RESIDUAL FINITENESS OF FUNDAMENTAL n-QUANDLES OF LINKS

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Abstract. In this paper, we investigate residual finiteness and subquandle separability of quandles. The existence of these finiteness properties implies the solvability of the word problem and the generalised word problem for quandles. We prove that the fundamental n-quandle of any link in the 3-sphere is residually finite for each $n \geq 2$. This supplements the recent result on residual finiteness of link quandles and the classification of links whose fundamental n-quandles are finite for some n. We also establish several general results on these finiteness properties and give many families of quandles admitting them.

1. Introduction

Let S be a subset of a group G. Then S is said to be separable in G if given any element $x \in G \setminus S$, there exists a finite group F and a group homomorphism $\phi : G \to F$ such that $\phi(x) \notin \phi(S)$. The group G is residually finite if the trivial subgroup is separable in G, and it is subgroup separable if every finitely generated subgroup is separable in G.

Subgroup separability has applications in combinatorial group theory and low dimensional topology. If a finitely presented group G is residually finite, then it has the solvable word problem. More generally, if G is a finitely presented subgroup separable group, then the generalised word problem is solvable in G. In the context of 3-manifold topology, subgroup separability has been used to solve immersion to embedding problems. For instance, it is known due to Thurston that subgroup separability allows passage from immersed incompressible surfaces to embedded incompressible surfaces in finite covers.

The aim of this paper is to investigate these properties in the category of quandles with a focus on fundamental quandles of links in $\mathbb{S}^3$. Quandles are right distributive algebraic structures that appear naturally as strong invariants of links and as non-degenerate set-theoretical solutions to the Yang-Baxter equation. More formally, a quandle is a set with a binary operation that satisfies three axioms modelled on the three Reidemeister moves of planar diagrams of links in $\mathbb{S}^3$. Joyce [13] and Matveev [16] independently proved that each oriented diagram $D(L)$ of a link L gives rise to a quandle $Q(L)$, called the fundamental quandle, which is independent of the diagram $D(L)$. Further, they showed that if K_1 and K_2 are oriented knots with $Q(K_1) \cong Q(K_2)$, then there is a homeomorphism of $\mathbb{S}^3$ mapping K_1 onto K_2, not necessarily preserving the orientation of the ambient space. Although, the fundamental quandle is a strong invariant for knots, it is usually difficult to check whether two quandles are isomorphic. This has motivated the search for newer properties of these structures. Since fundamental quandles of links in $\mathbb{S}^3$ are always infinite, except for the

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case when it is the unknot or the Hopf link, it is reasonable to look for residual finiteness of these quandles. In this direction, it has been proved in [6, 7] that, along with many classes of quandles arising from groups, the fundamental quandles of oriented links in $\mathbb{S}^3$ are residually finite. Since link quandles are finitely presented, it follows that they have the solvable word problem.

In this paper, we carry out this study further in two directions. Firstly, we consider the residual finiteness of some canonical quotients of fundamental quandles of oriented links in $\mathbb{S}^3$, called fundamental n-quandles, where $n \geq 2$. Using Thurston's geometrisation theorem, Hoste-Shanahan [12] derived the complete list of links which have a finite fundamental n-quandle for some $n \geq 2$. It turns out that most of the links have infinite fundamental n-quandles for almost all values of n. Thus, it is reasonable to ask whether these n-quandles are residually finite. Employing consequences of Thurston's geometrisation theorem and related results, we first prove that if L is an oriented link and $\widetilde{M_n}(L)$ is the n-fold cyclic branched cover of $\mathbb{S}^3$, then $\pi_1(\widetilde{M_n}(L))$ is abelian subgroup separable (Theorem 3.9). Using this result and a description of $\pi_1(\widetilde{M_n}(L))$ as a subgroup of a canonical quotient of the link group $\pi_1(\mathbb{S}^3 \setminus L)$, we prove that the fundamental n-quandle of any oriented link is residually finite for each $n \geq 2$ (Theorem 3.12).

Secondly, we develop a general theory of subquandle separability of quandles, which implies the solvability of the generalised word problem for these algebraic structures. Among other results, we prove that certain subquandles of quandles arising from subgroup separable groups are separable (Proposition 5.3). We also establish subquandle separability of certain twisted unions of subquandle separable quandles (Proposition 5.6), abelian quandles generated by two elements (Theorem 5.8), and finitely generated free abelian quandles (Theorem 5.9).

2. Preliminaries

This section reviews the essential preliminary material that will be employed throughout the paper. To set our convention, recall that, a *quandle* is a set X with a binary operation $*$ that satisfies the following axioms:

(1) $x * x = x$ for all $x \in X$.
(2) Given $x, y \in X$, there exists a unique $z \in X$ such that $x = z * y$.
(3) $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in X$.

Analogous to groups, quandles can be represented by their presentations.

Example 2.1. Let L be an oriented link in $\mathbb{S}^3$. Then the *fundamental quandle* $Q(L)$ of L can be constructed from a regular diagram $D(L)$ of L. Suppose that $D(L)$ has s arcs and t crossings. We assign labels $x_1, x_2, \ldots, x_s$ to arcs of $D(L)$, and then introduce the relation r_l given by $x_k * x_j = x_i$ at the l-th crossing of $D(L)$ as shown in Figure 1. It is known due to [13, 16] that

$$Q(L) \cong \langle x_1, x_2, \ldots, x_s \mid r_1, r_2, \ldots, r_t \rangle,$$

and it is an invariant of the isotopy type of the link L.

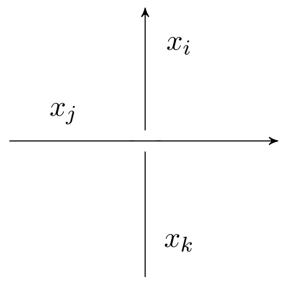

FIGURE 1. Quandle relation at a crossing.

Example 2.2. Though links in the 3-sphere are rich sources of quandles, many interesting examples arise particularly from groups, some of them will be used in later sections.

- If G is a group, then the set G equipped with the binary operation $x * y = y^{-1}xy$ gives a quandle structure on G, called the *conjugation quandle*, and denoted by $\mathrm{Conj}(G)$.
- Let H be a subgroup of a group G and $\alpha \in \mathrm{Aut}(G)$ that acts trivially on H. Then, the set G/H of right cosets becomes a quandle with the binary operation
$$Hx * Hy = H\alpha(xy^{-1})y.$$
In particular, if α is the inner automorphism of G induced by an element x_0 in the centraliser of H in G, then the quandle operation on G/H becomes
$$Hx * Hy = Hx_0^{-1}xy^{-1}x_0y,$$
and we denote this quandle by $(G/H, x_0)$.
- The preceding example can be extended as follows. Let G be a group, $\{x_i \mid i \in I\}$ be a set of elements of G, and $\{H_i \mid i \in I\}$ a set of subgroups of G such that $H_i \leq \mathrm{C}_G(x_i)$ for each i. Then, we can define a quandle structure on the disjoint union $\sqcup_{i \in I} G/H_i$ by
$$H_ix * H_jy = H_ix_i^{-1}xy^{-1}x_jy,$$
and denote this quandle by $\sqcup_{i\in I}(G/H_i, x_i)$.

