous;
- (C) for all  $\varepsilon > 0$ , there is some  $\theta \geq 0$  such that

$$|\omega(T)| \leq \varepsilon \mathbf{N}(T) + \theta \mathbf{F}(T)$$

holds for any normal current  $T \in \mathbf{N}_m(K)$ .

One clearly has (A)  $\iff$  (B) and (C)  $\implies$  (A). The only non trivial implication (A)  $\implies$  (C) can be derived as a short consequence of the compactness theorem. Indeed, suppose by contradiction that (A) holds and (C) is false. In this case, there is  $\varepsilon > 0$  and a sequence  $(T_n)$  of normal currents supported in  $K$ , with normal masses  $\mathbf{N}(T_n) = 1$  such that

$$|\omega(T_n)| > n \mathbf{F}(T_n) + \varepsilon \quad (2)$$

for all  $n$ . Some subsequence  $(T_{n_k})$  converges to a normal current  $T \in \mathbf{N}_m(K)$  in flat norm. Property (A) then implies that  $\omega(T_{n_k}) \rightarrow \omega(T)$ . Consequently,  $\mathbf{F}(T_{n_k}) \leq n_k^{-1} |\omega(T_{n_k})|$  tends to 0 as  $k \rightarrow \infty$ , which implies that  $T = 0$  and  $\omega(T_{n_k}) \rightarrow 0$ . This is in contradiction with (2).

3.2. — The space of  $m$ -charges over  $K$  is denoted  $\mathbf{CH}^m(K)$ . As  $\mathbf{F} \leq \mathbf{N}$ , the continuity property (C) above implies that charges are **N**-continuous, i.e.  $\mathbf{CH}^m(K)$  is a subspace of the dual  $\mathbf{N}_m(K)^*$ . We set

$$\|\omega\|_{\mathbf{CH}^m(K)} := \sup \{ \omega(T) : T \in \mathbf{N}_m(K) \text{ and } \mathbf{N}(T) \leq 1 \}.$$

In fact,  $\mathbf{CH}^m(K)$  is a closed subspace of  $\mathbf{N}_m(K)^*$ , for if  $(\omega_n)$  is a sequence in  $\mathbf{CH}^m(K)$  converging towards  $\omega \in \mathbf{N}_m(K)^*$ , then  $\omega_n \rightarrow \omega$  uniformly on the unit ball of  $\mathbf{N}_m(K)$ . We conclude by (B) that  $\omega$  is a charge.

In addition, there is also a notion of **weak convergence of charges**: we say that  $\omega_n \rightarrow \omega$  weakly whenever  $\omega_n(T) \rightarrow \omega(T)$  for all  $T \in \mathbf{N}_m(K)$ .

3.3 (Exterior derivative). — Operations on normal currents, such as pushforwards by Lipschitz maps, taking the boundary, have a counterpart in term of charges, defined by duality. Here we focus only on defining the **exterior derivative**  $d\omega \in \mathbf{CH}^{m+1}(K)$ , by setting

$$d\omega(T) := \omega(\partial T)$$

for all  $T \in \mathbf{N}_m(K)$ . That  $d\omega$  is continuous, in the sense of charges, is a consequence of the identities

$$\mathbf{N}(\partial T) \leq \mathbf{N}(T), \quad \mathbf{F}(\partial T) \leq \mathbf{F}(T) \quad (3)$$

that furthermore implies that  $d: \mathbf{CH}^m(K) \rightarrow \mathbf{CH}^{m+1}(K)$  is bounded. As  $\partial \circ \partial = 0$ , we have  $d \circ d = 0$ . In other words,  $(\mathbf{CH}^\bullet(K), d)$  is a cochain complex. In fact, the De Pauw-Moonens-Pfeffer representation theorem expresses that this is the smallest cochain complex spanned by continuous differential forms.

Before we state this result properly, we will first identify 0-charges with continuous functions. To each 0-charge  $\omega$ , we associate the function  $\Gamma(\omega) \in C(K)$  defined by  $\Gamma(\omega)(x) = \omega(\llbracket x \rrbracket)$ . The continuity of  $\Gamma(\omega)$  is a consequence of that of  $\omega$ .

**3.4. THEOREM.** —  $\Gamma: \mathbf{CH}^0(K) \rightarrow C(K)$  is a Banach space isomorphism.

*Proof.* First we check that  $\Gamma$  is a continuous. For all  $x \in K$ , we have

$$|\Gamma(\omega)(x)| \leq \|\omega\|_{\mathbf{CH}^0(K)} \mathbf{M}(\llbracket x \rrbracket) = \|\omega\|_{\mathbf{CH}^0(K)}.$$

Thus  $\|\Gamma(\omega)\|_\infty \leq \|\omega\|_{\mathbf{CH}^0(K)}$ .

Next we claim that  $\Gamma$  is injective. Let us call  $\mathbf{P}_0(K)$  the space of polyhedral 0-currents supported in  $K$ , i.e the linear subspace of  $\mathcal{D}_0(\mathbb{R}^d)$  spanned by the  $\llbracket x \rrbracket$ ,  $x \in K$ . By an easy corollary of the deformation theorem [4, 4.2.9], every  $T \in \mathbf{N}_0(K)$  is the  $\mathbf{F}$ -limit of a sequence  $(T_n)$  in  $\mathbf{P}_0(K)$  such that  $\mathbf{M}(T_n) \uparrow \mathbf{M}(T)$ .

Let  $\omega \in \ker \Gamma$ , so that  $\omega$  vanishes on  $\mathbf{P}_0(K)$ . By the preceding result and the continuity property of charges,  $\omega = 0$ . This proves that  $\Gamma$  is injective.

Next we prove the surjectivity of  $\Gamma$ . Let  $g \in C(K)$ . We define the function  $\omega$ , on polyhedral 0-chains, by

$$\omega \left( \sum_{k=1}^n a_k \llbracket x_k \rrbracket \right) = \sum_{k=1}^n a_k g(x_k).$$

Let  $\varepsilon > 0$ . There is a Lipschitz function  $f \in \text{Lip}(K)$  such that  $\|f - g\|_\infty \leq \varepsilon$ . Setting  $\theta = \max\{\|f\|_\infty, \text{Lip } f\}$ , we have

$$\begin{aligned} |\omega(T)| &\leq \left| \sum_{k=1}^n a_k f(x_k) \right| + \left| \sum_{k=1}^n a_k (f - g)(x_k) \right| \\ &\leq \theta \mathbf{F}(T) + \varepsilon \mathbf{M}(T) \end{aligned} \quad (4)$$

for every 0-polyhedral chain  $T$ . We extend  $\omega$  to  $\mathbf{M}_0(K)$  with

$$\omega(T) = \lim_{n \rightarrow \infty} \omega(T_n)$$

where  $T \in \mathbf{M}_0(K)$  and  $(T_n)$  is any sequence of polyhedral 0-chains that  $\mathbf{F}$ -converges to  $T$ , with  $\mathbf{M}(T_n) \uparrow \mathbf{M}(T)$ . By (4), the quantity  $\omega(T)$ , thus defined, does not depend on the approximating sequence. It is also straightforward that  $\omega$  is linear and (4) holds now for any  $T \in \mathbf{M}_0(K)$ . Hence  $\omega \in \mathbf{CH}^0(K)$  and  $\Gamma(\omega) = g$ . This proves that  $\Gamma$  is onto.

Finally,  $\Gamma^{-1}$  is continuous by the open mapping theorem.  $\square$

**3.5 (Continuous differential forms).** — A continuous  $m$ -form  $\omega \in C(K, \wedge^m \mathbb{R}^d)$  act on a normal current  $T \in \mathbf{N}_m(K)$  by means of

$$\omega(T) := \int_{\text{spt } T} \langle \omega(x), \vec{T}(x) \rangle d\|T\|(x)$$

We argue that this formula makes  $\omega$  into an  $m$ -charge (we are committing a slight abuse of notation by using the same symbol  $\omega$  for both the continuous form and the corresponding charge, even though the obvious mapping  $C(K, \wedge^m \mathbb{R}^d) \rightarrow \mathbf{CH}^m(K)$  may be not injective). Linearity is clear. As for continuity, let us fix  $\varepsilon > 0$  and choose a compactly supported

smooth form  $\phi: \mathbb{R}^d \rightarrow \wedge^m \mathbb{R}^d$  such that  $|\omega(x) - \phi(x)| \leq \varepsilon$  for all  $x \in K$ . Then, for all  $T \in \mathbf{N}_m(K)$ ,

$$\begin{aligned} |\omega(T)| &\leq \left| \int_{\text{spt } T} \langle \omega(x) - \phi(x), \vec{T}(x) \rangle d\|T\|(x) \right| + |T(\phi)| \\ &\leq \varepsilon \mathbf{M}(T) + \max\{\|\phi\|_\infty, \|d\phi\|_\infty\} \mathbf{F}(T) \\ &\leq \varepsilon \mathbf{N}(T) + \theta \mathbf{F}(T) \end{aligned}$$

for  $\theta = \max\{\|\phi\|_\infty, \|d\phi\|_\infty\}$ .

We now state a criterion for relative compactness in  $\mathbf{CH}^m(K)$ . It will prove useful in the next section for establishing the basic properties of the space of fractional charges.

**3.6. THEOREM.** — *Let  $\Omega \subseteq \mathbf{CH}^m(K)$ . The following are equivalent:*

- (A)  $\Omega$  is relatively compact;
- (B) the continuity inequality

$$|\omega(T)| \leq \varepsilon \mathbf{N}(T) + \theta \mathbf{F}(T)$$

holds for all  $\omega \in \Omega$  and  $T \in \mathbf{N}_m(K)$ , with a  $\theta = \theta(\varepsilon) \geq 0$  that can be chosen independently of  $\omega$ .

