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A (ϕ_n, ϕ)-POINCARÉ INEQUALITY ON JOHN DOMAIN

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ABSTRACT. Given a bounded domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$, let ϕ is a Young function satisfying the doubling condition with the constant $K_\phi < 2^n$. If Ω is a John domain, we show that Ω supports a (ϕ_n, ϕ)-Poincaré inequality. Conversely, assume additionally that Ω is simply connected domain when $n = 2$ or a bounded domain which is quasiconformally equivalent to some uniform domain when $n \geq 3$. If Ω supports a (ϕ_n, ϕ)-Poincaré inequality, we show that it is a John domain.

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^n$ with $n \geq 2$. Assume ϕ is a Young function in $[0, \infty)$, that is, $\phi \in C[0, \infty)$ is convex and satisfies $\phi(0) = 0, \phi(t) > 0$ for $t > 0$ and $\lim_{t\to\infty} \phi(t) = \infty$. Recall the Orlicz space $L^\phi(\Omega)$ as the collection of all measurable functions u in Ω with the semi-norm

$$\|u\|_{L^\phi(\Omega)} := \inf\left\{\lambda > 0 : \int_\Omega \phi\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1\right\} < \infty.$$

The classical Orlicz-Sobolev space $W^{1,\phi}(\Omega)$ consists of all measurable functions $u \in L^\phi(\Omega)$ and $\nabla u \in L^\phi(\Omega)$, whose norm is

$$\|u\|_{W^{1,\phi}(\Omega)} := \|u\|_{L^\phi(\Omega)} + \|\nabla u\|_{L^\phi(\Omega)}.$$

Sometimes we consider the homogeneous Orlicz-Sobolev space $\dot{W}^{1,\phi}(\Omega)$ with its norm $\|u\|_{W^{1,\phi}(\Omega)} := \|\nabla u\|_{L^\phi(\Omega)}$, whose sharp embedding has been solved in [11](see also [3] for an alternate formulation of the solution). The detailed description is as follows.

Theorem 1.1. *Let Ω is an open bounded domain in $\mathbb{R}^n$ with finite measure and ϕ is a Young function satisfying*

$$\int_0^t \left(\frac{\tau}{\phi(\tau)}\right)^{\frac{1}{n-1}} d\tau < \infty. \tag{1}$$

Define $\phi_n := \phi \circ H^{-1}$, where

$$H(t) = \left[\int_0^t \left(\frac{\tau}{\phi(\tau)}\right)^{\frac{1}{n-1}} d\tau\right]^{\frac{n-1}{n}} \quad \forall\ t \geq 0. \tag{2}$$

Then $\dot{W}^{1,\phi}(\Omega) \subset L^{\phi_n}(\Omega)$, that is, for any $u \in \dot{W}^{1,\phi}(\Omega)$, one has $u \in L^{\phi_n}(\Omega)$ with $\|u\|_{L^{\phi_n}(\Omega)} \leq C\|u\|_{\dot{W}^{1,\phi}(\Omega)}$, where C is a constant independent of u.

We are interested in bounded domains which supports the imbedding $\dot{W}^{1,\phi}(\Omega) \subset L^{\phi_n}(\Omega)$ or (ϕ_n, ϕ)-Poincaré inequality, that is, there exists a constant $C \geq 1$ such that

$$\|u - u_\Omega\|_{L^{\phi_n}(\Omega)} \leq C\|u\|_{\dot{W}^{1,\phi}(\Omega)}, \tag{3}$$

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where $u_\Omega = \fint_\Omega u = \frac{1}{|\Omega|}\int_\Omega u dx$ denotes the average of u in the set of Ω with $|\Omega| > 0$.

The primary goal of this paper is to effectively characterizes supports the imbedding $\dot{W}^{1,\phi}(\Omega) \subset L^{\phi_n}(\Omega)$ via John domains under certain doubling assumption in ϕ; see Theorem 1.2 below. Recall that a bounded domain $\Omega \subset \mathbb{R}^n$ is called as a c-John domain with respect to some $x_0 \in \Omega$ for some $c > 0$ if for any $x \in \Omega$, there is a rectifiable curve $\gamma : [0, T] \to \Omega$ parameterized by arc-length such that $\gamma(0) = x$, $\gamma(T) = x_0$ and $d(\gamma(t), \Omega^{\complement}) > ct$ for all $t > 0$. We could refer to [6, 7, 8, 9, 36, 37, 38] and references therein for more study about c-John domains. Moreover, a Young function ϕ has the doubling property ($\phi \in \Delta_2$) if

$$K_\phi := \sup_{x>0} \frac{\phi(2t)}{\phi(t)} < \infty. \tag{4}$$

It is well known that if a Young function $\phi \in \Delta_2$ with $K_\phi < 2^n$, then ϕ satisfies (1), see Lemma 2.2.

Theorem 1.2. *Let ϕ be a Young function and $\phi \in \Delta_2$ with $K_\phi < 2^n$ in* (4)*.*

(i) *If $\Omega \subset \mathbb{R}^n$ is a c-John domain, then Ω supports the (ϕ_n, ϕ)-Poincaré inequality* (3) *with the constant C depending on n, c and K_ϕ.*
(ii) *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded simply connected planar domain, or a bounded domain which is a quasiconformally equivalent to some uniform domain when $n \geq 3$. If Ω supports the (ϕ_n, ϕ)- Poincaré inequality, then Ω is a c-John domain, where the constant c depend on n, C, K_ϕ and Ω.*

Remark 1.1. (i) Putting $\phi(t) = t^p$ for some $p \in [1, n)$, we know $L^\phi(\Omega) = L^p(\Omega)$ and $\phi_n(t) = Ct^{\frac{np}{n-p}}$, namely, (ϕ_n, ϕ)-Poincaré inequality (3) equals to Sobolev $\dot{W}^{1,p}$-imbedding or $(\frac{np}{n-p}, p)$-Poincaré inequality: for any $u \in \dot{W}^{1,p}(\Omega)$, there exists a constant $C > 0$ such that

$$\|u - u_\Omega\|_{L^{np/(n-p)}(\Omega)} \leq C\|u\|_{\dot{W}^{1,p}(\Omega)}, \tag{5}$$

where the constant C depends on n, p and c. Noted that c-John domain Ω supports $(\frac{np}{n-p}, p)$-Poincaré inequality, details see Reshetnyak [38] and Martio [37] for $1 < p < n$ and Borjarski [5] (and also Hajlasz [24]) for $p = 1$. On the other hand, additionally assume that Ω is a bounded simply connected planar domain or a domain that is quasiconformally equaivalently to some uniform domain when $n \geq 3$, Buckley and Koskela [7] proved that if (5) holds, then Ω is a c-John domain.

The paper is organized as follows. The proof of Theorem 1.2(i) is proven in Section 2 using Boman's chain property, the embedding $\dot{W}^{1,\phi}(Q) \subset L^{\phi_n}(Q)$ for cube Q and the vector-valued inequality in Orlicz norms for the Hardy-Littlewood maximum operators. Section 2 also contains some property of the doubling Young function. Conversely, together with the aid of some ideas from [7, 25, 34, 40, 41], we obtain the $LLC(2)$ property of Ω, and then prove Theorem 1.2(ii) by a capacity argument; see Section 3 for details.

2. Proof of Theorem 1.2(i)

First we give the embedding $C_c^\infty(\Omega) \subset \dot{W}^{1,\phi}(\Omega)$, which means that $\dot{W}^{1,\phi}(\Omega)$ contains basic functions. In some terms, $\dot{W}^{1,\phi}(\Omega)$ is useful.

Lemma 2.1. *Let ϕ be a Young function. For any bounded domain $\Omega \subset \mathbb{R}^n$, we have $C_c^\infty(\Omega) \subset \dot{W}^{1,\phi}(\Omega)$.*

Proof. Write $L := \|Du\|_{L^\infty(\Omega)}$ and choose $\mathrm{supp} u \subset W \subset \Omega$ such that $|\nabla u(x)| = 0$ for $x \in \Omega \backslash W$.

For any $u \in C_c^1(\Omega)$, we know

$$H := \int_\Omega \phi\left(\frac{|\nabla u(x)|}{\lambda}\right) dx \le \int_W \phi\left(\frac{L}{\lambda}\right) dx = \phi\left(\frac{L}{\lambda}\right)|W|.$$

If $\lambda = (L+1)\left(\phi^{-1}\left(\frac{1}{|W|}\right)\right)^{-1}$, we have $H \le 1$. Hence $u \in \dot{W}^{1,\phi}(\Omega)$. Moreover, $C_c^\infty(\Omega) \subset C_c^1(\Omega)$, we get the desired result. □

Now we give some lemmas of Young function ϕ with the doubling property.