If X is a quandle and $x \in X$, then the map $S_x : X \to X$ given by $S_x(y) = y * x$ is an automorphism of X fixing x. The group $\mathrm{Inn}(X)$ generated by such automorphisms is called the *inner automorphism group* of X. The quandle X is called *connected* if $\mathrm{Inn}(X)$ acts transitively on X.

Using the defining axioms [22, Lemma 4.4.7], any element of a quandle X can be written in a left-associated product of the form
$$((\cdots((x_0 *^{\epsilon_1} x_1) *^{\epsilon_2} x_2) *^{\epsilon_3} \cdots) *^{\epsilon_{n-1}} x_{n-1}) *^{\epsilon_n} x_n,$$
where $x_i \in X$ and $\epsilon_i \in \{1, -1\}$. For simplicity, we write the preceding expression as
$$x_0 *^{\epsilon_1} x_1 *^{\epsilon_2} \cdots *^{\epsilon_n} x_n.$$

A quandle X is called an *n-quandle* if each S_x has order dividing n. In other words, if X is an n-quandle, then

$$x *^n y := x * \underbrace{y * y * \cdots * y}_{n \text{ times}} = x$$

for all $x, y \in X$.

2.1. **Enveloping group.** To each quandle X, we associate its *enveloping group* $\mathrm{Env}(X)$, which is given by the presentation

$$\mathrm{Env}(X) = \langle e_x,\ x \in X \mid e_{x*y} = e_y^{-1} e_x e_y \text{ for all } x, y \in X \rangle. \tag{2.1.1}$$

The association $X \mapsto \mathrm{Env}(X)$ defines a functor from the category of quandles to that of groups, which is left adjoint to the functor $G \mapsto \mathrm{Conj}(G)$ from the category of groups to that of quandles. Analogously, there is a functor from the category of groups to the category of n-quandles for each $n \geq 2$. To be precise, given a group G, we consider the set

$$Q_n(G) = \{x \in G \mid x^n = 1\}$$

equipped with the binary operation of conjugation, which is clearly an n-quandle. In the reverse direction, given an n-quandle X, we define its *n-enveloping group* to be

$$\mathrm{Env}_n(X) = \langle \mathbf{e}_x,\ x \in X \mid \mathbf{e}_x^n = 1,\ \mathbf{e}_{x*y} = \mathbf{e}_y^{-1} \mathbf{e}_x \mathbf{e}_y \text{ for all } x, y \in X \rangle.$$

It follows from [22, Theorem 5.1.7] that if a quandle X has the presentation

$$X = \langle x_1, x_2, \ldots, x_s \mid r_1, r_2, \ldots, r_t \rangle,$$

then $\mathrm{Env}(X)$ has the presentation

$$\mathrm{Env}(X) = \langle e_{x_1}, e_{x_2}, \ldots, e_{x_s} \mid \bar{r}_1, \bar{r}_2, \ldots, \bar{r}_t \rangle,$$

where each relation $\bar{r}_i$ is obtained from the relation r_i by replacing each expression $x * y$ by $e_y^{-1} e_x e_y$ and $x *^{-1} y$ by $e_y e_x e_y^{-1}$. Furthermore, if X is an n-quandle, then it follows that $\mathrm{Env}_n(X)$ has the presentation

$$\mathrm{Env}_n(X) = \left\langle \mathbf{e}_{x_1}, \mathbf{e}_{x_2}, \ldots, \mathbf{e}_{x_s} \mid \mathbf{e}_{x_1}^n = 1, \mathbf{e}_{x_2}^n = 1, \ldots, \mathbf{e}_{x_s}^n = 1, \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_t \right\rangle,$$

where each relation $\mathbf{r}_i$ is obtained from the relation r_i by replacing each expression $x * y$ by $\mathbf{e}_y^{-1} \mathbf{e}_x \mathbf{e}_y$ and $x *^{-1} y$ by $\mathbf{e}_y \mathbf{e}_x \mathbf{e}_y^{-1}$.

Given a quandle X and an integer $n \geq 2$, the n-quandle X_n of X is defined as the quotient of X by the relations

$$x *^n y := x * \underbrace{y * y * \cdots * y}_{n \text{ times}} = x$$

for all $x, y \in X$. It follows that

$$\mathrm{Env}_n(X) \cong \mathrm{Env}_n(X_n)$$

for each $n \geq 2$.

2.2. **Homogeneous representation of n-quandles.** Given a quandle X, there is a right action of $\mathrm{Env}(X)$ on X, which on generators of $\mathrm{Env}(X)$ is given by

$$x \cdot e_y = x * y$$

for $x, y \in X$. Recall that

$$\mathrm{Env}_n(X) \cong \mathrm{Env}(X)/\langle\!\langle e_y^n,\ y \in X \rangle\!\rangle.$$

Let $g = e_{y_1}^{\epsilon_1} e_{y_2}^{\epsilon_2} \cdots e_{y_r}^{\epsilon_r}$ be an element of $\mathrm{Env}(X)$, where $y_i \in X$ and $\epsilon_i \in \{1, -1\}$. If X is an n-quandle, then for each $x \in X$, we have

$$\begin{aligned} x \cdot (g e_y^n g^{-1}) &= x *^{\epsilon_1} y_1 *^{\epsilon_2} y_2 * \cdots *^{\epsilon_r} y_r * \underbrace{y * y * \cdots * y}_{n-\text{times}} *^{-\epsilon_r} y_r * \cdots *^{-\epsilon_2} y_2 *^{-\epsilon_1} y_1 \\ &= x *^{\epsilon_1} y_1 *^{\epsilon_2} y_2 * \cdots *^{\epsilon_r} y_r *^{-\epsilon_r} y_r * \cdots *^{-\epsilon_2} y_2 *^{-\epsilon_1} y_1 \\ &= x, \end{aligned}$$

and hence the action descends to an action of $\mathrm{Env}_n(X)$ on X. The following result can be proved easily, and we present a proof for the benefit of the reader.