*Proof.* (A)  $\implies$  (B). We prove this implication by contradiction. Suppose there are  $\varepsilon > 0$  and two sequences  $(\omega_n)$  in  $\Omega$  and  $(T_n)$  in  $\mathbf{N}_m(K)$  such that

$$|\omega_n(T_n)| > \varepsilon \mathbf{N}(T_n) + n \mathbf{F}(T_n)$$

for all integers  $n$ . We can also suppose  $\mathbf{N}(T_n) = 1$  for all  $n$ . As  $\Omega$  is relatively compact, it is bounded, consequently

$$n \mathbf{F}(T_n) < \sup_{\omega \in \Omega} \|\omega\|_{\mathbf{CH}^m(K)} < \infty$$

which implies that  $(T_n)$  converges to 0 in flat norm. On the other side, there is a subsequence  $(\omega_{n_k})$  that converges to  $\omega \in \mathbf{CH}^m(K)$ . Hence

$$|\omega_{n_k}(T_{n_k})| \leq |\omega(T_{n_k})| + \|\omega - \omega_{n_k}\|_{\mathbf{CH}^m(K)} \rightarrow 0$$

which contradicts that  $|\omega_{n_k}(T_{n_k})| > \varepsilon$ .

(B)  $\implies$  (A). Denote by  $B_{\mathbf{N}_m(K)}$  the unit ball of  $\mathbf{N}_m(K)$ , metrized by  $\mathbf{F}$ , and let  $\iota: \mathbf{CH}^m(K) \rightarrow C(B_{\mathbf{N}_m(K)})$  be the linear map that sends a charge to its restriction to  $B_{\mathbf{N}_m(K)}$ . Here,  $C(B_{\mathbf{N}_m(K)})$  is given the supremum norm, so that  $\iota$  is an isometric embedding.

Since  $\mathbf{CH}^m(K)$  is a Banach space,  $\iota(\mathbf{CH}^m(K))$  is closed. We only need to show that  $\iota(\Omega)$  is relatively compact in  $C(B_{\mathbf{N}_m(K)})$ .

First, the inequality in (B) (for  $\varepsilon = 1$ ) entails that  $\iota(\Omega)$  is pointwise bounded. Now, for an arbitrary  $\varepsilon > 0$  there is  $\theta \geq 0$  as in (B). If  $T, S \in B_{\mathbf{N}_m(K)}$  satisfy  $\mathbf{F}(T - S) \leq \varepsilon/\theta$ , then for any  $\omega \in \Omega$ , one has

$$|\iota(\omega)(T) - \iota(\omega)(S)| \leq \theta \mathbf{F}(T - S) + \varepsilon \mathbf{N}(T - S) \leq 3\varepsilon.$$

This proves that  $\iota(\Omega)$  is equicontinuous, thus relatively compact by the Arzelà-Ascoli theorem. The proof is then finished.  $\square$

**3.7 (Charges over  $\mathbb{R}^d$ ).** — It is possible, as was done in [3], to extend the notion of  $m$ -charge over arbitrary subsets of  $\mathbb{R}^d$ . In this article, we will not attempt to be as general as possible, but rather concentrate on charges defined over  $\mathbb{R}^d$ , a domain particularly suited for performing convolutions or introducing the so-called Littlewood-Paley decomposition of Section 5.

We call  $m$ -charge over  $\mathbb{R}^d$  a linear functional  $\omega: \mathbf{N}_m(\mathbb{R}^d) \rightarrow \mathbb{R}$  whose restriction to  $\mathbf{N}_m(K)$  is an element of  $\mathbf{CH}^m(K)$ , for all compact subsets  $K \subseteq \mathbb{R}^d$ . The space of  $m$ -charges is denoted  $\mathbf{CH}^m(\mathbb{R}^d)$ , and we equip it with the Fréchet topology induced by the

family of seminorms

$$\|\omega\|_{\mathbf{CH}^m(K)} = \sup \{ \omega(T) : T \in \mathbf{N}_m(K) \text{ and } \mathbf{N}(T) \leq 1 \}$$

where  $K$  ranges over all compact subsets of  $\mathbb{R}^d$ .

The results proven in this section carry over to  $\mathbb{R}^d$ . Specifically,

- (A) the definition of weak convergence, that of the exterior derivative in  $\mathbf{CH}^m(\mathbb{R}^d)$  is unchanged.
- (B) The map  $\Gamma$  that sends  $\omega \in \mathbf{CH}^m(\mathbb{R}^d)$  to the continuous function  $x \mapsto \omega(\llbracket x \rrbracket)$  is a Fréchet space isomorphism. Here, the topology  $C(\mathbb{R}^d)$  is induced by the seminorms  $\|\cdot\|_{\infty, K}$ , where  $K \subseteq \mathbb{R}^d$  is compact.
- (C) Any continuous form  $\omega \in C(\mathbb{R}^d; \wedge^m \mathbb{R}^d)$  can be regarded as a charge.
- (D) A subset  $\Omega \subseteq \mathbf{CH}^m(\mathbb{R}^d)$  is relatively compact if and only if, for every compact set  $K \subseteq \mathbb{R}^d$  and every  $\varepsilon > 0$ , there is  $\theta = \theta(K, \varepsilon)$  such that for all  $T \in \mathbf{N}_m(K)$ , we have  $|\omega(T)| \leq \varepsilon \mathbf{N}(T) + \theta \mathbf{F}(T)$ .

3.8 (Regularization by convolution). — Let  $\phi \in C_c^\infty(\mathbb{R}^d)$ . We define the **convolution** of a charge  $\omega \in \mathbf{CH}^m(\mathbb{R}^d)$  with  $\phi$  by

$$(\omega * \phi)(T) = \omega(T * \check{\phi}) \quad \text{for all } T \in \mathbf{N}_m(\mathbb{R}^d).$$

The next proposition shows that this construction yields a smooth form.

3.9. PROPOSITION. — Let  $\omega \in \mathbf{CH}^m(\mathbb{R}^d)$  and  $\phi \in C_c^\infty(\mathbb{R}^d)$ . Then  $\omega * \phi \in C^\infty(\mathbb{R}^d; \wedge^m \mathbb{R}^d)$ . Explicitly,

$$(\omega * \phi)(z) = \sum_{I \in \Lambda(d, m)} \omega \left( \mathcal{L}^d \wedge \phi(z - \cdot) \mathbf{e}_I \right) dx_I \text{ for all } z \in \mathbb{R}^d.$$

*Proof.* Call  $\tilde{\omega}(z)$  the right-hand side. First we check that  $\tilde{\omega}$  is a smooth  $m$ -form. This is done by ensuring that, for all  $1 \leq i \leq d$  and for any sequence  $(h_n)$  of nonzero real numbers tending to 0,

$$\frac{\mathcal{L}^d \wedge \phi(z + h_n \mathbf{e}_i - \cdot) \mathbf{e}_I - \mathcal{L}^d \wedge \phi(z - \cdot) \mathbf{e}_I}{h_n} \rightarrow \mathcal{L}^d \wedge \frac{\partial \phi}{\partial x_i}(z - \cdot) \mathbf{e}_I$$

in flat norm with uniformly bounded normal masses.

Next, in order to prove that the charges  $\omega * \phi$  and  $\tilde{\omega}$  coincide, we need only do so on currents of the form  $\mathcal{L}^d \wedge \xi$ , where  $\xi = \sum_{I \in \Lambda(d, m)} \xi_I \mathbf{e}_I$  is a compactly supported smooth  $m$ -vectorfield. This is because, for all  $T \in \mathbf{N}_m(\mathbb{R}^d)$ ,

$$(\omega * \phi - \tilde{\omega})(T) = \lim_{\varepsilon \rightarrow 0} (\omega * \phi - \tilde{\omega})(T * \Phi_\varepsilon)$$

by Proposition 2.4(C) and (E), and  $T * \Phi_\varepsilon$  has the form  $\mathcal{L}^d \wedge \xi$  by [4, 4.1.2].

