A location Invariant Statistic-Based Consistent Estimation Method for Three-Parameter Generalized Exponential Distribution

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Abstract

In numerous instances, the generalized exponential distribution can be used as an alternative to the gamma distribution or the Weibull distribution when analyzing lifetime or skewed data. This article offers a consistent method for estimating the parameters of a three-parameter generalized exponential distribution that sidesteps the issue of an unbounded likelihood function. The method is hinged on a maximum likelihood estimation of shape and scale parameters that uses a location-invariant statistic. Important estimator properties, such as uniqueness and consistency, are demonstrated. In addition, quantile estimates for the lifetime distribution are provided. We present a Monte Carlo simulation study along with comparisons to a number of well-known estimation techniques in terms of bias and root mean square error. For illustrative purposes, a real-world lifetime data set is analyzed.

Key words: non-regular family; maximum likelihood estimation; modified maximum likelihood estimation; consistency; unimodal; confidence interval

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1 Introduction

The most widely used and well-liked distributions for analyzing any skewed data or lifetime data are the gamma and Weibull distributions with three parameters. These three parameters, which stand for location, scale, and shape, give the distributions a lot of flexibility when it comes to analyzing skewed data. Unfortunately, both distributions have some flaws as well. Additionally, as a special case, the exponentiated Weibull distribution reduces to the three-parameter generalized exponential (GE) distribution when its location parameter is not present. The exponentiated Weibull distribution was first proposed by Mudholkar et al. (1995). In many instances, it has been demonstrated that the GE model is applicable as a replacement for the gamma model or the Weibull model; for more information, see Gupta and Kundu (1999, 2001, 2007).

Consider a three-parameter GE distribution denoted by GE(α, β, γ), where α, β and γ represent, respectively, the scale, shape, and location parameters. For $\alpha > 0,\ \beta > 0,\ \gamma \in \mathbb{R}$, a random variable that follows the GE(α, β, γ) has the cumulative distribution function (CDF) $F(.; \alpha, \beta, \gamma)$,

$$F(x; \alpha, \beta, \gamma) = \begin{cases} 0, & \text{if } x < \gamma \\ \left(1 - e^{-\frac{x-\gamma}{\alpha}}\right)^{\beta}, & \text{if } x > \gamma. \end{cases} \tag{1}$$

Here, all three parameters are undetermined. For the other similar three-parameter distributions, such as the lognormal, gamma, Weibull, and inverse Gaussian, etc., in which the location parameter is unknown, it is well known that the regularity conditions are not satisfied for the estimation method of the widely recognized maximum likelihood (ML) because the support of the probability density function (PDF) depends on the unknown location parameter, and thus the ML estimation may encounter difficulties. In the majority of time, the maximum likelihood estimator (MLE) does not exist for a particular range of the parameter space. In such situations, the likelihood becomes unbounded. Some additional issues with MLEs for non-regular distributions (when they exist): Asymptotic normality of the MLE and its functions cannot be used; the Fisher information matrix cannot be used for asymptotic variances and covariances due to the failure of the regularity conditions. Several authors, including Cohen and Whitten (1980, 1982); Cohen et al. (1984); Hall and Wang (2005); Nagatsuka and Balakrishnan (2012); Nagatsuka et al. (2013, 2014); Nagatsuka and Balakrishnan (2015); Prajapat et al. (2021); Basu and Kundu (2023) have explored this issue. The identical issue arises in the case of three-parameter GE distribution. Gupta and Kundu (1999) elaborated on the ML estimation for the three-parameter GE distribution. When the shape parameter $\beta < 1$, it has been demonstrated that the MLE does not exist because the likelihood function becomes unbounded when the location parameter γ is approximately equal to the smallest observation in the observed sample; whereas asymptotic results have been presented only for the range of $\beta > 2$. Taking this problem into account, Prajapat et al. (2021) and Basu and Kundu (2023) offered other procedures for estimating the parameters of three-parameter GE for its entire parameter space.

Most recent and newly proposed estimation methods in this direction include the location-scale-parameter-free (LSPF) and location-parameter-free (LPF) methods. As the names of the methods imply, the LSPF method is based on a location and scale invariant statistic, whereas the LPF method uses a location invariant statistic; consequently, the likelihood functions based on the invariant statistics for these methods are one-dimensional and two-dimensional, respectively. Uniqueness and the consistency properties of LPF estimators have not yet been demonstrated analytically for any of the well-known three-parameter distributions examined in the literature. We assume this since the resultant likelihood function has a complicated form. Consequently, an attempt has been made for the hypothesized GE distribution, and consistency is proved whereas unimodality has been established for some cases.

This study shows that the LPF approach has an advantage over the LSPF method in that it requires less time to execute the simulations and has less computational complexity. The main explanation of this is the reduction in the number of integration as compare to the LSPF method. One of the LSPF method's drawbacks over the LPF method is that it uses the estimates from the preceding sequence to construct estimates separately and sequentially. This might lead to a significant build-up of biases in the sequences. The LPF, on the other hand, produces estimates through two-dimensional optimization by reducing number of steps in the estimation procedure. Moreover, LPF method works with a reduced number of degenerate random variables as compared to the LSPF method, which possibly may encourage it to better perform in some situations.

Therefore, the major purpose of the study is to develop the LPF method of estimation in detail for the parameters and quantiles of the three-parameter GE distribution. The second key purpose of the paper is to demonstrate the estimators' uniqueness and consistency. Because proving the uniqueness analytically is challenging, therefore proofs are provided for some particular cases. To obtain bias and root mean square error (RMSE) of the estimators, a Monte Carlo simulation is implemented. On the basis of the biases and RMSE of the estimators, we also conduct a comparative study with some prominent methods.

The remaining sections are structured as follows. The estimation procedure based on the proposed LPF method for the three-parameter GE distribution is detailed in Section 2. In this section, we also discuss the estimators' properties, such as their uniqueness and consistency. In Section 3, a Monte Carlo simulation study is conducted for the purpose of evaluating the LPF method and making comparisons to some existing methods. In Section 4, the LPF method is illustrated using a real-lifetime data set, and Section 5 presents concluding remarks.

2 Proposed Estimators and their Properties

Assume $X_1, X_2, \ldots, X_n$ are n random variables that are *i.i.d.* following the three-parameter GE distribution with the common CDF defined in equation (1). Throughout the paper, it is presumed that $n \geq 3$. Consider the order statistics of $X_1, X_2, \ldots, X_n$ to be $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$. To begin the estimation procedure, we will consider the following statistic:

$$V_{(i)} = X_{(i)} - X_{(1)},\ i = 1, 2, \ldots, n. \tag{2}$$

$V_{(i)}$'s probability distribution is independent of the location parameter. It is worth noting that $V_{(1)} = 0$. The scale and shape parameters are therefore estimated based on the transformed data $V_{(1)}, V_{(2)}, \ldots, V_{(n)}$, whose joint probability distribution primarily depends on scale and shape parameters.

In this estimating sequence according to the developed LPF approach, estimators of the scale and shape parameter are then being used to estimate the location parameter. The estimation of parameters is discussed in detail in Subsection 2.1, while the estimation of quantiles for the lifetime distribution is elaborated on in Subsection 2.2.

2.1 Parameter Estimation

We estimate the scale and shape parameters α and β based on the random variables $V_{(1)}, V_{(2)}, \ldots, V_{(n)}$. As the likelihood function of α and β based on the transformed data is not dependent on the location parameter, it is bounded. Let $v_1, v_2, \ldots, v_n$ represent the respective realizations of $V_{(1)}, V_{(2)}, \ldots, V_{(n)}$. Note that v_1 must be 0.

Theorem 2.1. *The likelihood function of α and β, given $v_2, v_3, \ldots, v_n$, is given by*

$$\ell_v(\alpha, \beta | v_2, \ldots, v_n) = n!\Big(\frac{\beta}{\alpha}\Big)^n \int_0^\infty e^{-\frac{1}{\alpha}\sum_{i=1}^n (u+v_i)} \prod_{i=1}^n \big(1 - e^{-\frac{u+v_i}{\alpha}}\big)^{\beta-1} du, \ \alpha > 0, \ \beta > 0, \tag{3}$$

with $0 < v_2 < \cdots < v_n < \infty, \ v_1 = 0$.

Proof. See Appendix A. □

In Theorem 2.1, the likelihood function is a bounded and differentiable function with respect to the parameters α and β. These properties' proofs are listed in Appendix B. In order to maintain simplicity, we will refer to $\ell_v(\alpha, \beta | v_2, \ldots, v_n)$ as $\ell_v(\alpha, \beta)$ from now on. Due to the complexity of the likelihood function, we were unable to prove the unimodality of the bivariate function $\ell_v(\alpha, \beta)$, but we could establish the unimodality of the likelihood function $\ell_v(\alpha, \beta)$ when one of the parameters α and β is fixed. Now, we present two main findings in the subsequent theorems, the first of which relates to unimodality and the second to the consistency of the unique maximum. In Theorem 2.2, the unimodality of the likelihood function is emphasized.

Theorem 2.2. *For every $0 < v_2 < \cdots < v_n < \infty, v_1 = 0$, the likelihood function $\ell_v(\alpha, \beta)$ is unimodal function of $\alpha > 0$ (or $\beta > 0$) whenever β (or α) is fixed.*

Proof. See Appendix C. □

Now, in Theorem 2.3, we demonstrate the consistency of the estimators of α and β obtained by maximizing $\ell_v(\alpha, \beta)$.

Theorem 2.3. *Estimators based on the maximization of the likelihood $\ell_v(\alpha, \beta)$ are consistent estimators for $\alpha > 0$ and $\beta > 0$.*

Proof. See Appendix D. $\square$

We already have estimators based on the maximization of the likelihood function $\ell_v(\alpha, \beta)$ using the location-invariant data $v_2, v_3, \ldots, v_n$. Let us refer to the estimators of α and β based on the location-invariant data as $\widehat{\alpha}_v$ and $\widehat{\beta}_v$, respectively.

In addition, it is appropriate to use $X_{(1)}$ as an estimator for the location parameter γ. Let us represent it as $\widehat{\gamma}_{init}$ *i.e.*, $\widehat{\gamma}_{init} = X_{(1)}$. Because $E(X_{(1)}) = \gamma + \alpha \int_0^\infty (1 - F_Y(y; \beta))^n \, dy$ with $Y \sim \text{GE}(1, \beta, 0)$, it should be emphasized that $\widehat{\gamma}_{init}$ is a biased estimator of the parameter γ. It is therefore feasible to look at a bias-corrected estimate of γ, denoted by $\widehat{\gamma}_v$, as $X_{(1)} - \widehat{\alpha}_v \int_0^\infty (1 - F_Y(y; \widehat{\beta}_v))^n \, dy$.

Now, we establish that $\widehat{\gamma}_v$ is a consistent estimator of γ. To explain the consistency of $\widehat{\gamma}_{init}$, consider the following probability for an arbitrary $\epsilon > 0$:

$$\text{P}(|\widehat{\gamma}_{init} - \gamma| > \epsilon) = \text{P}(X_{(1)} - \gamma > \epsilon) = \text{P}(n^{1/\beta} \frac{X_{(1)} - \gamma}{\alpha} > n^{1/\beta} \frac{\epsilon}{\alpha}) \tag{4}$$

Using the Theorem 8 of Gupta and Kundu (1999), it can be observed that the probability aforesaid in (4) is less than $e^{-n(\epsilon/\alpha)^\beta}$ and converges to 0 as sample size n approaches ∞, proving that $\widehat{\gamma}_{init}$ is consistent for γ. Recall, in order to show the consistency of the bias-corrected estimator, that $\widehat{\gamma}_v = X(1) - \widehat{\alpha}_v \int_0^\infty (1 - (1 - e^{-y})^{\widehat{\beta}_v})^n \, dy$. Using Slutsky's theorem and the facts that $\widehat{\alpha}_v$ and $\widehat{\beta}_v$ are consistent for α and β, respectively, it can be shown that $\widehat{\alpha}_v \int_0^\infty (1 - (1 - e^{-y})^{\widehat{\beta}_v})^n \, dy$ converges to 0 in probability. Consequently, applying Slutsky's theorem once more, $\widehat{\gamma}_v$ is consistent for γ.

The above-mentioned detailed estimation approach under the LPF method described in this section is quickly summarized as follows:

Step 1. Obtaining $\widehat{\alpha}_v$ and $\widehat{\beta}_v$, take $\widehat{\gamma}_{init} = X_{(1)}$.

Step 2. Utilize $\widehat{\alpha}_v$, $\widehat{\beta}_v$ and $\widehat{\gamma}_{init}$ of Step 1 to obtain the bias-corrected $\widehat{\gamma}_v$ as follows:

$$\widehat{\gamma}_v = X_{(1)} - \widehat{\alpha}_v \int_0^\infty (1 - (1 - e^{-y})^{\widehat{\beta}_v})^n \, dy.$$

2.2 Quantile Estimation

Consider the estimation of the quantile of the three-parameter GE distribution using the CDF in (1). In this instance, the ζ-th quantile for $0 < \zeta < 1$ is indeed the solution of the equation $F(x_\zeta; \alpha, \beta, \gamma) = \zeta$ with respect to x_ζ, which is given by

$$x_\zeta = \gamma - \alpha \ln(1 - \zeta^{1/\beta}), \quad (5)$$

in which the estimators of α, β and γ are substituted for the parameters.

3 Simulation Study

We evaluate the performance of the proposed estimator through a Monte Carlo simulation study. The proposed method of estimation is the LPF method. We compare the performance of the LPF method to the LSPF method and two other modified maximum likelihood estimation methods, because these methods provide estimators for the entire parameter space. Since the MLE does not exist when $\beta < 1$, the LPF method is compared to the MLE when the shape parameter is assumed to be greater than or equal to 1. The comparisons are based on the estimators' biases and the RMSE. These two MMLEs are MMLE I and MMLE III, that are discussed in Section 3 of Prajapat et al. (2021). Recall that MMLE I is the most convenient estimation method, which has been recently implemented by Pasari and Dikshit (2014) and Raqab et al. (2008), whereas the MMLE III method was proposed by Hall and Wang (2005). In this article, the MMLE I method is modified using the bias-corrected estimator of γ before being implemented.