Proposition 2.3. *Let X be an n-quandle and $\{x_i \mid i \in I\}$ a set of representatives of orbits of X under the action of $\mathrm{Env}(X)$. Let H_i be the stabiliser of x_i in $\mathrm{Env}_n(X)$ under the above action. Then H_i lies in the centraliser of $\mathbf{e}_{x_i}$ in $\mathrm{Env}_n(X)$ and the orbit map induces an isomorphism $\sqcup_{i \in I}(\mathrm{Env}_n(X)/H_i, \mathbf{e}_{x_i}) \cong X$ of quandles.*

Proof. Let $h \in H_i$ such that $h = \mathbf{e}_{x_1}^{\epsilon_1} \mathbf{e}_{x_2}^{\epsilon_2} \cdots \mathbf{e}_{x_r}^{\epsilon_r}$ for some $x_j \in X$ and $\epsilon_j \in \{1, -1\}$. Then, we see that

$$\begin{aligned} h^{-1} \mathbf{e}_{x_i} h &= \mathbf{e}_{x_r}^{-\epsilon_r} \mathbf{e}_{x_{r-1}}^{-\epsilon_{r-1}} \cdots \mathbf{e}_{x_1}^{-\epsilon_1} \mathbf{e}_{x_i} \mathbf{e}_{x_1}^{\epsilon_1} \mathbf{e}_{x_2}^{\epsilon_2} \cdots \mathbf{e}_{x_r}^{\epsilon_r} \\ &= \mathbf{e}_{x_i *^{\epsilon_1} x_1 *^{\epsilon_2} x_2 * \cdots *^{\epsilon_r} x_r} \\ &= \mathbf{e}_{x_i \cdot h} \\ &= \mathbf{e}_{x_i}, \end{aligned}$$

and hence H_i lies in the centraliser of $\mathbf{e}_{x_i}$ in $\mathrm{Env}_n(X)$. Thus, we obtain the quandle $\sqcup_{i \in I}(\mathrm{Env}_n(X)/H_i, \mathbf{e}_{x_i})$. Since $\mathrm{Env}(X)$ acts transitively on connected components of X, the induced action of $\mathrm{Env}_n(X)$ is also transitive on connected components of X. Further, since H_i is the stabiliser of x_i in $\mathrm{Env}_n(X)$, we have a bijection

$$\phi : \sqcup_{i \in I}(\mathrm{Env}_n(X)/H_i, \mathbf{e}_{x_i}) \to X$$

induced by the orbit map $H_i g \mapsto x_i \cdot g$. It remains to check that ϕ is a quandle homomorphism. Indeed, for $u, v \in \mathrm{Env}_n(X)$, we have

$$\begin{aligned} \phi(H_i u * H_j v) &= \phi(H_i \mathbf{e}_{x_i}^{-1} u v^{-1} \mathbf{e}_{x_j} v) \\ &= x_i \cdot (\mathbf{e}_{x_i}^{-1} u v^{-1} \mathbf{e}_{x_j} v) \\ &= x_i \cdot (u v^{-1} \mathbf{e}_{x_j} v), \text{ since } x_i \cdot \mathbf{e}_{x_i}^{-1} = x_i *^{-1} x_i = x_i \\ &= (((x_i \cdot u) \cdot v^{-1}) * x_j) \cdot v \\ &= (x_i \cdot u) * (x_j \cdot v) \\ &= \phi(H_i u) * \phi(H_j v), \end{aligned}$$

which completes the proof. □

As a consequence, we generalise a result of Hoste and Shanahan [12, Theorem 3.2] to arbitrary n-quandles. See also [4, Proposition 3.1] and [13, Section 3.5] for the one way implication.

Corollary 2.4. *Let X be an n-quandle for some $n \geq 2$. Then X is finite if and only if $\mathrm{Env}_n(X)$ is finite.*

Proof. By [12, Theorem 3.2], if X is finite, then $\mathrm{Env}_n(X)$ is finite. The converse follows from Proposition 2.3. □

3. Residual finiteness and fundamental n-quandles of links

3.1. **Residual finiteness and subquandle separability of quandles.** We begin by recalling the definition of a subgroup separable group.

Definition 3.1. A subset S of a group G is said to be *separable* in G if for each $x \in G \setminus S$, there exists a finite group F and a group homomorphism $\phi : G \to F$ such that $\phi(x) \notin \phi(S)$. If the singleton set S consisting of only the identity element is separable, then G is called *residually finite*. If each finitely generated subgroup of G is separable, then G is called *subgroup separable*.

Recall that, the *profinite topology* on a group G has a basis consisting of right cosets of all finite index subgroups of G. By definition, every right coset of a finite index subgroup is closed in the profinite topology. An easy check shows that a subgroup H of G is separable in G if and only if H is closed in the profinite topology on G [15, 21]. We will use this equivalent definition of subgroup separability to prove the following result.

Proposition 3.2. *Let H and K be subgroups of G such that $[G : K]$ is finite. Let $L = H \cap K$ such that $[H : L]$ is finite and L is a separable subgroup of K. Then H is a separable subgroup of G.*

Proof. Since L is a separable subgroup of K, it is closed in the profinite topology on K. Since $[G : K]$ is finite, it follows that K is closed in the profinite topology on G. Consequently, L is closed in the profinite topology on G. Since $[H : L]$ is finite, H is a finite union of right cosets of L, and hence it is closed in the profinite topology on G. Thus, it follows that H is a separable subgroup of G. □

As a consequence of Proposition 3.2, we recover the following well-known result.

Corollary 3.3. *Let G be a group admitting a residually finite subgroup of finite index. Then G is residually finite.*

In analogy with groups, we introduce the following definition for quandles.

Definition 3.4. A subset S of a quandle X is said to be *separable* in X if for each $x \in X \setminus S$, there exists a finite quandle F and a quandle homomorphism $\phi : X \to F$ such that $\phi(x) \notin \phi(S)$. If each singleton set is separable, then X is called *residually finite*. If each finitely generated subquandle of X is separable, then X is called *subquandle separable*.

We note that residual finiteness of fundamental quandles of links has been established recently in [6, 7].

Let G be a group and H a finitely generated subgroup of G. The *generalised word problem* is the problem of deciding for an arbitrary element w in G whether or not w lies in H. Let X be a quandle and Y its finitely generated subquandle. We can define the generalised word problem for quandles as the problem of deciding for an arbitrary element w in X whether or not w lies in Y. The following is an analogue of the corresponding result for groups.

Proposition 3.5. *A finitely presented subquandle separable quandle has the solvable generalised word problem.*

Proof. Let $X = \langle S \mid R \rangle$ be a finitely presented subquandle separable quandle and Y be its finitely generated subquandle. Let x be an element of X. We describe two procedures to determine whether x is in Y or not. The first procedure lists all the elements obtained from elements of Y using the relations in R. If, at some stage, x turns up as one of these elements, then $x \in Y$.

The second procedure lists all the finite quandles. Since X is finitely generated, for each finite quandle F, the set $\mathrm{Hom}(X, F)$ of all quandle homomorphisms is finite. Now, for each homomorphism $\phi \in \mathrm{Hom}(X, F)$, we look for $\phi(x)$ and $\phi(Y)$ in F and check whether or not $\phi(x) \in \phi(Y)$. If, at some stage, $\phi(x) \notin \phi(Y)$, then $x \notin Y$. Since X is a finitely presented subquandle separable quandle and Y is a finitely generated subquandle of X, one of the above procedures must stop in finite time. $\square$

The following result from [7, Proposition 3.4] will be used later in proving our main result on fundamental n-quandles of links.