We begin by evaluating

$$(\mathcal{L}^d \wedge \xi)(\tilde{\omega}) = \sum_{I \in \Lambda(d, m)} \int_{\mathbb{R}^d} \omega(\mathcal{L}^d \wedge \phi(z - \cdot) \mathbf{e}_I) \xi_I(z) dz.$$

On the other hand, one has

$$\omega(T * \check{\phi}) = \sum_{I \in \Lambda(d, m)} \omega \left( \mathcal{L}^d \wedge \xi_I * \check{\phi} \mathbf{e}_I \right).$$

Following [4, 4.1.2], we introduce, for every  $n \geq 1$ , a partition  $A_{n,1}, \dots, A_{n,p_n}$  of  $\text{spt } \xi$  into Borel sets of diameter less than  $n^{-1}$  and choose points  $z_{n,k} \in A_{n,k}$  for  $1 \leq k \leq p_n$ . Then

$$\sum_{k=1}^{p_n} \xi_I(z_{n,k}) \left( \mathcal{L}^d \wedge \phi(z_{n,k} - \cdot) \mathbf{e}_I \right) \mathcal{L}^d(A_{n,k}) \rightarrow \mathcal{L}^d \wedge \xi_I * \check{\phi} \mathbf{e}_I$$



in flat norm with uniformly bounded normal masses. Thus,

$$\begin{aligned}\omega(T * \check{\phi}) &= \lim_{n \rightarrow \infty} \sum_{I \in \Lambda(d, m)} \sum_{k=1}^{p_n} \xi_I(z_{n, k}) \omega(\mathcal{L}^d \wedge \phi(z_{n, k} - \cdot) \mathbf{e}_I) \mathcal{L}^d(A_{n, k}) \\ &= \sum_{I \in \Lambda(d, m)} \int_{\mathbb{R}^d} \xi_I(z) \omega(\mathcal{L}^d \wedge \phi(z - \cdot) \mathbf{e}_I) dz \\ &= (\mathcal{L}^d \wedge \xi)(\tilde{\omega}).\end{aligned}$$

□

#### 4. $\alpha$ -FRACTIONALITY

4.1. — Let  $\alpha \in (0, 1]$ . An  $\alpha$ -**fractional  $m$ -charge** over a compact set  $K \subseteq \mathbb{R}^d$  is a linear functional  $\omega: \mathbf{N}_m(K) \rightarrow \mathbb{R}$  for which there is a constant  $C \geq 0$  such that

$$|\omega(T)| \leq C \mathbf{N}(T)^{1-\alpha} \mathbf{F}(T)^\alpha \text{ for all } T \in \mathbf{N}_m(K).$$

It is clear that the above requirement is a stronger condition than the charge continuity property stated in Paragraph 3.1.

We adopt the notation  $\mathbf{CH}^{m, \alpha}(K)$  to represent the space of  $\alpha$ -fractional  $m$ -charges, normed by

$$\|\omega\|_{\mathbf{CH}^{m, \alpha}(K)} = \inf \{C \geq 0 : |\omega(T)| \leq C \mathbf{N}(T)^{1-\alpha} \mathbf{F}(T)^\alpha \text{ for all } T \in \mathbf{N}_m(K)\}. \quad (5)$$

We also define  $\|\omega\|_{\mathbf{CH}^{m, \alpha}(K)} = \infty$  if  $\omega \in \mathbf{CH}^m(K) \setminus \mathbf{CH}^{m, \alpha}(K)$ .

The parameter  $\alpha$  represents regularity. Equivalently, an  $m$ -charge  $\omega$  is  $\alpha$ -fractional whenever its restriction to the unit ball of  $\mathbf{N}_m(K)$ , endowed with the distance inherited from  $\mathbf{F}$ , is  $\alpha$ -Hölder continuous. One clearly has inclusions

$$\mathbf{CH}^{m, \beta}(K) \subseteq \mathbf{CH}^{m, \alpha}(K) \subseteq \mathbf{CH}^m(K)$$

(that are continuous) whenever  $\beta \geq \alpha$ . In addition, the reader may use the continuity of the second embedding and the lower semicontinuity of  $\|\cdot\|_{\mathbf{CH}^{m, \alpha}(K)}$  with respect to weak convergence to check that  $\mathbf{CH}^{m, \alpha}(K)$  is a Banach space.

When  $\alpha = 1$  and  $K$  is a Lipschitz neighborhood retract, we encounter a well-known object. Indeed, in this case, a 1-fractional charge  $\omega$  is  $\mathbf{F}$ -continuous and  $\mathbf{N}_m(K)$  is  $\mathbf{F}$ -dense in  $\mathbf{F}_m(K)$ . As such,  $\omega$  can be uniquely extended so as to become an element of  $\mathbf{F}_m(K)^*$ , the space of **flat  $m$ -cochains over  $K$** , introduced by H. Whitney.

More generally, we can think of  $\alpha$ -fractionality as a regularity that is intermediate between that of mere charges and that of flat cochains. We observe that, as a consequence of (3), the exterior derivative of an  $\alpha$ -fractional  $m$ -charge is again  $\alpha$ -fractional (and the map  $d: \mathbf{CH}^{m, \alpha}(K) \rightarrow \mathbf{CH}^{m+1, \alpha}(K)$  is continuous).

We can of course define  $\alpha$ -fractional  $m$ -charges over the whole space  $\mathbb{R}^d$ . They are by definition the charges  $\omega \in \mathbf{CH}^m(\mathbb{R}^d)$  whose restrictions to  $\mathbf{N}_m(K)$  belong to  $\mathbf{CH}^{m, \alpha}(K)$ , for each compact subset  $K$  of  $\mathbb{R}^d$ . The space they form is denoted  $\mathbf{CH}^{m, \alpha}(\mathbb{R}^d)$ , and we give it the locally convex topology induced by the seminorms  $\|\cdot\|_{\mathbf{CH}^{m, \alpha}(K)}$  defined as in (5), where  $K$  ranges over all compact sets. The elements of  $\mathbf{CH}^{m, 1}(\mathbb{R}^d)$  correspond to the locally flat  $m$ -cochains over  $\mathbb{R}^d$  described in [5, Section 4].

It may seem strange that the coefficient quantifying the regularity of a fractional charge is not diminished by 1 when the exterior derivative is applied. This is why we introduced the term  $\alpha$ -fractionality. It is already the case that the exterior derivative of a flat cochain remains a flat cochain. The distinction between working with generalized differential forms  $\omega$  and, say, Sobolev functions, whose high-order distributional derivatives are increasingly poorly controlled, is that  $d^k \omega = 0$  for  $k \geq 2$ .

In the next paragraphs we will exhibit two important examples of fractional charges, which served as motivations for the definition.

4.2 (Relationship with Hölder charges). — Here we look at the zero-codimensional case  $m = d$  and  $K = [0, 1]^d$ . We say that a measurable set  $E \subseteq [0, 1]^d$  has finite perimeter whenever  $\llbracket E \rrbracket \in \mathbf{N}_d(K)$  and we denote by  $\mathcal{BV}(K)$  the algebra of such sets. To each  $\alpha$ -fractional  $d$ -charge  $\omega$ , we associate the map  $\Upsilon(\omega) : \mathcal{BV}(K) \rightarrow \mathbb{R}$  that sends  $E$  to  $\omega(\llbracket E \rrbracket)$ . We set

$$\gamma := \frac{d-1}{d} + \frac{\alpha}{d} \in \left( \frac{d-1}{d}, 1 \right].$$

The map  $\mu = \Upsilon(\omega)$  satisfies the following properties

- (A) Finite additivity: for disjoint sets with finite perimeters  $E, F \in \mathcal{BV}(K)$ , we have  $\mu(E \cup F) = \mu(E) + \mu(F)$ ;
- (B) Continuity: if  $(E_n)$  is a sequence in  $\mathcal{BV}(K)$  with uniformly bounded perimeters  $\sup_n \mathbf{M}(\partial \llbracket E_n \rrbracket) < \infty$  and such that  $\mathcal{L}^d(E_n) \rightarrow 0$ , we have  $\mu(E_n) \rightarrow 0$ ;
- (C) Hölder control over dyadic cubes: there is a constant  $C \geq 0$  such that  $|\mu(Q)| \leq C \mathcal{L}^d(Q)^\gamma$  for all dyadic cubes  $Q \subseteq K$ .

The last property comes from

$$\begin{aligned} |\omega(\llbracket Q \rrbracket)| &\leq \|\omega\|_{\mathbf{CH}^{d,\alpha}(K)} \mathbf{N}(\llbracket Q \rrbracket)^{1-\alpha} \mathbf{F}(\llbracket Q \rrbracket)^\alpha \\ &\leq (1+2d)^{1-\alpha} \|\omega\|_{\mathbf{CH}^{d,\alpha}(K)} \mathcal{L}^d(Q)^{(1-\alpha)\frac{d-1}{d}} \mathcal{L}^d(Q)^\alpha. \end{aligned}$$

The maps  $\mu : \mathcal{BV}(K) \rightarrow \mathbb{R}$  that satisfy (A), (B) and (C) appeared in [2, 1] under the name  **$\gamma$ -Hölder charges**. The article [2] exhibits some examples of Hölder charges derived from stochastic processes, whereas [1] used Hölder charges as integrators for Young-type multidimensional integrals. The space of  $\gamma$ -Hölder charges is designated  $\text{sch}^\gamma(K)$ . We claim that

$$\Upsilon : \mathbf{CH}^{d,\alpha}(K) \rightarrow \text{sch}^\gamma(K)$$

is a one-to-one correspondence. Note that this paragraph violates our promise of a self-contained article, since it relies on material from [1]. However, the result we demonstrate here, and even more so, the notion of (top-dimensional) Hölder charge, will not be utilized further in the article.

First we prove that  $\Upsilon$  is injective. We recall that each normal current in  $\mathbf{N}_d(K)$  has the form  $\llbracket K \rrbracket \llcorner f$ , where  $f$  is a function with bounded variation supported in  $K$ . We can show the existence of a sequence of functions  $(f_n)$  supported in  $K$ , each  $f_n$  being constant on dyadic cubes of side length  $2^{-n}$ , such that  $\sup_n \mathbf{N}(\llbracket K \rrbracket \llcorner f_n) < \infty$  and  $\mathbf{F}(\llbracket K \rrbracket \llcorner f - \llbracket K \rrbracket \llcorner f_n) \rightarrow 0$ , see for example [2, Lemma 4.8]. If  $\omega$  is an  $\alpha$ -fractional  $m$ -charge in the kernel of  $\Upsilon$ , then  $\omega(\llbracket K \rrbracket \llcorner f_n) = 0$  for each  $n$ . By letting  $n \rightarrow \infty$ , we obtain that  $\omega$  vanishes on a general normal current  $\llbracket K \rrbracket \llcorner f$ .

