STABLE NON-HAUSDORFF RIEMANNIAN FOLIATIONS

STEPHANE GEUDENS

Abstract. This paper aims to construct foliations $\mathcal{F}$ that are stable, meaning that any other foliation close to $\mathcal{F}$ is conjugate to $\mathcal{F}$ by a diffeomorphism. Our main result yields examples of stable Riemannian foliations $\mathcal{F}$ with dense leaves. To our knowledge, these are the first examples of stable Riemannian foliations that are not Hausdorff.

Contents

1. Introduction

A foliation $\mathcal{F}$ on a compact manifold M is called stable if any other foliation sufficiently close to $\mathcal{F}$ in the C^{∞}-topology is conjugate to $\mathcal{F}$ under a diffeomorphism of M. Here the C^{∞}-topology on the space of k-dimensional foliations is induced by the inclusion

$$\mathrm{Fol}_k(M) \hookrightarrow \Gamma(\mathrm{Gr}_k(M)) : \mathcal{F} \mapsto T\mathcal{F},$$

where $\mathrm{Gr}_k(M) \to M$ is the Grassmannian fiber bundle of k-planes on M. Unpublished work by Hamilton [8] contains a stability result for Hausdorff foliations, i.e. those foliations $\mathcal{F}$ for which the leaf space $M/\mathcal{F}$ is Hausdorff when endowed with the quotient topology. The leaves of such a foliation are compact, and if M is connected then there exists a generic leaf L such that all the leaves of $\mathcal{F}$ contained in a saturated dense open are diffeomorphic to L.

Theorem 1.1 (Hamilton [8]). *Let M be a compact, connected manifold with a Hausdorff foliation $\mathcal{F}$. If the generic leaf L satisfies $H^1(L) = 0$, then $\mathcal{F}$ is stable.*

In fact, various versions of this result have been obtained independently throughout the years, using a variety of techniques. Langevin and Rosenberg [10] established C^1-stability of fibrations whose fiber L satisfies $H^1(L) = 0$. Epstein and Rosenberg [5] proved the analog of Thm. 1.1 for Hausdorff C^k-foliations with compact leaves on manifolds that are not necessarily compact. Finally, Del Hoyo and Fernandes [2] proved the statement of Thm. 1.1 for smooth paths of foliations $\mathcal{F}_t$ deforming $\mathcal{F}$, rather than foliations $\mathcal{F}'$ close to $\mathcal{F}$.

Aside from the Hausdorff foliations appearing in Thm. 1.1, very few examples of stable foliations are known. El Kacimi Alaoui and Nicolau [4] established stability for a particular class of non-Hausdorff foliations obtained by suspending linear foliations on tori by suitable linear Anosov diffeomorphisms. Their proof is an ad hoc application of the method used by Hamilton to prove Thm. 1.1.

Recent work by the author and Zeiser [7] aimed to establish a stability result for foliations that extends Thm. 1.1. To explain the main result of [7], recall that for any foliation $\mathcal{F}$ there is a canonical representation ∇ of the Lie algebroid $T\mathcal{F}$ on the normal bundle $N\mathcal{F}$, called Bott connection. Hence, one obtains a complex $\left(\Omega^\bullet(\mathcal{F}, N\mathcal{F}), d_\nabla\right)$ with cohomology groups $H^\bullet(\mathcal{F}, N\mathcal{F})$. Heitsch [9] showed that first order deformations of $\mathcal{F}$, modulo those obtained by applying an isotopy to $\mathcal{F}$, are given by elements in $H^1(\mathcal{F}, N\mathcal{F})$. Therefore, vanishing of $H^1(\mathcal{F}, N\mathcal{F})$ can be interpreted heuristically as the infinitesimal requirement for stability of $\mathcal{F}$. The main result in [7] states that vanishing of $H^1(\mathcal{F}, N\mathcal{F})$ actually implies that $\mathcal{F}$ is stable, at least when the foliation $\mathcal{F}$ is Riemannian. Recall here that $\mathcal{F}$ is Riemannian if the manifold M admits a Riemannian metric that is bundle-like with respect to $\mathcal{F}$ [17]. Equivalently, the foliation $\mathcal{F}$ is locally defined by Riemannian submersions.