All results are obtained using the Monte Carlo simulation with the shape parameter β set to $0.50,\ 0.75,\ 1.00,\ 1.50,\ 2.00,\ 3.00$ and the sample size n set to $20,\ 50,\ 100$ and 200. All simulation results in terms of biases and RMSEs are provided based on $10,000$ simulations. We equate the scale parameter α to 1 and the location parameter γ to 0. We present bias and RMSE for the estimators of location, scale, and shape parameters in Subsection 3.1 and for the estimators of quantiles in Subsection 3.2. In the case of the LSPF method, numerical results for the sample size $n = 200$ are neglected due to a computational issue encountered during

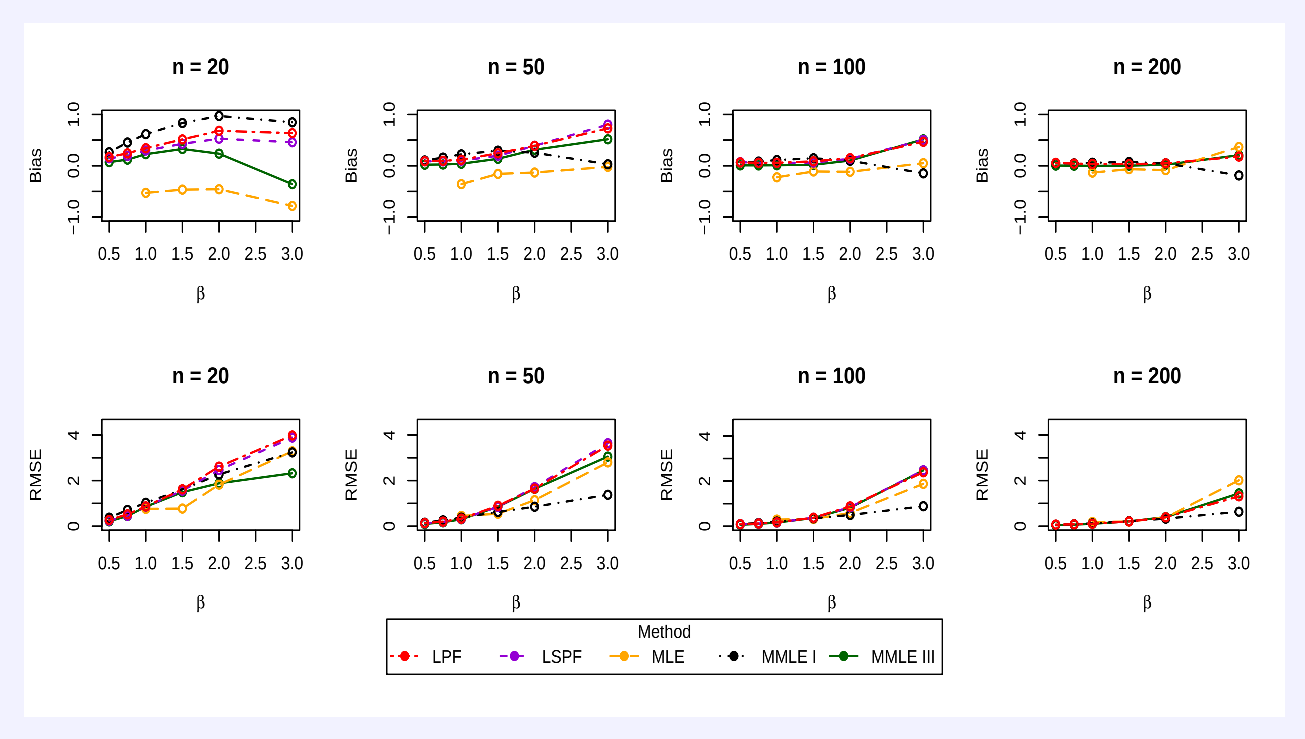


Figure 1: Plots for bias and RMSE of the estimator of shape parameter based on various estimation methods varying β values.

simulations.

3.1 Evaluation of Parameter Estimation

In Tables 4-5, all numerical simulations for estimating the parameters are presented. The shape parameter is assumed to be greater than 1 in Table 5 so that the MLEs are computed appropriately. To get a clear picture of the performance of the estimates of all three parameters, their biases and RMSEs are plotted in Figures 1-3 against varying β values. Now, we will attempt to summarize the results of the simulation study performed. The following observations are based on the simulation study reported in the first two tables:

- As the value of the shape parameter β approaches zero or as the sample grows in size, the performance of each method for estimating all parameters improves.
- MMLE I: When $\beta \leq 1$, it underestimates the scale parameter, and when $\beta > 1$, it overestimates the scale parameter. Also, the shape parameter is underestimated when $\beta > 1$.
- LSPF and LPF methods are nearly always similar in performance when estimating the

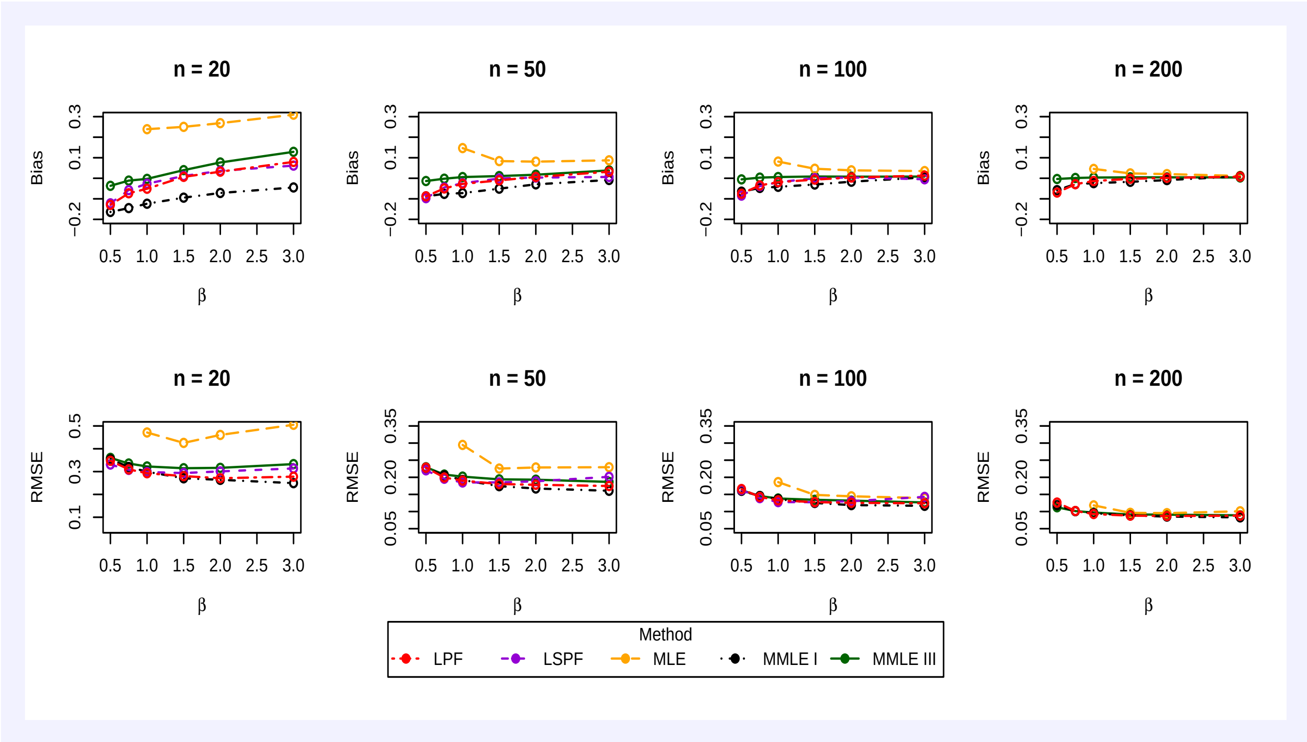


Figure 2: Plots for bias and RMSE of the estimator of scale parameter based on various estimation methods varying β values.

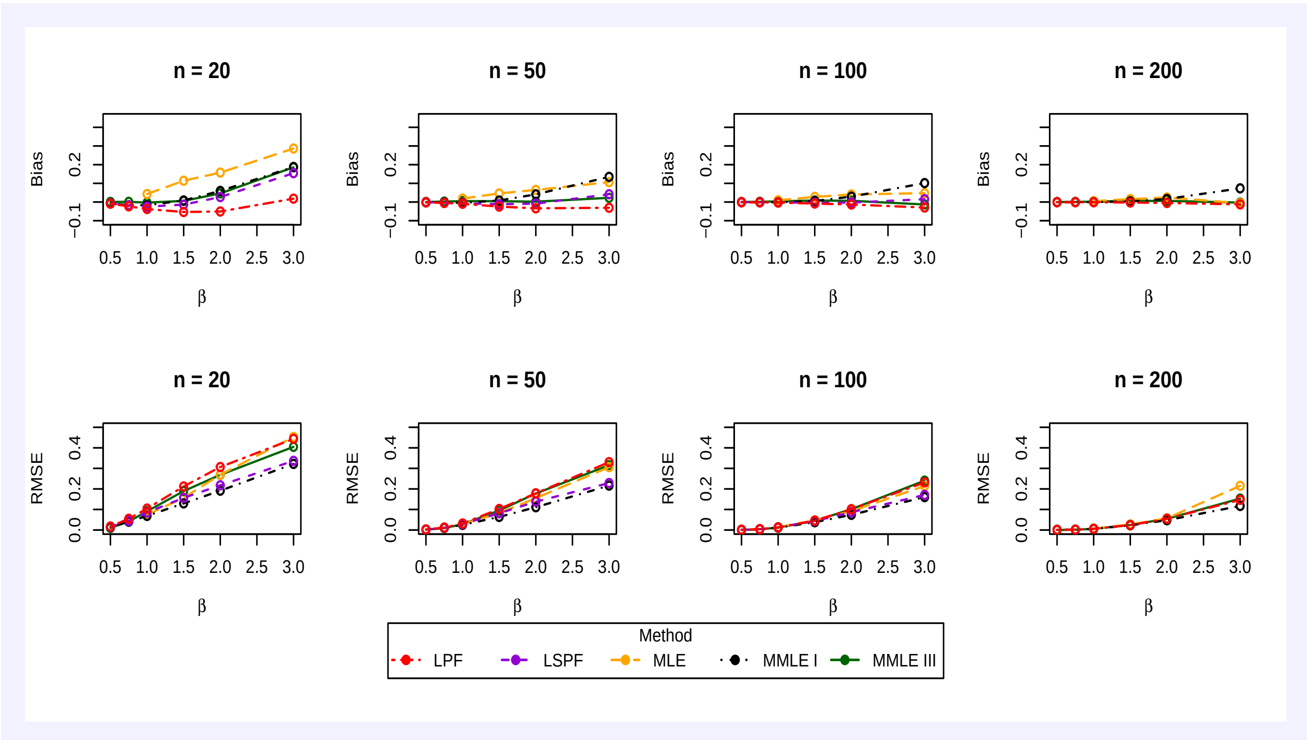


Figure 3: Plots for bias and RMSE of the estimator of location parameter based on various estimation methods varying β values.

shape and scale parameters.

- To estimate the shape parameter:
  - If $\beta > 1$, it is recommended to use MMLE III for $n \le 20$, MLE for $20 < n \le 100$, and either LPF or MMLE III for $n > 100$.
  - If $\beta \le 1$, it is best advised to use MMLE III.
- To estimate the scale parameter:
  - When $\beta > 1$, LPF is suggested as the first choice due to its small bias and RMSE, followed by MMLE III as the second choice because it performs slightly worse than LPF.
  - If $\beta \le 1$, it is best advised to use MMLE III.
  - For small sample sizes, such as $n \le 20$, we strongly discourage the use of MLE. In this situation, LPF or MMLE I is preferable if RMSE is the prime concern while MMLE III is preferable if bias is the prime concern.
  - LSPF and LPF perform similar and outperform all other methods.

■ To estimate the location parameter:

- LPF and LSPF perform similar in terms of biases and RMSEs for $\beta \le 2$. As β moves away from 2 when $n \le 50$, LPF method surpasses the LSPF method in terms of bias whereas LSPF outperforms LPF in terms of RMSE. The explanation for this might be that the LSPF produces estimates individually and sequentially, using the estimates obtained in the previous sequence, resulting in a substantial accumulation of biases in the sequences. This is one disadvantage of the LSPF approach over the LPF method.
- LPF consistently performs better than all other methods in terms of RMSE and bias, followed by MMLE III, when estimating location parameters. We do not advise MMLE I.

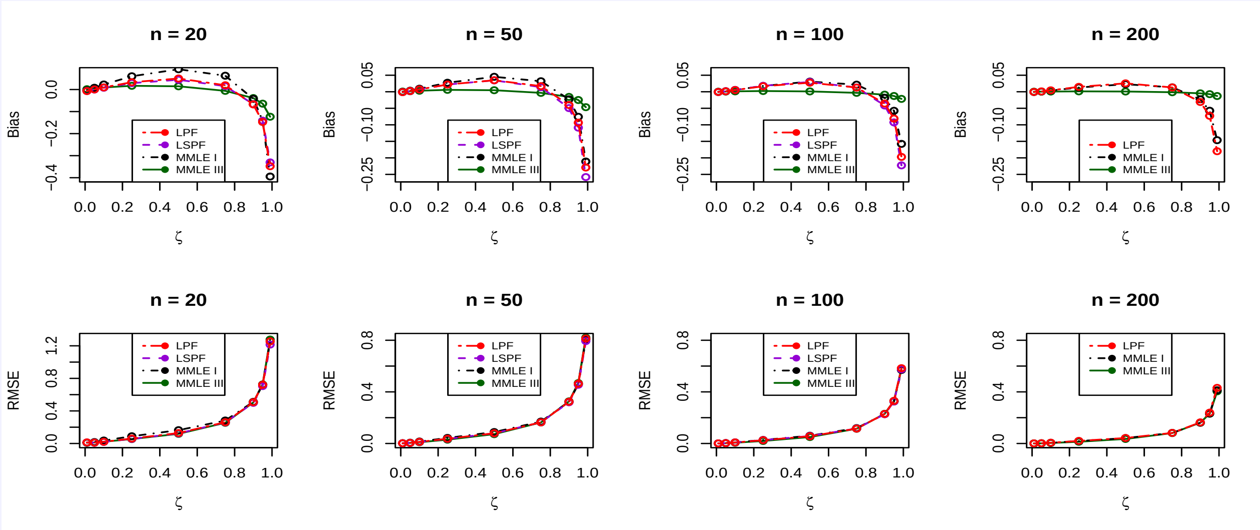
■ MMLE I is the only method that estimates parameters with a proportion of rejection of approximately zero, regardless of β value and sample size. On the basis of the reported proportion of rejections, we recommend using this method as a first preference. If we only consider the LSPF and LPF methods, LSPF has a lower proportion of rejection than LPF. However, if we compare LPF to all other methods including LSPF, we notice the following: After MMLE I, the second preference is LPF for $0.5 \le \beta < 3$ and MLE for relatively large values of ($\beta \ge 3$); when β is very small ($\beta \approx 0.5$), it has been observed that all the methods have a very small proportion of rejection (≈ 0) for sample size $n \ge 50$ and therefore any method can be recommended but when $n \le 20$, we advise to use LPF.

3.2 Evaluation of Quantile Estimation

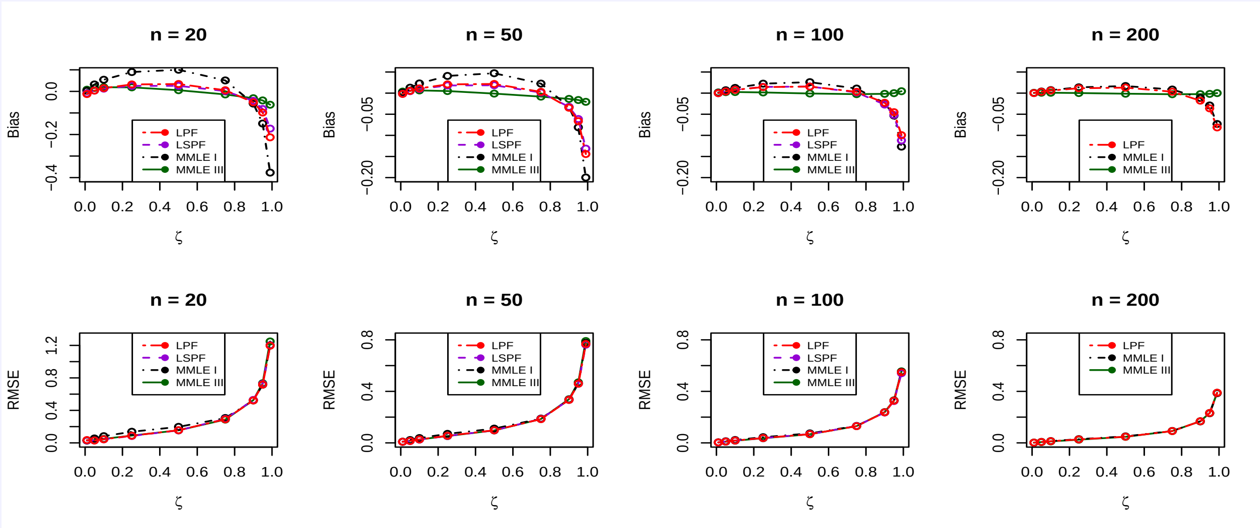
In practice, it is essential to estimate all three parameters in such a way that their combined performance is satisfactory. One may consider the quantile estimation as a possible solution to this problem, in which all parameters are utilized to estimate a qantile of the distribution. We therefore present quantile estimates for $\zeta = 0.01,\ 0.05,\ 0.10,\ 0.25,\ 0.50,\ \ 0.75,\ \ 0.90,\ \ 0.95,\ \ 0.99$ using equation (5) based on Monte Carlo simulation for the same values of shape parameter and sample size as in Subsection 3.1.