Proposition 3.6. *Let G be a group, $\{x_i \mid i \in I\}$ be a finite set of elements of G, and $\{H_i \mid i \in I\}$ a finite set of subgroups of G such that $H_i \leq \mathrm{C}_G(x_i)$ for each i. If each H_i is separable in G, then the quandle $\sqcup_{i \in I}(G/H_i, x_i)$ is residually finite.*

3.2. **Residual finiteness of fundamental n-quandles of links.** Let L be an oriented link in $\mathbb{S}^3$ with components $K_1, K_2, \ldots, K_m$. Then, as in Example 2.2, we can associate the *fundamental quandle* $Q(L)$ to the link L, which is constructed from a regular diagram $D(L)$ of L and admits the presentation

$$Q(L) = \langle x_1, x_2, \ldots, x_s \mid r_1, r_2, \ldots, r_t \rangle,$$

where each x_i is an arc of $D(L)$, and each relation r_l is given by $x_k * x_j = x_i$ as per the corresponding crossing in $D(L)$. The preceding presentation yields the presentation of $\mathrm{Env}(Q(L))$, which is precisely the Wirtinger presentation of the link group $\pi_1(\mathbb{S}^3 \setminus L)$. Hence, $\mathrm{Env}(Q(L)) \cong \pi_1(\mathbb{S}^3 \setminus L)$ for any link L.

A less sensitive, but presumably more tractable invariant of a link L is the *fundamental n-quandle* $Q_n(L)$ defined for each natural number $n \geq 2$ as the quandle with the presentation

$$Q_n(L) = \langle x_1, x_2, \ldots, x_s \mid r_1, r_2, \ldots, r_t, u_1, u_2, \ldots, u_k \rangle,$$

where each relation u_ℓ is of the form $x_i *^n x_j = x_i$ for distinct generators x_i and x_j. It follows from [8, Proposition 3.1] that the additional relations $u_1, u_2, \ldots, u_k$ suffice to make $Q_n(L)$ an n-quandle. If L is a link with more than one component, then both $Q(L)$ and $Q_n(L)$ are disconnected with one component $Q^i(L)$ and $Q^i_n(L)$, respectively, for each component K_i of L.

Passing from the presentation of $Q_n(L)$ to the presentation for $\mathrm{Env}_n(Q_n(L))$, we see that $\mathrm{Env}_n(Q_n(L))$ is a quotient of $\mathrm{Env}(Q(L))$. In fact, we may present $\mathrm{Env}_n(Q_n(L))$ by adjoining the relations $x^n = 1$ for each Wirtinger generator x of the link group $\pi_1(\mathbb{S}^3 \setminus L) \cong \mathrm{Env}(Q(L))$. While the fundamental quandle of a non-trivial knot, except the Hopf link, is always infinite, its corresponding fundamental n-quandle can be finite. In fact, it is known due to Hoste and Shanahan [12, Theorem 3.1] that, if $\widetilde{M}_n(L)$ is the n-fold cyclic branched cover of $\mathbb{S}^3$, branched over the link L, then $Q_n(L)$ is finite if and only if $\pi_1(\widetilde{M}_n(L))$ is finite.

Proposition 3.7. *If L is an oriented link and $n \geq 2$, then $\mathrm{Env}_n(Q_n(L))$ is a residually finite group.*

Proof. In view of [22, Remark 5.1.5, Theorem 5.2.2], for each $n \geq 2$, we have

$$\pi_1\left(\widetilde{M}_n(L)\right) \cong E^0_n,$$

where E^0_n is the subgroup of $\mathrm{Env}_n(Q_n(L))$ consisting of all elements whose total exponent sum equals to zero modulo n. Since fundamental groups of 3-manifolds are residually finite [11], it follows that $\pi_1(\widetilde{M}_n(L))$, and hence E^0_n is residually finite. By [12, Section 3], the subgroup E^0_n is of finite index in $\mathrm{Env}_n(Q_n(L))$. Hence, by Corollary 3.3, $\mathrm{Env}_n(Q_n(L))$ is residually finite. □

Proposition 3.8. *If K is an oriented knot and $n \geq 2$, then $\mathrm{Env}(Q_n(K)) \cong \pi_1(\widetilde{M}_n(K)) \rtimes \mathbb{Z}$. Moreover, $\mathrm{Env}(Q_n(K))$ is residually finite.*

Proof. Since K is a knot, the fundamental n-quandle $Q_n(K)$ is connected and $E^0_n = [\mathrm{Env}_n(Q_n(K)), \mathrm{Env}_n(Q_n(K))]$. By [4, Corollary 4.2], we have

$$\mathrm{Env}(Q_n(K)) \cong [\mathrm{Env}_n(Q_n(K)), \mathrm{Env}_n(Q_n(K))] \rtimes \mathbb{Z}.$$

Further, by [22, Remark 5.1.5, Theorem 5.2.2], we have

$$[\mathrm{Env}_n(Q_n(K)), \mathrm{Env}_n(Q_n(K))] \cong \pi_1(\widetilde{M}_n(K))$$

and hence $\mathrm{Env}(Q_n(K)) \cong \pi_1(\widetilde{M}_n(K)) \rtimes \mathbb{Z}$. For the second assertion, recall from [17, Theorem 7] that semi-direct products of finitely generated residually finite groups are residually finite. Since both $\pi_1(\widetilde{M}_n(K))$ and $\mathbb{Z}$ are residually finite, it follows that $\mathrm{Env}(Q_n(K))$ is residually finite. □

Theorem 3.9. *If L is an oriented link and $n \geq 2$, then $\pi_1(\widetilde{M}_n(L))$ is abelian subgroup separable.*

Proof. By the Prime Decomposition Theorem [18, Theorem 1], every compact connected orientable 3-manifold without boundary which is not the 3-sphere is homeomorphic to a connected sum of prime 3-manifolds. Thus, we have $\widetilde{M}_n(L) = N_1 \# N_2 \# \cdots \# N_q$, where each N_i is a compact connected orientable prime 3-manifold without boundary. The van-Kampen Theorem gives

$$\pi_1(\widetilde{M}_n(L)) \cong \pi_1(N_1) * \pi_1(N_2) * \cdots * \pi_1(N_q).$$

Recall that a 3-manifold is irreducible if every embedded 2-sphere bounds a 3-ball. It is known that, with the exception of 3-manifolds $\mathbb{S}^3$ and $\mathbb{S}^1 \times \mathbb{S}^2$, an orientable manifold is prime if and only if it is irreducible [18, Lemma 1]. By [10, Proposition 6], a free product of abelian subgroup separable groups is abelian subgroup separable. Thus, it suffices to prove that $\pi_1(N)$ is abelian subgroup separable for each compact connected irreducible orientable 3-manifold N without boundary.

By the Geometrization Theorem [2, Theorem 1.7.6], if N is such a 3-manifold, then there exists a (possibly empty) collection of disjointly embedded incompressible tori $T_1, \ldots, T_p$ in N such that each component of N cut along $T_1 \cup \cdots \cup T_p$ is hyperbolic or Seifert fibered.