Lemma 2.2. *Let $\phi \in \Delta_2$ be a Young function satisfying $K_\phi < 2^n$, then ϕ satisfy* (1).

Proof. Since $\phi \in \Delta_2$ with $\phi(2t) \le K_\phi \phi(t)$, we have

$$\int_{\frac{t}{2}}^{t} \left(\frac{\tau}{\phi(\tau)}\right)^{\frac{1}{n-1}} d\tau = 2\int_{\frac{t}{4}}^{\frac{t}{2}} \left(\frac{2\tau}{\phi(2\tau)}\right)^{\frac{1}{n-1}} d\tau \ge 2\int_{\frac{t}{4}}^{\frac{t}{2}} \left(\frac{2\tau}{K_\phi \phi(\tau)}\right)^{\frac{1}{n-1}} d\tau,$$

that is,

$$\int_{\frac{t}{4}}^{\frac{t}{2}} \left(\frac{\tau}{\phi(\tau)}\right)^{\frac{1}{n-1}} d\tau \le \frac{K_\phi^{\frac{1}{n-1}}}{2^{\frac{n}{n-1}}} \int_{\frac{t}{2}}^{t} \left(\frac{\tau}{\phi(\tau)}\right)^{\frac{1}{n-1}} d\tau.$$

Therefore,

$$\int_{\frac{t}{2^m}}^{\frac{t}{2^{m-1}}} \left(\frac{\tau}{\phi(\tau)}\right)^{\frac{1}{n-1}} d\tau \le \frac{K_\phi^{\frac{1}{n-1}}}{2^{\frac{n}{n-1}}} \int_{\frac{t}{2^{m-1}}}^{\frac{t}{2^{m-2}}} \left(\frac{\tau}{\phi(\tau)}\right)^{\frac{1}{n-1}} d\tau \le \left(\frac{K_\phi^{\frac{1}{n-1}}}{2^{\frac{n}{n-1}}}\right)^{m-1} \int_{\frac{t}{2}}^{t} \left(\frac{\tau}{\phi(\tau)}\right)^{\frac{1}{n-1}} d\tau.$$

Using $K_\phi < 2^n$, there exists a constant $C > 0$ such that $\sum_{m=1}^{\infty} \left(\frac{K_\phi^{\frac{1}{n-1}}}{2^{\frac{n}{n-1}}}\right)^{m-1} \le C$. Hence

$$\int_0^t \left(\frac{\tau}{\phi(\tau)}\right)^{\frac{1}{n-1}} d\tau \le \sum_{m=1}^{\infty} \left(\frac{K_\phi^{\frac{1}{n-1}}}{2^{\frac{n}{n-1}}}\right)^{m-1} \int_{\frac{t}{2}}^{t} \left(\frac{\tau}{\phi(\tau)}\right)^{\frac{1}{n-1}} d\tau \le C \int_{\frac{t}{2}}^{t} \left(\frac{\tau}{\phi(\tau)}\right)^{\frac{1}{n-1}} d\tau.$$

On the other hand, because $\frac{t}{\phi(t)}$ is decreasing, we know

$$\int_{\frac{t}{2}}^{t} \left(\frac{\tau}{\phi(\tau)}\right)^{\frac{1}{n-1}} d\tau \le \left(\frac{\frac{t}{2}}{\phi(\frac{t}{2})}\right)^{\frac{1}{n-1}} \frac{t}{2} < \infty.$$

As $t > 0$ could be any positive number, we conclude

$$\int_0^t \left(\frac{\tau}{\phi(\tau)}\right)^{\frac{1}{n-1}} d\tau \le C \left(\frac{\frac{t}{2}}{\phi(\frac{t}{2})}\right)^{\frac{1}{n-1}} \frac{t}{2} < \infty.$$

□

Lemma 2.3. *Let $\phi \in \Delta_2$ be a Young function satisfying $K_\phi < 2^n$. Then there exists a constant $C > 0$ such that*

$$\frac{H(A)}{A} \le \frac{C}{\phi(A)^{\frac{1}{n}}}. \tag{6}$$

Proof. Applying Lemma 2.2,

$$\int_0^t \left(\frac{\tau}{\phi(\tau)}\right)^{\frac{1}{n-1}} d\tau \leq C \int_{\frac{t}{2}}^t \left(\frac{\tau}{\phi(\tau)}\right)^{\frac{1}{n-1}} d\tau \leq C \left(\frac{\frac{t}{2}}{\phi(\frac{t}{2})}\right)^{\frac{1}{n-1}} \frac{t}{2}.$$

Together with $K_\phi < 2^n$, we get

$$\frac{H(A)}{A} = \frac{\left(\int_0^A \left(\frac{\tau}{\phi(\tau)}\right)^{\frac{1}{n-1}} d\tau\right)^{\frac{n-1}{n}}}{A} \leq \frac{\left(C\left(\frac{\frac{A}{2}}{\phi(\frac{A}{2})}\right)^{\frac{1}{n-1}} \frac{A}{2}\right)^{\frac{n-1}{n}}}{A}$$

$$\leq \frac{\left(C\left(\frac{\frac{A}{2}}{\frac{1}{K_\phi}\phi(A)}\right)^{\frac{1}{n-1}} \frac{A}{2}\right)^{\frac{n-1}{n}}}{A} \leq \frac{C}{\phi(A)^{\frac{1}{n}}}$$

as desired. □

The Young function ϕ is in ∇_2 ($\phi \in \nabla_2$) if there exists a constant $a > 1$ such that for any $x \geq 0$,

$$\phi(x) \leq \frac{1}{2a}\phi(ax).$$

Lemma 2.4. *If $\phi \in \Delta_2$ be a Young function satisfying $K_\phi < 2^n$, then $\phi_n \in \Delta_2 \cap \nabla_2$.*

Proof. Write

$$H(2t) = \left(\int_0^{2t} \left(\frac{\tau}{\phi(\tau)}\right)^{\frac{1}{n-1}} d\tau\right)^{\frac{n-1}{n}} \geq \left(\int_0^t \left(\frac{2\tau}{K_\phi\phi(\tau)}\right)^{\frac{1}{n-1}} 2d\tau\right)^{\frac{n-1}{n}} = \frac{2}{K_\phi^{\frac{1}{n}}} H(t).$$

Letting $2y = H(2t)$, we know $K_\phi^{\frac{1}{n}} y \geq H\left(\frac{H^{-1}(2y)}{2}\right)$. Therefore,

$$H^{-1}(2y) \leq 2H^{-1}(K_\phi^{\frac{1}{n}} y) \leq 2^2 H^{-1}(K_\phi^{\frac{1}{n}} \frac{K_\phi^{\frac{1}{n}}}{2} y) \leq ... \leq 2^{m+1} H^{-1}(K_\phi^{\frac{1}{n}} \left(\frac{K_\phi^{\frac{1}{n}}}{2}\right)^m y).$$

Because of the range of K, we get $\frac{K_\phi^{\frac{1}{n}}}{2} < 1$. Putting m big enough so that $K_\phi^{\frac{1}{n}} \left(\frac{K_\phi^{\frac{1}{n}}}{2}\right)^m < 1$, we have $H^{-1}(2y) < CH^{-1}(y)$. Hence $H^{-1} \in \Delta_2$ and $\phi_n = \phi \circ H^{-1} \in \Delta_2$.

By the decreasing property of $\frac{\tau}{\phi(\tau)}$,

$$H(2^n x) = \left(\int_0^{2^n x} \left(\frac{\tau}{\phi(\tau)}\right)^{\frac{1}{n-1}} d\tau\right)^{\frac{n-1}{n}} = \left(\int_0^x \left(\frac{2^n\tau}{\phi(2^n\tau)}\right)^{\frac{1}{n-1}} 2^n d\tau\right)^{\frac{n-1}{n}}$$

$$\leq \left(\int_0^x \left(\frac{\tau}{\phi(\tau)}\right)^{\frac{1}{n-1}} 2^n d\tau\right)^{\frac{n-1}{n}} = 2^{n-1} H(x).$$

Hence $2^n x \leq H^{-1}(2^{n-1}H(x))$, it means that $2^n H^{-1}(x) \leq H^{-1}(2^{n-1}x)$. Moreover,

$$2^n \phi \circ H^{-1}(x) \leq \phi(2^n H^{-1}(x)) \leq \phi \circ H^{-1}(2^{n-1}x).$$

Letting $a = 2^{n-1} > 1$, we have $\phi_n(x) \leq \frac{1}{2a}\phi_n(ax)$ and $\phi_n \in \nabla_2$. □

To prove Theorem 1.2(i), we also need the following result.