In Tables 6-9, the biases and RMSEs of the quantile estimation are shown. Figures 4-5 exhibit plots of the biases and RMSEs related to each method as the parameter value β is varied for each of the sample sizes previously studied. The following can be deduced from these results:

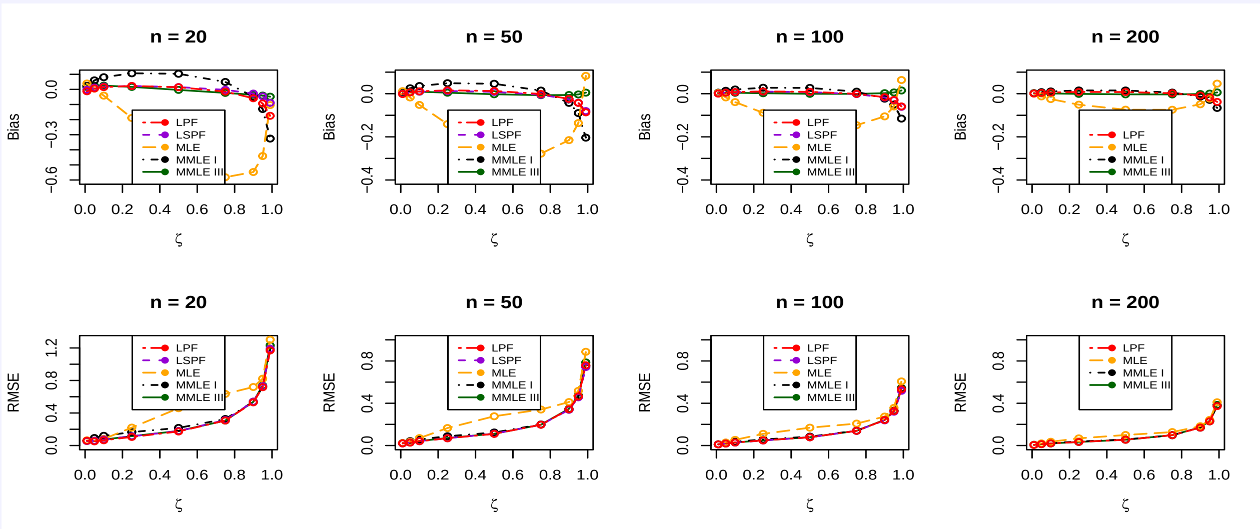
- The absolute value of the bias grows as ζ approaches towards 0 or 1 from 0.5, whereas the RMSE increases as ζ approaches to 1 from 0.
- All methods are equivalent in terms of RMSE, particularly when $n > 20$.
- LPF and LSPF comparisons: LPF and LSPF methods yield similar performances almost everywhere except for the situation when $\beta > 2$ and $n < 100$. For this exception, we recommend LPF when ζ is small (< 0.2), LSPF when ζ is close to 1 (> 0.8) and any of the LPF or LSPF when $0.2 \le \zeta \le 0.8$.
- In terms of bias and RMSE, LPF performs much better than all other methods when $\beta \ge 1$. It has the lowest bias values of the quantile estimates. Consequently, LPF is strongly advised when $\beta \ge 1$.
- When $\beta \le 1$, MMLE III provides the highest performance for any choice of $0 < \zeta < 1$, although LPF performs very well for $0 < \zeta < 0.7$ and much better as n grows. In this



(a) β = 0.5

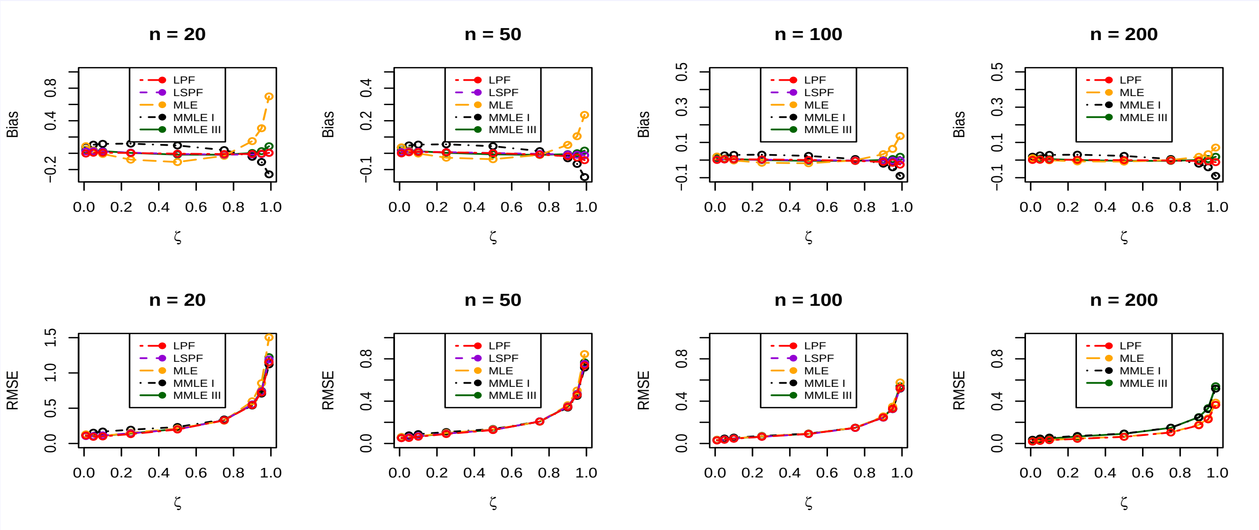


(b) β = 0.75

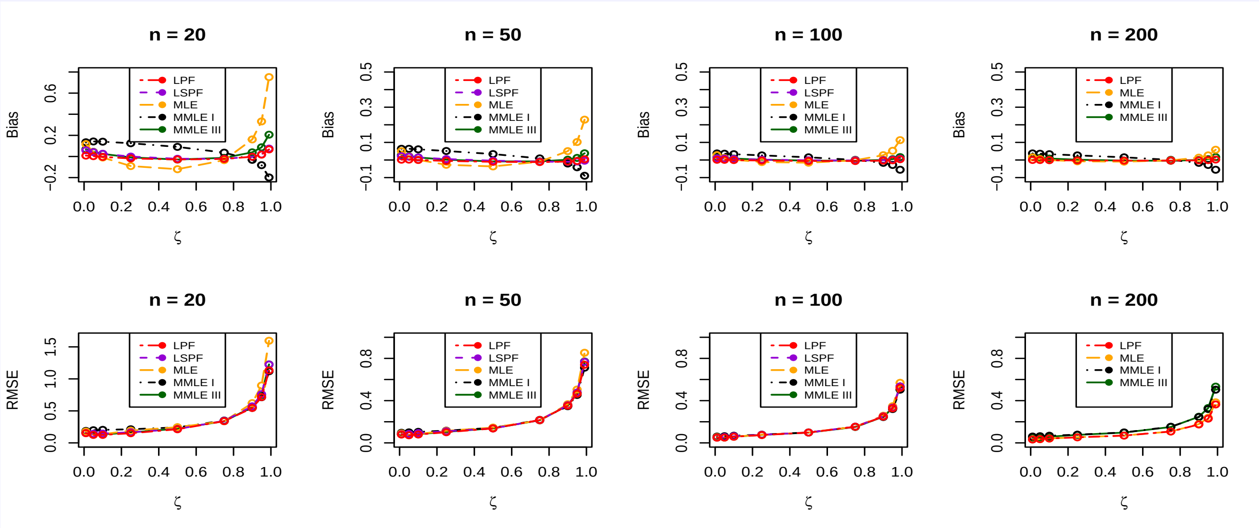


(c) β = 1.00

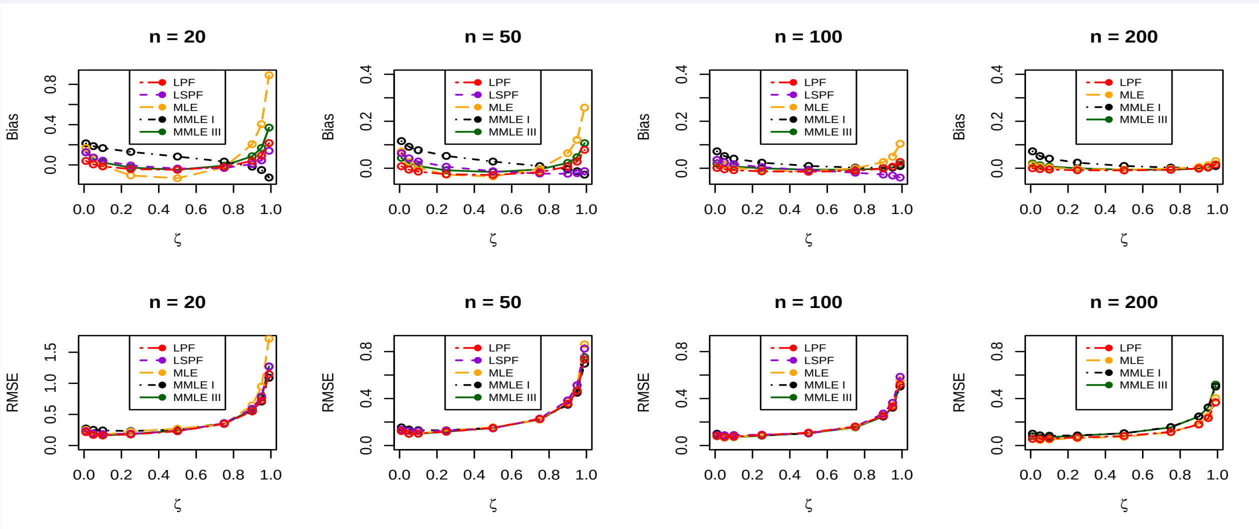
Figure 4: Plots for bias and RMSE based on various estimation methods when β = 0.50, 0.75, 1.00.



(a) $\beta = 1.5$



(b) $\beta = 2.0$



(c) $\beta = 3.0$

Figure 5: Plots for bias and RMSE based on various estimation methods when $\beta = 1.5,\ 2.0,\ 3.0.$

instance, we advise LPF for mainly sample sizes $n > 20$.

4 Data Analysis

We consider a data analysis for a set of real data that was initially reported by Bain and Engelhardt (1988) (see example 4.6.3 on page 162) and analyzed by Prajapat et al. (2021). The data indicate the observed lifetimes in months of a random sample of 40 electrical components. Histogram plot of the data set reveals that the shape of the density is reversed 'J' shaped and a simple data analysis provides the descriptive statistics as follows: mean = 93.12, sd = 96.70, median = 69.09, mad = 61.52, min = 0.15, max = 409.97, range = 409.82, skew = 1.74, , kurtosis = 2.49. In Table 1, the parameters' estimates are presented. For each method,

Table 1: Data analysis: Estimates of the parameters.

Method	Shape	Scale	Location	KS-Statistic	p-value	CvM-Statistic	p-value
LPF	1.0821	91.1620	-2.7991	0.0836	0.9323	0.0414	0.9258
LSPF	1.0799	91.2659	-2.7673	0.0838	0.9311	0.0415	0.9252
MMLE I	1.2564	85.0422	-2.7578	0.0897	0.8899	0.0543	0.8495
MMLE III	1.0408	92.1752	-1.4577	0.0419	0.9229	0.0419	0.9229

Table 2: Data analysis:Quantile estimates.

Method	ζ								
	0.01	0.05	0.10	0.25	0.50	0.75	0.90	0.95	0.99
LPF	-1.4970	3.1096	8.7601	26.8607	65.4500	129.8222	213.9440	277.3149	424.1757
LSPF	-1.4750	3.1138	8.7517	26.8349	65.4270	129.8431	214.0460	277.4846	424.5093
MMLE I	-0.5531	5.4629	12.0672	31.5204	70.2034	132.1558	211.5682	270.9731	408.1978
MMLE III	-0.3469	3.8768	9.2265	26.7915	65.0078	129.5150	214.2825	278.2687	426.6927

KS test and CvM test are implemented and the KS distance statistic, CvM test statistic, and corresponding p-values are reported. To illustrate the LPF method, we maximize the likelihood function $\ell_v(\alpha, \beta)$ in equation (3) with respect to the parameters α and β and estimate α and β to be 91.1620 and 1.0821, respectively. Using the estimates of α and β, we estimate γ to be -2.7991, by following the steps outlined in Section 2. According to the results reported in Table 1 based on the CvM test, LPF and LSPF perform exceptionally well for this particular data set, while MMLE III performs second best. In Table 2, qantiles estimates are reported for this particular data set. The plot in Figure 6 can be used to find the initial values of the pa-

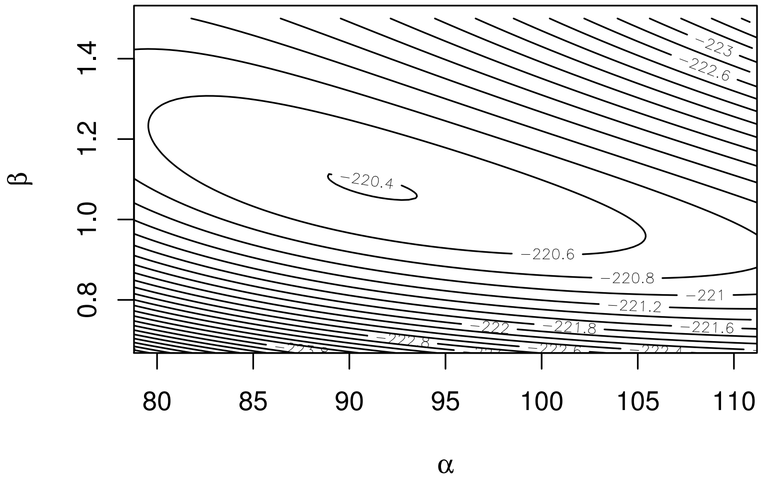


Figure 6: Data analysis: Plot of the log-likelihood function $\ln(\ell_v(\alpha, \beta))$.