Conversely, for each  $\mu \in \text{sch}^\gamma(K)$  and each nonnegative function  $f$  with bounded variation supported in  $K$ , we set

$$\Upsilon^{-1}(\mu)(\llbracket K \rrbracket \llcorner f) = \int_0^\infty \mu(\{f > t\}) dt.$$

For a general  $f$ , we define

$$\Upsilon^{-1}(\mu)(\llbracket K \rrbracket \llcorner f) = \Upsilon^{-1}(\mu)(\llbracket K \rrbracket \llcorner f^+) - \Upsilon^{-1}(\mu)(\llbracket K \rrbracket \llcorner f^-)$$

where  $f^+$  and  $f^-$  are the positive and negative parts of  $f$ . We claim that  $\Upsilon^{-1}(\mu)$  is an  $\alpha$ -fractional  $m$ -charge. This results essentially comes from [1, Theorem 3.10], where it was proven that

$$\mu(E) \leq C \mathbf{M}(\partial \llbracket E \rrbracket)^{1-\alpha} \mathcal{L}^d(E)^\alpha$$

for some constant  $C = C(\mu)$  and for any  $E \in \mathcal{BV}(K)$ . Applying Young's inequality, we obtain

$$\mu(E) \leq C \left( (1-\alpha) \lambda \mathbf{M}(\partial \llbracket E \rrbracket) + \frac{\alpha}{\lambda^{\frac{1}{\alpha}-1}} \mathcal{L}^d(E) \right)$$

for all  $\lambda > 0$ . Let  $f$  be a nonnegative function with bounded variation supported in  $K$ . By the coarea formula,

$$\begin{aligned} |\Upsilon^{-1}(\mu)(\llbracket K \rrbracket \llcorner f)| &\leq C(1-\alpha)\lambda \int_0^\infty \mathbf{M}(\partial(\llbracket K \rrbracket \llcorner \{f > t\})) \, dt \\ &\quad + \frac{C\alpha}{\lambda^{\frac{1}{\alpha}-1}} \int_0^\infty \mathcal{L}^d(\{f > t\}) \, dt \\ &\leq C(1-\alpha)\lambda \mathbf{N}(\llbracket K \rrbracket \llcorner f) + \frac{C\alpha}{\lambda^{\frac{1}{\alpha}-1}} \mathbf{F}(\llbracket K \rrbracket \llcorner f) \end{aligned}$$

Choosing

$$\lambda = \left( \frac{\mathbf{F}(\llbracket K \rrbracket \llcorner f)}{\mathbf{N}(\llbracket K \rrbracket \llcorner f)} \right)^\alpha.$$

we obtain

$$|\Upsilon^{-1}(\mu)(\llbracket K \rrbracket \llcorner f)| \leq C \mathbf{N}(\llbracket K \rrbracket \llcorner f)^{1-\alpha} \mathbf{F}(\llbracket K \rrbracket \llcorner f)^\alpha.$$

An inequality of the type above is easily obtained as well when we remove the restriction on the sign of  $f$ . This ends the proof that  $\Upsilon^{-1}(\mu) \in \mathbf{CH}^{d,\alpha}(K)$ . We let the reader check that  $\Upsilon$  and  $\Upsilon^{-1}$  are reciprocal maps.

4.3 (Hölder differential form). — We let  $\text{Lip}_{\text{loc}}^\alpha(\mathbb{R}^d, \wedge^m \mathbb{R}^d)$  the space of locally  $\alpha$ -Hölder continuous  $m$ -forms. It is a Fréchet space, when given the family of seminorms

$$\|\omega\|_{\text{Lip}^\alpha(K, \wedge^m \mathbb{R}^d)} := \max \{ \|\omega\|_{\infty, K}, \text{Lip}^\alpha(\omega; K) \}$$

indexed over all compact subsets  $K$  of  $\mathbb{R}^d$ .

We claim that, for  $\omega \in \text{Lip}_{\text{loc}}^\alpha(\mathbb{R}^d, \wedge^m \mathbb{R}^d)$ , the corresponding charge is  $\alpha$ -fractional. We recall that  $\Phi: \mathbb{R}^d \rightarrow \mathbb{R}$  be a smooth nonnegative even function, supported in the closed unit ball, with  $\int_{\mathbb{R}^d} \Phi = 1$ . For all  $\varepsilon \in (0, 1]$ , we set  $\Phi_\varepsilon(x) = \frac{1}{\varepsilon^d} \Phi\left(\frac{x}{\varepsilon}\right)$  and

$$\omega_\varepsilon(x) = \omega * \Phi_\varepsilon(x) = \int_{\mathbb{R}^d} \omega(y) \Phi_\varepsilon(x-y) \, dy.$$

For each  $x$  in a compact set  $K$ , we have

$$\|\omega - \omega_\varepsilon\|_{\infty, K} \leq \text{Lip}^\alpha(\omega; K_1) \varepsilon^\alpha$$

where  $K_1 := \{x \in \mathbb{R}^d : \text{dist}(x, K) \leq 1\}$ . Moreover,  $\omega_\varepsilon$  is smooth, and for all  $i \in \{1, \dots, d\}$ , we have

$$\begin{aligned} \partial_i \omega_\varepsilon(x) &= \frac{1}{\varepsilon^{d+1}} \int_{\mathbb{R}^d} \omega(y) \partial_i \Phi\left(\frac{x-y}{\varepsilon}\right) \, dy \\ &= \frac{1}{\varepsilon^{d+1}} \int_{\mathbb{R}^d} (\omega(y) - \omega(x)) \partial_i \Phi\left(\frac{x-y}{\varepsilon}\right) \, dy \end{aligned}$$

from which we infer that

$$|\partial_i \omega_\varepsilon(x)| \leq \text{Lip}^\alpha(\omega; K_1) \left( \int_{\mathbb{R}^d} |\partial_i \Phi| \right) \frac{1}{\varepsilon^{1-\alpha}}.$$

and thus,

$$\|\text{d}\omega_\varepsilon\|_{\infty, K} \leq C \text{Lip}^\alpha(\omega, K_1) \frac{1}{\varepsilon^{1-\alpha}}$$

for some constant depending on  $d$ . Next, for  $T \in \mathbf{N}_m(K)$ , we estimate

$$\omega(T) = \int_K \langle \omega(x) - \omega_\varepsilon(x), \vec{T}(x) \rangle \, \text{d}\|T\| + T(\omega_\varepsilon)$$

From the above inequalities, we can control the first term

$$\left| \int_K \langle \omega(x) - \omega_\varepsilon(x), \vec{T}(x) \rangle \, \text{d}\|T\| \right| \leq \text{Lip}^\alpha(\omega; K_1) \varepsilon^\alpha \mathbf{N}(T)$$

whereas, if we suppose furthermore that  $K$  is a 1-Lipschitz retract, the second term is controlled by

$$\begin{aligned} |T(\omega_\varepsilon)| &\leq \max\{\|\omega_\varepsilon\|_{\infty, K}, \|\mathrm{d}\omega_\varepsilon\|_{\infty, K}\} \mathbf{F}(T) \\ &\leq \frac{1}{\varepsilon^{1-\alpha}} \max\{\|\omega\|_{\infty, K_1}, C \operatorname{Lip}^\alpha(\omega; K_1)\} \mathbf{F}(T) \end{aligned}$$

By choosing  $\varepsilon = \mathbf{F}(T)^\alpha / \mathbf{N}(T)^\alpha$  (which is indeed less than 1) and combining the preceding inequalities, we obtain

$$|\omega(T)| \leq C_K \mathbf{N}(T)^{1-\alpha} \mathbf{F}(T)^\alpha,$$

with  $C_K := \operatorname{Lip}^\alpha(\omega; K_1) + \max\{\|\omega\|_{\infty, K_1}, C \operatorname{Lip}^\alpha(\omega; K_1)\}$ .

Next we will identify  $\alpha$ -fractional 0-charges with (locally) Hölder continuous functions.

**4.4. PROPOSITION.** — *The map  $\Gamma$  restricts to a Banach space isomorphism between  $\mathbf{CH}^{0,\alpha}(K)$  and  $\operatorname{Lip}^\alpha(K)$ .*

*Proof.* If  $\omega$  is  $\alpha$ -fractional, then

$$\begin{aligned} |\Gamma(\omega)(x) - \Gamma(\omega)(y)| &\leq \|\omega\|_{\mathbf{CH}^{0,\alpha}(K)} \mathbf{F}(\llbracket x \rrbracket - \llbracket y \rrbracket)^\alpha \mathbf{M}(\llbracket x \rrbracket - \llbracket y \rrbracket)^{1-\alpha} \\ &\leq 2^{1-\alpha} \|\omega\|_{\mathbf{CH}^{0,\alpha}(K)} |x - y|^\alpha \end{aligned}$$

for all  $x, y \in K$ . Furthermore,  $\|\Gamma(\omega)\|_\infty \leq \|\omega\|_{\mathbf{CH}^0(K)} \leq \|\omega\|_{\mathbf{CH}^{0,\alpha}(K)}$ . This also shows that  $\Gamma: \mathbf{CH}^{0,\alpha}(K) \rightarrow \operatorname{Lip}^\alpha(K)$  is continuous.