Theorem 1.2 (Geudens-Zeiser [7])**.** *Let M be a compact manifold and $\mathcal{F}$ a Riemannian foliation on M such that $H^1(\mathcal{F}, N\mathcal{F}) = 0$. Then $\mathcal{F}$ is stable.*

To relate this result with Thm. 1.1, note that on a compact manifold M, Hausdorff foliations are exactly the Riemannian foliations with all leaves compact. Moreover, if the generic leaf L of a Hausdorff foliation $\mathcal{F}$ satisfies $H^1(L) = 0$, then also the cohomology group $H^1(\mathcal{F}, N\mathcal{F})$ vanishes [8]. So Thm. 1.2 implies Thm. 1.1. However, it was still unclear whether Thm. 1.2 admits new examples that are not already covered by Thm. 1.1. This note aims to fill that gap. We construct examples of Riemannian foliations $\mathcal{F}$ with dense leaves for which $H^1(\mathcal{F}, N\mathcal{F})$ vanishes, showing that Thm. 1.2 is indeed more general than Thm. 1.1. To our knowledge, these are the first examples of stable Riemannian foliations that are not Hausdorff – note that the stable foliations in [4] are not Riemannian.

Main Theorem. *Let B be a connected, compact manifold such that $\pi_1(B)$ has Kazhdan's property (T) as a discrete group. Assume that there is a connected, compact Lie group G and a group homomorphism $\varphi : \pi_1(B) \to G$ such that $\varphi(\pi_1(B)) \subset G$ is dense. Then the suspension of φ gives a Riemannian foliation $\mathcal{F}$ with dense leaves on a compact manifold, such that $H^1(\mathcal{F}, N\mathcal{F}) = 0$. Consequently, the foliation $\mathcal{F}$ is stable.*

Since the foliations appearing in the Main Theorem are transversely modeled on the Lie group G, they are a special kind of Riemannian foliations called Lie foliations. Molino's structure theory for Riemannian foliations [15] reduces the study of Riemannian foliations on compact manifolds to that of Lie foliations with dense leaves. From this point of view, our Main Theorem yields stable Riemannian foliations of the most elementary kind.

Explicit examples of the Main Theorem can be constructed from arithmetic lattices Γ in semisimple Lie groups H. Such lattices sometimes admit a dense embedding into a compact Lie group G. Moreover, since they are finitely presented, they arise as fundamental groups of compact 4-manifolds B. We work out an explicit example in §4.

The proof of the Main Theorem relies on recent work by El Kacimi Alaoui [3] which computes the cohomology group $H^1(\mathcal{F}, N\mathcal{F})$ for developable foliations $\mathcal{F}$. These are foliations $\mathcal{F}$ whose lift $\widehat{\mathcal{F}}$ to some covering $\widehat{M}$ of M is given by a fibration. Note that the suspension foliation $\mathcal{F}$ of a homomorphism $\varphi : \pi_1(B) \to G$ is of this type. The main result of [3] shows that $H^1(\mathcal{F}, N\mathcal{F})$ is given by the first group cohomology $H^1\left(\pi_1(B), \mathfrak{X}(G)\right)$. Here the discrete group $\pi_1(B)$ acts on G by left translations (through the homomorphism φ) and therefore on $\mathfrak{X}(G)$ by pushforward. Endowing G with a left invariant metric, the representation of $\pi_1(B)$ on $\mathfrak{X}(G)$ is orthogonal for the induced L^2-inner product. The assumption that $\pi_1(B)$ has Kazhdan's property (T) ensures that the cohomology group $H^1\left(\pi_1(B), V\right)$ vanishes whenever V is a real Hilbert space and the representation of $\pi_1(B)$ on V is orthogonal [1, Chapter 2]. This does not immediately imply the vanishing of $H^1(\pi_1(B), \mathfrak{X}(G))$ but rather that of $H^1\big(\pi_1(B), L^2\mathfrak{X}(G)\big)$, where $L^2\mathfrak{X}(G)$ is the Hilbert space of square integrable vector fields on G for the L^2-inner product. However, the fact that $\varphi(\pi_1(B)) \subset G$ is dense implies that the action of $\pi_1(B)$ on G is ergodic, hence a regularity result by Lubotzky-Zimmer [12] can be applied. It shows that a 1-cocycle $\pi_1(B) \to \mathfrak{X}(G)$, which a priori has a primitive defined in terms of an element in $L^2\mathfrak{X}(G)$, admits a primitive coming from a smooth vector field. Consequently, the first group cohomology $H^1(\pi_1(B), \mathfrak{X}(G))$ vanishes.