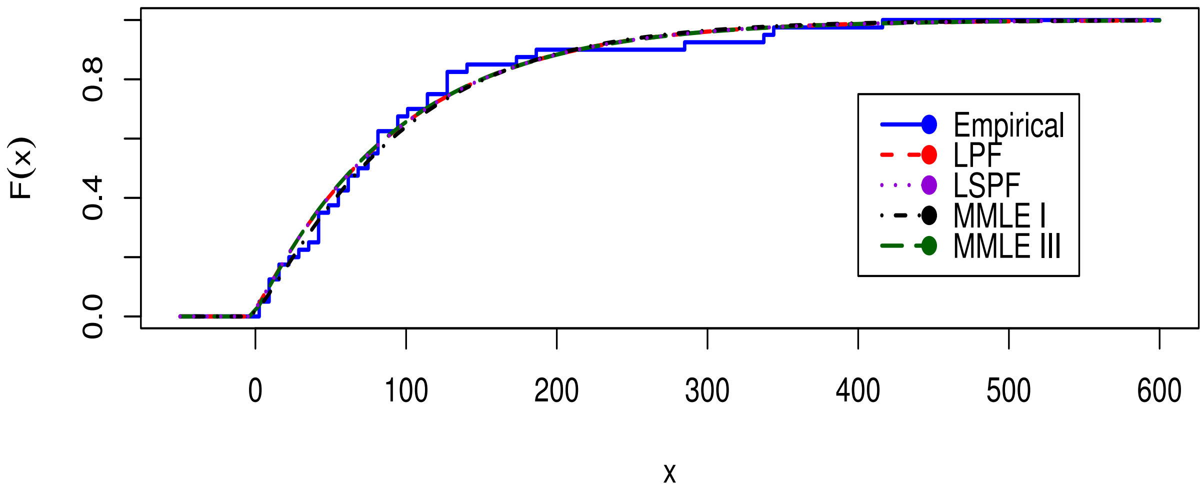


Figure 7: Data analysis: Fitted CDF plots along with the empirical CDF of data set.

rameters. Additionally, it indicates that the likelihood function is unimodal; thus, the obtained estimates maximize the likelihood globally. Figure 7 demonstrates the plots of the fitted CDF and the empirical CDF. To find the bootstrap confidence intervals (CIs), we set $\beta_U = 12$. Based on this data set, Table 3 presents 95% and 99% bootstrap CIs for each method.

Table 3: Data analysis: Bootstrap CIs based on 10000 simulations.

CI	Method	Shape	Scale	Location	p
95%	LPF	(0.7102, 2.3575)	(56.0280, 132.2849)	(-14.1289, 4.4647)	0.0046
	LSPF	(0.7462, 2.3137)	(56.7866, 130.8569)	(-12.8434, 4.5948)	0.0020
	MMLE I	(0.8165, 2.3900)	(53.5170, 123.5312)	(-6.4196, 7.4623)	0.0000
	MMLE III	(0.6568, 2.1126)	(57.0883, 137.9524)	(-8.8587, 6.7738)	0.0014
99%	LPF	(0.6400, 3.4396)	(48.6631, 149.7184)	(-22.8317, 8.7151)	0.0046
	LSPF	(0.6765, 3.7231)	(48.3591, 147.1999)	(-19.4118, 8.3977)	0.0020
	MMLE I	(0.7230, 3.1111)	(46.0003, 138.4216)	(-9.3808, 11.2007)	0.0000
	MMLE III	(0.5831, 3.1162)	(48.3295, 157.8311)	(-17.9056, 10.3762)	0.0014

5 Summary

In this paper, we consider an earlier-proposed estimation method for a three-parameter GE distribution known as the LPF method. We discussed some properties of the estimators, such as uniqueness and consistency. A Monte Carlo simulation study has been conducted to evaluate the performance of the LPF method comparative to other existing methods. In numerical simulations, we reported bias and RMSE for estimators of all three parameters and quantiles of the GE distribution. For $\beta > 1$, the LPF method performs better than all other methods for estimating scale parameters in terms of bias and RMSE. In addition to other methods such as MMLE I, MLE, etc., MMLE III has excellent and consistent performance when estimating shape parameter. When $\beta < 1$, the LPF method is recommended for the estimation of the shape parameter when the sample size is small. When $n \geq 50$, MMLE III is advisable if β is sufficiently small ($\beta \approx 0.5$) and any method is acceptable if $0.5 < \beta \leq 1$ as they all have similar performances for all parameters. The LPF method is recommended for estimating quantiles, especially when $\beta > 1$. Quantile estimation based on the LPF method appeared performing very good even for large values of the shape parameter. Based on the reported results of the proportion of rejected samples during the simulation study, it is recommended that LSPF and MMLE I be used for data analysis before any other methods. A real lifetime data set has been used to illustrate the LPF method.

According to this study, the LPF method is superior to the LSPF method in a number of aspects and offers certain advantages: (1) It consumes less time to run simulations and has a lower computational complexity. The primary cause for this is the decrease in the number of integration; (2) One disadvantage of the LSPF approach over the LPF method is that it employs estimates from the previous step in order to generate estimates separately in steps.

This might result in a considerable accumulation of biases in the steps. The LPF, on the other hand, creates estimates by reducing the number of steps in the estimating approach; (3) the LPF method is based on $n-1$ observations, whereas the LSPF method makes use of just $n-2$ observations.

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A Proof of Theorem 2.1

Using the transformation of random variables, we find the joint PDF of the random variables $V_{(2)}, V_{(3)}, \ldots, V_{(n)}$ for $\alpha > 0,\ \beta > 0$. Define $Z_{(i)} = X_{(i)} - \gamma, i = 1, 2, \ldots, n$. $Z_1, Z_2, \ldots, Z_n$ are *i.i.d.* random variables from the GE distribution $\text{GE}(\alpha, \beta, 0)$ with the shape parameter β and the scale parameter α. Denote the PDF and the CDF of $\text{GE}(\alpha, \beta, 0)$ by $g(.; \alpha, \beta)$ and $G(.; \alpha, \beta)$, respectively, for convenience. Now, $V_{(i)}$ in equation (2) can be rewritten in terms of $Z_{(i)}$'s as follows:

$$V_{(i)} = Z_{(i)} - Z_{(1)},\ i = 2, 3, \ldots, n \implies Z_{(i)} = Z_{(1)} + V_{(i)},\ i = 2, 3, \ldots, n. \qquad .$$

Let us assume that $U = Z_{(1)}$. Therefore $Z_{(i)} = U + V_{(i)},\ i = 2, 3, \ldots, n$. It can be shown that the Jacobian of the transformation $J = \frac{\partial(Z_{(1)}, Z_{(2)}, \ldots, Z_{(n-1)}, Z_{(n)})}{\partial(U, V_{(2)}, \ldots, V_{(n-1)}, V_{(n)})} = 1$. If we use a notation $f_{Y_1, Y_2, \ldots, Y_p}(.)$ to denote the joint PDF of $Y_1, Y_2, \ldots, Y_p$, we have

$$\begin{aligned} f_{U, V_{(2)}, \ldots, V_{(n)}}(u, v_2, \ldots, v_n; \alpha, \beta) &= |J| f_{Z_{(1)}, Z_{(2)}, \ldots, Z_{(n-1)}, Z_{(n)}}(u, u + v_2, \ldots, u + v_n; \alpha, \beta) \\ &= n! g(u; \alpha, \beta) \Big\{ \prod_{i=2}^{n} g(u + v_i; \alpha, \beta) \Big\}, \end{aligned}$$

$0 < v_2 < \cdots < v_n < \infty$ and $0 < u < \infty$. Therefore

$$\begin{aligned} & f_{V_{(2)}, \ldots, V_{(n)}}(v_2, \ldots, v_n; \alpha, \beta) \\ &\quad = \int_0^\infty n! g(u; \alpha, \beta) \Big\{ \prod_{i=2}^{n} g(u + v_i; \alpha, \beta) \Big\}\, du \\ &\quad = n! \Big(\frac{\beta}{\alpha}\Big)^n \int_0^\infty e^{-\frac{1}{\alpha}\sum_{i=1}^{n}(u+v_i)} \prod_{i=1}^{n} \big(1 - e^{-\frac{u+v_i}{\alpha}}\big)^{\beta-1}\, du, \end{aligned} \tag{6}$$

with $v_1 = 0$ and hence, the likelihood function of (α, β) given $v_2, v_3, \ldots, v_n$ is given by

$$\ell_v(\alpha, \beta | v_2, \ldots, v_n) = f_{V_{(2)}, \ldots, V_{(n)}}(v_2, \ldots, v_n; \alpha, \beta),\ \alpha > 0,\ \beta > 0$$

which proves the theorem.

B Boundedness and Differentiablity of the Likelihood Function $\ell_v(\alpha, \beta)$

B.1 Boundedness

By equation (6), we have

$$\ell_v(\alpha, \beta) = n! \int_0^\infty g(u; \alpha, \beta) \prod_{i=2}^{n} g(u + v_i; \alpha, \beta)\, du, \;\; \alpha > 0, \;\; \beta > 0. \tag{7}$$

It can be shown for $\alpha > 0, \;\beta > 0, \;0 < v_2 < \cdots < v_n < \infty$ and $0 < u < \infty$ that $\frac{(n-1)!\prod_{i=2}^{n} g(u+v_i;\alpha,\beta)}{(1-G(u;\alpha,\beta))^{n-1}}$ is bounded , *i.e.*, $\exists$ an $M > 0$ such that $(n-1)!\prod_{i=2}^{n} g(u + v_i; \alpha, \beta) < M(1 - G(u; \alpha, \beta))^{n-1}$ $\forall$ $\alpha > 0, \;\beta > 0, \;0 < v_2 < \cdots < v_n < \infty$ and $0 < u < \infty$. It implies:

$$\int_0^\infty n! g(u; \alpha, \beta) \prod_{i=2}^{n} g(u + v_i; \alpha, \beta)\, du < M \int_0^\infty n g(u; \alpha, \beta)(1 - G(u; \alpha, \beta))^{n-1}\, du = M,$$

B.2 Differentiablity

First we show differentiablity of the likelihood function $\ell_v(\alpha, \beta)$ in equation (3) with respect to β. In order to do so, let us rewrite the likelihood function as follows:

$$\ell_v(\alpha, \beta) = n! \int_0^\infty e^{h_v(\alpha,\beta;u)}\, du, \;\; \alpha > 0, \;\; \beta > 0, \tag{8}$$

where

$$h_v(\alpha, \beta; u) = n \ln(\beta) - n \ln(\alpha) - \frac{1}{\alpha} \sum_{i=1}^{n} (u + v_i) + (\beta - 1) \sum_{i=1}^{n} \ln(1 - e^{-\frac{u+v_i}{\alpha}}). \tag{9}$$

To show that the likelihood function is differentiable, we need to show that the partial derivative, with respect to β, can be taken inside the integral in the equation (3). Given $0 < v_2 < \cdots < v_n < \infty, \;v_1 = 0$ and $\alpha > 0$, we show:

1. $\frac{\partial}{\partial\beta} e^{h_v(\alpha,\beta;u)}$ exists,

2. $\left|\frac{\partial}{\partial\beta}e^{h_v(\alpha,\beta;u)}\right| < h_2(u)$ for some positive function h_2 and $\forall\ u \in (0,\infty)$ such that $\int_0^\infty h_2(u)\,du < \infty$, *i.e.* $\frac{\partial}{\partial\beta}e^{h_v(\alpha,\beta;u)}$ is an integrable function with respect to the variable u.

Since exponential, logarithmic and polynomials are well-known smooth functions, therefore $e^{h_v(\alpha,\beta;u)}$ is differentiable with respect to β and it is given by

$$\frac{\partial}{\partial\beta}e^{h_v(\alpha,\beta;u)} = e^{h_v(\alpha,\beta;u)}\frac{\partial}{\partial\beta}h_v(\alpha,\beta;u) = \Big(\frac{n}{\beta} + \sum_{i=1}^{n}\ln\big(1 - e^{-\frac{u+v_i}{\alpha}}\big)\Big)e^{h_v(\alpha,\beta;u)}. \tag{10}$$

Equation (10) can be simplified as follows:

$$\frac{\partial}{\partial\beta}e^{h_v(\alpha,\beta;u)} = \frac{n}{\beta}e^{h_v(\alpha,\beta;u)} + \Big(\frac{\beta}{\alpha}\Big)^n\Big(\sum_{i=1}^{n}\ln\big(1 - e^{-\frac{u+v_i}{\alpha}}\big)\Big)\prod_{i=1}^{n}\Big\{e^{-\frac{u+v_i}{\alpha}}\big(1 - e^{-\frac{u+v_i}{\alpha}}\big)^{\beta-1}\Big\}. \tag{11}$$

It is clear that first term of the equation (11) is integrable with respect to the variable u. Now, we show integrability of the second term and in order to do so, let us assume that

$$\mathfrak{H}_1(u) = \Big(\sum_{i=1}^{n}\ln\big(1 - e^{-\frac{u+v_i}{\alpha}}\big)\Big)\prod_{i=1}^{n}\Big\{e^{-\frac{u+v_i}{\alpha}}\big(1 - e^{-\frac{u+v_i}{\alpha}}\big)^{\beta-1}\Big\}.$$

For every $\alpha > 0,\ \beta > 0,\ 0 < v_2 < \cdots < v_n < \infty,\ v_1 = 0$ and $0 < u < \infty$, $\mathfrak{H}_1(u)$ tends to zero as $u \to \infty$. Also, $\mathfrak{H}_1(u) < h_3(u)$, where $h_3(u) = C_1e^{-\frac{u}{\alpha}}\big(1-e^{-\frac{u}{\alpha}}\big)^{\beta-1}\big(C_2+\ln\big(1-e^{-\frac{u}{\alpha}}\big)\big)$ with some finite constants C_1 and C_2. Now it is enough to show that $h_3(u)$ is integrable. Therefore, consider the quantity

$$\begin{aligned}\int_0^\infty h_3(u)\,du &= \int_0^\infty C_1e^{-\frac{u}{\alpha}}\big(1 - e^{-\frac{u}{\alpha}}\big)^{\beta-1}\big(C_2 + \ln\big(1 - e^{-\frac{u}{\alpha}}\big)\big)\,du\\ &= C_1C_2\int_0^\infty e^{-\frac{u}{\alpha}}\big(1 - e^{-\frac{u}{\alpha}}\big)^{\beta-1}\,du + C_1\int_0^\infty e^{-\frac{u}{\alpha}}\big(1 - e^{-\frac{u}{\alpha}}\big)^{\beta-1}\ln\big(1 - e^{-\frac{u}{\alpha}}\big)\,du\\ &= C_1C_2\alpha\int_0^1 u_1^{\beta-1}\,du_1 + C_1\alpha\int_0^1 u_1^{\beta-1}\ln(u_1)\,du_1\end{aligned}$$

Now, when $\beta \le 1$, we see that the integrand of the second term has singularity point at the boundary of the integration domain, but the integrand is integrable because

$$\lim_{\epsilon\to0^+}\int_\epsilon^1 u^{\beta-1}\ln(u)\,du = \lim_{\epsilon\to0^+}\int_\epsilon^1\ln(u)\,du = -1.$$

and hence, after simplification

$$\int_0^\infty h_3(u)\,du = C_1C_2\frac{\alpha}{\beta} - C_1\frac{\alpha}{\beta^2} < \infty.$$

Hence, part (*ii*) of Theorem 16.8 of Billingsley (1994) implies that the likelihood function $\ell_v(\alpha,\beta)$ is differentiable with respect to β and its derivative is given by

$$\begin{aligned}\frac{\partial}{\partial\beta}\ell_v(\alpha,\beta) &= n!\int_0^\infty \frac{\partial}{\partial\beta}e^{h_v(\alpha,\beta;u)}\,du\\ &= n!\Big(\frac{\beta}{\alpha}\Big)^n\int_0^\infty\Big(\frac{n}{\beta}+\sum_{i=1}^n\ln\big(1-e^{-\frac{u+v_i}{\alpha}}\big)\Big)e^{-\frac{1}{\alpha}\sum_{i=1}^n(u+v_i)}\prod_{i=1}^n\big(1-e^{-\frac{u+v_i}{\alpha}}\big)^{\beta-1}. \end{aligned} \tag{12}$$

Now, in a similar manner, we step forward to show that $\ell_v(\alpha,\beta)$ is differentiable with respect to α. Given $0<v_2<\cdots<v_n<\infty,\ v_1=0$ and $\beta>0$, we show:

1. $\frac{\partial}{\partial\alpha}e^{h_v(\alpha,\beta;u)}$ exists,
2. $\frac{\partial}{\partial\alpha}e^{h_v(\alpha,\beta;u)}$ is integrable function with respect to the variable u.