Conversely, let  $f \in \operatorname{Lip}^\alpha(K)$  and  $\omega = \Gamma^{-1}(f)$  the associated 0-charge. Let  $\varepsilon > 0$ . The function

$$f_\varepsilon: x \in K \mapsto \min \left\{ f(y) + \frac{\operatorname{Lip}^\alpha f}{\varepsilon^{1-\alpha}} d(x, y) : y \in K \right\}$$

belongs to  $\operatorname{Lip}(K)$ , and  $\operatorname{Lip} f_\varepsilon \leq (\operatorname{Lip}^\alpha f) \varepsilon^{\alpha-1}$ . It is also clear that  $f_\varepsilon(x) \leq f(x)$  for all  $x \in K$ . If  $d(x, y) \geq \varepsilon$ , then

$$f(y) + \frac{\operatorname{Lip}^\alpha f}{\varepsilon^{1-\alpha}} d(x, y) \geq f(y) + (\operatorname{Lip}^\alpha f) d(x, y)^\alpha \geq f(x)$$

which implies that the minimum in the definition of  $f_\varepsilon(x)$  is attained at a point  $y \in K$  such that  $d(x, y) \leq \varepsilon$ , and

$$f_\varepsilon(x) = f(y) + \frac{\operatorname{Lip}^\alpha f}{\varepsilon^{1-\alpha}} d(x, y) \geq f(y) \geq f(x) - (\operatorname{Lip}^\alpha f) \varepsilon^\alpha.$$

This shows that  $\|f_\varepsilon - f\|_\infty \leq \operatorname{Lip}^\alpha(f) \varepsilon^\alpha$ .

Let  $T = \sum_{k=1}^n a_k \llbracket x_k \rrbracket$  be a 0-polyhedral chain. Then

$$\omega(T) = \sum_{k=1}^n a_k f_\varepsilon(x_k) + \sum_{k=1}^n a_k (f(x_k) - f_\varepsilon(x_k)).$$

Approximating  $f_\varepsilon$  with a smooth function, we prove that

$$\left| \sum_{k=1}^n a_k f_\varepsilon(x_k) \right| \leq \max\{\operatorname{Lip} f_\varepsilon, \|f_\varepsilon\|_\infty\} \mathbf{F}(T).$$

Henceforth, if  $\varepsilon \leq 1$ ,

$$\begin{aligned} |\omega(T)| &\leq \max\{\operatorname{Lip} f_\varepsilon, \|f_\varepsilon\|_\infty\} \mathbf{F}(T) + \|f - f_\varepsilon\|_\infty \mathbf{M}(T) \\ &\leq (\|f\|_\infty + \operatorname{Lip}^\alpha(f)) \left( \frac{1}{\varepsilon^{1-\alpha}} \mathbf{F}(T) + \varepsilon^\alpha \mathbf{M}(T) \right). \end{aligned}$$

By choosing  $\varepsilon = \mathbf{F}(T)/\mathbf{M}(T)$  (which is less than or equal to 1), we obtain

$$|\omega(T)| \leq 2 (\|f\|_\infty + \operatorname{Lip}^\alpha(f)) \mathbf{F}(T)^\alpha \mathbf{M}(T)^{1-\alpha}.$$

The preceding identity also holds for any chain  $T \in \mathbf{M}_0(K)$ , as it is the  $\mathbf{F}$ -limit of a sequence  $(T_n)$  of polyhedral 0-chains such that  $\mathbf{M}(T_n) \uparrow \mathbf{M}(T)$ , and therefore,  $\omega \in \mathbf{CH}^{0,\alpha}(K)$ .

Finally, by the open mapping theorem,  $\Gamma$  is a Banach space isomorphism between  $\mathbf{CH}^{0,\alpha}(K)$  and  $\text{Lip}^\alpha(K)$ .  $\square$

4.5. COROLLARY. — *The map  $\Gamma$  (from 3.7(B)) restricts to a Fréchet space isomorphism between  $\mathbf{CH}^{0,\alpha}(\mathbb{R}^d)$  and  $\text{Lip}_{\text{loc}}^\alpha(\mathbb{R}^d)$ .*

4.6 (An open problem). — Proposition 4.4 and Corollary 4.5 completely describe  $\alpha$ -fractional charges in the case  $m = 0$ . In the zero-codimensional case  $m = d$ , there is a representation theorem for  $\alpha$ -fractional charges over  $[0, 1]^d$  in [1, Theorem 7.2], which characterizes them as the exterior derivatives of  $\alpha$ -Hölder  $(d - 1)$ -forms over  $K$  (a similar representation theorem for  $d$ -charges over  $\mathbb{R}^d$  can be obtained by using partitions of unity).

Given those results as well as the De Pauw-Moonens-Pfeffer representation theorem, one can conjecture that, in the middle cases  $1 \leq m \leq d - 1$ , an arbitrary  $\alpha$ -fractional charge  $\omega \in \mathbf{CH}^{m,\alpha}(\mathbb{R}^d)$  can be (non uniquely) decomposed as a sum

$$\omega = \eta_1 + d\eta_2, \text{ where } \eta_1 \in \text{Lip}_{\text{loc}}^\alpha\left(\mathbb{R}^d; \bigwedge^m \mathbb{R}^d\right) \text{ and } \eta_2 \in \text{Lip}_{\text{loc}}^\alpha\left(\mathbb{R}^d; \bigwedge^{m-1} \mathbb{R}^d\right).$$

4.7. PROPOSITION (Compactness). — *Any bounded sequence in  $\mathbf{CH}^{m,\alpha}(K)$  has a subsequence that converges in  $\mathbf{CH}^m(K)$  to an  $\alpha$ -fractional charge.*

*Proof.* First observe that  $\|\cdot\|_{\mathbf{CH}^{m,\alpha}(K)}$  (defined on  $\mathbf{CH}^m(K)$  with values in  $[0, \infty]$ ) is lower semi-continuous with respect to weak convergence. Therefore, we only need to check that a sequence  $(\omega_n)$  that satisfies

$$M := \sup_n \|\omega_n\|_{\mathbf{CH}^{m,\alpha}(K)} < \infty$$

has a convergent subsequence in  $\mathbf{CH}^m(K)$ . This is an easy consequence of the compactness criterion (Theorem 3.6), as for any  $T \in \mathbf{N}_m(K)$ , any integer  $n$  and  $\varepsilon > 0$ , one has, by Young inequality,

$$|\omega_n(T)| \leq \varepsilon \mathbf{N}(T) + \frac{M^{1/\alpha}}{\varepsilon^{(1-\alpha)/\alpha}} \mathbf{F}(T). \quad \square$$

4.8. — We say that a sequence  $(\omega_n)$  in  $\mathbf{CH}^{m,\alpha}(K)$  converges weakly-\* to  $\omega \in \mathbf{CH}^{m,\alpha}(K)$  whenever

- (A)  $(\omega_n)$  is bounded in  $\mathbf{CH}^{m,\alpha}(K)$ ;
- (B)  $\omega_n \rightarrow \omega$  in  $\mathbf{CH}^m(K)$ .

Using the preceding proposition, it is easy to prove that condition (B) can be substituted with “ $\omega_n \rightarrow \omega$  weakly”.

There is a similar compactness result for charges over  $\mathbb{R}^d$ . Recall that the topology of  $\mathbf{CH}^{m,\alpha}(\mathbb{R}^d)$  is induced that the family of seminorms  $\|\cdot\|_{\mathbf{CH}^{m,\alpha}(K)}$ . A sequence  $(\omega_n)$  in  $\mathbf{CH}^{m,\alpha}(\mathbb{R}^d)$  is bounded whenever  $\sup_n \|\omega_n\|_{\mathbf{CH}^{m,\alpha}(K)} < \infty$  for every compact  $K \subseteq \mathbb{R}^d$ . It is enough to consider a countable family of compact subsets  $K$  (for example the closed balls centered at the origin with an integer radius). By a diagonal argument, any bounded sequence in  $\mathbf{CH}^{m,\alpha}(\mathbb{R}^d)$  has a subsequence that converges in  $\mathbf{CH}^m(\mathbb{R}^d)$  to some  $\alpha$ -fractional charge. The notion of weak\* convergence in  $\mathbf{CH}^{m,\alpha}(\mathbb{R}^d)$  is easily adapted. We will use the compactness theorem in the following form.

4.9. PROPOSITION. — *Let  $(\omega_n)$  be a sequence in  $\mathbf{CH}^{m,\alpha}(\mathbb{R}^d)$  that is bounded and such that  $\lim_n \omega_n(T)$  exists for all  $T \in \mathbf{N}_m(\mathbb{R}^d)$ . Then  $(\omega_n)$  converges weakly-\* to an  $\alpha$ -fractional charge.*

The smoothing of charges provides an example of weak\* convergence. Precise estimates are given in the next proposition.

4.10. PROPOSITION. — *Let  $\omega \in \mathbf{CH}^{m,\alpha}(\mathbb{R}^d)$ , let  $K \subseteq \mathbb{R}^d$  be compact and  $\varepsilon \in (0, 1]$ . We have*

- (A)  $\|\omega * \Phi_\varepsilon\|_{\mathbf{CH}^{m,\alpha}(K)} \leq \|\omega\|_{\mathbf{CH}^{m,\alpha}(K_\varepsilon)}$ ;
- (B)  $\|\omega * \Phi_\varepsilon\|_{\mathbf{CH}^{m,1}(K)} \leq C\varepsilon^{\alpha-1}\|\omega\|_{\mathbf{CH}^{m,\alpha}(K_\varepsilon)}$ ;
- (C)  $\|\omega - \omega * \Phi_\varepsilon\|_{\mathbf{CH}^m(K)} \leq C\varepsilon^\alpha\|\omega\|_{\mathbf{CH}^{m,\alpha}(K_\varepsilon)}$ .