Acknowledgements. I acknowledge support from the UCL Institute for Mathematical and Statistical Sciences (IMSS). I would like to thank Ioan Mărcuţ for useful discussions about stability of foliations, and Aziz El Kacimi Alaoui for e-mail exchanges about his paper [3].

2. Suspension foliations

The examples of stable Riemannian foliations that we will construct are obtained by the suspension method, which we now briefly recall. Let B and T be compact, connected manifolds and assume that we are given a group homomorphism

$$\varphi : \pi_1(B) \to \mathrm{Diff}(T),$$

where $\mathrm{Diff}(T)$ denotes the diffeomorphism group of T. Let $\widehat{B}$ be the universal cover of B, and consider the product manifold $\widehat{B} \times T$. We get an action $\pi_1(B) \curvearrowright \widehat{B} \times T$ defined by

$$\gamma \cdot (\tilde{b}, t) = \big(\gamma \star \tilde{b}, \varphi(\gamma)(t)\big), \tag{1}$$

where $\star$ denotes the natural action of $\pi_1(B)$ on $\widehat{B}$. The action $\pi_1(B) \curvearrowright \widehat{B} \times T$ is free and properly discontinuous, hence the quotient $B_\varphi := (\widehat{B} \times T)/\pi_1(B)$ is smooth and the quotient map is a covering map

$$\pi : \widehat{B} \times T \to B_\varphi.$$

The quotient manifold B_φ is a fiber bundle over B with fiber T. Note that the foliation by slices $\widehat{B} \times \{t\}$ on $\widehat{B} \times T$ is invariant under the action (1), hence it descends to a foliation $\mathcal{F}$ on B_φ transverse to the fibers of $B_\varphi \to B$. The leaves of $\mathcal{F}$ are of the form

$$(\widehat{B} \times \mathrm{Orb}(t))/\pi_1(B) \cong \widehat{B}/\mathrm{Stab}(t) \qquad \text{for } t \in T,$$

where $\mathrm{Orb}(t) \subset T$ is the orbit through t and $\mathrm{Stab}(t) \subset \pi_1(B)$ is the stabilizer of t. This shows in particular that compact leaves of $\mathcal{F}$ correspond with finite orbits in T. In the following, we refer to the foliated manifold $\big(B_\varphi, \mathcal{F}\big)$ as the suspension of $\varphi : \pi_1(B) \to \mathrm{Diff}(T)$.

In this note, we are interested in Riemannian foliations $\mathcal{F}$. The suspension foliation of a homomorphism $\varphi : \pi_1(B) \to \mathrm{Diff}(T)$ is Riemannian whenever φ takes values in the isometry group $\mathrm{Isom}(T, g) \subset \mathrm{Diff}(T)$ of (T, g) for some choice of Riemannian metric g. A particular case arises when the manifold T is a compact Lie group G. Endowing G with a left invariant metric g, the group G embeds into the isometry group $\mathrm{Isom}(G, g)$ as the subgroup of left translations. Consequently, given a group homomorphism

$$\varphi : \pi_1(B) \to G,$$

the associated suspension $(B_\varphi, \mathcal{F})$ is a Riemannian foliation (in fact, a Lie G-foliation). We will only consider homomorphisms φ for which the image $H := \varphi(\pi_1(B))$ is a dense subgroup of G. In that case, the suspension foliation $\mathcal{F}$ on B_φ is a Lie G-foliation with dense leaves. This follows immediately from the fact that $\mathcal{F}$ is a transversely parallelizable foliation on a compact connected manifold, all of whose basic functions are constant [14, Thm. 4.24]. To see this, note that a frame of left invariant vector fields on G gives a transverse parallelism for the horizontal foliation on $\widehat{B} \times G$, which descends to a transverse parallelism for $\mathcal{F}$ on B_φ. Moreover, the $\mathcal{F}$-basic functions on B_φ are constant since

$$C^\infty_{bas}(B_\varphi) = C^\infty(G)^H = C^\infty(G)^G = \mathbb{R},$$

where $C^\infty(G)^H$ and $C^\infty(G)^G$ denote the spaces of H-invariant and G-invariant smooth functions on G. Here we made use of the fact that the subgroup $H \subset G$ is dense.

3. Proof of the Main Theorem

The proof of the Main Theorem relies on two auxiliary results. We will first use that the deformation cohomology of a developable foliation can be expressed in terms of group cohomology of a suitable discrete group [3]. We will then use a vanishing result for group cohomology, which holds when the discrete group acting has Kazhdan's property (T) [12].