The first point is obvious to satisfy as it is a function of exponential, logarithmic and polynomial functions which are well-known smooth functions and the derivative is given by

$$\begin{aligned}\frac{\partial}{\partial\alpha}e^{h_v(\alpha,\beta;u)} &= e^{h_v(\alpha,\beta;u)}\frac{\partial}{\partial\alpha}h_v(\alpha,\beta;u)\\ &= \Big(-\frac{n}{\alpha}+\frac{1}{\alpha}\sum_{i=1}^n\frac{u+v_i}{\alpha}-\frac{(\beta-1)}{\alpha}\sum_{i=1}^n\frac{u+v_i}{\alpha}\Big(\frac{e^{-\frac{u+v_i}{\alpha}}}{1-e^{-\frac{u+v_i}{\alpha}}}\Big)\Big)e^{h_v(\alpha,\beta;u)}.\end{aligned} \tag{13}$$

Equation (13) is simplified as follows:

$$\begin{aligned}\frac{\partial}{\partial\alpha}e^{h_v(\alpha,\beta;u)} &= -\frac{n}{\alpha}e^{h_v(\alpha,\beta;u)}+\frac{1}{\alpha}\Big(\sum_{i=1}^n\frac{u+v_i}{\alpha}\Big)e^{h_v(\alpha,\beta;u)}-\frac{(\beta-1)}{\alpha}\Big(\sum_{i=1}^n\frac{u+v_i}{\alpha}\Big(\frac{e^{-\frac{u+v_i}{\alpha}}}{1-e^{-\frac{u+v_i}{\alpha}}}\Big)\Big)e^{h_v(\alpha,\beta;u)}\\ &= -\frac{n}{\alpha}e^{h_v(\alpha,\beta;u)}+\frac{\beta}{\alpha}\Big(\sum_{i=1}^n\frac{u+v_i}{\alpha}\Big)e^{h_v(\alpha,\beta;u)}-\frac{(\beta-1)}{\alpha}\Big(\sum_{i=1}^n\frac{u+v_i}{\alpha}\Big(\frac{1}{1-e^{-\frac{u+v_i}{\alpha}}}\Big)\Big)e^{h_v(\alpha,\beta;u)}\\ &= -\frac{n}{\alpha}e^{h_v(\alpha,\beta;u)}+\frac{\beta}{\alpha^2}\Big(\sum_{i=1}^n v_i\Big)e^{h_v(\alpha,\beta;u)}+\frac{n\beta}{\alpha^2}ue^{h_v(\alpha,\beta;u)}\\ &\quad -\frac{(\beta-1)}{\alpha^2}\Big(\sum_{i=1}^n\Big(\frac{1}{1-e^{-\frac{u+v_i}{\alpha}}}\Big)\Big)ue^{h_v(\alpha,\beta;u)}-\frac{(\beta-1)}{\alpha^2}\Big(\sum_{i=2}^n v_i\Big(\frac{1}{1-e^{-\frac{u+v_i}{\alpha}}}\Big)\Big)e^{h_v(\alpha,\beta;u)}.\end{aligned} \tag{14}$$

It is clear that first two terms of equation (14) are integrable as $e^{h_v(\alpha,\beta;u)}$ is integrable with respect to u. Now, we show integrablity of the third term and the proofs for the remaining terms goes along the same line and hence proofs are omitted. Assume that

$$\mathfrak{H}_2(u) = ue^{h_v(\alpha,\beta;u)} = \Big(\frac{\beta}{\alpha}\Big)^n u \prod_{i=1}^{n} \Big\{e^{-\frac{u+v_i}{\alpha}}\big(1-e^{-\frac{u+v_i}{\alpha}}\big)^{\beta-1}\Big\}.$$

For every $\alpha > 0,\ \beta > 0,\ 0 < v_2 < \cdots < v_n < \infty,\ v_1 = 0$ and $0 < u < \infty$, $\mathfrak{H}_2(u)$ tends to zero as $u \to \infty$. Also, $\mathfrak{H}_2(u) < h_4(u)$, where $h_4(u) = C_3 u e^{-\frac{u}{\alpha}}\big(1-e^{-\frac{u}{\alpha}}\big)^{\beta-1}$ with some finite constant C_3. Now it is enough to show that $h_4(u)$ is integrable. Therefore, we consider the quantity

$$\begin{aligned}
\int_0^\infty h_4(u)\,du &= \int_0^\infty C_3 u e^{-\frac{u}{\alpha}}\big(1-e^{-\frac{u}{\alpha}}\big)^{\beta-1}\,du \\
&= -C_3\alpha^2 \int_0^1 u_1^{\beta-1}\ln(1-u_1)\,du_1 \\
&= -C_3\alpha^2 \int_0^1 (1-y)^{\beta-1}\ln(y)\,dy \\
&= -C_3\alpha^2\Big(\int_0^{1/2}(1-y)^{\beta-1}\ln(y)\,dy + \int_{1/2}^1 (1-y)^{\beta-1}\ln(y)\,dy\Big).
\end{aligned}$$

We denote the first and the second terms of the above equation by I_1 and I_2, respectively. Here

$$I_1 = \int_0^{1/2}(1-y)^{\beta-1}\ln(y)\,dy < \int_0^{1/2}\ln(y)\,dy = \frac{1}{2}(\ln(1/2)-1) < \infty$$

and

$$\begin{aligned}
I_2 = \int_{1/2}^1 (1-y)^{\beta-1}\ln(y)\,dy &= \frac{1}{2^\beta}\ln(1/2) + \int_{1/2}^1 \frac{(1-y)^\beta}{y}\,dy \\
&< \frac{1}{2^\beta}\ln(1/2) + 2\int_{1/2}^1 (1-y)^\beta\,dy = \frac{1}{2^\beta}(\ln(1/2) - \frac{1}{\beta}) < \infty.
\end{aligned}$$

Therefore, $\int_0^\infty h_4(u)\,du < \infty$. Hence, it implies that the likelihood function $\ell_v(\alpha,\beta)$ is differentiable with respect to α and its derivative is given by

$$\begin{aligned}
\frac{\partial}{\partial\alpha}\ell_v(\alpha,\beta) &= n!\int_0^\infty \frac{\partial}{\partial\alpha}e^{h_v(\alpha,\beta;u)}\,du \\
&= n!\Big(\frac{\beta}{\alpha}\Big)^n \int_0^\infty \Big(-\frac{n}{\alpha} + \frac{1}{\alpha}\sum_{i=1}^n \frac{u+v_i}{\alpha} - \frac{(\beta-1)}{\alpha}\sum_{i=1}^n \frac{u+v_i}{\alpha}\Big(\frac{e^{-\frac{u+v_i}{\alpha}}}{1-e^{-\frac{u+v_i}{\alpha}}}\Big)\Big) \\
&\quad \times e^{-\frac{1}{\alpha}\sum_{i=1}^n (u+v_i)} \prod_{i=1}^n \big(1-e^{-\frac{u+v_i}{\alpha}}\big)^{\beta-1}. \qquad (15)
\end{aligned}$$

C Proof of the Theorem 2.2

Let us recall the likelihood function in (8) and rewrite the function $h_v(\alpha, \beta; u)$:

$$\begin{aligned}
\ell_v(\alpha, \beta) =& n! \int_0^\infty e^{h_v(\alpha,\beta;u)}\, du, \;\; \alpha > 0, \;\; \beta > 0, \\
\text{where } h_v(\alpha, \beta; u) =& n \ln(\beta) - n\ln(\alpha) - \frac{1}{\alpha}\sum_{i=1}^{n} c_i + (\beta - 1)\sum_{i=1}^{n} \ln(1 - e^{-c_i/\alpha})
\end{aligned}$$

with $c_i = u + v_i$. Since the likelihood is differentiable with respect to α and β, therefore let us recall its derivatives, from equations (10) and (15), which are as follows.

$$\frac{\partial \ell_v(\alpha, \beta)}{\partial \alpha} = n! \int_0^\infty \frac{\partial h_v(\alpha, \beta; u)}{\partial \alpha} e^{h_v(\alpha,\beta;u)}\, du, \tag{16}$$

with $\frac{\partial h_v(\alpha,\beta;u)}{\partial \alpha} = -\frac{n}{\alpha} + \frac{1}{\alpha^2}\sum\limits_{i=1}^{n} c_i\Big(\frac{1-\beta e^{-c_i/\alpha}}{1-e^{-c_i/\alpha}}\Big) = -\frac{n}{\alpha} + \frac{1}{\alpha^2}\sum\limits_{i=1}^{n} c_i\Big(1 + \frac{1-\beta}{e^{c_i/\alpha}-1}\Big)$, and

$$\frac{\partial \ell_v(\alpha, \beta)}{\partial \beta} = n! \int_0^\infty \frac{\partial h_v(\alpha, \beta; u)}{\partial \beta} e^{h_v(\alpha,\beta;u)}\, du, \tag{17}$$

with $\frac{\partial h_v(\alpha,\beta;u)}{\partial \beta} = \frac{n}{\beta} + \sum\limits_{i=1}^{n} \ln(1 - e^{-c_i/\alpha})$.

First we show that $\ell_v(\alpha, \beta)$ is a unimodal function of β for a fixed α, and then its unimodality with respect to α when β is fixed. Therefore, let us first assume that α is fixed. Note that, for any choice of u and v_i's, $e^{h_v(\alpha,\beta;u)} > 0$ and also the integrals are on positive support, hence the change in the sign of $\frac{\partial \ell_v(\alpha,\beta)}{\partial \beta}$ in (17) directly depends on the change in the sign of $\frac{\partial h_v(\alpha,\beta;u)}{\partial \beta}$. $\frac{\partial h_v(\alpha,\beta;u)}{\partial \beta} \to \infty$ as $\beta \downarrow 0$ and $\frac{\partial h_v(\alpha,\beta;u)}{\partial \beta} < 0$ as $\beta \to \infty$. Moreover, $\frac{\partial^2 h_v(\alpha,\beta;u)}{\partial \beta^2} = -\frac{n}{\beta^2} < 0 \;\forall\; \beta$, *i.e.*, $\frac{\partial h_v(\alpha,\beta;u)}{\partial \beta}$ changes sign from positive to negative and the change in sign is only once. Hence, for a fixed α, $\frac{\partial \ell_v(\alpha,\beta)}{\partial \beta}$ changes sign in the similar way and $\frac{\partial \ell_v(\alpha,\beta)}{\partial \beta} = 0$ has a unique solution which maximizes the likelihood function $\ell_v(\alpha, \beta)$ with respect to β. Now, we move forward to show that $\ell_v(\alpha, \beta)$ is a unimodal function of α for a fixed β. With the same argument as given above, sign of $\frac{\partial \ell_v(\alpha,\beta)}{\partial \alpha}$ in (16) directly depends on the sign of $\frac{\partial \ell_v(\alpha,\beta)}{\partial \alpha}$ or else, we can say that sign of $\frac{\partial \ell_v(\alpha,\beta)}{\partial \alpha}$ is inversely proportional to the sign of $\alpha - \frac{1}{n}\sum\limits_{i=1}^{n} c_i\Big(1 + \frac{1-\beta}{e^{c_i/\alpha}-1}\Big)$.

Say

$$H_{1,n}(\alpha) = \frac{1}{n}\sum_{i=1}^{n} c_i\Big(1 + \frac{1-\beta}{e^{c_i/\alpha} - 1}\Big). \tag{18}$$

When $\beta > 1$, $H_{1,n}(\alpha)$ is a strictly decreasing function in α that decreases from $\frac{1}{n}\sum_{i=1}^{n} c_i$ to $-\infty$. $H_{1,n}(\alpha)$ is a constant function of α whenever $\beta = 1$ taking value $\frac{1}{n}\sum_{i=1}^{n} c_i$. It is easy to see that $H_{1,n}(\alpha)$ and α meet exactly once whenever $\beta \geq 1$. Therefore, $\alpha - H_{1,n}(\alpha)$ changes sign from negative to positive and change in sign is only once whenever $\beta \geq 1$. Again, when $\beta < 1$, $H_{1,n}(\alpha)$ is strictly increasing in α and it increases from $\frac{1}{n}\sum_{i=1}^{n} c_i$ to ∞. Now, let us consider the quantity

$$\frac{\partial H_{1,n}(\alpha)}{\partial \alpha} = \frac{1-\beta}{n\alpha^2}\sum_{i=1}^{n}\left(\frac{c_i^2 e^{-c_i/\alpha}}{(1-e^{-c_i/\alpha})^2}\right).$$

Since $\frac{s^2 e^{-s}}{(1-e^{-s})^2} < 1 \; \forall \; s > 0$ (see part (i) of Lemma 2 in Ghitany et al. (2013)), we have $\frac{\partial H_{1,n}(\alpha)}{\partial \alpha} < 1$. Hence, α and $H_{1,n}$ have to meet exactly once. This implies that $\alpha - H_{1,n}(\alpha)$ changes its sign only once from negative to positive for $\beta < 1$ also. Since the sign of $\frac{\partial \ell_v(\alpha,\beta)}{\partial \alpha}$ is inversely proportional to the sign of $\alpha - H_{1,n}(\alpha)$, it is clear that change in sign of $\frac{\partial \ell_v(\alpha,\beta)}{\partial \alpha}$ from positive to negative and it is only once. It implies that $\ell_v(\alpha,\beta)$ is unimodal with respect to α for a fixed β.