Lemma 2.5. *Let ϕ be a Young function satisfying* (1). *Then for any cube $Q \subset \mathbb{R}^n$, $u \in \dot{W}^{1,\phi}(Q)$ and $\lambda \geq C_1\|\nabla u\|_{L^\phi(Q)}$, there exists a constant $C_1 = C_1(n) > 0$ such that*

$$\int_Q \phi_n\left(\frac{|u(x) - u_Q|}{\lambda}\right) dx \leq \int_Q \phi\left(\frac{C_1|\nabla u(x)|}{\lambda}\right) dx. \tag{7}$$

Proof. Let $m = 1, A = \phi$ and $\Omega = Q(O, 1)$, where

$$Q(a, b) = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : |x_i - a_i| < b, a = (a_1, a_2, ..., a_n) \in \mathbb{R}^n, b > 0\}.$$

By theorem 4.3 in [14], there exists constants C_1 and c such that

$$\int_{Q(0,1)} \phi_n\left(\frac{|u(x) - c|}{C_1\left(\int_{Q(0,1)} \phi(|\nabla u(x)|)dx\right)^{\frac{1}{n}}}\right) dx \leq \int_{Q(0,1)} \phi\left(|\nabla u(x)|\right) dx.$$

In fact, the above inequality holds if $c = u_{Q(0,1)}$.

If a cube centered at a with sides of length $2l$ paralleled to the axes, there exists an orthogonal transformation T such that $T(Q - a) = Q(O, l)$. For any $u \in \dot{W}^{1,\phi}(Q)$, put $v(x) = \frac{u(T^{-1}(lx)+a)}{\lambda l}$ where $x \in Q(0, 1)$. Then $v \in \dot{W}^{1,\phi}(Q(0, 1))$. Hence

$$\int_{Q(0,1)} \phi_n\left(\frac{|\frac{u(T^{-1}(lx)+a)}{\lambda l} - c|}{C_1\left[\int_{Q(0,1)} \phi\left(l\left|\nabla\left(\frac{u(T^{-1}(lx)+a)}{\lambda l}\right)\right|\right) dx\right]^{\frac{1}{n}}}\right) dx$$
$$\leq \int_{Q(0,1)} \phi\left(l\left|\nabla\left(\frac{u(T^{-1}(lx) + a)}{\lambda l}\right)\right|\right) dx.$$

Using $y = T^{-1}(lx) + a$, we know

$$\int_Q \phi_n\left(\frac{|\frac{u(y)}{\lambda l} - c|}{C_1\left[\int_Q \phi\left(l\left|\nabla\left(\frac{u(y)}{\lambda l}\right)\right|\right) \frac{dy}{l^n}\right]^{\frac{1}{n}}}\right) \frac{dy}{l^n} \leq \int_Q \phi\left(l\left|\nabla\left(\frac{u(y)}{\lambda l}\right)\right|\right) \frac{dy}{l^n},$$

that is,

$$\int_Q \phi_n\left(\frac{|u(y) - \lambda lc|}{C_1\lambda\left(\int_Q \phi\left(\frac{|\nabla u(y)|}{\lambda}\right) dy\right)^{\frac{1}{n}}}\right) dy \leq \int_Q \phi\left(\frac{|\nabla u(y)|}{\lambda}\right) dy.$$

Since $c = v_{Q(O,1)}$, we get $\lambda lc = u_Q$. By variable substitution $u = C_1 u$, we have

$$\int_Q \phi_n\left(\frac{|u(y) - u_Q|}{\lambda(\int_Q \phi\left(\frac{C_1|\nabla u(y)|}{\lambda}\right) dy)^{\frac{1}{n}}}\right) dy \leq \int_Q \phi\left(\frac{C_1|\nabla u(y)|}{\lambda}\right) dy.$$

If $\lambda \geq C_1\|\nabla u\|_{L^\phi(Q)}$, then we get

$$\int_Q \phi_n\left(\frac{|u(y) - u_Q|}{\lambda}\right) dy \leq \int_Q \phi_n\left(\frac{|u(y) - u_Q|}{\lambda\left[\int_Q \phi\left(\frac{C_1|\nabla u(y)|}{\lambda}\right) dy\right]^{\frac{1}{n}}}\right) dy$$

$$\leq \int_Q \phi(\frac{C_1|\nabla u(y)|}{\lambda})dy \leq 1$$

as desired. □

Recalled the Fefferman-Stein type verct-valued inequality for Hardy-Littlewood maximum operator in Orlicz space. Denote by $\mathcal{M}$ the Hardy-Littlewood maximum operator,

$$\mathcal{M}(g)(x) = \sup_{x\in Q} \fint_Q |g|dx$$

with the supremum taken over all cubes $Q \subset \mathbb{R}^n$ containing x.

Lemma 2.6 ([16])**.** *Let $\psi \in \Delta_2 \cap \nabla_2$ be a Young function. For any $0 < q < \infty$, there exists a constant $C > 1$ depending on n, q, K_ψ and a such that for all sequences $\{f_j\}_{j\in\mathbb{N}}$, we have*

$$\int_{\mathbb{R}^n} \psi\left(\left[\sum_{j\in\mathbb{N}} (\mathcal{M}(f_j))^2\right]^{\frac{1}{q}}\right) dx \leq C(n, K_\psi, a) \int_{\mathbb{R}^n} \psi\left(\left[\sum_{j\in\mathbb{N}} (f_j)^2\right]^{\frac{1}{q}}\right) dx.$$

Lemma 2.7. *For any constant $k \geq 1$, sequence $\{a_j\}_{j\in\mathbb{N}}$, and cubes $\{Q_j\}_{j\in\mathbb{N}}$ with $\sum_j \chi_{Q_j} \leq k$, we have*

$$\sum_j |a_j|\chi_{kQ_j} \leq C(k, n) \sum_j [\mathcal{M}(|a_j|^{\frac{1}{2}}\chi_{Q_j})]^2.$$

Proof. By the definition of $\mathcal{M}$, we know

$$\chi_{kQ_j} \leq k^n \mathcal{M}(\chi_{Q_j}).$$

So

$$\sum_j |a_j|\chi_{kQ_j} = \sum_j (|a_j|^{\frac{1}{2}}\chi_{kQ_j})^2 \leq k^{2n} \sum_j [\mathcal{M}(|a_j|^{\frac{1}{2}}\chi_{Q_j})]^2.$$

□

Now we begin to prove Theorem 1.2(i).

Proof of Theorem 1.2(i). Let Ω be a c-John domain. By Boman [6] and Buckley [9], Ω enjoys the following chain property: for every integer $\kappa > 1$, there exist a positive constant $C(\kappa, \Omega)$ and a collection $\mathcal{F}$ of the cubes such that

(i) $Q \subset \kappa Q \subset \Omega$ for all $Q \in \mathcal{F}, \Omega = \cup_{Q\in\mathcal{F}} Q$ and

$$\sum_{Q\in\mathcal{F}} \chi_{\kappa Q} \leq C_{\kappa,c}\chi_\Omega.$$

(ii) $Q_0 \in \mathcal{F}$ is a fixed cube. For any other $Q \in \mathcal{F}$, there exist a subsequence $\{Q_j\}_{j=1}^N \subset \mathcal{F}$, satisfying that $Q = Q_N \subset C_{\kappa,c}Q_j$, $C_{\kappa,c}^{-1}|Q_{j+1}| \leq |Q_j| \leq C_{\kappa,c}|Q_{j+1}|$ and $|Q_j \cap Q_{j+1}| \geq C_{\kappa,c}^{-1} \min\{|Q_j|, |Q_{j+1}|\}$ for all $j = 0, \ldots, N-1$.

Let $\kappa = 5n$, by (i) $Q \subset 5nQ \subset \Omega$ for each $Q \in \mathcal{F}$,

$$d(Q, \partial\Omega) \geq d(Q, \partial(5nQ) \geq \frac{5n-1}{2} l(Q) \geq 2nl(Q),$$

and hence

$$|x-y| \leq \sqrt{n}l(Q) \leq nl(Q) \leq \frac{1}{2}d(Q, \partial\Omega) \leq \frac{1}{2}d(x, \partial\Omega),\ \forall x, y \in Q \in \mathcal{F}.$$

Let $u \in \dot{W}^{1,\phi}(\Omega)$. Up to approximating by $\min\{\max\{u, -N\}, N\}$, we may assume that $u \in L^\infty(\Omega)$, and by the boundedness of Ω, $u \in L^1(\Omega)$.