*Proof.* Let  $T \in \mathbf{N}_m(K)$  be arbitrary. Regarding (A), we have

$$|\omega * \Phi_\varepsilon(T)| = |\omega(T * \Phi_\varepsilon)| \leq \|\omega\|_{\mathbf{CH}^{m,\alpha}(K_\varepsilon)} \mathbf{N}(T * \Phi_\varepsilon)^{1-\alpha} \mathbf{F}(T * \Phi_\varepsilon)^\alpha \quad (6)$$

because  $\text{spt}(T * \Phi_\varepsilon) \subseteq K_\varepsilon$ . By Proposition 2.4(C),

$$|\omega * \Phi_\varepsilon(T)| \leq \|\omega\|_{\mathbf{CH}^{m,\alpha}(K_\varepsilon)} \mathbf{N}(T)^{1-\alpha} \mathbf{F}(T)^\alpha.$$

Therefore  $\|\omega * \Phi_\varepsilon\|_{\mathbf{CH}^{m,\alpha}(K)} \leq \|\omega\|_{\mathbf{CH}^{m,\alpha}(K_\varepsilon)}$ .

(B) is obtained by combining (6) with Proposition 2.4(D).

(C). This time,

$$\begin{aligned} |(\omega - \omega * \Phi_\varepsilon)(T)| &= |\omega(T - T * \Phi_\varepsilon)| \\ &\leq \|\omega\|_{\mathbf{CH}^{m,\alpha}(K_\varepsilon)} \mathbf{F}(T - T * \Phi_\varepsilon)^\alpha \mathbf{N}(T - T * \Phi_\varepsilon)^{1-\alpha} \\ &\leq C\|\omega\|_{\mathbf{CH}^{m,\alpha}(K_\varepsilon)} \varepsilon^\alpha \mathbf{N}(T)^\alpha (\mathbf{N}(T) + \mathbf{N}(T * \Phi_\varepsilon))^{1-\alpha} \quad \text{Prop. 2.4(E)} \\ &\leq C\|\omega\|_{\mathbf{CH}^{m,\alpha}(K_\varepsilon)} \mathbf{N}(T) \quad \text{Prop. 2.4(C)} \end{aligned}$$

We conclude with the arbitrariness of  $T$ .  $\square$

We end this section with a (technical) proposition that gives the 1-fractional norm of a smooth form.

**4.11. PROPOSITION.** — *Suppose  $\omega \in C^\infty(\mathbb{R}^d; \wedge^m \mathbb{R}^d)$  and  $K \subseteq \mathbb{R}^d$  is a compact set such that*

- (A) *Every nonempty open subset of  $K$  has positive Lebesgue measure;*
- (B)  *$K$  is a 1-Lipschitz retract of  $\mathbb{R}^d$ .*

*Then  $\|\omega\|_{\mathbf{CH}^{m,1}(K)} = \max\{\|\omega\|_{\infty,K}, \|\mathrm{d}\omega\|_{\infty,K}\}$ .*

*Proof.* Hypothesis (B) guarantees that  $\mathbf{F}(T) = \mathbf{F}_K(T)$  for  $T \in \mathbf{N}_m(K)$ . Therefore, we consider  $A \in \mathbf{N}_m(K)$  and  $B \in \mathbf{N}_{m+1}(K)$  such that  $T = A + \partial B$  and compute

$$|\omega(T)| = |A(\omega)| + |B(\mathrm{d}\omega)| \leq (\mathbf{M}(A) + \mathbf{M}(B)) \max\{\|\omega\|_{\infty,K}, \|\mathrm{d}\omega\|_{\infty,K}\}.$$

Taking the infimum over  $A, B$ , one obtains that  $|\omega(T)| \leq \mathbf{F}(T) \max\{\|\omega\|_{\infty,K}, \|\mathrm{d}\omega\|_{\infty,K}\}$ . This means that  $\|\omega\|_{\mathbf{CH}^{m,1}(K)} \leq \max\{\|\omega\|_{\infty,K}, \|\mathrm{d}\omega\|_{\infty,K}\}$ .

Hypothesis (A) implies that the  $\|\cdot\|_{\infty,K}$  seminorms can be replaced with essential suprema. One has then

$$\begin{aligned} \|\omega\|_{\infty,K} &= \sup_{\zeta} \int \langle \omega(x), \zeta(x) \rangle \mathrm{d}x = \sup_{\zeta} \omega(\mathcal{L}^d \wedge \zeta) \\ \|\mathrm{d}\omega\|_{\infty,K} &= \sup_{\xi} \int \langle (\mathrm{d}\omega)(x), \xi(x) \rangle \mathrm{d}x = \sup_{\xi} \omega(\partial(\mathcal{L}^d \wedge \xi)) \end{aligned}$$

where  $\zeta$  (resp.  $\xi$ ) ranges over the summable  $m$ -vectorfields (resp.  $(m+1)$ -vectorfields) supported in  $K$  of  $L^1$ -norm 1. As  $\mathbf{F}(\mathcal{L}^d \wedge \zeta) \leq 1$  and  $\mathbf{F}(\partial(\mathcal{L}^d \wedge \xi)) \leq 1$ , one finally proves the desired inequality.  $\square$

**4.12. COROLLARY.** — *If  $\omega \in C^\infty(\mathbb{R}^d; \wedge^m \mathbb{R}^d)$ ,  $\eta \in C^\infty(\mathbb{R}^d; \wedge^{m'} \mathbb{R}^d)$  and  $K$  is a compact set that satisfies (A) and (B), then  $\|\omega \wedge \eta\|_{\mathbf{CH}^{m+m',1}(K)} \leq C\|\omega\|_{\mathbf{CH}^{m,1}(K)}\|\eta\|_{\mathbf{CH}^{m',1}(K)}$ , for some constant  $C$ .*

*Proof.* It is a consequence of the identity  $\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^m \omega \wedge \mathrm{d}\eta$ . The reader interested in computing the constant may consult [4, 1.8.1].  $\square$

## 5. A LITTLEWOOD-PALEY TYPE DECOMPOSITION OF FRACTIONAL CHARGES

We wish to introduce a decomposition result for  $\alpha$ -fractional charges into much more regular components (that are at least 1-fractional). It will be analogous to the decomposition of Hölder functions that is well described in [6, Appendix B, 2.6] that we briefly recall here. Any  $\alpha$ -Hölder continuous function  $f: X \rightarrow \mathbb{R}$  (for  $0 < \alpha < 1$ ) defined on a metric space can be decomposed into Lipschitz parts  $f = \sum_{n=0}^{\infty} f_n$ , where, for all  $n \geq 0$ ,

- (A)  $\|f_n\|_{\infty} \leq C2^{-n\alpha}$ ;
- (B)  $\text{Lip } f_n \leq C2^{n(1-\alpha)}$ ;
- (C)  $\sum_{n=0}^{\infty} f_n(x)$  converges for some  $x \in X$ .

Here  $C$  is a constant independent of  $n$ . We can think of  $f_n$  as being the part of  $f$  whose frequencies are localized around  $2^n$ . The estimates (A) and (B) guarantee that  $f_n$  is in  $\text{Lip}^{\alpha}(X)$ , with  $\text{Lip}^{\alpha} f_n \leq 2C$ . In our future decomposition of charges, the smooth components will be flat cochains. We begin our study when the domain is a compact subset  $K$  of  $\mathbb{R}^d$ , with an elementary lemma, that estimates the  $\alpha$ -fractional norm of such a component.

5.1. LEMMA. — Let  $\omega \in \mathbf{CH}^{m,1}(K)$ ,  $C \geq 0$  and  $\varepsilon > 0$ . Suppose that

$$\|\omega\|_{\mathbf{CH}^{m,1}(K)} \leq \frac{C}{\varepsilon^{1-\alpha}} \text{ and } \|\omega\|_{\mathbf{CH}^m(K)} \leq C\varepsilon^{\alpha}. \quad (7)$$

Then  $\|\omega\|_{\mathbf{CH}^{m,\alpha}(K)} \leq C$ .

*Proof.* For any  $T \in \mathbf{N}_m(K)$ ,

$$\begin{aligned} |\omega(T)| &\leq |\omega(T)|^{\alpha} |\omega(T)|^{1-\alpha} \\ &\leq \left( \frac{C}{\varepsilon^{1-\alpha}} \right)^{\alpha} C^{1-\alpha} \varepsilon^{\alpha(1-\alpha)} \mathbf{F}(T)^{\alpha} \mathbf{N}(T)^{1-\alpha} \\ &\leq C \mathbf{F}(T)^{\alpha} \mathbf{N}(T)^{1-\alpha}. \end{aligned} \quad \square$$

5.2. PROPOSITION. — Suppose  $0 < \alpha < 1$ . Let  $(\omega_n)$  be a sequence in  $\mathbf{CH}^{m,1}(K)$  such that

$$\|\omega_n\|_{\mathbf{CH}^{m,1}(K)} \leq C2^{n(1-\alpha)} \text{ and } \|\omega_n\|_{\mathbf{CH}^m(K)} \leq \frac{C}{2^{n\alpha}}$$

for some  $C \geq 0$  and for all  $n$ . Then  $\sum_{n=0}^{\infty} \omega_n$  converges weakly-\* to a charge in  $\mathbf{CH}^{m,\alpha}(K)$  and

$$\left\| \sum_{n=0}^{\infty} \omega_n \right\|_{\mathbf{CH}^{m,\alpha}(K)} \leq C_{5.2} C$$

where  $C_{5.2} = C_{5.2}(\alpha)$  is a constant.