D Proof of Theorem 2.3

To show the consistency of the estimators of (α,β) obtained by maximizing the likelihood function $\ell_v(\alpha,\beta)$, it suffices to prove the following result:

For any fixed $\alpha \neq \alpha_0$ and $\beta \neq \beta_0$, where α_0 and β_0 are true values of the parameters α and β,

$$\lim_{n\to\infty} P\Big(\frac{\ell_v(\alpha,\beta;V_{(2)},\dots,V_{(n)})}{\ell_v(\alpha_0,\beta_0;V_{(2)},\dots,V_{(n)})} < 1\Big) = 1.$$

To proceed further, we recall the conditional joint PDF of the order statistics of $V_{(2)},\dots,V_{(n)}$ given $Z_{(1)} = u$ (See Appendix A). Here $Z_{(1)} = X_{(1)} - \gamma_0$ and γ_0 is true value of the parameter γ. The conditional joint PDF is as follows:

$$\begin{aligned}
f_{V_{(2)},\dots,V_{(n)}|Z_{(1)}=u}(v_2,\dots,v_n|u) &= \frac{f_{Z_{(1)},V_{(2)},\dots,V_{(n)}}(u,v_2,\dots,v_n)}{f_{Z_{(1)}}(u)} \\
&= \frac{n!g(u;\alpha_0,\beta_0)\Big\{\prod_{i=2}^{n} g(u+v_i;\alpha_0,\beta_0)\Big\}}{ng(u;\alpha_0,\beta_0)(1-G(u;\alpha_0,\beta_0))^{n-1}} \\
&= (n-1)!\prod_{i=2}^{n}\frac{g(u+v_i;\alpha_0,\beta_0)}{1-G(u;\alpha_0,\beta_0)}
\end{aligned}$$

$$= (n-1)! \prod_{i=2}^{n} f_{V_i|Z_{(1)}=u}(v_i|u),$$

where $f_{V_i|Z_{(1)}=u}(v_i|u) = \frac{g(u+v_i;\alpha_0,\beta_0)}{1-G(u;\alpha_0,\beta_0)}$. Here, $V_i's$ are *i.i.d.* conditional on $Z_{(1)} = u$, with a common conditional PDF $f_{V_i|Z_{(1)}=u}(.|u)$. Define

$$\ell_u(\alpha,\beta;V_{(2)},\ldots,V_{(n)}) = (n-1)! \prod_{i=2}^{n} \frac{g(u+V_i;\alpha,\beta)}{1-G(u;\alpha,\beta)}$$

and then, consider the following quantity for fixed $u > 0$, conditioned on $Z_{(1)} = u$. For every $\alpha \neq \alpha_0$ and $\beta \neq \beta_0$,

$$\frac{1}{n-1} \ln\left\{\frac{\ell_u(\alpha,\beta;V_{(2)},\ldots,V_{(n)})}{\ell_u(\alpha_0,\beta_0;V_{(2)},\ldots,V_{(n)})}\right\} = \frac{1}{n-1}\sum_{i=2}^{n} \ln\left\{\frac{g(u+V_i;\alpha,\beta)/(1-G(u;\alpha,\beta))}{g(u+V_i;\alpha_0,\beta_0)/(1-G(u;\alpha_0,\beta_0))}\right\}. \quad (19)$$

By the law of large numbers, right hand side of the equation 19 converges to

$$E\left(\ln\left\{\frac{g(u+V;\alpha,\beta)/(1-G(u;\alpha,\beta))}{g(u+V;\alpha_0,\beta_0)/(1-G(u;\alpha_0,\beta_0))}\right\}\middle| Z_{(1)} = u\right).$$

Here, V has a conditional PDF $f_{V|Z_{(1)}=u}(.|u)$ given $Z_{(1)} = u$. Now, by Jensen's inequality,

$$\begin{aligned}
&E\left(\ln\left\{\frac{g(u+V;\alpha,\beta)/(1-G(u;\alpha,\beta))}{g(u+V;\alpha_0,\beta_0)/(1-G(u;\alpha_0,\beta_0))}\right\}\middle| Z_{(1)} = u\right) \\
&\leq \ln\left\{E\left(\frac{g(u+V;\alpha,\beta)/(1-G(u;\alpha,\beta))}{g(u+V;\alpha_0,\beta_0)/(1-G(u;\alpha_0,\beta_0))}\middle| Z_{(1)} = u\right)\right\} \\
&= \ln\left\{\int_0^\infty \frac{g(u+v;\alpha,\beta)}{(1-G(u;\alpha,\beta))}\, dv\right\} = 0.
\end{aligned}$$

Therefore, it implies that

$$\begin{aligned}
&\lim_{n\to\infty} P\left(\frac{1}{n-1}\ln\left\{\frac{\ell_u(\alpha,\beta;V_{(2)},\ldots,V_{(n)})}{\ell_u(\alpha_0,\beta_0;V_{(2)},\ldots,V_{(n)})}\right\} < 0 \;\middle|\; Z_{(1)} = u\right) = 1 \\
\Longrightarrow\; &\lim_{n\to\infty} P\left(\frac{\ell_u(\alpha,\beta;V_{(2)},\ldots,V_{(n)})}{\ell_u(\alpha_0,\beta_0;V_{(2)},\ldots,V_{(n)})} < 1 \;\middle|\; Z_{(1)} = u\right) = 1 \\
\Longrightarrow\; &\lim_{n\to\infty} P\big(\ell(\alpha,\beta;V_{(2)},\ldots,V_{(n)}) < \ell(\alpha_0,\beta_0;V_{(2)},\ldots,V_{(n)})|Z_{(1)} = u\big) = 1. \quad (20)
\end{aligned}$$

Moreover

$$
\begin{aligned}
&P(\ell(\alpha,\beta;V_{(2)},\ldots,V_{(n)}) < \ell(\alpha_0,\beta_0;V_{(2)},\ldots,V_{(n)})) \\
&= \int_0^\infty P\big(\ell(\alpha,\beta;V_{(2)},\ldots,V_{(n)}) < \ell(\alpha_0,\beta_0;V_{(2)},\ldots,V_{(n)}) | Z_{(1)} = u\big) f_{Z_{(1)}}(u)\, du
\end{aligned}
$$

Therefore, using the Lebesgue's dominated convergence theorem and (20)

$$
\begin{aligned}
&\lim_{n\to\infty} P\big(\ell(\alpha,\beta;V_{(2)},\ldots,V_{(n)}) < \ell(\alpha_0,\beta_0;V_{(2)},\ldots,V_{(n)})\big) \\
&= \int_0^\infty \lim_{n\to\infty} P\big(\ell(\alpha,\beta;V_{(2)},\ldots,V_{(n)}) < \ell(\alpha_0,\beta_0;V_{(2)},\ldots,V_{(n)}) | Z_{(1)} = u\big) f_{Z_{(1)}}(u)\, du \\
&= \int_0^\infty \lim_{n\to\infty} f_{Z_{(1)}}(u)\, du = \lim_{n\to\infty} \int_0^\infty f_{Z_{(1)}}(u)\, du = 1. \qquad (21)
\end{aligned}
$$

Hence the result is proved.

E Tables

§ β_U is an upper bound for the estimates of shape parameter such that the estimates greater than β_U are rejected and the rejected proportion is p reported in the last column of the tables.

Table 4: Biases and RMSEs of the estimators while varying sample size n based on 10000 simulations.

β	β_U[§]	n	Method	Shape		Scale		Location		p
				Bias	RMSE	Bias	RMSE	Bias	RMSE	
0.50	2	20	LPF	0.1706	0.2750	-0.1296	0.3473	-0.0089	0.0171	0.0047
			LSPF	0.1468	0.2450	-0.1220	0.3310	-0.0078	0.0154	0.0036
			MMLE I	0.2642	0.3742	-0.1640	0.3538	-0.0080	0.0146	0.0076
			MMLE III	0.0709	0.2128	-0.0368	0.3601	0.0005	0.0103	0.0039
		50	LPF	0.0885	0.1335	-0.0879	0.2282	-0.0017	0.0026	0.0000
			LSPF	0.0885	0.1265	-0.0968	0.2204	-0.0016	0.0025	0.0000
			MMLE I	0.1029	0.1526	-0.0881	0.2272	-0.0015	0.0025	0.0000
			MMLE III	0.0217	0.0977	-0.0134	0.2294	0.0003	0.0017	0.0000
		100	LPF	0.0650	0.0898	-0.0752	0.1658	-0.0009	0.0010	0.0000
			LSPF	0.0701	0.0904	-0.0843	0.1631	-0.0009	0.0010	0.0000
			MMLE I	0.0646	0.0909	-0.0653	0.1605	-0.0009	0.0010	0.0000
			MMLE III	0.0086	0.0631	-0.0054	0.1623	0.0001	0.0004	0.0000
		200	LPF	0.0572	0.0705	-0.0695	0.1262	-0.0009	0.0009	0.0000
			MMLE I	0.0493	0.0643	-0.0579	0.1186	-0.0009	0.0009	0.0000
			MMLE III	0.0046	0.0427	-0.0035	0.1125	0.0000	0.0001	0.0000
0.75	5	20	LPF	0.2421	0.5214	-0.0730	0.3099	-0.0231	0.0547	0.0097
			LSPF	0.1916	0.4600	-0.0576	0.3076	-0.0174	0.0445	0.0094
			MMLE I	0.4540	0.7109	-0.1452	0.3201	-0.0171	0.0407	0.0023
			MMLE III	0.1202	0.4467	-0.0113	0.3354	0.0010	0.0413	0.0099
		50	LPF	0.0967	0.2004	-0.0495	0.1993	-0.0047	0.0118	0.0000
			LSPF	0.0857	0.1898	-0.0446	0.1958	-0.0042	0.0113	0.0000
			MMLE I	0.1587	0.2548	-0.0759	0.2056	-0.0036	0.0106	0.0000
			MMLE III	0.0253	0.1703	-0.0021	0.2080	0.0022	0.0096	0.0000
		100	LPF	0.0569	0.1189	-0.0347	0.1426	-0.0015	0.0041	0.0000
			LSPF	0.0586	0.1169	-0.0384	0.1392	-0.0015	0.0041	0.0000
			MMLE I	0.0813	0.1414	-0.0470	0.1454	-0.0012	0.0039	0.0000
			MMLE III	0.0076	0.1033	0.0032	0.1453	0.0011	0.0036	0.0000
		200	LPF	0.0420	0.0807	-0.0288	0.1016	-0.0008	0.0016	0.0000
			MMLE I	0.0466	0.0859	-0.0281	0.1010	-0.0007	0.0015	0.0000
			MMLE III	0.0024	0.0687	0.0014	0.1013	0.0005	0.0014	0.0000
1.00	8	20	LPF	0.3425	0.8685	-0.0510	0.2928	-0.0375	0.1042	0.0262
			LSPF	0.2975	0.8620	-0.0264	0.2980	-0.0246	0.0853	0.0252
			MLE	-0.5280	0.7605	0.2389	0.4714	0.0429	0.0747	0.0096
			MMLE I	0.6165	1.0248	-0.1239	0.2995	-0.0172	0.0687	0.0023
			MMLE III	0.2259	0.8532	-0.0030	0.3225	-0.0031	0.0911	0.0280
		50	LPF	0.1170	0.3318	-0.0275	0.1903	-0.0092	0.0316	0.0001
			LSPF	0.1001	0.3174	-0.0241	0.1849	-0.0072	0.0288	0.0002
			MLE	-0.3572	0.4509	0.1469	0.2946	0.0183	0.0287	0.0003
			MMLE I	0.2219	0.3751	-0.0721	0.1922	-0.0045	0.0250	0.0000
			MMLE III	0.0398	0.3037	0.0059	0.2020	0.0047	0.0281	0.0001
		100	LPF	0.0581	0.1722	-0.0191	0.1328	-0.0030	0.0126	0.0000
			LSPF	0.0534	0.1649	-0.0192	0.1276	-0.0023	0.013	0.0000
			MLE	-0.2239	0.2926	0.0813	0.1860	0.0093	0.0137	0.0000
			MMLE I	0.1119	0.2067	-0.0415	0.1362	-0.0015	0.0117	0.0000
			MMLE III	0.0104	0.1616	0.0060	0.1381	0.0033	0.0118	0.0000
		200	LPF	0.0360	0.1083	-0.0133	0.0928	-0.0011	0.0058	0.0000
			MLE	-0.1329	0.1827	0.0455	0.1181	0.0048	0.0070	0.0000
			MMLE I	0.0584	0.1252	-0.0237	0.0951	-0.0006	0.0055	0.0000
			MMLE III	-0.0006	0.1043	0.0036	0.0970	0.0020	0.0057	0.0000

Table 5: Biases and RMSEs of the estimators while varying sample size n based on 10000 simulations.

β	β_U	n	Method	Shape		Scale		Location		p
				Bias	RMSE	Bias	RMSE	Bias	RMSE	
1.5	12	20	LPF	0.5140	1.6165	0.0070	0.2802	-0.0533	0.2121	0.0752
			LSPF	0.4277	1.5597	0.0111	0.2937	-0.014	0.1566	0.0756
			MLE	-0.4644	0.7732	0.2503	0.4257	0.1141	0.1571	0.0700
			MMLE I	0.8335	1.6122	-0.0944	0.2706	0.0084	0.1295	0.0041
			MMLE III	0.3272	1.4925	0.0396	0.3148	0.0060	0.1893	0.1010
		50	LPF	0.2435	0.8945	-0.0101	0.1804	-0.0245	0.1024	0.0040
			LSPF	0.1952	0.8489	-0.0014	0.1851	-0.0116	0.0816	0.0036
			MLE	-0.1575	0.5464	0.0836	0.2255	0.0460	0.0820	0.0066
			MMLE I	0.2948	0.6252	-0.0507	0.1741	0.0081	0.0637	0.0000
			MMLE III	0.1359	0.8503	0.0107	0.1940	0.0045	0.0949	0.0047
		100	LPF	0.0865	0.3755	-0.0065	0.1275	-0.0081	0.0460	0.0000
			LSPF	0.0630	0.3743	0.0004	0.1274	-0.0041	0.0434	0.0000
			MLE	-0.1108	0.3225	0.0464	0.1486	0.0274	0.0469	0.0010
			MMLE I	0.1459	0.3493	-0.0304	0.1251	0.0055	0.0379	0.0000
			MMLE III	0.0196	0.3634	0.0086	0.1343	0.0081	0.0445	0.0000
		200	LPF	0.0318	0.2081	-0.0026	0.0880	-0.0025	0.0244	0.0000
			MLE	-0.0655	0.2048	0.0237	0.0963	0.0156	0.0269	0.0000
			MMLE I	0.0739	0.2185	-0.0172	0.0887	0.0035	0.0230	0.0000
			MMLE III	0.0026	0.2116	0.0046	0.0914	0.0059	0.0245	0.0000
2.0	20	20	LPF	0.6811	2.6117	0.0321	0.2710	-0.0507	0.3066	0.1209
			LSPF	0.5276	2.4641	0.0361	0.3009	0.0266	0.2175	0.1310
			MLE	-0.4570	1.8233	0.2683	0.4606	0.1577	0.2692	0.0748
			MMLE I	0.9708	2.2697	-0.0716	0.2640	0.0594	0.1915	0.0035
			MMLE III	0.2357	1.8799	0.0769	0.3164	0.0462	0.2685	0.1851
		50	LPF	0.3885	1.6485	0.0063	0.1782	-0.0338	0.1786	0.0175
			LSPF	0.3895	1.7103	0.0035	0.1884	-0.0086	0.1380	0.0198
			MLE	-0.1315	1.1409	0.0810	0.2287	0.0648	0.1552	0.0168
			MMLE I	0.2545	0.8535	-0.0296	0.1673	0.0411	0.1109	0.0000
			MMLE III	0.3126	1.6384	0.0172	0.1930	0.0018	0.1778	0.0238
		100	LPF	0.1481	0.8780	0.0039	0.1255	-0.0133	0.1010	0.0009
			LSPF	0.1358	0.8422	0.0040	0.1302	-0.0029	0.0867	0.0010
			MLE	-0.1168	0.5871	0.0386	0.1445	0.0405	0.0917	0.0050
			MMLE I	0.0990	0.4971	-0.0167	0.1188	0.0288	0.0739	0.0000
			MMLE III	0.1020	0.8177	0.0078	0.1319	0.0066	0.1012	0.0009
		200	LPF	0.0470	0.3897	0.0022	0.0871	-0.0049	0.0546	0.0000
			MLE	-0.0842	0.3770	0.0203	0.0952	0.0244	0.0570	0.0000
			MMLE I	0.0488	0.3317	-0.0088	0.0853	0.0163	0.0478	0.0000
			MMLE III	0.0210	0.3996	0.0045	0.0909	0.0061	0.0545	0.0000
3.0	30	20	LPF	0.6359	3.9790	0.0796	0.2779	0.0181	0.4438	0.2037
			LSPF	0.4588	3.8851	0.0617	0.3138	0.1546	0.3366	0.2088
			MLE	-0.7827	3.2839	0.3096	0.5051	0.2866	0.4529	0.1408
			MMLE I	0.8472	3.2347	-0.0447	0.2494	0.1879	0.3217	0.0031
			MMLE III	-0.3581	2.3205	0.1287	0.3328	0.1860	0.4048	0.3161
		50	LPF	0.7287	3.5317	0.0319	0.1746	-0.0307	0.3304	0.0715
			LSPF	0.8014	3.6369	0.0064	0.2014	0.0416	0.2287	0.0744
			MLE	-0.0205	2.7964	0.0873	0.2295	0.1066	0.3061	0.0554
			MMLE I	0.0295	1.3769	-0.0083	0.1606	0.1347	0.2161	0.0000
			MMLE III	0.5161	3.0517	0.0382	0.1866	0.0225	0.3155	0.1033
		100	LPF	0.4671	2.3920	0.0128	0.1241	-0.0296	0.2317	0.0151
			LSPF	0.5027	2.4756	-0.0048	0.1428	0.0150	0.1708	0.0174
			MLE	0.0504	1.8775	0.0350	0.1393	0.0475	0.2143	0.0156
			MMLE I	-0.1470	0.8864	0.0042	0.1167	0.1014	0.1600	0.0000
			MMLE III	0.5147	2.4750	0.0082	0.1266	-0.0131	0.2404	0.0191
		200	LPF	0.1794	1.3109	0.0075	0.0879	-0.0128	0.1468	0.0008
			MLE	0.3662	2.0130	0.0114	0.1008	-0.0062	0.2152	0.0011
			MMLE I	-0.1878	0.6348	0.0096	0.0830	0.0732	0.1170	0.0000
			MMLE III	0.2055	1.4385	0.0040	0.0890	-0.0031	0.1529	0.0013