- If N is Seifert fibered, then by [20, Corollary 5.1], $\pi_1(N)$ is double coset separable, and hence it is abelian subgroup separable.
- If N is hyperbolic, then $\pi_1(N)$ is subgroup separable by [2, Corollary 4.2.3], and hence it is abelian subgroup separable.
- If N admits an incompressible torus, then it is Haken [2, p.45, A.10]. It follows from [10, Theorem 1] that $\pi_1(N)$ is abelian subgroup separable.

This completes the proof of the theorem. □

Let L be a link with components $K_1, K_2, \ldots, K_m$ and $n \geq 2$. For each i, let m_i and ℓ_i be the fixed longitude and the meridian of the component K_i, respectively. By abuse of notation, we also denote by m_i and ℓ_i their images in the quotient $\mathrm{Env}_n(Q_n(L))$. In view of the isomorphism $\pi_1\left(\widetilde{M}_n(L)\right) \cong E_n^0$, we can further view each ℓ_i as an element of $\pi_1\left(\widetilde{M}_n(L)\right)$.

Proposition 3.9 leads to the following result.

Corollary 3.10. *Let L be an oriented link with components $K_1, K_2, \ldots, K_m$ and $n \geq 2$. Then, the subgroup $\langle \ell_i \rangle$ generated by the fixed longitude ℓ_i of K_i is subgroup separable in $\pi_1(\widetilde{M}_n(L))$ for each i.*

Corollary 3.11. *Let L be an oriented link with components $K_1, K_2, \ldots, K_m$ and $n \geq 2$. For each i, let m_i and ℓ_i be the fixed meridian and the longitude of K_i, respectively. Then, $P_i = \langle m_i, \ell_i \rangle$ is a separable subgroup of $\mathrm{Env}_n(Q_n(L))$ for each i.*

Proof. By [12, Section 3], the subgroup $\pi_1(\widetilde{M}_n(L))$ is of finite index in $\mathrm{Env}_n(Q_n(L))$. Since $\ell_i \in \pi_1\left(\widetilde{M}_n(L)\right) \cong E_n^0$, it follows that $P_i \cap \pi_1(\widetilde{M}_n(L)) = \langle \ell_i \rangle$. Also, we have $[P_i : \ell_i] = n$. Further, by Corollary 3.10, $\langle \ell_i \rangle$ is a separable subgroup of $\pi_1\left(\widetilde{M}_n(L)\right)$. Hence, by Proposition 3.2, P_i is a separable subgroup of $\mathrm{Env}_n(Q_n(L))$. □

We can now deduce the main result of this section.

Theorem 3.12. *If L is an oriented link and $n \geq 2$, then the fundamental n-quandle $Q_n(L)$ is residually finite.*

Proof. Let L be an oriented link with components $K_1, K_2, \dots, K_m$. Let m_i and ℓ_i be the fixed meridian and the longitude of K_i, respectively. Then, by [12, Theorem 1.1] or Proposition 2.3, we can write

$$Q_n(L) \cong \sqcup_{i=1}^{m} (\mathrm{Env}_n(Q_n(L))/P_i, m_i),$$

where $P_i = \langle m_i, \ell_i \rangle$. Corollary 3.11 implies that each P_i is a separable subgroup of $\mathrm{Env}_n(Q_n(L))$. The result now follows from Proposition 3.6. $\square$

By [6, Theorem 5.11], every finitely presented residually finite quandle has the solvable word problem. Thus, the preceding theorem leads to the following corollary.

Corollary 3.13. *If L is an oriented link and $n \geq 2$, then the fundamental n-quandle $Q_n(L)$ has the solvable word problem.*

We conclude this section with the following natural problem.

Problem 3.14. Classify links in $\mathbb{S}^3$ whose fundamental quandles and fundamental n-quandles (for $n \geq 2$) are subquandle separable. As expected, the problem is intimately related to subgroup separability of link groups.

4. Residual finiteness of general quandles

In this section, we establish residual finiteness of some classes of quandles.

Definition 4.1. Let S be a non-empty set and $n \geq 2$. A quandle $FQ_n(S)$ containing S is called a *free n-quandle* on the set S, if given any map $\phi : S \to X$, where X is an n-quandle, there is a unique quandle homomorphism $\bar{\phi} : FQ_n(S) \to X$ such that $\bar{\phi}|_S = \phi$.

Proposition 4.2. *Let S be a non-empty set and $n \geq 2$. Then $FQ_n(S) \cong FQ(S)_n$, where the latter is the n-quandle of the free quandle $FQ(S)$ on S. Further, $FQ_n(S)$ is residually finite.*

Proof. Let $\phi : S \to X$ be a map, where X is an n-quandle. Then, by the universal property of free quandles, we have a unique quandle homomorphism $\tilde{\phi} : FQ(S) \to X$ such that $\tilde{\phi}|_S = \phi$. Since X is an n-quandle, the homomorphism $\tilde{\phi}$ factors through the n-quandle $FQ(S)_n$ of $FQ(S)$. That is, there exists a unique quandle homomorphism $\bar{\phi} : FQ(S)_n \to X$ such that $\bar{\phi}|_S = \phi$. Thus, by the uniqueness of universal objects, $FQ(S)_n \cong FQ_n(S)$.

Consider the free product $G = *_S \mathbb{Z}_n$ of cyclic groups of order n, one for each element of S. Since free products of residually finite groups are residually finite, G is a residually finite group, and hence $\mathrm{Conj}(G)$ is a residually finite quandle. By [13, Section 2.11, Corollary 2], $FQ_n(S)$ is a subquandle of $\mathrm{Conj}(G)$, and hence it is also residually finite. $\square$

Let G be a group and $\alpha \in \mathrm{Aut}(G)$. Then the *twisted conjugation quandle* $\mathrm{Conj}(G, \alpha)$ is the set G equipped with the quandle operation

$$x * y = \alpha(y^{-1}x)y.$$

These structures appeared in Andruskiewitsch-Graña [1, Section 1.3.7] as twisted homogeneous crossed sets. We prefer calling them twisted conjugation quandles since $\mathrm{Conj}(G, \alpha) = \mathrm{Conj}(G)$ when α is the identity map.

Proposition 4.3. *Let G be a finitely generated residually finite group and $\alpha \in \mathrm{Aut}(G)$. Then the twisted conjugation quandle $\mathrm{Conj}(G, \alpha)$ is residually finite.*

Proof. By [3], there is an embedding of quandles $\mathrm{Conj}(G, \alpha) \hookrightarrow \mathrm{Conj}(G \rtimes_\alpha \mathbb{Z})$, where the action of $\mathbb{Z}$ on G is defined via the automorphism α. By [17, Theorem 7, p.29], a split extension of a residually finite group by a finitely generated residually finite group is again residually finite. Thus, $G \rtimes_\alpha \mathbb{Z}$ is a residually finite group, and hence $\mathrm{Conj}(G \rtimes_\alpha \mathbb{Z})$ is a residually finite quandle. Since subquandle of a residually finite quandle is residually finite, $\mathrm{Conj}(G, \alpha)$ is residually finite. $\square$

A quandle X is said to be *abelian* if $\mathrm{Inn}(X)$ is an abelian group. Equivalently, X is abelian if $(x * y) * z = (x * z) * y$ for all $x, y, z \in X$. In [14], a description of all finite quandles with abelian enveloping groups has been given, and it has been proved that any such quandle must be abelian.