3.1. **Infinitesimal deformations of developable foliations.** The cohomology group $H^1(\mathcal{F}, N\mathcal{F})$ governing the deformations of a foliation $\mathcal{F}$ is very hard to compute in general. The situation is better for developable foliations, thanks to work by El Kacimi Alaoui [3].

Definition 3.1. A foliation $\mathcal{F}$ on a connected manifold M is developable if there exists a connected, normal covering $\pi : \widehat{M} \to M$ such that the pullback foliation $\widehat{\mathcal{F}}$ on $\widehat{M}$ is given by the fibers of a locally trivial fibration $D : \widehat{M} \to W$.

The deck transformation group Γ of the covering $\pi : \widehat{M} \to M$ acts freely and properly discontinuously on $\widehat{M}$, and since the covering is normal we have that $\widehat{M}/\Gamma \cong M$. Moreover, since the action $\Gamma \curvearrowright \widehat{M}$ preserves the fibers of the developing map $D : \widehat{M} \to W$, there is an induced action $\Gamma \curvearrowright W$. In particular, the space of vector fields $\mathfrak{X}(W)$ is a Γ-module.

Example 3.2. Given a homomorphism $\varphi : \pi_1(B) \to \mathrm{Diff}(T)$, where B and T are connected compact manifolds, the suspension foliation $\mathcal{F}$ on B_φ is developable. Indeed, we have a connected, normal covering $\pi : \widehat{B} \times T \to B_\varphi$, and the pullback foliation $\widehat{\mathcal{F}}$ is given by the fibers of the second projection $\widehat{B} \times T \to T$. The deck transformation group is $\Gamma := \pi_1(B)$.

Given an arbitrary foliation $\mathcal{F}$ on a compact manifold M, recall that there is a canonical $T\mathcal{F}$-representation ∇ on the normal bundle $N\mathcal{F} := TM/T\mathcal{F}$, given by

$$\nabla_X \overline{Y} = \overline{[X, Y]},$$

for $X \in \Gamma(T\mathcal{F})$ and $\overline{Y} \in \Gamma(N\mathcal{F})$. This representation is called Bott connection. Accordingly, we get a differential d_∇ on the graded vector space $\Omega^\bullet(\mathcal{F}, N\mathcal{F}) := \Gamma(\wedge^\bullet T^*\mathcal{F} \otimes N\mathcal{F})$ of foliated forms with coefficients in $N\mathcal{F}$, given by the usual Koszul formula

$$\begin{aligned} d_\nabla \eta(V_1, \ldots, V_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i+1} \nabla_{V_i}\big(\eta(V_1, \ldots, V_{i-1}, \widehat{V_i}, V_{i+1}, \ldots, V_{k+1})\big) \\ &+ \sum_{i<j} (-1)^{i+j} \eta\big([V_i, V_j], V_1, \ldots, \widehat{V_i}, \ldots, \widehat{V_j}, \ldots, V_{k+1}\big). \end{aligned}$$

Heitsch [9] showed that infinitesimal deformations of the foliation $\mathcal{F}$ are one-cocycles in the complex $\big(\Omega^\bullet(\mathcal{F}; N\mathcal{F}), d_\nabla\big)$. Moreover, if a smooth deformation of $\mathcal{F}$ is obtained applying an isotopy to $\mathcal{F}$, then the corresponding infinitesimal deformation is a one-coboundary in $\big(\Omega^\bullet(\mathcal{F}; N\mathcal{F}), d_\nabla\big)$. Hence, vanishing of the cohomology group $H^1(\mathcal{F}, N\mathcal{F})$ can be interpreted heuristically as the infinitesimal requirement for stability of the foliation $\mathcal{F}$. In case the foliation $\mathcal{F}$ is developable, then $H^1(\mathcal{F}, N\mathcal{F})$ can be expressed in terms of group cohomology, as shown by El Kacimi Alaoui [3]. We now recall this result.