Table 6: Biases of the quantile estimators while varying sample size n based on 10000 simulations.

β	n	Method	ζ								
			0.01	0.05	0.10	0.25	0.50	0.75	0.90	0.95	0.99
0.50	20	LPF	-0.0070	0.0005	0.0098	0.0332	0.0501	0.0195	-0.0675	-0.1473	-0.3479
		LSPF	-0.0063	0.0001	0.0082	0.0292	0.0442	0.0153	-0.0666	-0.1417	-0.3306
		MMLE I	-0.0044	0.0081	0.0229	0.0602	0.0921	0.0627	-0.0424	-0.1418	-0.3946
		MMLE III	0.0015	0.0052	0.0094	0.0169	0.0147	-0.0060	-0.0387	-0.0641	-0.1233
	50	LPF	-0.0012	0.0021	0.0071	0.0218	0.0348	0.0170	-0.0401	-0.0935	-0.2292
		LSPF	-0.0012	0.0020	0.0070	0.0218	0.0345	0.0138	-0.0495	-0.1086	-0.2581
		MMLE I	-0.0008	0.0032	0.0092	0.0272	0.0452	0.0318	-0.0229	-0.0758	-0.2111
		MMLE III	0.0005	0.0016	0.0031	0.0057	0.0046	-0.0034	-0.0156	-0.0250	-0.0467
	100	LPF	-0.0007	0.0014	0.0051	0.0166	0.0274	0.0127	-0.0356	-0.0812	-0.1971
		LSPF	-0.0007	0.0016	0.0055	0.0180	0.0296	0.0128	-0.0415	-0.0927	-0.2227
		MMLE I	-0.0007	0.0015	0.0053	0.0175	0.0307	0.0214	-0.0186	-0.0576	-0.1577
		MMLE III	0.0002	0.0006	0.0012	0.0022	0.0010	-0.0033	-0.0088	-0.0128	-0.0216
	200	LPF	-0.0007	0.0010	0.0042	0.0148	0.0254	0.0130	-0.0310	-0.0729	-0.1799
		MMLE I	-0.0008	0.0007	0.0035	0.0128	0.0228	0.0133	-0.0228	-0.0575	-0.1465
		MMLE III	0.0001	0.0003	0.0006	0.0012	0.0007	-0.0015	-0.0048	-0.0073	-0.0129
0.75	20	LPF	-0.0124	0.0037	0.0152	0.0323	0.0343	0.0055	-0.0507	-0.0981	-0.2131
		LSPF	-0.009	0.0038	0.0129	0.0257	0.0257	0.0014	-0.0439	-0.0816	-0.1726
		MMLE I	0.0027	0.0321	0.0541	0.0898	0.1004	0.0509	-0.0565	-0.1492	-0.3768
		MMLE III	0.0079	0.0158	0.0194	0.0188	0.0056	-0.0140	-0.0316	-0.0419	-0.0619
	50	LPF	-0.0018	0.0053	0.0112	0.0207	0.0217	0.0030	-0.0346	-0.0664	-0.1442
		LSPF	-0.0016	0.0047	0.0098	0.0181	0.0186	0.0014	-0.0326	-0.0613	-0.1314
		MMLE I	0.0012	0.0129	0.0229	0.0407	0.0471	0.0225	-0.0328	-0.0810	-0.1998
		MMLE III	0.0036	0.0057	0.0064	0.0049	-0.0011	-0.0083	-0.0137	-0.0163	-0.0206
	100	LPF	-0.0001	0.0040	0.0078	0.0143	0.0155	0.0031	-0.0229	-0.0451	-0.0995
		LSPF	-0.0001	0.0041	0.0080	0.0147	0.0156	0.0017	-0.0271	-0.0517	-0.112
		MMLE I	0.0008	0.0068	0.0123	0.0224	0.0260	0.0108	-0.0234	-0.0532	-0.1267
		MMLE III	0.0016	0.0024	0.0026	0.0016	-0.0009	-0.0024	-0.0017	-0.0001	0.0045
	200	LPF	0.0001	0.0031	0.0061	0.0115	0.0130	0.0033	-0.0178	-0.0361	-0.0812
		MMLE I	0.0003	0.0038	0.0071	0.0137	0.0168	0.0085	-0.0115	-0.0292	-0.0731
		MMLE III	0.0007	0.0010	0.0009	0.0002	-0.0014	-0.0026	-0.0024	-0.0018	0.0001
1.00	20	LPF	-0.0124	0.0061	0.0151	0.0227	0.0143	-0.0146	-0.0587	-0.0933	-0.1749
		LSPF	-0.0026	0.0121	0.0185	0.0226	0.0154	-0.0029	-0.0277	-0.0462	-0.0889
		MLE	0.0390	0.0051	-0.0414	-0.1891	-0.4239	-0.5816	-0.5472	-0.4413	-0.1038
		MMLE I	0.0236	0.0598	0.0807	0.1067	0.1035	0.0494	-0.0487	-0.1299	-0.3257
		MMLE III	0.0159	0.0244	0.0250	0.0164	-0.0033	-0.0231	-0.0357	-0.0409	-0.0482
	50	LPF	-0.0016	0.0066	0.0111	0.0157	0.0128	-0.0012	-0.0241	-0.0425	-0.0863
		LSPF	-0.0005	0.0064	0.0098	0.0126	0.0085	-0.0049	-0.0256	-0.0419	-0.0804
		MLE	0.0118	-0.0174	-0.0520	-0.1409	-0.2437	-0.2777	-0.2152	-0.1361	0.0826
		MMLE I	0.0086	0.0252	0.0356	0.0488	0.0461	0.0145	-0.0425	-0.0898	-0.2037
		MMLE III	0.0087	0.0103	0.0095	0.0047	-0.0025	-0.0068	-0.0060	-0.0033	0.0050
	100	LPF	0.0004	0.0049	0.0076	0.0105	0.0088	-0.0005	-0.0161	-0.0287	-0.0589
		LSPF	0.0008	0.0049	0.0073	0.0097	0.0076	-0.0019	-0.0177	-0.0304	-0.0609
		MLE	0.0036	-0.0171	-0.0389	-0.0884	-0.1375	-0.1460	-0.1050	-0.0592	0.0635
		MMLE I	0.0045	0.0136	0.0196	0.0276	0.0269	0.0095	-0.0230	-0.0500	-0.1154
		MMLE III	0.0046	0.0050	0.0044	0.0022	-0.0005	-0.0007	0.0025	0.0059	0.0150
	200	LPF	0.0008	0.0037	0.0057	0.0082	0.0078	0.0021	-0.0084	-0.0170	-0.0380
		MLE	0.0006	-0.0124	-0.0248	-0.0509	-0.0741	-0.0744	-0.0490	-0.0226	0.0467
		MMLE I	0.0024	0.0073	0.0106	0.0152	0.0150	0.0052	-0.0132	-0.0286	-0.0659
		MMLE III	0.0023	0.0019	0.0012	-0.0008	-0.0029	-0.0033	-0.0015	0.0005	0.0060

Table 7: Biases of the quantile estimators while varying sample size n based on 10000 simulations.

β	n	Method	ζ								
			0.01	0.05	0.10	0.25	0.50	0.75	0.90	0.95	0.99
		LPF	-0.0030	0.0068	0.0077	0.0033	-0.0050	-0.0098	-0.0086	-0.0054	0.0046
	20	LSPF	0.0235	0.0251	0.0208	0.0077	-0.0082	-0.0165	-0.0147	-0.0096	0.0063
		MLE	0.0873	0.0333	-0.0083	-0.0783	-0.1076	-0.0311	0.1494	0.3078	0.6989
		MMLE I	0.0788	0.1057	0.1156	0.1189	0.0961	0.0395	-0.0431	-0.1074	-0.2584
		MMLE III	0.0399	0.0354	0.0263	0.0045	-0.0164	-0.0177	0.0037	0.0265	0.0866
		LPF	-0.0007	0.0060	0.0076	0.0065	0.0009	-0.0080	-0.0184	-0.0258	-0.0423
1.5	50	LSPF	0.0070	0.0105	0.0103	0.0071	0.0015	-0.0038	-0.0073	-0.0090	-0.0118
		MLE	0.0366	0.0145	-0.0016	-0.0271	-0.0365	-0.0096	0.0516	0.1048	0.2358
		MMLE I	0.0351	0.0483	0.0534	0.0552	0.0434	0.0134	-0.0307	-0.0652	-0.1462
		MMLE III	0.0189	0.0167	0.0127	0.0030	-0.0066	-0.0098	-0.0054	0.0004	0.0164
		LPF	0.0008	0.0038	0.0045	0.0038	0.0007	-0.0045	-0.0108	-0.0154	-0.0259
	100	LSPF	0.0026	0.0039	0.0036	0.0019	-0.0005	-0.0024	-0.0031	-0.0032	-0.0028
		MLE	0.0196	0.0066	-0.0022	-0.0151	-0.0182	-0.0014	0.0336	0.0635	0.1364
		MMLE I	0.0189	0.0264	0.0293	0.0305	0.0237	0.0059	-0.0204	-0.0410	-0.0896
		MMLE III	0.0111	0.0084	0.0055	0.0000	-0.0045	-0.0044	0.0006	0.0056	0.0187
		LPF	0.0008	0.0018	0.0019	0.0013	-0.0004	-0.0027	-0.0055	-0.0074	-0.0117
	200	MLE	0.0106	0.0034	-0.0012	-0.0076	-0.0088	0.0001	0.0182	0.0335	0.0708
		MMLE I	0.0102	0.0142	0.0157	0.0163	0.0123	0.0022	-0.0128	-0.0244	-0.0520
		MMLE III	0.0067	0.0051	0.0036	0.0010	-0.0010	-0.0007	0.0021	0.0049	0.0119
		LPF	0.0081	0.0030	-0.0042	-0.0185	-0.0286	-0.0233	-0.0024	0.0172	0.0668
	20	LSPF	0.0583	0.0404	0.0257	-0.0010	-0.0218	-0.0221	-0.0015	0.0197	0.0748
		MLE	0.1168	0.0418	-0.0099	-0.0906	-0.1206	-0.0347	0.1614	0.3320	0.7519
		MMLE I	0.1345	0.1422	0.1401	0.1253	0.0909	0.0364	-0.0319	-0.0824	-0.1984
		MMLE III	0.0685	0.0440	0.0246	-0.0086	-0.0288	-0.0127	0.0393	0.0869	0.2062
		LPF	0.0017	0.0021	0.0001	-0.0052	-0.0104	-0.0118	-0.0089	-0.0054	0.0041
2.0	50	LSPF	0.0215	0.0186	0.0144	0.0054	-0.0040	-0.0094	-0.0101	-0.0088	-0.0041
		MLE	0.0501	0.0199	0.0007	-0.0272	-0.0366	-0.0097	0.0504	0.1022	0.2292
		MMLE I	0.0637	0.0636	0.0607	0.0512	0.0338	0.0091	-0.0201	-0.0413	-0.0895
		MMLE III	0.0285	0.0212	0.0144	0.0018	-0.0080	-0.0080	0.0018	0.0119	0.0382
		LPF	0.0013	0.0010	-0.0003	-0.0031	-0.0056	-0.0059	-0.0038	-0.0016	0.0043
	100	LSPF	0.0096	0.0080	0.0060	0.0019	-0.0018	-0.0028	-0.0010	0.0012	0.0071
		MLE	0.0277	0.0113	0.0015	-0.0118	-0.0154	-0.0016	0.0275	0.0523	0.1129
		MMLE I	0.0374	0.0357	0.0331	0.0263	0.0153	6e-04	-0.0162	-0.0282	-0.0554
		MMLE III	0.0163	0.0119	0.0082	0.0016	-0.0036	-0.0040	0.0002	0.0048	0.0166
		LPF	0.0002	-0.0002	-0.0010	-0.0024	-0.0036	-0.0035	-0.0022	-0.0008	0.0026
	200	MLE	0.0149	0.0052	-0.0003	-0.0075	-0.0095	-0.0021	0.0133	0.0264	0.0583
		MMLE I	0.0206	0.0196	0.0183	0.0147	0.0089	0.0012	-0.0077	-0.0140	-0.0283
		MMLE III	0.0081	0.0053	0.0033	-0.0001	-0.0025	-0.0022	0.0006	0.0033	0.0102
		LPF	0.0392	0.005	-0.0149	-0.0416	-0.0486	-0.0199	0.0403	0.0917	0.2168
	20	LSPF	0.1266	0.0730	0.0414	-0.0063	-0.0368	-0.0311	0.0076	0.0449	0.1401
		MLE	0.1699	0.0577	-0.0095	-0.1049	-0.1326	-0.0251	0.2060	0.4044	0.8903
		MMLE I	0.2137	0.1864	0.1670	0.1293	0.0829	0.0328	-0.0173	-0.0511	-0.1252
		MMLE III	0.1247	0.0610	0.0243	-0.0272	-0.0474	-0.0075	0.0865	0.1684	0.3699
		LPF	0.0085	-0.0059	-0.0144	-0.0256	-0.0285	-0.0169	0.0074	0.0280	0.0783
3.0	50	LSPF	0.0640	0.0423	0.0287	0.0065	-0.0126	-0.0225	-0.0235	-0.0211	-0.0124
		MLE	0.0712	0.0270	0.0031	-0.0277	-0.0350	-0.0026	0.0641	0.1205	0.2580
		MMLE I	0.1156	0.0914	0.0769	0.0525	0.0285	0.0093	-0.0048	-0.0125	-0.0273
		MMLE III	0.0441	0.0222	0.0093	-0.0086	-0.0161	-0.0049	0.0228	0.0470	0.1068
		LPF	0.0013	-0.0044	-0.0081	-0.0132	-0.0150	-0.0108	-0.0013	0.0068	0.0268
	100	LSPF	0.0359	0.0248	0.0175	0.0048	-0.0083	-0.0189	-0.0268	-0.0312	-0.0396
		MLE	0.0349	0.0143	0.0035	-0.0100	-0.0134	-0.0004	0.0264	0.0490	0.1041
		MMLE I	0.0721	0.0519	0.0408	0.0238	0.0099	0.0028	0.0020	0.0035	0.0092
		MMLE III	0.0201	0.0126	0.0076	-0.0004	-0.0060	-0.0062	-0.0015	0.0033	0.0159
		LPF	-0.0001	-0.0036	-0.0057	-0.0085	-0.0093	-0.0067	-0.0010	0.0038	0.0155
	200	MLE	0.0149	0.0073	0.0030	-0.0031	-0.0057	-0.0026	0.0056	0.0128	0.0305
		MMLE I	0.0441	0.0286	0.0206	0.0094	0.0021	0.0018	0.0073	0.0129	0.0276
		MMLE III	0.0106	0.0065	0.0040	0.0000	-0.0028	-0.0029	-0.0006	0.0017	0.0079

Table 8: RMSEs of the quantile estimators while varying sample size n based on 10000 simulations.