Proposition 4.4. *If X is a finitely generated abelian quandle, then $\mathrm{Env}(X)$ is a residually finite group.*

Proof. We have the central extension

$$1 \to \ker(\psi_X) \to \mathrm{Env}(X) \xrightarrow{\psi_X} \mathrm{Inn}(X) \to 1,$$

where $\psi_X(e_x) = S_x$ for each $x \in X$. Since X is finitely generated, $\mathrm{Env}(X)$ is finitely generated. Further, since X is abelian, $\mathrm{Inn}(X)$ is an abelian group. Thus, $\mathrm{Env}(X)$ is a finitely generated metabelian group. It follows from the well-known result of Hall [9, Theorem 1] that $\mathrm{Env}(X)$ is residually finite. $\square$

Proposition 4.5. *A finitely generated abelian quandle is residually finite.*

Proof. Let X be a finitely generated abelian quandle. Let $\{x_i \mid i \in I\}$ be a finite set of representatives of orbits of X under the action of $\mathrm{Inn}(X)$, and let H_i be the stabiliser of x_i under this action. Since $H_i \leq \mathrm{C}_{\mathrm{Inn}(X)}(S_{x_i})$, arguments in the proof of proposition in [13, Section 2.4] shows that

$$X \cong \sqcup_{i \in I} (\mathrm{Inn}(X)/H_i, S_{x_i})$$

as quandles. Since $\mathrm{Inn}(X)$ is abelian, each of its subgroup, in particular, each H_i is separable in $\mathrm{Inn}(X)$. Thus, by Proposition 3.6, X is a residually finite quandle. $\square$

5. Subquandle separability of general quandles

In this section, we explore subquandle separability of some classes of quandles.

Proposition 5.1. *A trivial quandle is subquandle separable.*

Proof. Let X be a trivial quandle and S a finitely generated subquandle of X. If X has only one element, then there is nothing to prove. Suppose that X has at least two elements. Consider $x \in X \setminus S$ and let $\{a, b\}$ be a trivial quandle. Define $\phi : X \to \{a, b\}$ by $\phi(x) = a$ and $\phi(z) = b$ for all $z \neq x$. It is easy to see that ϕ is a quandle homomorphism with $\phi(x) \notin \phi(S)$, and hence X is subquandle separable. $\square$

Proposition 5.2. *If X is a residually finite quandle, then every finite subquandle of X is separable.*

Proof. Let $S = \{x_1, x_2, \ldots, x_k\}$ be a finite subquandle of X. For each $z \in X \setminus S$, there exists a finite quandle Y_i and a quandle homomorphism $\phi_i : X \to Y_i$ such that $\phi_i(z) \neq \phi_i(x_i)$. Define $\Phi : X \to \prod_{i=1}^{k} Y_i$ by $\Phi(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_k(x))$. Then, we have $\Phi(z) \notin \Phi(S)$, which is desired. □

If G is a group and $\alpha \in \mathrm{Aut}(G)$, then the binary operation

$$x * y = \alpha(xy^{-1})y$$

gives a quandle structure on G, denoted by $\mathrm{Alex}(G, \alpha)$. These quandles are called *generalized Alexander quandles*. If G is abelian, then $\mathrm{Alex}(G, \alpha)$ is precisely the twisted conjugation quandle $\mathrm{Conj}(G, \alpha)$. The following results generalise [6, Proposition 4.1 and Proposition 4.2].

Proposition 5.3. *Let G be a subgroup separable group, H a finitely generated subgroup of G and α an inner automorphism of G such that $\alpha(H) = H$. Then the following assertions hold:*

(1) $\mathrm{Alex}(H, \alpha|_H)$ *is a separable subquandle of* $\mathrm{Alex}(G, \alpha)$.
(2) $\mathrm{Conj}(H, \alpha|_H)$ *is a separable subquandle of* $\mathrm{Conj}(G, \alpha)$.

Proof. Let α be the inner automorphism induced by $g \in G$, and let $x \in \mathrm{Alex}(G, \alpha) \setminus \mathrm{Alex}(H, \alpha|_H)$. By subgroup separability of G, there exists a finite group F and a group homomorphism $\phi : G \to F$ such that $\phi(x) \notin \phi(H)$. Let β be the inner automorphism of F induced by $\phi(g)$. It follows that ϕ viewed as a map $\mathrm{Alex}(G, \alpha) \to \mathrm{Alex}(F, \beta)$ is a quandle homomorphism with $\phi(x) \notin \phi(\mathrm{Alex}(H, \alpha|_H))$. Hence, $\mathrm{Alex}(H, \alpha|_H)$ is a separable subquandle of $\mathrm{Alex}(G, \alpha)$, which proves (1). The proof of assertion (2) is analogous. □

If G is a group, then the binary operation

$$x * y = yx^{-1}y$$

gives the quandle $\mathrm{Core}(G)$, called the *core quandle* of G.

Corollary 5.4. *Let G be a subgroup separable group and $H \leq G$. Then the following assertions hold:*

(1) If $\mathrm{Conj}(H)$ *is a finitely generated subquandle of* $\mathrm{Conj}(G)$*, then it is a separable subquandle of* $\mathrm{Conj}(G)$.
(2) If $\mathrm{Core}(H)$ *is a finitely generated subquandle of* $\mathrm{Core}(G)$*, then it is a separable subquandle of* $\mathrm{Core}(G)$.

Proof. Since $\mathrm{Conj}(H)$ is finitely generated as a subquandle, it follows that H is finitely generated as a subgroup of G. The first assertion follows by taking α to be the identity map in Proposition 5.3(2). For the second assertion, since $\mathrm{Core}(H)$ is a finitely generated subquandle, it follows that H is a finitely generated subgroup of G. Now, for any $x \in G \setminus H$, there is a finite group F and a group homomorphism $\phi : G \to F$ such that $\phi(x) \notin \phi(H)$. Viewing $\phi : \mathrm{Core}(G) \to \mathrm{Core}(F)$ shows that $\phi(x) \notin \phi(\mathrm{Core}(H))$, which is desired. □

Proposition 5.5. *Let X be a quandle. Then the following assertions hold:*

(1) *If $\{X_i \mid i \in I\}$ is a family of separable subquandles of X, then $\cap_{i\in I} X_i$ is a separable subquandle of X.*

(2) *If $\{\alpha_i \mid i \in I\}$ is a family of automorphisms of X, then $\cap_{i\in I}\mathrm{Fix}(\alpha_i)$ is a separable subquandle of X. Here, $\mathrm{Fix}(\alpha_i) = \{x \in X \mid \alpha_i(x) = x\}$ for each i.*

Proof. If $x \in X \setminus \cap_{i\in I} X_i$, then there exists j such that $x \in X \setminus X_j$. Since X_j is separable in X, we have a surjective quandle homomorphism $\phi : X \to Y$, where Y is finite, such that $\phi(x) \notin \phi(X_j)$. Thus, $\phi(x) \notin \phi(\cap_{i\in I} X_i)$, which is desired.