Let Γ be a discrete group acting on a vector space V. For each integer $n \geq 1$, denote by $C^n(\Gamma, V)$ the vector space of maps $\Gamma^n \to V$. By convention, $C^0(\Gamma, V) = V$. There is a differential d on the graded vector space $C^\bullet(\Gamma, V)$, defined by

$$\begin{aligned} d\psi(\gamma_1, \dots, \gamma_{n+1}) = \gamma_1 \cdot \psi(\gamma_2, \dots, \gamma_{n+1}) \\ + \sum_{i=1}^{n} (-1)^i \psi(\gamma_1, \dots, \gamma_{i-1}, \gamma_i\gamma_{i+1}, \gamma_{i+2}, \dots, \gamma_{n+1}) \\ + (-1)^{n+1} \psi(\gamma_1, \dots, \gamma_n). \end{aligned}$$

We denote the associated cohomology groups by $H^\bullet(\Gamma, V)$. Of particular interest to us is the first cohomology group $H^1(\Gamma, V)$, which is given by

$$H^1(\Gamma, V) = \frac{\{\psi : \Gamma \to V : \ \psi(\gamma_1\gamma_2) = \psi(\gamma_1) + \gamma_1 \cdot \psi(\gamma_2)\}}{\{\chi : \Gamma \to V : \ \chi(\gamma) = \gamma \cdot v - v \ \text{ for some } v \in V\}}.$$

In other words, $H^1(\Gamma, V)$ is the space of crossed homomorphisms $\Gamma \to V$ modulo those induced by elements of V. We can now state the following result from [3].

Proposition 3.3. *Let $\mathcal{F}$ be a developable foliation on a connected manifold M with normal covering $\pi : \widehat{M} \to M$, deck transformation group Γ and developing map $D : \widehat{M} \to W$. Assume that the fiber $\widehat{L}$ of D satisfies $H^1(\widehat{L}) = 0$. Then $H^1(\mathcal{F}, N\mathcal{F}) \cong H^1(\Gamma, \mathfrak{X}(W))$.*

Example 3.4. Let $\varphi : \pi_1(B) \to \mathrm{Diff}(T)$ be a group homomorphism, where B and T are compact connected manifolds. The suspension foliation $\mathcal{F}$ on B_φ lifts under $\pi : \widehat{B} \times T \to B_\varphi$ to the foliation by fibers of $\widehat{B} \times T \to T$. Since the universal cover $\widehat{B}$ is simply connected, we have $H^1(\widehat{B}) = 0$. Hence, Prop. 3.3 implies that $H^1(\mathcal{F}, N\mathcal{F}) \cong H^1\big(\pi_1(B), \mathfrak{X}(T)\big)$.

Now consider the special case in which $T = G$ is a compact Lie group and $\varphi : \pi_1(B) \to G$. We then have $\mathfrak{X}(G) \cong C^\infty(G) \otimes \mathfrak{g}$, and since $\pi_1(B)$ acts by left translations it is clear that the action is trivial on $\mathfrak{g}$. It follows that the suspension foliation $\mathcal{F}$ of φ satisfies

$$H^1(\mathcal{F}, N\mathcal{F}) \cong H^1\big(\pi_1(B), \mathfrak{X}(G)\big) \cong H^1\big(\pi_1(B), C^\infty(G)\big) \otimes \mathfrak{g}. \tag{2}$$

3.2. **Actions of Kazhdan groups.** In the previous subsection, we reduced the computation of the cohomology group $H^1(\mathcal{F}, N\mathcal{F})$ for a suspension foliation $\mathcal{F}$ to group cohomology of the discrete group $\pi_1(B)$. We will now invoke a vanishing result for cohomology of discrete groups with Kazhdan's property (T). For an extensive treatment of property (T) groups, we refer to the book [1]. Let us just recall the definition here.

Definition 3.5. Let G be a topological group and $\mathcal{H}$ a complex Hilbert space. Assume that $\rho : G \to U(\mathcal{H})$ is a unitary representation. For a subset $Q \subset G$ and a constant $\epsilon > 0$, a vector $\xi \in \mathcal{H}$ is (Q, ϵ)-invariant if

$$\sup_{g \in Q} \|\rho(g)\xi - \xi\| < \epsilon \|\xi\|.$$

Definition 3.6. Let G be a topological group.

i) A subset $Q \subset G$ is a Kazhdan set if there exists $\epsilon > 0$ with the following property: every unitary representation $(\rho, \mathcal{H})$ of G which has a (Q, ϵ)-invariant vector also has a non-zero invariant vector.

ii) The group G has Kazhdan's property (T) if G has a compact Kazhdan set.