β	n	Method	ζ								
			0.01	0.05	0.10	0.25	0.50	0.75	0.90	0.95	0.99
0.50	20	LPF	0.0134	0.0096	0.0207	0.0609	0.1284	0.2614	0.5069	0.7214	1.2550
		LSPF	0.0126	0.0096	0.0192	0.0566	0.1243	0.2596	0.4993	0.7054	1.2152
		MMLE I	0.0101	0.0171	0.0362	0.0889	0.1635	0.2822	0.5117	0.7217	1.2559
		MMLE III	0.0089	0.0121	0.0217	0.0543	0.1185	0.2540	0.5049	0.7255	1.2771
	50	LPF	0.0020	0.0045	0.0122	0.0367	0.0794	0.1638	0.3232	0.4635	0.8136
		LSPF	0.0020	0.0042	0.0115	0.0356	0.0789	0.1638	0.3194	0.4552	0.7936
		MMLE I	0.0018	0.0062	0.0152	0.0430	0.0886	0.1697	0.3232	0.4606	0.8065
		MMLE III	0.0017	0.0043	0.0097	0.0295	0.0722	0.1629	0.3263	0.4681	0.8209
	100	LPF	0.0008	0.0029	0.0080	0.0255	0.0561	0.1152	0.2288	0.3300	0.5837
		LSPF	0.0008	0.0029	0.0082	0.0262	0.0577	0.1159	0.2267	0.3258	0.5748
		MMLE I	0.0008	0.0032	0.0086	0.0272	0.0602	0.1191	0.2281	0.3253	0.5697
		MMLE III	0.0005	0.0022	0.0057	0.0194	0.0498	0.1142	0.2300	0.3304	0.5800
	200	LPF	0.0008	0.0019	0.0059	0.0198	0.0425	0.0819	0.1637	0.2391	0.4309
		MMLE I	0.0008	0.0018	0.0054	0.0187	0.0418	0.0822	0.1607	0.2320	0.4123
		MMLE III	0.0002	0.0013	0.0037	0.0134	0.0352	0.0806	0.1613	0.2311	0.4042
0.75	20	LPF	0.0347	0.0290	0.0440	0.0881	0.1556	0.2904	0.5222	0.7176	1.1965
		LSPF	0.0307	0.0300	0.0441	0.0852	0.1539	0.2939	0.5283	0.7239	1.2006
		MMLE I	0.0285	0.0544	0.0825	0.1369	0.1971	0.3060	0.5227	0.7164	1.2023
		MMLE III	0.0301	0.0350	0.0495	0.0895	0.1546	0.2868	0.5265	0.7340	1.2485
	50	LPF	0.0085	0.0147	0.0268	0.0551	0.0979	0.1844	0.3338	0.4596	0.7678
		LSPF	0.0083	0.0143	0.0258	0.0534	0.0973	0.1858	0.3347	0.4591	0.7625
		MMLE I	0.0085	0.0220	0.0374	0.0699	0.1108	0.1887	0.3345	0.4614	0.7763
		MMLE III	0.0092	0.0161	0.0264	0.0529	0.0966	0.1852	0.3391	0.4697	0.7908
	100	LPF	0.0035	0.0098	0.0181	0.0374	0.0674	0.1297	0.2373	0.3277	0.5486
		LSPF	0.0035	0.0098	0.0180	0.0374	0.0683	0.1308	0.2366	0.3250	0.5407
		MMLE I	0.0038	0.0126	0.0224	0.0441	0.0737	0.1317	0.2371	0.3277	0.5513
		MMLE III	0.0040	0.0095	0.0169	0.0359	0.0671	0.1302	0.2388	0.3305	0.5551
	200	LPF	0.0017	0.0069	0.0129	0.0267	0.0479	0.0917	0.1679	0.2322	0.3895
		MMLE I	0.0018	0.0076	0.0141	0.0289	0.0504	0.0926	0.1669	0.2301	0.3857
		MMLE III	0.0019	0.0061	0.0114	0.0248	0.0470	0.0918	0.1679	0.2320	0.3888
1.00	20	LPF	0.0595	0.0514	0.0646	0.1056	0.1725	0.3072	0.5310	0.7171	1.1701
		LSPF	0.0551	0.0589	0.0744	0.1139	0.1784	0.3129	0.5396	0.7286	1.1893
		MLE	0.0638	0.0566	0.0812	0.2203	0.4586	0.6350	0.7197	0.8220	1.3018
		MMLE I	0.0585	0.0923	0.1193	0.1651	0.2165	0.3261	0.5375	0.7216	1.1781
		MMLE III	0.0569	0.0588	0.0737	0.1151	0.1788	0.3065	0.5376	0.7370	1.2308
	50	LPF	0.0202	0.0272	0.0404	0.0685	0.1102	0.1969	0.3430	0.4644	0.7594
		LSPF	0.0202	0.0284	0.0415	0.0695	0.1116	0.1967	0.3385	0.4562	0.7425
		MLE	0.0237	0.0360	0.0704	0.1653	0.2769	0.3405	0.4112	0.5172	0.8867
		MMLE I	0.0227	0.0417	0.0583	0.0879	0.1235	0.1993	0.3394	0.4594	0.7546
		MMLE III	0.0221	0.0305	0.0430	0.0710	0.1117	0.1970	0.3474	0.4746	0.7863
	100	LPF	0.0101	0.0185	0.0283	0.0482	0.0778	0.1390	0.2413	0.3261	0.5321
		LSPF	0.0107	0.0184	0.0278	0.0477	0.0782	0.1391	0.2382	0.3198	0.5177
		MLE	0.0117	0.0288	0.0535	0.1103	0.1688	0.2074	0.2738	0.3564	0.6066
		MMLE I	0.0118	0.0248	0.0362	0.0573	0.0842	0.1406	0.2415	0.3271	0.5369
		MMLE III	0.0113	0.0196	0.0290	0.0492	0.0786	0.1392	0.2434	0.3308	0.5443
	200	LPF	0.0055	0.0127	0.0197	0.0337	0.0545	0.0977	0.1695	0.2289	0.3729
		MLE	0.0065	0.0203	0.0356	0.0678	0.0996	0.1273	0.1828	0.2426	0.4092
		MMLE I	0.0063	0.0154	0.0232	0.0378	0.0571	0.0976	0.1689	0.2289	0.3757
		MMLE III	0.0062	0.0130	0.0198	0.0339	0.0547	0.0981	0.1720	0.2337	0.3841

Table 9: RMSEs of the quantile estimators while varying sample size n based on 10000 simulations.

β	n	Method	ζ								
			0.01	0.05	0.10	0.25	0.50	0.75	0.90	0.95	0.99
		LPF	0.1122	0.0944	0.1000	0.1327	0.1979	0.3294	0.5437	0.7210	1.1528
	20	LSPF	0.1055	0.1052	0.1146	0.1438	0.2004	0.3301	0.5523	0.7378	1.1904
		MLE	0.1284	0.1014	0.1050	0.1531	0.2118	0.3227	0.5982	0.8551	1.5038
		MMLE I	0.1260	0.1527	0.1692	0.195	0.2337	0.3402	0.5386	0.7073	1.1213
		MMLE III	0.1132	0.0999	0.1070	0.1432	0.2060	0.3293	0.5518	0.7447	1.2240
		LPF	0.0508	0.0521	0.0637	0.0893	0.1269	0.2075	0.3447	0.4591	0.7374
1.5	50	LSPF	0.0507	0.0581	0.0697	0.0930	0.1278	0.2083	0.3489	0.4663	0.7520
		MLE	0.0605	0.0571	0.0686	0.0992	0.1329	0.2061	0.3618	0.5003	0.8451
		MMLE I	0.0601	0.0766	0.0885	0.1081	0.1365	0.2089	0.3396	0.4494	0.7172
		MMLE III	0.0529	0.0554	0.0673	0.0941	0.1295	0.2066	0.3481	0.4691	0.7668
		LPF	0.0288	0.0352	0.0447	0.0630	0.0898	0.1482	0.2466	0.3279	0.5252
	100	LSPF	0.0296	0.0368	0.0461	0.0638	0.0894	0.1467	0.2444	0.3255	0.5225
		MLE	0.0348	0.0376	0.0475	0.0684	0.0932	0.1488	0.2561	0.3487	0.5769
		MMLE I	0.0352	0.0467	0.0556	0.0708	0.0937	0.1491	0.2452	0.3249	0.5184
		MMLE III	0.0309	0.0374	0.0472	0.0660	0.0915	0.1484	0.2491	0.3339	0.5409
		LPF	0.0179	0.0243	0.0312	0.0438	0.0626	0.1036	0.1719	0.2282	0.3645
	200	MLE	0.0207	0.0257	0.0330	0.0469	0.0649	0.1048	0.1767	0.2374	0.3858
		MMLE I	0.0211	0.0296	0.0364	0.0477	0.064	0.1032	0.1714	0.2279	0.3652
		MMLE III	0.0188	0.0256	0.0328	0.0459	0.0639	0.1041	0.1737	0.2317	0.3729
		LPF	0.1511	0.1243	0.1243	0.1504	0.2125	0.3394	0.5453	0.7159	1.1317
	20	LSPF	0.1524	0.1423	0.1452	0.1652	0.2158	0.3449	0.5712	0.7608	1.2238
		MLE	0.1762	0.1314	0.1359	0.1917	0.2451	0.3377	0.6211	0.8959	1.5959
		MMLE I	0.1868	0.1976	0.2033	0.2126	0.2422	0.3490	0.5470	0.7133	1.1187
		MMLE III	0.1610	0.1318	0.1316	0.1625	0.2223	0.3391	0.5563	0.7475	1.2263
		LPF	0.0788	0.0717	0.0791	0.1006	0.1365	0.2163	0.3519	0.4647	0.7393
2.0	50	LSPF	0.0816	0.0852	0.0932	0.1097	0.1373	0.2149	0.3569	0.4761	0.7666
		MLE	0.0888	0.0745	0.0842	0.1144	0.1452	0.2127	0.3663	0.5053	0.8535
		MMLE I	0.0956	0.0997	0.1046	0.1157	0.1425	0.2177	0.3470	0.4537	0.7120
		MMLE III	0.0829	0.0744	0.0822	0.1048	0.1378	0.2138	0.3546	0.4749	0.7707
		LPF	0.0492	0.0501	0.0577	0.0737	0.0977	0.1529	0.2482	0.3277	0.5211
	100	LSPF	0.0503	0.0552	0.0629	0.0771	0.0983	0.1532	0.2517	0.3341	0.5348
		MLE	0.0544	0.0509	0.0587	0.0769	0.0991	0.1527	0.2572	0.3472	0.5689
		MMLE I	0.0599	0.0631	0.0677	0.0779	0.0988	0.1528	0.2447	0.3204	0.5038
		MMLE III	0.0509	0.0507	0.0587	0.0754	0.0976	0.1503	0.2475	0.3301	0.5326
		LPF	0.0321	0.0355	0.0414	0.0524	0.0689	0.1076	0.1741	0.2294	0.3638
	200	MLE	0.0335	0.0346	0.0410	0.0534	0.0692	0.1074	0.1779	0.2377	0.3842
		MMLE I	0.0369	0.0407	0.0450	0.0536	0.069	0.1081	0.1744	0.2289	0.3608
		MMLE III	0.0311	0.0347	0.0410	0.0526	0.0687	0.1076	0.1765	0.2340	0.3741
		LPF	0.2147	0.1710	0.1625	0.1772	0.2294	0.3479	0.5517	0.7238	1.1473
	20	LSPF	0.2298	0.1940	0.1845	0.1899	0.2317	0.3590	0.5915	0.7881	1.2697
		MLE	0.2525	0.1743	0.1704	0.2255	0.2704	0.3452	0.6464	0.9475	1.7161
		MMLE I	0.2731	0.2520	0.2420	0.2326	0.2552	0.3600	0.5496	0.7072	1.0899
		MMLE III	0.2350	0.1757	0.1627	0.1839	0.2396	0.3516	0.5726	0.7712	1.2724
		LPF	0.1230	0.1004	0.1013	0.1170	0.1503	0.2255	0.3553	0.4642	0.7310
3.0	50	LSPF	0.1340	0.1240	0.1233	0.1263	0.1460	0.2293	0.3841	0.5130	0.8249
		MLE	0.1335	0.0997	0.1026	0.1273	0.1550	0.2204	0.3733	0.5125	0.8614
		MMLE I	0.1543	0.1364	0.1307	0.1297	0.1510	0.2229	0.3470	0.4492	0.6965
		MMLE III	0.1268	0.1001	0.1006	0.1181	0.1496	0.2210	0.3538	0.4685	0.7522
		LPF	0.0824	0.0726	0.0762	0.0883	0.1096	0.1614	0.2533	0.3308	0.5205
	100	LSPF	0.0894	0.0875	0.0893	0.0922	0.1038	0.1617	0.2720	0.3637	0.5852
		MLE	0.0829	0.0683	0.0728	0.0880	0.1079	0.1574	0.2562	0.3421	0.5546
		MMLE I	0.1004	0.0863	0.0834	0.0860	0.1042	0.1574	0.2477	0.3220	0.5017
		MMLE III	0.0792	0.0667	0.0707	0.0839	0.1044	0.1544	0.2467	0.3254	0.5188
		LPF	0.0570	0.0547	0.0589	0.0680	0.0814	0.1155	0.1790	0.2333	0.3671
	200	MLE	0.0561	0.0466	0.0513	0.0619	0.0747	0.1120	0.1859	0.2492	0.4043
		MMLE I	0.0650	0.0549	0.0543	0.0590	0.0735	0.1109	0.1745	0.2271	0.3547
		MMLE III	0.0523	0.0466	0.0504	0.0598	0.0745	0.1114	0.1778	0.2337	0.3703