Using the convexity of ϕ_n, we have

$$\begin{aligned} I := & \int_\Omega \phi_n\left(\frac{|u(z)-u_\Omega|}{\lambda}\right)dz \\ \le & \int_\Omega \phi_n\left[\frac{1}{2}\left(\frac{2|u(z)-u_{Q_0}|+2|u_\Omega-u_{Q_0}|}{\lambda}\right)\right]dz \\ \le & \frac{1}{2}\left[\int_\Omega \phi_n\left(\frac{2|u(z)-u_{Q_0}|}{\lambda}\right)dz+|\Omega|\phi_n\left(\frac{2|u_\Omega-u_{Q_0}|}{\lambda}\right)\right]. \end{aligned}$$

By Jensen inequality,

$$|\Omega|\phi_n\left(\frac{2|u_\Omega-u_{Q_0}|}{\lambda}\right)\le \int_\Omega \phi_n\left(\frac{2|u(z)-u_{Q_0}|}{\lambda}\right)dz.$$

Since $\chi_\Omega \le \sum_{Q\in\mathcal{F}}\chi_Q$ as given (i) above,

$$\begin{aligned} I \le & \int_\Omega \phi_n\left(\frac{2|u(z)-u_{Q_0}|}{\lambda}\right)dz \\ \le & \sum_{Q\in\mathcal{F}}\int_Q \phi_n\left(\frac{2|u(z)-u_{Q_0}|}{\lambda}\right)dz \\ \le & \frac{1}{2}\sum_{Q\in\mathcal{F}}\int_Q \phi_n\left(\frac{4|u(z)-u_Q|}{\lambda}\right)dz+\frac{1}{2}\sum_{Q\in\mathcal{F}\setminus\{Q_0\}}|Q|\phi_n\left(\frac{4|u_Q-u_{Q_0}|}{\lambda}\right) \\ := & \frac{1}{2}I_1+\frac{1}{2}I_2. \end{aligned}$$

Then it suffices to show that

$$I_i \le \int_\Omega \phi\left(\frac{|\nabla u(x)|}{\lambda/C(n,C_{\kappa,c},K_\phi)}\right)dx \quad \text{for } i=1,2.$$

To bound I_1, for any $Q\in\mathcal{F}$, applying inequality (7),

$$I_1 \le \sum_{Q\in\mathcal{F}}\int_Q \phi\left(\frac{|\nabla u(x)|}{\frac{\lambda}{4C_1}}\right)dx.$$

Together with $\sum\chi_Q(x)\le C_{\kappa,c}\chi_\Omega(x)$ as in (i) above, by the convexity we know

$$I_1 \le C_{\kappa,c}\int_\Omega \phi\left(\frac{|\nabla u(x)|}{\frac{\lambda}{4C_1}}\right)dx \le \int_\Omega \phi\left(\frac{|\nabla u(x)|}{\lambda/C(n,C_{\kappa,c})}\right)dx.$$

To estimate I_2, for each $Q\in\mathcal{F}$, applying the chain property given in (ii) above, for any $Q\in\mathcal{F}$ with $Q\ne Q_0$, we obtain

$$\begin{aligned} |u_Q-u_{Q_0}| \le & \sum_{j=0}^{N-1}|u_{Q_j}-u_{Q_{j+1}}| \\ \le & \sum_{j=0}^{N-1}(|u_{Q_j}-u_{Q_{j+1}\cap Q_j}|+|u_{Q_{j+1}}-u_{Q_{j+1}\cap Q_j}|). \end{aligned}$$

For adjacent cubes Q_j, Q_{j+1}, one has $|Q_j \cap Q_{j+1}| \geq C_{\kappa,c}^{-1} \min\{|Q_j|, |Q_{j+1}|\}$ and $C_{\kappa,c}^{-1}|Q_{j+1}| \leq |Q_j| \leq C_{\kappa,c}|Q_{j+1}|$. This implies

$$\begin{aligned}|u_{Q_j} - u_{Q_j\cap Q_{j+1}}| &\leq \frac{1}{|Q_j \cap Q_{j+1}|}\int_{Q_j} |u(v) - u_{Q_j}|\, dv\\ &\leq \frac{C_{\kappa,c}}{\min\{|Q_j|, |Q_{j+1}|\}}\int_{Q_j} |u(v) - u_{Q_j}|\, dv\\ &\leq \frac{C_{\kappa,c}^2}{|Q_j|}\int_{Q_j} |u(v) - u_{Q_j}|\, dv,\end{aligned}$$

and also similar estimate for $|u_{Q_{j+1}} - u_{Q_j\cap Q_{j+1}}|$. Therefore we get

$$|u_Q - u_{Q_0}| \leq 2C_{\kappa,c}^2 \sum_{i=1}^{N} \fint_{Q_j} |u(v) - u_{Q_j}|\, dv.$$

For each cube Q_j, by Jessen inequality, one has

$$\begin{aligned}\fint_{Q_j} \frac{|u(v) - u_{Q_j}|}{\lambda} dv &= {\phi_n}^{-1} \circ \phi_n\left(\fint_{Q_j} \frac{|u(v) - u_{Q_j}|}{\lambda} dv\right)\\ &\leq {\phi_n}^{-1}\left(\fint_{Q_j} \phi_n\left(\frac{|u(v) - u_{Q_j}|}{\lambda}\right) dv\right).\end{aligned}$$

Using inequality (7),

$$\fint_{Q_j} \frac{|u(v) - u_{Q_j}|}{\lambda} dv \leq {\phi_n}^{-1}\left(\int_{Q_j} \phi\left(\frac{|\nabla u(v)|}{\lambda/C_1}\right) dv\right) := {\phi_n}^{-1}\left(\int_{Q_j} f(v) dv\right).$$

Therefore,

$$\frac{4|u_Q - u_{Q_0}|}{\lambda} \leq 8{C_{\kappa,c}}^2 \sum_{j=0}^{N} {\phi_n}^{-1}\left(\fint_{Q_j} f(v) dv\right).$$

Since $\phi_n \in \triangle_2$, we know $\phi_n(tx) \geq t^{K_{\phi_n}-1}\phi_n(x)$ for all $t \in [1,\infty)$ and $x \in \mathbb{R}$. Together with Lemma 2.4, we get

$$\phi_n\left(8{C_{\kappa,c}}^2 \sum_{j=0}^{N} {\phi_n}^{-1}\left(\fint_{Q_j} f(v) dv\right)\right) \leq C(C_{\kappa,c}, K_\phi)\phi_n\left(\sum_{j=0}^{N} {\phi_n}^{-1}\left(\fint_{Q_j} f(v) dv\right)\right).$$

Applying $Q = Q_N \subset C_{\kappa,c}Q_j$ given in (ii), one has

$$|Q|\phi_n\left(\sum_{j=0}^{N} {\phi_n}^{-1}\left(\fint_{Q_j} f(v) dv\right)\right) \leq \int_Q \phi_n\left(\sum_{P\in\mathcal{F}} {\phi_n}^{-1}\left(\fint_P f(v) dv\right)\chi_{C_{\kappa,c}P}\right)(x) dx.$$

By $\sum_{Q\in\mathcal{F}} \chi_Q \leq \sum_{Q\in\mathcal{F}} \chi_{\kappa Q} \leq C_{\kappa,c}\chi_\Omega$ as given (i) above,

$$\begin{aligned}I_2 &\leq C(C_{\kappa,c}, K_\phi) \sum_{Q\in\mathcal{F}} \int_Q \phi_n\left(\sum_{P\in\mathcal{F}} {\phi_n}^{-1}\left(\fint_P f(v) dv\right)\chi_{C_{\kappa,c}P}\right)(x) dx\\ &\leq C(C_{\kappa,c}, K_\phi) \int_\Omega \phi_n\left(\sum_{P\in\mathcal{F}} {\phi_n}^{-1}\left(\fint_P f(v) dv\right)\chi_{C_{\kappa,c}P}\right)(x) dx.\end{aligned}$$

Using Lemma 2.7, we know

$$I_2 \le C(C_{\kappa,c},K_\phi)\int_\Omega \phi_n\left(\sum_{P\in\mathcal{F}}\left\{\mathcal{M}\left[\left(\phi_n{}^{-1}(\fint_P f(v)dv)\right)^{\frac{1}{2}}\chi_P\right]\right\}^2\right)(x)dx.$$