Compact topological groups have property (T) [1, Prop. 1.1.5], while $\mathbb{R}^n$ and $\mathbb{Z}^n$ do not [1, Ex. 1.1.7]. Of special interest to us are semisimple Lie groups G with property (T). A connected semisimple Lie group G with Lie algebra $\mathfrak{g}$ has property (T) if and only if no simple factor of $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(n,1)$ or $\mathfrak{su}(n,1)$ [1, Thm. 3.5.4].

Kazhdan's property (T) is relevant for our purpose because of the following: a countable discrete group Γ has property (T) exactly when the first group cohomology $H^1(\Gamma, V)$ vanishes for every orthogonal representation of Γ on a real Hilbert space V [1, Chapter 2]. This statement is not directly applicable in our situation because the representations that we encounter are no Hilbert spaces, see (2). To overcome this issue, we need to invoke the following result by Lubotzky-Zimmer [12, Thm. 4.1]. We only state a particular case of their result, which holds under more general assumptions that are implied by property (T).

Proposition 3.7. *Let Γ be a finitely generated discrete group acting smoothly on a compact manifold M. Suppose that Γ has Kazhdan's property (T), and that the Γ-action is isometric and ergodic. Then both $H^1(\Gamma, \mathfrak{X}(M))$ and $H^1(\Gamma, C^\infty(M))$ vanish.*

In the above, say that the Γ-action is isometric with respect to the Riemannian metric g on M. Denoting by μ the Riemannian measure coming from g, it follows that Γ acts by measure preserving automorphisms of (M, μ). Such an action $\Gamma \curvearrowright (M, \mu)$ is called ergodic if for every Γ-invariant measurable subset $N \subset M$, one either has $\mu(N) = 0$ or $\mu(N) = 1$.

3.3. **Proof of Main Theorem.** Now all preliminaries are in place for us to prove our Main Theorem. Let us state it again for convenience.

Theorem 3.8. *Let B be a connected, compact manifold such that $\pi_1(B)$ has Kazhdan's property (T) as a discrete group. Assume that there is a connected, compact Lie group G and a group homomorphism $\varphi : \pi_1(B) \to G$ such that $\varphi(\pi_1(B)) \subset G$ is dense. Then the suspension of φ gives a Riemannian foliation $\mathcal{F}$ with dense leaves on a compact manifold, such that $H^1(\mathcal{F}, N\mathcal{F}) = 0$. Consequently, the foliation $\mathcal{F}$ is stable.*

Proof. We already argued in §2 that $\mathcal{F}$ is a Riemannian foliation (indeed, a Lie G-foliation) with dense leaves. By Example 3.4, we know that

$$H^1(\mathcal{F}, N\mathcal{F}) \cong H^1\big(\pi_1(B), \mathfrak{X}(G)\big) \cong H^1\big(\pi_1(B), C^\infty(G)\big) \otimes \mathfrak{g}. \tag{3}$$

The action of $\pi_1(B)$ on G is isometric with respect to any left invariant Riemannian metric on G. Moreover, it is well-known that if $\Gamma \subset G$ is a dense subgroup of a compact Lie group, then the action by left translations $\Gamma \curvearrowright G$ is ergodic with respect to the normalized Haar measure on G [16, Lemma 2.4.1]. Since by assumption $\varphi(\pi_1(B)) \subset G$ is dense, it follows that the action $\pi_1(B) \curvearrowright (G, \mu)$ is ergodic. Moreover, the fundamental group $\pi_1(B)$ of a compact manifold B is finitely generated. Applying Prop. 3.7, we see that the cohomology group (3) vanishes. Stability of $\mathcal{F}$ immediately follows from Thm. 1.2 in the Introduction. ☐

4. An example

The aim of this section is to construct a discrete group Γ which can serve as the fundamental group $\pi_1(B)$ in Thm. 3.8. We will obtain Γ as a suitable arithmetic lattice in $SO(3,2)$ which can be realized as a dense subgroup of $SO(5)$. The lattice in question was considered before in connection with the Ruziewicz problem, which concerns rotationally invariant measures on spheres [11, §3.4].