Note that the convergence of the series  $\sum_{n=0}^{\infty} \omega_n$  is not strong in  $\mathbf{CH}^{m,\alpha}(K)$ , but only weak\*, a fact that should evoke some sort of orthogonality of the components  $\omega_n$ .

*Proof.* Let  $T \in \mathbf{N}_m(K)$  be nonzero and  $p \geq 1$ . We have

$$\left| \sum_{n=0}^{p-1} \omega_n(T) \right| \leq C \mathbf{F}(T) \sum_{n=0}^{p-1} 2^{n(1-\alpha)} \leq \frac{C2^{p(1-\alpha)}}{2^{1-\alpha} - 1} \mathbf{F}(T).$$

On the other hand,

$$\left| \sum_{n=p}^{\infty} \omega_n(T) \right| \leq C \mathbf{N}(T) \sum_{n=p}^{\infty} 2^{-n\alpha} \leq \frac{C2^{-p\alpha}}{1 - 2^{-\alpha}} \mathbf{N}(T)$$

Now, choose  $p \geq 1$  such that

$$2^{p-1} \leq \frac{\mathbf{N}(T)}{\mathbf{F}(T)} \leq 2^p.$$

This is possible because  $\mathbf{N}(T) \geq \mathbf{F}(T)$ . Then

$$\left| \sum_{n=0}^{\infty} \omega_n(T) \right| \leq \left( \frac{2^{1-\alpha}}{2^{1-\alpha} - 1} + \frac{1}{1 - 2^{-\alpha}} \right) C \mathbf{F}(T)^{\alpha} \mathbf{N}(T)^{1-\alpha}$$

This shows that  $\sum_{n=0}^{\infty} \omega_n \in \mathbf{CH}^{m,\alpha}(K)$ . The preceding argument shows that the sequence of partial sums  $\left( \sum_{n=0}^p \omega_n \right)_{p \geq 0}$  is bounded in  $\mathbf{CH}^{m,\alpha}(K)$ , and from there, it is easy to see that the convergence is weak-\*.  $\square$

When adapted to charges over  $\mathbb{R}^d$ , the preceding proposition takes the following form.

5.3. COROLLARY. — Suppose  $0 < \alpha < 1$ . Let  $(\omega_n)$  be a sequence in  $\mathbf{CH}^{m,1}(\mathbb{R}^d)$  such that, for each compact  $K \subseteq \mathbb{R}^d$ , there is  $C_K \geq 0$  such that

$$\|\omega_n\|_{\mathbf{CH}^{m,1}(K)} \leq C_K 2^{n(1-\alpha)} \text{ and } \|\omega_n\|_{\mathbf{CH}^m(K)} \leq \frac{C_K}{2^{n\alpha}}$$

for all  $n$ . Then  $\sum_{n=0}^{\infty} \omega_n$  converges weakly-\* to a charge in  $\mathbf{CH}^{m,\alpha}(\mathbb{R}^d)$ .

5.4. — Of course, it is enough to check the hypothesis of Corollary 5.3 when  $K$  ranges over (non degenerate) closed balls. A Littlewood-Paley type decomposition of a fractional charge  $\omega \in \mathbf{CH}^{m,\alpha}(\mathbb{R}^d)$  can be obtained by convolution

$$\omega = \omega * \Phi_1 + \sum_{n=0}^{\infty} (\omega * \Phi_{2^{-(n+1)}} - \omega * \Phi_{2^{-n}}).$$

We claim that the weak\* convergence above is ensured by Corollary 5.3 and the estimates of Proposition 4.10. This decomposition will play a pivotal role in the forthcoming proof of Theorem 6.1.

## 6. MAIN RESULT

6.1. THEOREM. — Let  $\alpha, \beta$  be parameters such that  $0 < \alpha, \beta \leq 1$  and  $\alpha + \beta > 1$ . There is a unique map

$$\wedge : \mathbf{CH}^{m,\alpha}(\mathbb{R}^d) \times \mathbf{CH}^{m',\beta}(\mathbb{R}^d) \rightarrow \mathbf{CH}^{m+m',\alpha+\beta-1}(\mathbb{R}^d)$$

such that

- (A)  $\wedge$  extends the pointwise exterior product between smooth forms;
- (B) Weak\*-to-weak\* continuity: if  $(\omega_n)$  and  $(\eta_n)$  are two sequences that converge weakly-\* to  $\omega$  and  $\eta$  in  $\mathbf{CH}^{m,\alpha}(\mathbb{R}^d)$  and  $\mathbf{CH}^{m',\beta}(\mathbb{R}^d)$  respectively, then  $\omega_n \wedge \eta_n$  converge weakly-\* to  $\omega \wedge \eta$  in  $\mathbf{CH}^{m+m',\alpha+\beta-1}(\mathbb{R}^d)$ .

Moreover,  $\wedge$  is continuous.

*Proof.* We proved in Proposition 4.10 that  $\omega * \Phi_{\varepsilon} \rightarrow \omega$  weakly-\* in  $\mathbf{CH}^{m,\alpha}(\mathbb{R}^d)$ , and similarly,  $\eta * \Phi_{\varepsilon} \rightarrow \eta$  weakly-\* in  $\mathbf{CH}^{m',\beta}(\mathbb{R}^d)$ . Taking into account that  $\omega * \Phi_{\varepsilon}$  and  $\eta * \Phi_{\varepsilon}$  are smooth, the uniqueness of the map  $\wedge$  follows.

We now address the issue of existence. First we treat the case  $(\alpha, \beta) \neq (1, 1)$ . Abbreviate  $\omega_n = \omega * \Phi_{2^{-n}}$  and  $\eta_n = \eta * \Phi_{2^{-n}}$ . We shall prove that the weak\* limit of  $(\omega_n \wedge \eta_n)$  exists in  $\mathbf{CH}^{m+m',\alpha+\beta-1}(\mathbb{R}^d)$ . This will be achieved if we manage to prove that the following series converges weakly-\*

$$\omega_0 \wedge \eta_0 + \sum_{n=0}^{\infty} (\omega_{n+1} \wedge (\eta_{n+1} - \eta_n) + (\omega_{n+1} - \omega_n) \wedge \eta_n) \quad (8)$$

in  $\mathbf{CH}^{m+m',\alpha+\beta-1}(\mathbb{R}^d)$ . Let us note in passing that both sums  $\sum_{n=0}^{\infty} \omega_{n+1} \wedge (\eta_{n+1} - \eta_n)$  and  $\sum_{n=0}^{\infty} (\omega_{n+1} - \omega_n) \wedge \eta_n$  can be interpreted as paraproducts. The weak\* convergence



will follow from an application of Corollary 5.3. To this end, we let  $K$  be a closed (non degenerate) ball and estimate

$$\|\eta_{n+1} - \eta_n\|_{\mathbf{CH}^{m'}(K)} \leq \|\eta_{n+1} - \eta\|_{\mathbf{CH}^{m'}(K)} + \|\eta_n - \eta\|_{\mathbf{CH}^{m'}(K)} \leq C2^{-n\beta} \|\eta\|_{\mathbf{CH}^{m',\beta}(K_1)}$$

by Proposition 4.10 and similarly

$$\|\omega_{n+1} - \omega_n\|_{\mathbf{CH}^m(K)} \leq C2^{-n\alpha} \|\omega\|_{\mathbf{CH}^{m,\alpha}(K_1)}.$$

Next we wish to control the  $\mathbf{CH}^{m+m'}(K)$  seminorm of the exterior product  $\omega_{n+1} \wedge (\eta_{n+1} - \eta_n)$ . Let  $T \in \mathbf{N}_{m+m'}(K)$ . We recall that

$$\partial(T \lrcorner \omega_{n+1}) = (-1)^m (\partial T) \lrcorner \omega_{n+1} + (-1)^{m+1} T \lrcorner d\omega_{n+1}.$$

Hence

$$\begin{aligned} \mathbf{N}(T \lrcorner \omega_{n+1}) &\leq C\mathbf{N}(T) \max\{\|\omega_{n+1}\|_{K,\infty}, \|d\omega_{n+1}\|_{K,\infty}\} \\ &\leq C\mathbf{N}(T) \|\omega_{n+1}\|_{\mathbf{CH}^{m,1}(K)} && \text{by Proposition 4.11} \\ &\leq C\mathbf{N}(T) \|\omega\|_{\mathbf{CH}^{m,\alpha}(K_1)} 2^{n(1-\alpha)} && \text{by Proposition 4.10(B)} \end{aligned}$$

We deduce that

$$\begin{aligned} |\omega_{n+1} \wedge (\eta_{n+1} - \eta_n)(T)| &= |(\eta_{n+1} - \eta_n)(T \lrcorner \omega_{n+1})| \\ &\leq \|\eta_{n+1} - \eta_n\|_{\mathbf{CH}^{m'}(K)} \mathbf{N}(T \lrcorner \omega_{n+1}) \\ &\leq C \|\omega\|_{\mathbf{CH}^{m,\alpha}(K_1)} \|\eta\|_{\mathbf{CH}^{m',\beta}(K_1)} 2^{n(1-\alpha-\beta)} \mathbf{N}(T). \end{aligned}$$

As a result,

$$\|\omega_{n+1} \wedge (\eta_{n+1} - \eta_n)\|_{\mathbf{CH}^m(K)} \leq C \|\omega\|_{\mathbf{CH}^{m,\alpha}(K_1)} \|\eta\|_{\mathbf{CH}^{m',\beta}(K_1)} 2^{n(1-\alpha-\beta)}.$$