By [6, Proposition 6.4], each $\mathrm{Fix}(\alpha_i)$ is a separable subquandle of X. It now follows from assertion (1) that $\cap_{i\in I}\mathrm{Fix}(\alpha_i)$ is a separable subquandle of X. □

Let $(X_1, \star_1)$ and $(X_2, \star_2)$ be quandles, $f \in \mathrm{C}_{\mathrm{Aut}(X_1)}(\mathrm{Inn}(X_1))$ and $g \in \mathrm{C}_{\mathrm{Aut}(X_2)}(\mathrm{Inn}(X_2))$. Then, by [5, Section 9], $X := X_1 \sqcup X_2$ turns into a quandle with the operation $*$ defined as

$$x * y = \begin{cases} x \star_1 y & \text{if } x, y \in X_1, \\ x \star_2 y & \text{if } x, y \in X_2, \\ f(x) & \text{if } x \in X_1 \text{ and } y \in X_2, \\ g(x) & \text{if } x \in X_2 \text{ and } y \in X_1. \end{cases}$$

Proposition 5.6. *Let $(X_1, \star_1)$ and $(X_2, \star_2)$ be subquandle separable quandles, $f \in \mathrm{Z}(\mathrm{Inn}(X_1))$ and $g \in \mathrm{Z}(\mathrm{Inn}(X_2))$. Then $(X, *)$ is subquandle separable.*

Proof. Let Y be a subquandle of X and $z \in X \setminus Y$. Without loss of generality, we can assume that $z \in X_1$. Since $z \in X_1$ and $Y \cap X_1$ is a separable subquandle of X_1, there is a surjective quandle homomorphism $\phi : X_1 \to F$, where $(F, \circ)$ is a finite quandle, such that $\phi(z) \notin \phi(Y \cap X_1)$. Let p be a symbol disjoint from F and $F' = F \sqcup \{p\}$. Define a binary operation $\circ'$ on F' as

$$x \circ' y = \begin{cases} x \circ y & \text{if } x, y \in F, \\ p & \text{if } x = p, \\ \phi(f(z)) & \text{if } y = p, \end{cases}$$

where $z \in X_1$ is such that $\phi(z) = x$. We claim that $(F', \circ')$ is a quandle. Suppose that $z_1, z_2 \in X_1$ such that $\phi(z_1) = \phi(z_2) = x$. Since $f \in \mathrm{Inn}(X_1)$, we see that $\phi(f(z_1)) = \phi(f(z_2))$, and the binary operation $\circ'$ is indeed well-defined. Let $S_p : F' \to F'$ be the right multiplication by p. For arbitrary $x_1, x_2 \in F$, let $z_1, z_2 \in X_1$ such that $\phi(z_1) = x_1$ and $\phi(z_2) = x_2$. Suppose that $S_p(x_1) = S_p(x_2)$, that is, $x_1 \circ' p = x_2 \circ' p$. This gives $\phi(f(z_1)) = \phi(f(z_2))$. Since $f \in \mathrm{Inn}(X_1)$, it follows that $x_1 = \phi(z_1) = \phi(z_2) = x_2$. Thus, S_p is injective, and finiteness of F' implies that it is a bijection of F'. Further, we see that

$$S_p(x_1 \circ' x_2) = S_p(x_1 \circ x_2) = \phi f(z_1 \star_1 z_2) = \phi f(z_1) \circ' \phi f(z_2) = S_p(x_1) \circ' S_p(x_2),$$

$$S_p(x_1 \circ' p) = S_p(\phi f(z_1)) = \phi f(f(z_1)) = \phi f(z_1) \circ' p = S_p(x_1) \circ' S_p(p)$$

and

$$S_p(p \circ' x_2) = S_p(p) = p = p \circ' S_p(x_2) = S_p(p) \circ' S_p(x_2).$$

Using the fact that $f \in \mathrm{Z}(\mathrm{Inn}(X_1))$, we have

$$S_{x_1}(x_2 \circ' p) = S_{x_1}(\phi f(z_2)) = \phi S_{z_1} f(z_2) = \phi f S_{z_1}(z_2) = S_{x_1}(x_2) \circ' p = S_{x_1}(x_2) \circ' S_{x_1}(p)$$

and

$$S_{x_1}(p \circ' x_2) = S_{x_1}(p) = p = S_{x_1}(p) \circ' S_{x_1}(x_2).$$

This proves our claim. It is easy to see that the map $\Phi : X \to F'$ defined by $\Phi(X_2) = p$ and $\Phi(y) = \phi(y)$ for $y \in X_1$, is a quandle homomorphism. Further, $\Phi(z) \notin \Phi(Y)$, and the result follows. $\square$

The following observation will be used to establish subquandle separability of some abelian quandles.

Proposition 5.7. *Let X be an abelian quandle. Then the following assertions hold:*

1. *The number of orbits of X equals the cardinality of a minimal generating set for X.*
2. *If X is finitely generated, then its n-quandle X_n is finite for each $n \geq 2$.*

Proof. Let S be a minimal generating set for X. Then the number of orbits of X under the action of $\mathrm{Inn}(X)$ is atmost $|S|$. Suppose that there exist $x, y \in S$ such that they are in the same orbit, that is, there is an element $\eta \in \mathrm{Inn}(X)$ such that $\eta(x) = y$. Since $\mathrm{Inn}(X)$ is an abelian group, we can write $\eta = S_{x_1}^{\epsilon_1} S_{x_2}^{\epsilon_2} \ldots S_{x_r}^{\epsilon_r}$, where $\epsilon_i \in \mathbb{Z}$ and $x_i \in S$ are distinct generators. If $x_i = x$ or y for some i, then by reordering, we can assume that $x_1 = y$ or $x_r = x$. This gives $S_{x_2}^{\epsilon_2} S_{x_3}^{\epsilon_3} \ldots S_{x_{r-1}}^{\epsilon_{r-1}}(x) = y$, where none of the x_i equals x or y. Thus, the generator y can be written as a product of other generators from S, which contradicts the minimality of S. This proves assertion (1).