4.1. **Arithmetic lattices.** We start by recalling an arithmetic construction of lattices, following [13, Chapter IX, §1.7]. Fix an integer $n \geq 3$ and a subfield $K \subset \mathbb{C}$ which is a finite extension of $\mathbb{Q}$. Denote by $d = [K : \mathbb{Q}]$ the degree of the extension. Assume that

$$\Phi = \sum_{1 \leq i,j \leq n} a_{ij} x_i x_j$$

is a non-degenerate quadratic form in n variables with coefficients in K. Recall that there are d distinct field embeddings $\sigma_1, \dots, \sigma_d : K \to \mathbb{C}$ with $\sigma_1 = \mathrm{Id}$. We call such an embedding σ real if $\sigma(K) \subset \mathbb{R}$, otherwise it is called imaginary. Two embeddings of K into $\mathbb{C}$ are called equivalent if one is the complex conjugate of the other. Choosing a representative in each equivalence class of embeddings, we obtain the set

$$\mathcal{R} := \{\sigma_1, \dots, \sigma_l\}.$$

For every $\sigma \in \mathcal{R}$, we define $k_\sigma := \mathbb{R}$ if σ is real and $k_\sigma := \mathbb{C}$ if σ is imaginary. Each field embedding $\sigma \in \mathcal{R}$ gives rise to a new quadratic form

$$\Phi_\sigma = \sum_{1 \leq i,j \leq n} \sigma(a_{ij}) x_i x_j \tag{4}$$

and we set

$$\mathcal{T} = \{\sigma \in \mathcal{R} : \ k_\sigma = \mathbb{R} \text{ and } \Phi_\sigma \text{ is either positive definite or negative definite}\}.$$

Choose a subset $\mathcal{S} \subset \mathcal{R}$ such that $\mathcal{R} \setminus \mathcal{T} \subset \mathcal{S}$. Denote by SO_Φ the special orthogonal group of the quadratic form Φ, consisting of complex matrices with determinant 1 preserving Φ. It is an algebraic group defined over K. We let $\mathcal{O}$ be the ring of integers in K and denote by $SO_\Phi(\mathcal{O})$ the group of $\mathcal{O}$-valued matrices in SO_Φ. Similarly, we define for each $\sigma \in \mathcal{S}$ the group $SO_{\Phi_\sigma}(k_\sigma)$ of k_σ-valued matrices preserving the quadratic form Φ_σ defined in (4). In the following, we identify $SO_\Phi(\mathcal{O})$ with its image under the embedding

$$\prod_{\sigma \in \mathcal{S}} \sigma : SO_\Phi(\mathcal{O}) \to \prod_{\sigma \in \mathcal{S}} SO_{\Phi_\sigma}(k_\sigma).$$

Proposition 4.1. [13, Chapter IX, §1.7] *Assume that $\mathcal{T} \neq \mathcal{R}$ and that the group SO_Φ is almost K-simple. Then $SO_\Phi(\mathcal{O})$ is an arithmetic lattice in $\prod_{\sigma \in \mathcal{S}} SO_{\Phi_\sigma}(k_\sigma)$.*

Almost K-simplicity means that proper algebraic K-closed normal subgroups of SO_Φ are finite. This condition is equivalent to that of either $n \neq 4$ or $n = 4$ and the discriminant of the quadratic form Φ is not a square in K.

4.2. **An example.** We let $n = 5$ and consider the field extension $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2})$ of degree 2. Consider the non-degenerate quadratic form

$$\Phi = x_1^2 + x_2^2 + x_3^2 - \sqrt{2} x_4^2 - \sqrt{2} x_5^2$$

in 5 variables with coefficients in $\mathbb{Q}(\sqrt{2})$. There are 2 field embeddings $\sigma_1, \sigma_2 : \mathbb{Q}(\sqrt{2}) \to \mathbb{C}$, namely $\sigma_1 = \mathrm{Id}$ and σ_2 given by

$$\sigma_2(a + b\sqrt{2}) = a - b\sqrt{2}, \qquad a, b \in \mathbb{Q}.$$

Both are real, and we have $\mathcal{R} = \{\sigma_1, \sigma_2\}$. The embedding σ_2 defines a new quadratic form

$$\Phi_{\sigma_2} = x_1^2 + x_2^2 + x_3^2 + \sqrt{2} x_4^2 + \sqrt{2} x_5^2,$$

and in this case we have $\mathcal{T} = \{\sigma_2\}$. Let us choose $\mathcal{S} = \{\sigma_1, \sigma_2\}$. Applying Prop. 4.1 gives an arithmetic lattice

$$\Gamma := SO_\Phi(\mathbb{Z}[\sqrt{2}]) \subset SO(3,2) \times SO(5).$$

The first projection in $SO(3,2)\times SO(5)$ realizes Γ as an arithmetic lattice in $SO(3,2)$, while the second projection in $SO(3,2)\times SO(5)$ realizes Γ as a subgroup of the compact Lie group $SO(5)$. We now check that this lattice can be used to construct an example of Thm. 3.8.