By Lemma 2.4, we know $\phi_n\in\Delta_2\cap\nabla_2$. Then $\phi_n(t^2)\in\Delta_2\cap\nabla_2$ Applying Lemma 2.6 to $q=2$ and $\psi(t):=\phi_n(t^2)$, we obtain

$$I_2 \le CC(C_{\kappa,c},K_\phi,a)\int_\Omega \phi_n\left(\sum_{P\in\mathcal{F}}\left(\phi_n{}^{-1}(\fint_P f(v)dv)\right)\chi_P\right)(x)dx.$$

Let $a_P=|P|^{\frac{\beta}{n}}\fint_P f(v)\,dv$. For each $x\in\Omega$, by the increasing property of ϕ_n and the convexity of ϕ_n, we have

$$\begin{aligned}
\phi_n\left(\sum_{P\in\mathcal{F}}\left(\phi_n{}^{-1}(a_P)\right)\chi_P(x)\right) &= \phi_n\left(\frac{\sum_{P\in\mathcal{F}}\chi_P(x)}{\sum_{P\in\mathcal{F}}\chi_P(x)}\sum_{P\in\mathcal{F}}\left(\phi_n{}^{-1}(a_P)\right)\chi_P(x)\right)\\
&\le \phi_n\left(\frac{C_{\kappa,c}}{\sum_{P\in\mathcal{F}}\chi_P(x)}\sum_{P\in\mathcal{F}}\left(\phi_n{}^{-1}(a_P)\right)\chi_P(x)\right)\\
&\le \sum_{P\in\mathcal{F}}\frac{\chi_P(x)}{\sum_{P\in\mathcal{F}}\chi_P(x)}\phi_n(C_{\kappa,c}\phi_n{}^{-1}(a_P)).
\end{aligned}$$

Applying $\phi_n(tx)\ge t^{K_{\phi_n}-1}\phi_n(x)$ for all $t\in[1,\infty)$ and $x\in\mathbb{R}$. and $\chi_\Omega\le\sum\chi_Q$ as given in (i) above, one gets

$$\begin{aligned}
\phi_n\left(\sum_{P\in\mathcal{F}}\left(\phi_n{}^{-1}(a_P)\right)\chi_P(x)\right) &\le \sum_{P\in\mathcal{F}}\frac{C(C_{\kappa,c},K_\phi)}{\chi_\Omega(x)}\phi_n\left(\phi_n^{-1}(a_P)\right)\chi_P(x)\\
&\le C(C_{\kappa,c},K_\phi)\sum_{P\in\mathcal{F}}\chi_P(x)a_P.
\end{aligned}$$

Using $\sum\chi_Q\le C_{\kappa,c}\chi_\Omega$ again, one gets

$$\begin{aligned}
I_2 &\le C(C_{\kappa,c},K_\phi,a)\int_\Omega\sum_{P\in\mathcal{F}}a_P\chi_P(x)dx\\
&\le C(C_{\kappa,c},K_\phi,a)\sum_{P\in\mathcal{F}}a_P|P| = C(C_{\kappa,c},K_\phi,a)\sum_{P\in\mathcal{F}}\int_P f(v)dv\\
&\le C(C_{\kappa,c},K_\phi,a)\int_\Omega\phi\left(\frac{|\nabla u(v)|}{\lambda/C_1}\right)dv.
\end{aligned}$$

By the convexity, one has

$$I_2\le\int_\Omega\phi\left(\frac{|\nabla u(v)|}{\lambda/C(n,C_{\kappa,c},K_\phi,a)}\right)dv.$$

Combing the estimates I_1 and I_2, we complete the proof. □

3. Proof of Theorem 1.2 (ii)

To prove Theorem 1.2 (ii), we need the following estimates and Lemmas which would be prove later.

Let $z \in \Omega$, $d(z, \partial\Omega) \le m <$ diam Ω. Denote $\Omega_{z,m}$ by a component of $\Omega \setminus \overline{B_\Omega(z,m)}$. For $t > r \ge m$ with $\Omega_{z,m} \neq \emptyset$, define $u_{z,r,t}$ in Ω as

$$u_{z,r,t}(y) = \begin{cases} 0, & y \in \Omega \setminus [\Omega_{z,m} \setminus B_\Omega(z,r)] \\ \frac{|y-z|-r}{t-r}, & y \in \Omega_{z,m} \cap [B(z,t) \setminus B(z,r)] \\ 1, & y \in \Omega_{z,m} \setminus B_\Omega(z,t), \end{cases} \tag{8}$$

where $B_\Omega(z,t) = B(z,t) \cap \Omega$.

It's not difficult to know that $u_{z,r,t}$ is Lipschitz with the Lipschitz constant $\frac{1}{t-r}$.

Lemma 3.1. *Let ϕ be a Young function. For any bounded domain $\Omega \subset \mathbb{R}^n$ and $z \in \Omega$ with $d(z, \partial\Omega) \le m <$ diam Ω. For $t > r \ge m$, we have $u_{z,r,t} \in \dot{W}^{1,\phi}(\Omega)$ with*

$$||u_{z,r,t}||_{\dot{W}^{1,\phi}(\Omega)} \le \left[\phi^{-1}\left(\frac{1}{|\Omega_{z,m} \setminus B(z,r)|}\right)(t-r)\right]^{-1}.$$

Proof. Noting that $u_{z,r,t}$ is Lipschitz with the Lipschitz constant $\frac{1}{t-r}$, then $\nabla u_{z,r,t}$ almost exists and $|\nabla u_{z,r,t}| \le \frac{1}{t-r}$. By the definition of $u_{z,r,t}$, we know $|\nabla u_{z,r,t}| = 0$ in $\Omega \setminus [\Omega_{z,m} \setminus B_\Omega(z,r)]$ and $\Omega_{z,m} \setminus B_\Omega(z,t)$. Hence

$$H := \int_\Omega \phi\left(\frac{|\nabla u_{z,r,t}(x)|}{\lambda}\right) dx \le \int_{\Omega_{z,m} \setminus B(z,r)} \phi\left(\frac{1}{\lambda(t-r)}\right) dx.$$

Letting $\lambda \ge \left[\phi^{-1}\left(\frac{1}{|\Omega_{z,m} \setminus B(z,r)|}\right)(t-r)\right]^{-1}$, we have $H \le 1$ as desired. □

For $x_0, z \in \Omega$, let $r > 0$ such that $d(z, \partial\Omega) < r < |x_0 - z|$. Define

$$\omega_{x_0,z,r}(y) = \frac{1}{r} \inf_{\gamma(x_0,y)} \ell(\gamma \cap B(z,r)), \quad \forall y \in \Omega,$$

where the infimum is taken over all rectifiable curves γ joining x_0 and y.

Lemma 3.2. *Let ϕ be a Young function. For any bounded domain $\Omega \subset \mathbb{R}^n$, $x_0, z \in \Omega$ and $r > 0$ satisfying $d(z, \partial\Omega) < r < |x_0 - z|$, we have $w_{x_0,z,r} \in \dot{W}^{1,\phi}(\Omega)$ with*

$$||\omega_{x_0,z,r}||_{\dot{W}^{1,\phi}(\Omega)} \le C\left[\phi^{-1}(r^{-n})\, r\right]^{-1}$$

where $C \ge 1$ is depending only on n, ω_n and ϕ.

Proof. Let $\gamma_{x,y}$ be the segment joining x, y. Noting that $l(\gamma_{x,y} \cap B(z,r)) \le |x-y|$ for any $x \in \Omega$ and $y \in \Omega$, together with a curve $\gamma(x_0,x) \cup \gamma_{x,y}$ joining x_0, x, we have

$$\omega_{x_0,z,r}(y) \le \omega_{x_0,z,r}(x) + \frac{1}{r}|x-y|.$$

Similarly, $\omega_{x_0,z,r}(x) \le \omega_{x_0,z,r}(y) + \frac{1}{r}|x-y|$. Therefore, we get $|\omega_{x_0,z,r}(y) - \omega_{x_0,z,r}(x)| \le \frac{1}{r}|x-y|$, that is, $\omega_{x_0,z,r}$ is Lipschitz and $\nabla\omega_{x_0,z,r}$ exists with $|\nabla\omega_{x_0,z,r}| \le \frac{1}{r}$.