If X is finitely generated and abelian, then so is X_n. Let $S = \{x_1, x_2, \ldots, x_r\}$ be a finite generating set for X_n. Then, any element of X_n can be written in the form $x_i *^{\epsilon_1} x_1 *^{\epsilon_2} x_2 \ldots *^{\epsilon_r} x_r$ for some $1 \leq i \leq r$ and $0 \leq \epsilon_j \leq n-1$ with $\epsilon_i = 0$. Thus, we have $|X_n| \leq r n^{r-1}$, which proves assertion (2). $\square$

Theorem 5.8. *An abelian quandle generated by two elements is subquandle separable.*

Proof. Let $X = \langle S \mid R \rangle$ be a presentation of X, where $S = \{x, y\}$. Since X is abelian, any element of X can be written in the form $x *^n y$ or $y *^m x$ for some $n, m \in \mathbb{Z}$. If X is finite, then there is nothing to prove. So, we assume that X is infinite. We claim that R must be a singleton set. Suppose that R contains elements from the orbits of both x and y. This implies that $\{x *^n y = x, y *^m x = y\} \subseteq R$ for some $n, m \in \mathbb{Z}$. In this case, X is an $\mathrm{lcm}(n, m)$-quandle. It follows from Proposition 5.7 that X must be a finite quandle, which is a contradiction. Thus, R contains elements from only one orbit, say, $R = \{x *^{n_1} y = x, x *^{n_2} y = x, \ldots, x *^{n_r} y = x\}$. An easy calculation shows that $X = \langle S \mid x *^n y = x \rangle$, where $n = \gcd(n_1, n_2, \ldots, n_r)$, and hence the claim holds.

We can now assume that $X = \langle x, y \mid x *^n y = x \rangle$ for some $n \in \mathbb{Z}$. Let Y be a finitely generated subquandle of X such that $Y \neq X$. If $x *^k y, y *^l x \in Y$ for some $k, l \in \mathbb{Z}$, then $x = (x *^k y) *^{-k} (y *^l x) \in Y$ and $y = (y *^l x) *^{-l} (x *^k y) \in Y$. This gives $Y = X$, which is a contradiction. Hence, Y contains elements from only one orbit, say, that of x. Consequently, Y is generated by $\{x *^{n_1} y, x *^{n_2} y, \ldots, x *^{n_r} y\}$ for some $r \geq 1$ and $n_i \in \mathbb{Z}$. Since X is abelian, it follows that Y is a finite trivial subquandle. The theorem now follows from Proposition 4.5 and Proposition 5.2 $\square$

Let $X = \langle x_1, x_2, \ldots, x_r \rangle$ be a finitely generated abelian quandle. Then, any element of X can be written in the form

$$x_i *^{n_1} x_1 *^{n_2} x_2 \cdots *^{n_r} x_r$$

for some $1 \leq i \leq r$ and $n_j \in \mathbb{Z}$ such that $n_i = 0$. Following [19], we denote the element $x_i *^{n_1} x_1 *^{n_2} x_2 \cdots *^{n_r} x_r$ of X by the tuple of integers $(i; n_1, n_2, \ldots, n_r)$. With this notation, the quandle operation in X is given by

$$(i; n_{11}, n_{12}, \ldots, n_{1r}) * (j; n_{21}, n_{22}, \ldots, n_{2r}) = (i; n_1, n_2, \ldots, n_r), \tag{5.0.1}$$

where $n_k = n_{1k}$ for $k \neq j$ and $n_j = n_{1j} + 1$ if $j \neq i$.

Theorem 5.9. *A finitely generated free abelian quandle is subquandle separable.*

Proof. Let X be a finitely generated free abelian quandle. Then X has a presentation

$$X = \langle x_1, x_2, \ldots, x_r \mid x_i * x_j * x_k = x_i * x_k * x_j \text{ for all } 1 \leq i, j, k \leq r \rangle.$$

Let Y be a subquandle of X such that $Y \neq X$. As in the proof of Theorem 5.8, it is clear that if Y contains elements from each orbit under the action of $\mathrm{Inn}(X)$, then the repeated use of equation (5.0.1) implies that Y contains all the generators of X. This gives $Y = X$, a contradiction. Now, suppose that Y is generated by the set

$$\begin{gathered}\{(i_1; n^1_{11}, n^1_{12}, \ldots, n^1_{1r}), (i_1; n^1_{21}, n^1_{22}, \ldots, n^1_{2r}), \ldots, (i_1; n^1_{k_11}, n^1_{k_12}, \ldots, n^1_{k_1r}),\\ (i_2; n^2_{11}, n^2_{12}, \ldots, n^2_{1r}), (i_2; n^2_{21}, n^2_{22}, \ldots, n^2_{2r}), \ldots, (i_2; n^2_{k_21}, n^2_{k_22}, \ldots, n^2_{k_2r}), \ldots,\\ (i_p; n^p_{11}, n^p_{12}, \ldots, n^p_{1r}), (i_p; n^p_{21}, n^p_{22}, \ldots, n^p_{2r}), \ldots, (i_p; n^p_{k_p1}, n^p_{k_p2}, \ldots, n^p_{k_pr})\}.\end{gathered}$$

Let us set $\{j_1, j_2, \ldots, j_q\} = \{1, 2, \ldots, r\} \setminus \{i_1, i_2, \ldots, i_p\}$. Let $x = (i; n_1, n_2, \ldots, n_r) \in X \setminus Y$. If $i \in \{j_1, j_2, \ldots, j_q\}$, then define the map $\eta : X \to \{a, b\}$ by $\eta(x_t) = a$ for $t \neq i$ and $\eta(x_i) = b$, where $\{a, b\}$ is a two element trivial quandle. Then, η is a quandle homomorphism with $\eta(x) \notin \eta(Y)$, and we are done. Next, let $i \in \{i_1, i_2, \ldots, i_p\}$, say $i = i_t$. Since X is free abelian, $x = (i; n_1, n_2, \ldots, n_r) \notin Y$ if and only if

$$(n_{j_1}, n_{j_2}, \ldots, n_{j_q}) \notin \{(n^{i_t}_{1j_1}, n^{i_t}_{1j_2}, \ldots, n^{i_t}_{1j_q}), (n^{i_t}_{2j_1}, n^{i_t}_{2j_2}, \ldots, n^{i_t}_{2j_q}), \ldots, (n^{i_t}_{k_tj_1}, n^{i_t}_{k_tj_2}, \ldots, n^{i_t}_{k_tj_q})\}.$$

We choose a sufficiently large $N \in \mathbb{N}$ such that

$$(n_{j_1}, n_{j_2}, \ldots, n_{j_q}) \notin \{(n^{i_t}_{1j_1}, n^{i_t}_{1j_2}, \ldots, n^{i_t}_{1j_q}), (n^{i_t}_{2j_1}, n^{i_t}_{2j_2}, \ldots, n^{i_t}_{2j_q}), \ldots, (n^{i_t}_{k_tj_1}, n^{i_t}_{k_tj_2}, \ldots, n^{i_t}_{k_tj_q})\} \mod N.$$

Let X_N be the corresponding N-quandle of X, which is finite by Proposition 5.7. Then, the quandle homomorphism $\eta : X \to X_N$ has the property that $\eta(x) \notin \eta(Y)$, and the proof is complete. $\square$

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6. Declaration

The authors declare that they have no conflicts of interest and that there is no data associated with this paper.

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