- Lattices in connected semisimple Lie groups are finitely presented [6, §1.1]. Viewing Γ as a lattice inside $SO(3,2)$, this implies that Γ is finitely presented, even though $SO(3,2)$ has 2 connected components. To see this, let us denote by $SO(3,2)^+$ its identity component, which is a normal index 2 subgroup $SO(3,2)^+ \subset SO(3,2)$. The fact that $SO(3,2)^+$ is an open subgroup implies that $\Gamma\cap SO(3,2)^+$ is a lattice inside $SO(3,2)^+$, hence $\Gamma \cap SO(3,2)^+$ is finitely presented by the above. Moreover, since
$$\frac{\Gamma}{\Gamma \cap SO(3,2)^+} \cong \frac{\Gamma SO(3,2)^+}{SO(3,2)^+}$$
is finite, it is also finitely presented. Hence, Γ is finitely presented as well. It follows that there is a compact, connected 4-dimensional manifold B with $\pi_1(B) = \Gamma$.
- We have the compact, connected Lie group $G := SO(5)$ and a group homomorphism $\varphi : \Gamma \to SO(5)$ given by the restriction of the second projection in $SO(3,2)\times SO(5)$. It was shown in [11, Prop. 3.4.3] that the projection of Γ in $SO(5)$ is dense.
- Finally, we argue that the discrete group Γ has property (T). We view Γ as a lattice inside $SO(3,2)$ and recall that property (T) is inherited by lattices [1, Thm. 1.7.1]. Hence, it suffices to show that $SO(3,2)$ has property (T). As before, denote by $SO(3,2)^+$ its identity component, which is a closed normal subgroup. It follows that $SO(3,2)$ has property (T) as soon as we show that $SO(3,2)^+$ and $SO(3,2)/SO(3,2)^+$ have property (T) [1, Prop. 1.7.6]. The identity component $SO(3,2)^+$ has property (T) because it is a connected simple Lie group with Lie algebra not of the form $\mathfrak{so}(n,1)$ or $\mathfrak{su}(n,1)$ [1, Thm. 3.5.4]. Next, the quotient $SO(3,2)/SO(3,2)^+$ is discrete because $SO(3,2)^+ \subset SO(3,2)$ is open. Since finite discrete groups are compact, it follows that also $SO(3,2)/SO(3,2)^+$ has property (T) [1, Prop. 1.1.5].

Applying Thm. 3.8, the suspension of $\varphi : \pi_1(B) \to SO(5)$ is a stable Riemannian foliation $\mathcal{F}$ with dense leaves on a compact manifold B_φ.

Remark 4.2. The above argument works for all values $n \geq 5$, considering the quadratic form
$$\Phi = x_1^2 + \cdots + x_{n-2}^2 - \sqrt{2}x_{n-1}^2 - \sqrt{2}x_n^2.$$
We obtain an arithmetic lattice $\Gamma \subset SO(n-2,2)\times SO(n)$ with property (T), which projects to a dense subgroup in $SO(n)$ [11, Prop. 3.4.3].

The method fails for $n < 5$. In fact, Thm. 3.8 admits no examples in which $G = SO(n)$ with $n < 5$. Indeed, if $\varphi : \Gamma \to SO(n)$ is a group homomorphism where Γ is a discrete Kazhdan group and $n < 5$, then the image $\varphi(\Gamma)$ is finite. This was shown by Zimmer in [18, Thm. 7] for $SO(3)$ and $SO(4)$. It is immediate in the case of $SO(2)$, because of the following. Since $SO(2)$ is abelian, the group homomorphism $\varphi : \Gamma \to SO(2)$ factors through the abelianization $\Gamma/[\Gamma,\Gamma]$. Because the abelianization of a discrete group with property (T) is finite [1, Cor. 1.3.6], it follows that the image $\varphi(\Gamma) \subset SO(2)$ is finite. This shows that the suspension foliation of any homomorphism $\varphi : \Gamma \to SO(n)$, with Γ a discrete Kazhdan group and $n < 5$, is Hausdorff since all of its leaves are compact (see §2).

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Department of Mathematics, University College London, 25 Gordon St, London WC1H0AY, UK
Email address: `s.geudens@ucl.ac.uk`