Noting that $d(x, \partial\Omega) \le |x-z| + d(z, \partial\Omega) \le |x-z| + r$ for $x \in \Omega \setminus B(z, 6r)$, $y \in B(x, \frac{1}{2}d(x, \partial\Omega))$, then we know

$$|y-z| \ge |x-z| - |y-x| \ge |x-z| - \frac{1}{2}(|x-z| + r) = \frac{1}{2}|x-z| - \frac{r}{2} \ge 3r - \frac{r}{2} \ge 2r.$$

that is, $B(x, \frac{1}{2}d(x, \partial\Omega)) \cap B(z, 2r) = \emptyset$. Let $\gamma_{x,y}$ is the segment joining x, y. Then $\gamma_{x,y}$ is in $B(x, \frac{1}{2}d(x, \partial\Omega))$. Moreover, $\gamma_{x,y} \subset \Omega \setminus B(z, r)$. For any curve $\gamma(x_0, x)$, $\gamma(x_0, x) \cup \gamma_{x,y}$ joining x_0 and y, we get

$$l((\gamma(x_0, x) \cup \gamma_{x,y}) \cap B(z, r)) = l(\gamma(x_0, x) \cap B(z, r)).$$

So $\omega_{x_0,z,r}(y) \leq \omega_{x_0,z,r}(x)$.

Similarly, $\omega_{x_0,z,r}(x) \leq \omega_{x_0,z,r}(y)$. Hence $\omega_{x_0,z,r}(x) = \omega_{x_0,z,r}(y), \forall x \in \Omega \setminus B(z, 6r), y \in B(x, \frac{1}{2}d(x, \partial\Omega))$. Since $\omega_{x_0,z,r}(x) = \omega_{x_0,z,r}(y)$ for any $x \in \Omega \setminus B(z, 6r), y \in B(x, \frac{1}{2}d(x, \partial\Omega))$ then $|\nabla\omega_{x_0,z,r}(x)| = 0$ for any $x \in \Omega \setminus B(z, 6r)$. Hence

$$\begin{aligned} H := \int_\Omega \phi\left(\frac{|\nabla\omega_{x_0,z,r}(x)|}{\lambda}\right) dx &= \int_{\Omega\cap B(z,6r)} \phi\left(\frac{|\nabla\omega_{x_0,z,r}(x)|}{\lambda}\right) dx \\ &\leq \int_{\Omega\cap B(z,6r)} \phi\left(\frac{1}{\lambda r}\right) dx \leq \omega_n(6r)^n \phi\left(\frac{1}{\lambda r}\right). \end{aligned}$$

If $\lambda = M\left[\phi^{-1}\left(r^{-n}\right) r\right]^{-1}$ with $M = \omega_n(6r)^n$, then $H \leq 1$. □

Lemma 3.3. *Let $\phi \in \Delta_2$ be a Young function with $K_\phi < 2^n$ in* (4) *and a bounded domain $\Omega \subset \mathbb{R}^n$ supports the (ϕ_n, ϕ)- Poincaré inequality* (3)*. Fix a point x_0 so that $r_0 := \max\{d(x, \partial\Omega) : x \in \Omega\} = d(x_0, \partial\Omega)$. Assume that $x, x_0 \in \Omega \setminus \overline{B(z, r)}$ for some $z \in \Omega$ and $r \in (0, 2\operatorname{diam}\Omega)$, there exists a positive constant b_0 that x, x_0 are contained in the same component of $\Omega \setminus \overline{B(z, b_0 r)}$.*

Proof. Let

$$b_{x,z,r} := \sup\{c \in (0, 1], x, x_0 \text{ are contained in the same component of } \Omega \setminus \overline{B(z, cr)}\}.$$

To get b_0, it sufficient to prove $b_{x,z,r}$ has a positive lower bound independent of x, z, r. We may assume $b_{x,z,r} \leq \frac{1}{10}$. Denote Ω_x as the component of $\Omega \setminus \overline{B(z, 2b_{x,z,r} r)}$ containing x. If exists a constant $C \geq 1$ independent of x, z, r such that

$$\frac{r}{C}\left(\frac{1}{2} - 2b_{x,z,r}\right) \leq |\Omega_x|^{\frac{1}{n}} \leq C2b_{x,z,r} r, \tag{9}$$

we know $b_{x,z,r} > \frac{1}{4(C^2+1)}$ as desired. Set $c_0 = 2b_{x,z,r} < \frac{1}{5}$. Denote by Ω_{x_0} the component of $\Omega \setminus \overline{B(z, c_0 r)}$ containing x_0. Observing

$$r_0 \leq \max_{y\in B(z,c_0r)} |x_0 - y| \leq r + c_0 r + d(x_0, B(z, r)) \leq \frac{6}{5}r + d(x_0, B(z, r))$$

and

$$d(x_0, B(z, c_0 r)) > |x_0 - z| - \frac{1}{5} = d(x_0, B(z, r)) + r - \frac{1}{5}r = d(x_0, B(z, r)) + \frac{4}{5}r,$$

we obtain $d(x_0, B(z, c_0 r)) \geq \frac{r_0}{2}$, hence

$$B(x_0, \frac{r_0}{2}) \subset \Omega_{x_0} \subset \Omega \setminus \Omega_x. \tag{10}$$

Define

$$w(y) = \frac{1}{c_0 r} \inf_{\gamma(x_0,y)} \ell(\gamma \cap B(z, c_0 r)), \quad \forall y \in \Omega,$$

where the infimum is taken over all rectifiable curves γ joining x_0 and y.

By Lemma 3.2,

$$\|\omega\|_{\dot{W}^{1,\phi}(\Omega)} \leq C\left[\phi^{-1}\left(\frac{1}{(c_0 r)^n}\right) c_0 r\right]^{-1},$$

together with (ϕ_n, ϕ)-Poincaré inequality (3), we know

$$\|\omega - \omega_\Omega\|_{L^{\phi_n}(\Omega)} \le C\|\omega\|_{\dot{W}^{1,\phi}(\Omega)} \le C\left[\phi^{-1}\left(\frac{1}{(c_0 r)^n}\right)\right]^{-1} \frac{1}{c_0 r}.$$

On the other hand, by (10), $y \in B(x_0, \frac{1}{2}r_0), \omega(y) = 0$. Since Ω is bounded, $r_0 > 0$, we have $\frac{|\operatorname{diam}\Omega|}{r_0^n} \le C$. Using the convexity of ϕ_n,

$$\int_\Omega \phi_n\left(\frac{|\omega(x)|}{\lambda}\right)dx \le \frac{1}{2}\int_\Omega \phi_n\left(\frac{|\omega(x) - \omega_\Omega|}{\lambda}\right)dx + \frac{|\Omega|}{2}\phi_n\left(\frac{|\omega_{B(x_0,\frac{1}{2}r_0)} - \omega_\Omega|}{\lambda}\right).$$

By the Jensen inequality,

$$\begin{aligned}
|\Omega|\phi_n\left(\frac{|\omega_{B(x_0,\frac{1}{2}r_0)} - \omega_\Omega|}{\lambda}\right) &\le |\Omega| ⨍_{B(x_0,\frac{1}{2}r_0)} \phi_n\left(\frac{|\omega(x) - \omega_\Omega|}{\lambda}\right)dx \\
&\le \frac{|\Omega|}{|B(x_0, \frac{1}{2}r_0)|}\int_\Omega \phi_n\left(\frac{|\omega(x) - \omega_\Omega|}{\lambda}\right)dx \\
&\le 2^n C^n \int_\Omega \phi_n\left(\frac{|\omega(x) - \omega_\Omega|}{\lambda}\right)dx.
\end{aligned}$$

Hence

$$\int_\Omega \phi_n\left(\frac{|\omega(x)|}{\lambda}\right)dx \le C\int_\Omega \phi_n\left(\frac{|\omega(x) - \omega_\Omega|}{\lambda}\right)dx,$$

furthermore, we get

$$\|\omega\|_{L^{\phi_n}(\Omega)} \le C\|\omega - \omega_\Omega\|_{L^{\phi_n}(\Omega)}. \tag{11}$$

Since for any $y \in \Omega_x,\ \omega(y) \ge 1$,

$$\int_\Omega \phi_n\left(\frac{|\omega(x)|}{\lambda}\right)dx \ge \phi_n\left(\frac{1}{\lambda}\right)|\Omega_x|,$$

then we know

$$\|\omega\|_{L^{\phi_n}(\Omega)} \ge \left[{\phi_n}^{-1}\left(\frac{1}{|\Omega_x|}\right)\right]^{-1}.$$

Therefore,

$$C\phi^{-1}\left[\frac{1}{(c_0 r)^n}\right](c_0 r) \le {\phi_n}^{-1}\left[\frac{1}{|\Omega_x|}\right].$$