Next we estimate the  $\mathbf{CH}^{m,1}(K)$  seminorm of  $\omega_{n+1} \wedge (\eta_{n+1} - \eta_n)$ . By Proposition 4.10(B),

$$\begin{aligned} \|\omega_{n+1}\|_{\mathbf{CH}^{m,1}(K)} &\leq C2^{n(1-\alpha)} \|\omega\|_{\mathbf{CH}^{m,\alpha}(K_1)} \\ \|\eta_{n+1} - \eta_n\|_{\mathbf{CH}^{m',1}(K)} &\leq \|\eta_{n+1}\|_{\mathbf{CH}^{m',1}(K_1)} + \|\eta_n\|_{\mathbf{CH}^{m',1}(K)} \leq C2^{n(1-\beta)} \|\eta\|_{\mathbf{CH}^{m',\beta}(K_1)} \end{aligned}$$

By Corollary 4.12,

$$\|\omega_{n+1} \wedge (\eta_{n+1} - \eta_n)\|_{\mathbf{CH}^{m,1}(K)} \leq C \|\omega\|_{\mathbf{CH}^{m,\alpha}(K_1)} \|\eta\|_{\mathbf{CH}^{m',\beta}(K_1)} 2^{n(1-(\alpha+\beta-1))}.$$

Similarly, we have

$$\begin{aligned} \|(\omega_{n+1} - \omega_n) \wedge \eta_n\|_{\mathbf{CH}^{m+m'}(K)} &\leq C \|\omega\|_{\mathbf{CH}^{m,\alpha}(K_1)} \|\eta\|_{\mathbf{CH}^{m',\beta}(K_1)} 2^{n(1-\alpha-\beta)} \\ \|(\omega_{n+1} - \omega_n) \wedge \eta_n\|_{\mathbf{CH}^{m+m',1}(K)} &\leq C \|\omega\|_{\mathbf{CH}^{m,\alpha}(K_1)} \|\eta\|_{\mathbf{CH}^{m',\beta}(K_1)} 2^{n(1-(\alpha+\beta-1))} \end{aligned}$$

Thus the series in (8) converges weakly-\* and we naturally define  $\omega \wedge \eta$  to be the weak\* limit of  $(\omega_n \wedge \eta_n)$ . By Proposition 5.2, we have

$$\|\omega \wedge \eta\|_{\mathbf{CH}^{m,\alpha+\beta-1}(K)} \leq C \|\omega\|_{\mathbf{CH}^{m,\alpha}(K_1)} \|\eta\|_{\mathbf{CH}^{m',\beta}(K_1)}. \quad (9)$$

Thus,  $\wedge$  is continuous, as desired.

Now we need to prove that (A) and (B) hold. (A) is easy, for if  $\omega$  and  $\eta$  are already smooth, then  $(\omega_n \wedge \eta_n)$  converges locally uniformly (and thus weakly) to the pointwise exterior product of  $\omega$  and  $\eta$ . Then the weak\* and weak limits coincide, so we can conclude that  $\omega \wedge \eta$  has its natural meaning.

Finally we prove (B). Let  $(\omega^{(p)})$  and  $(\eta^{(p)})$  two sequences, indexed by  $p \geq 0$ , that converge weakly-\* towards  $\omega \in \mathbf{CH}^{m,\alpha}(\mathbb{R}^d)$  and  $\eta \in \mathbf{CH}^{m',\beta}(\mathbb{R}^d)$  as  $p \rightarrow \infty$ . As before, we fix a closed ball  $K$  and we set  $\omega_n^{(p)} = \omega^{(p)} * \Phi_{2^{-n}}$  and  $\eta_n^{(p)} = \eta^{(p)} * \Phi_{2^{-n}}$  for all  $n, p$ . Inequality (9), applied to  $\omega^{(p)}$  and  $\eta^{(p)}$ , show that the sequence  $(\omega^{(p)} \wedge \eta^{(p)})$  is bounded in  $\mathbf{CH}^{m+m',\alpha+\beta-1}(\mathbb{R}^d)$ .

Proposition 3.9 entails that, for all integer  $n$ , the smooth forms  $\omega_n^{(p)}$  and  $\eta_n^{(p)}$  converge locally uniformly to  $\omega_n$  and  $\eta_n$  as  $p \rightarrow \infty$ . Thus, for a fixed normal current  $T$  with support in a non degenerate closed ball, we have

$$T(\omega_{n+1} \wedge (\eta_{n+1} - \eta_n) + (\omega_{n+1} - \omega_n) \wedge \eta_n) = \lim_{p \rightarrow \infty} T(\omega_{n+1}^{(p)} \wedge (\eta_{n+1}^{(p)} - \eta_n^{(p)}) + (\omega_{n+1}^{(p)} - \omega_n^{(p)}) \wedge \eta_n^{(p)}).$$

Likewise,

$$T(\omega_0 \wedge \eta_0) = \lim_{p \rightarrow \infty} T(\omega_0^{(p)} \wedge \eta_0^{(p)}).$$

Arguing as before, one has

$$\begin{aligned} & \left| T(\omega_{n+1}^{(p)} \wedge (\eta_{n+1}^{(p)} - \eta_n^{(p)}) + (\omega_{n+1}^{(p)} - \omega_n^{(p)}) \wedge \eta_n^{(p)}) \right| \\ & \leq C \|\omega^{(p)}\|_{\mathbf{CH}^{m,\alpha}(K_1)} \|\eta^{(p)}\|_{\mathbf{CH}^{m',\beta}(K_1)} 2^{n(1-\alpha-\beta)} \mathbf{N}(T). \end{aligned}$$

As the sequences  $(\omega^{(p)})$  and  $(\eta^{(p)})$  are bounded, the previous bound can be made uniform in  $p$ , allowing us to apply Lebesgue's dominated convergence theorem in

$$\begin{aligned} & \omega^{(p)} \wedge \eta^{(p)}(T) \\ &= \lim_{p \rightarrow \infty} \left( T(\omega_0^{(p)} \wedge \eta_0^{(p)}) + \sum_{n=0}^{\infty} T(\omega_{n+1}^{(p)} \wedge (\eta_{n+1}^{(p)} - \eta_n^{(p)}) + (\omega_{n+1}^{(p)} - \omega_n^{(p)}) \wedge \eta_n^{(p)}) \right) \\ &= \omega \wedge \eta(T) \end{aligned}$$

The case  $\alpha = \beta = 1$ , though simpler, requires special attention. The uniqueness of  $\wedge$  is already established. As previously, we define  $\omega \wedge \eta$  to be the weak\* limit  $\lim_n \omega_n \wedge \eta_n$ . It exists because of the compactness theorem. Indeed, for a suitable  $K$ , we have, by Corollary 4.12 and Proposition 4.10(A) (or 4.10(B))

$$\begin{aligned} \|\omega_n \wedge \eta_n\|_{\mathbf{CH}^{m+m',1}(K)} &\leq C \|\omega_n\|_{\mathbf{CH}^{m,1}(K)} \|\eta_n\|_{\mathbf{CH}^{m',1}(K)} \\ &\leq C \|\omega\|_{\mathbf{CH}^{m,1}(K_1)} \|\eta\|_{\mathbf{CH}^{m',1}(K_1)} \end{aligned}$$

It remains to show that  $(\omega_n \wedge \eta_n)$  converges weakly. This is the case because, for  $T \in \mathbf{N}_m(K)$ ,

$$T(\omega_N \wedge \eta_N) = T(\omega_0 \wedge \eta_0) + \sum_{n=0}^{N-1} T(\omega_{n+1} \wedge (\eta_{n+1} - \eta_n) + (\omega_{n+1} - \omega_n) \wedge \eta_n) \quad (10)$$

As before, we establish

$$|T(\omega_{n+1} \wedge (\eta_{n+1} - \eta_n) + (\omega_{n+1} - \omega_n) \wedge \eta_n)| \leq \frac{C \|\omega\|_{\mathbf{CH}^{m,1}(K_1)} \|\eta\|_{\mathbf{CH}^{m',1}(K_1)}}{2^n}$$

which ensures the absolute convergence of the series in (10).

Now that the exterior product is well-defined as a map  $\mathbf{CH}^{m,1}(\mathbb{R}^d) \times \mathbf{CH}^{m',1}(\mathbb{R}^d) \rightarrow \mathbf{CH}^{m+m',1}(\mathbb{R}^d)$ , properties (A) and (B) are shown as before.  $\square$

6.2 (Properties of  $\wedge$ ). — Using the weak\* density of smooth forms, it is easy to extend the well-known formulae of exterior calculus to fractional charges. Among them, we have, for  $\omega \in \mathbf{CH}^{m,\alpha}(\mathbb{R}^d)$ ,  $\eta \in \mathbf{CH}^{m',\beta}(\mathbb{R}^d)$  and  $\alpha + \beta > 1$ ,

- (A)  $\wedge$  is bilinear;
- (B)  $\omega \wedge \eta = (-1)^{mm'} \eta \wedge \omega$ ;
- (C)  $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^m \omega \wedge d\eta$ .

The last item uses the weak\*-to-weak\* continuity of the exterior derivative.

Regarding the product of many fractional charges, we notice that  $\omega_1 \wedge \cdots \wedge \omega_k$  makes sense (and the product is associative) as long as  $\omega_i$  is  $\alpha_i$ -fractional and  $\alpha_1 + \cdots + \alpha_k > k-1$ . In this case, the result is an  $(\alpha_1 + \cdots + \alpha_k - (k-1))$ -fractional charge.

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