By $\frac{H(A)}{A} \le C\frac{1}{\phi(A)^{\frac{1}{n}}}$ in (6), letting $A = \phi^{-1}\left[\frac{1}{(c_0 r)^n}\right]$, we have

$$\frac{{\phi_n}^{-1}\left[\frac{1}{(c_0 r)^n}\right]}{\phi^{-1}\left[\frac{1}{(c_0 r)^n}\right]} \le C(c_0 r),$$

that is,

$${\phi_n}^{-1}\left[\frac{1}{(c_0 r)^n}\right] \le C{\phi_n}^{-1}\left[\frac{1}{|\Omega_x|}\right].$$

By Lemma 2.4, $\phi_n \in \Delta_2$, and the fact $\phi_n(tx) \geq t^{K_{\phi_n}-1}\phi_n(x)$ for all $t \in [1, \infty)$ and $x \in \mathbb{R}$, we have $\frac{1}{(c_0 r)^n} \leq C\frac{1}{|\Omega_x|}$ and

$$|\Omega_x|^{\frac{1}{n}} \leq C(c_0 r). \tag{12}$$

For $j \geq 0$ with $\Omega_x \setminus \overline{B(z, c_j r)} \neq \emptyset$, define v_j in Ω as

$$v_j(y) = \begin{cases} 0 & y \in \Omega \setminus [\Omega_x \setminus B_\Omega(z, c_{j+1}r)] \\ \frac{|y-z|-c_j r}{c_{j+1}r - c_j r} & y \in \Omega_x \cap [B(z, c_j r) \setminus B(z, c_{j+1}r)], \\ 1 & y \in \Omega_x \setminus B_\Omega(z, c_j r), \end{cases}$$

Let $\Omega_{z,x} = \Omega_x$, $r = c_j r$ and $t = c_{j+1}r$, then $v_j(y) = u_{z,c_j r, c_{j+1}r}(y)$ where $u_{z,c_j r,c_{j+1}r}(y)$ is defined in (8). Applying Lemma 3.1, we have

$$\|v_j\|_{\dot{W}^{1,\phi}(\Omega)} \leq C\left[\phi^{-1}\left(\frac{1}{|\Omega_x \setminus B(z, c_j r)|}\right)(c_{j+1}r - c_j r)\right]^{-1}.$$

Applying (9), we have $v_j(y) = 0$ for $y \in B(x_0, \frac{1}{2}r_0)$. Similarly to (11), we get

$$\|v_j\|_{L^{\phi_n}(\Omega)} \leq C\|v_j - v_{j\Omega}\|_{L^{\phi_n}(\Omega)}. \tag{13}$$

And $v_j(y) = 1$ for $y \in \Omega_x \setminus B_\Omega(z, c_j r)$, then we have

$$\|v_j\|_{L^{\phi_n}(\Omega)} \geq \left[{\phi_n}^{-1}\left(\frac{1}{|\Omega_x \setminus B_\Omega(z, c_j r)|}\right)\right]^{-1}.$$

By the (ϕ_n, ϕ)-Poincaré inequality (3), we know

$${\phi_n}^{-1}\left(\frac{1}{|\Omega_x \setminus B_\Omega(z, c_j r)|}\right) \geq C\phi^{-1}\left(\frac{1}{|\Omega_x \setminus B(z, c_j r)|}\right)(c_{j+1}r - c_j r).$$

By $\frac{H(A)}{A} \leq C\frac{1}{\phi(A)^{\frac{1}{n}}}$ in (6), letting $A = \phi^{-1}\left(\frac{1}{|\Omega_x \setminus B(z, c_j r)|}\right)$, we get

$$c_{j+1}r - c_j r \leq C|\Omega_x \setminus B(z, c_j r)|^{\frac{1}{n}}.$$

Hence $c_{j+1} - c_j r \leq C|\Omega_x \setminus B(z, c_j r)|^{\frac{1}{n}} \leq C2^{-\frac{j}{n}}|\Omega_x|^{\frac{1}{n}}$.

Now we prove that $\sup\left\{c_j\right\} > 1$. Otherwise, we have $c_j \leq 1$ for all j. By $x \in \Omega \setminus \overline{B(x, r)}$, then there exists $\delta > 0$ such that

$$B(x, \delta) \subset \Omega \setminus \overline{B(x, r)} \subset \Omega \setminus \overline{B(x, c_0 r)}.$$

By the connectivity of the $B(x, \delta)$, we have $B(x, \delta) \subset \Omega_x$. Then

$$B(x, \delta) \subset \Omega_x \setminus \overline{B(x, r)} \subset \Omega_x \setminus B(x, c_j r),$$

and

$$0 < |B(x, \delta)| \leq |\Omega_x \setminus \overline{B(x, r)}| \leq |\Omega_x \setminus B(x, c_j r)| = 2^{-j}|\Omega_x|.$$

Letting $j \to \infty$ we get a contradiction, hence $\sup\left\{c_j\right\} > 1$. So there exists c_j such that $c_j \geq \frac{1}{2}$. Let $j_0 = \inf\left\{j \geq 1 : c_j \leq \frac{1}{2}\right\}$, then

$$(\frac{1}{2} - c_0)r \leq (c_{j_0} - c_0)r = \sum_{j=0}^{j_0-1}(c_{j+1} - c_j)r \leq C\sum_{j=0}^{j_0-1} 2^{-\frac{j}{n}}|\Omega_x|^{\frac{1}{n}} \leq 2C|\Omega_x|^{\frac{1}{n}}.$$

So $\frac{r}{C}(\frac{1}{2}-2b_{x,z,r})\leq|\Omega_x|^{\frac{1}{n}}$. By the (12), we have

$$\frac{r}{C}(\frac{1}{2}-2b_{x,z,r})\leq|\Omega_x|^{\frac{1}{n}}\leq C2b_{x,z,r}r,\ C\geq 1.$$

Then $b_{x,z,r}\geq\frac{1}{4(C^2+1)}$, which implies $b>0$. □

Lemma 3.4. *Let $s\in(0,1)$ and $\phi\in\Delta_2$ be a Young function with $K_\phi<2^n$ in* (4)*, a bounded domain $\Omega\subset\mathbb{R}^n$ supports the (ϕ_n,ϕ)- Poincaré inequality* (3)*, then the Ω has the LLC(2) property, that is, there exists a constant $b\in(0,1)$ such that for all $z\in\mathbb{R}^n$ and $r>0$, any pair of point in $\Omega\setminus\overline{B(z,r)}$ can be joined in $\Omega\setminus\overline{B(z,br)}$.*

Proof. Fix x_0 so that $r_0:=\max(d(x,\partial\Omega):x\in\Omega)=d(x_0,\partial\Omega))$ and b_0 is the constant in Lemma 3.3. Then we spilt into three cases to prove it.

Case 1. For $z\notin B\left(x_0,\frac{r_0}{8\operatorname{diam}\Omega}r\right)$, we consider the radius r.

If $r>\frac{16(\operatorname{diam}\Omega)^2}{r_0}$, then $\forall y\in\overline{B\left(z,\frac{r_0}{16\operatorname{diam}\Omega}r\right)}$, we have

$$|y-x_0|\geq|z-x_0|-|z-y|\geq\frac{r_0}{16\operatorname{diam}\Omega}r>\operatorname{diam}\Omega.$$

By $\Omega\subset B(x_0,\operatorname{diam}\Omega)$, we get $\Omega\cap\overline{B\left(z,\frac{r_0}{16\operatorname{diam}\Omega}r\right)}=\varnothing$. Here, any pair of point in $\Omega\setminus\overline{B(z,r)}$ can be joined in $\Omega\setminus\overline{B(z,\frac{r_0}{16\operatorname{diam}\Omega}r)}=\Omega$.

If $r\leq\frac{16(\operatorname{diam}\Omega)^2}{r_0}$ and $d(z,\partial\Omega)>\frac{b_0r_0}{32\operatorname{diam}\Omega}r$. When $z\notin\Omega$, then any pair of point in $\Omega\setminus\overline{B(z,r)}$ can be joined in $\Omega\setminus\overline{B\left(z,\frac{b_0r_0}{32\operatorname{diam}\Omega}r\right)}=\Omega$. When $z\in\Omega$, then $B\left(z,\frac{b_0r_0}{64\operatorname{diam}\Omega}r\right)\subset B\left(z,\frac{b_0r_0}{32\operatorname{diam}\Omega}r\right)\subset\Omega$. Similar to the process of proving $b_{x,z,r}>0$ in Lemma 3.3, we know $\Omega\setminus\overline{B\left(z,\frac{b_0r_0}{64\operatorname{diam}\Omega}r\right)}$ is a connected set. Here, any pair of point in $\Omega\setminus\overline{B(z,r)}$ can be joined in $\Omega\setminus\overline{B\left(z,\frac{b_0r_0}{64\operatorname{diam}\Omega}r\right)}$.

If $r\leq\frac{16(\operatorname{diam}\Omega)^2}{r_0}$ and $d(z,\partial\Omega)\leq\frac{b_0r_0}{32\operatorname{diam}\Omega}r$. Let $y\in B\left(z,\frac{b_0r_0}{16\operatorname{diam}\Omega}r\right)\cap\Omega$. By $B\left(y,(1-\frac{b_0}{2})\frac{r_0}{8\operatorname{diam}\Omega}r\right)\subset B\left(z,\frac{r_0}{8\operatorname{diam}\Omega}r\right)\subset B(z,r)$, we know

$$\forall x\in\Omega\setminus\overline{B(z,r)},x,x_0\in\Omega\setminus\overline{B\left(y,(1-\frac{b_0}{2})\frac{r_0}{8\operatorname{diam}\Omega}r\right)}.$$

By Lemma 3.3, x,x_0 are in the same component of $\Omega\setminus\overline{B\left(y,b_0(1-\frac{b_0}{2})\frac{r_0}{8\operatorname{diam}\Omega}r\right)}$. By

$$\forall w\in B\left(z,\frac{b_0(1-b_0)r_0}{16\operatorname{diam}\Omega}r\right),$$

we have

$$|w-y|\leq|w-z|+|z-y|<\frac{b_0(1-b_0)r_0}{16\operatorname{diam}\Omega}r+\frac{b_0r_0}{16\operatorname{diam}\Omega}r=b_0\left(1-\frac{b_0}{2}\right)\frac{r_0}{8\operatorname{diam}\Omega}r.$$

Then

$$B\left(z,\frac{b_0(1-b_0)r_0}{16\operatorname{diam}\Omega}r\right)\subset B\left(y,b_0\left(1-\frac{b_0}{2}\right)\frac{r_0}{8\operatorname{diam}\Omega}r\right),$$

and $\Omega\setminus\overline{B\left(y,b_0\left(1-\frac{b_0}{2}\right)\frac{r_0}{8\operatorname{diam}\Omega}r\right)}\subset\Omega\setminus\overline{B\left(z,\frac{b_0(1-b_0)r_0}{16\operatorname{diam}\Omega}r\right)}$. Here, any pair of point in $\Omega\setminus\overline{B(z,r)}$ can be joined in $\Omega\setminus\overline{B\left(z,\frac{b_0(1-b_0)r_0}{16\operatorname{diam}\Omega}r\right)}$.

Case 2. If $z \in B\left(x_0, \frac{r_0}{8\operatorname{diam}\Omega} r\right)$, for any $x \in \Omega \setminus \overline{B(z,r)}$,

$$r - \frac{r_0}{8\operatorname{diam}\Omega} r \leq |x-z| - |x_0 - z| \leq |x - x_0| \leq \operatorname{diam}\Omega,$$

so

$$r \leq \frac{\operatorname{diam}\Omega}{1 - \frac{r_0}{8\operatorname{diam}\Omega}} \leq 2\operatorname{diam}\Omega.$$

Then

$$B\left(z, \frac{r_0}{8\operatorname{diam}\Omega} r\right) \subset B\left(x_0, \frac{r_0}{4\operatorname{diam}\Omega} r\right) \subset B\left(x_0, \frac{r_0}{2}\right) \subset B(x_0, r_0) \subset \Omega$$

Similar to the process of proving $b_{x,z,r} > 0$ in Lemma 3.3, we have $\Omega \setminus \overline{B\left(z, \frac{r_0}{8\operatorname{diam}\Omega} r\right)}$ is a connected set. And by

$$\Omega \setminus \overline{B(z,r)} \subset \Omega \setminus \overline{B\left(z, \frac{r_0}{8\operatorname{diam}\Omega} r\right)},$$

we know any pair of point in $\Omega \setminus \overline{B(z,r)}$ can be joined in $\Omega \setminus \overline{B\left(z, \frac{r_0}{8\operatorname{diam}\Omega} r\right)}$.

Combining above cases, we get the desired result with

$$b = \min\left\{\frac{r_0}{16\operatorname{diam}\Omega}, \frac{b_0 r_0}{64\operatorname{diam}\Omega}, \frac{b_0(1-b_0)r_0}{16\operatorname{diam}\Omega}\right\}.$$

□

Proof of Theorem 1.2(ii). Let $\Omega \subset \mathbb{R}^n$ be a simply connected planar domain, or a bounded domain that is quasiconformally equivalent to some uniform domain when $n \leq 3$. Assume Ω supports the $(\phi_{\frac{n}{s}}, \phi)$-Poincaré inequality.

By [7, 8], Ω has a separation property with $x_0 \in \Omega$ and some constant $C_0 \geq 1$, that is $\forall x \in \Omega$, $\exists$ a curve $\gamma : [0,1] \to \Omega$, with $\gamma(0) = x, \gamma(1) = x_0$, and $\forall t \in [0,1]$, either $\gamma([0,1]) \subset \overline{B} := \overline{B(\gamma(t), C_0 d(\gamma(t), \Omega^{\complement}))}$, or $\forall y \in \gamma([0,1]) \setminus \overline{B}$ belongs to the different component of $\Omega \setminus \overline{B}$. For any $x \in \Omega$, let γ be a curve as above. By the arguments in [36], It suffices to prove there exists a constant $C > 0$ so that

$$d(\gamma(t), \Omega^{\complement}) \geq C \operatorname{diam}\ \gamma([0,t]),\ \forall t \in [0,1]. \tag{14}$$

Indeed, (14) could modify γ to get a John curve for x.

By Lemma 3.4, Ω has the LLC(2) property. Let $a = 2 + \frac{C_0}{b}$, where b is the constant in Lemma 3.4.

For $t \in [0,1]$. (1) If $d(\gamma(t), \Omega^{\complement}) \geq \frac{d(x_0, \Omega^{\complement})}{a}$, then

$$\gamma([0,t]) \subset \Omega \subset B\left(\gamma(t), \frac{a d(\gamma(t), \Omega^{\complement})}{d(x_0, \Omega^{\complement})} \operatorname{diam}\Omega\right).$$

So

$$\operatorname{diam}\ \gamma([0,t]) \leq \frac{2a d(\gamma(t), \Omega^{\complement})}{d(x_0, \Omega^{\complement})} \operatorname{diam}\Omega.$$

and

$$d(\gamma(t), \Omega^{\complement}) \geq \frac{d(x_0, \Omega^{\complement})}{2a \operatorname{diam}\Omega} \operatorname{diam}\ \gamma([0,t]).$$

(2) If $d(\gamma(t), \Omega^{\complement}) < \frac{d(x_0, \Omega^{\complement})}{a}$, we prove that

$$\gamma([0,t]) \subset \overline{B\left(\gamma(t), (a-1) d(\gamma(t), \Omega^{\complement})\right)}.$$

Otherwise, there exists $y \in \gamma([0,t]) \setminus \overline{B\left(\gamma(t),(a-1)d(\gamma(t),\Omega^{\complement})\right)}$. By

$$|x_0-\gamma(t)| \geq d(x_0,\Omega^{\complement}) - d(\gamma(t),\Omega^{\complement}) > (a-1)d(\gamma(t),\Omega^{\complement}),$$

we know $x_0, y \in \Omega \setminus \overline{B\left(\gamma(t),(a-1)d(\gamma(t),\Omega^{\complement}\right)}$, by Lemma 3.4, x_0 and y are contained in the same complement of $\Omega \setminus \overline{B\left(\gamma(t),b(a-1)d(\gamma(t),\Omega^{\complement}\right)}$. Since $b(a-1) \geq C_0$, then x_0 and y are contained in the same complement of $\Omega \setminus \overline{B\left(\gamma(t),C_0 d(\gamma(t),\Omega^{\complement}\right)}$, which is in contradiction with the separation property. Hence

$$\gamma([0,t]) \subset \overline{B\left(\gamma(t),(a-1)d(\gamma(t),\Omega^{\complement})\right)},$$

then

$$\operatorname{diam}\ \gamma([0,t]) \leq 2(a-1)d(\gamma(t),\Omega^{\complement}).$$

So

$$d(\gamma(t),\Omega^{\complement}) \geq \frac{1}{2(a-1)} \operatorname{diam}\ \gamma([0,t]).$$

Let $C = \min\left\{\frac{d(x_0,\Omega^{\complement})}{2a \operatorname{diam}\Omega}, \frac{1}{2(a-1)}\right\}$, then (14) holds. The proof is completed.

□

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